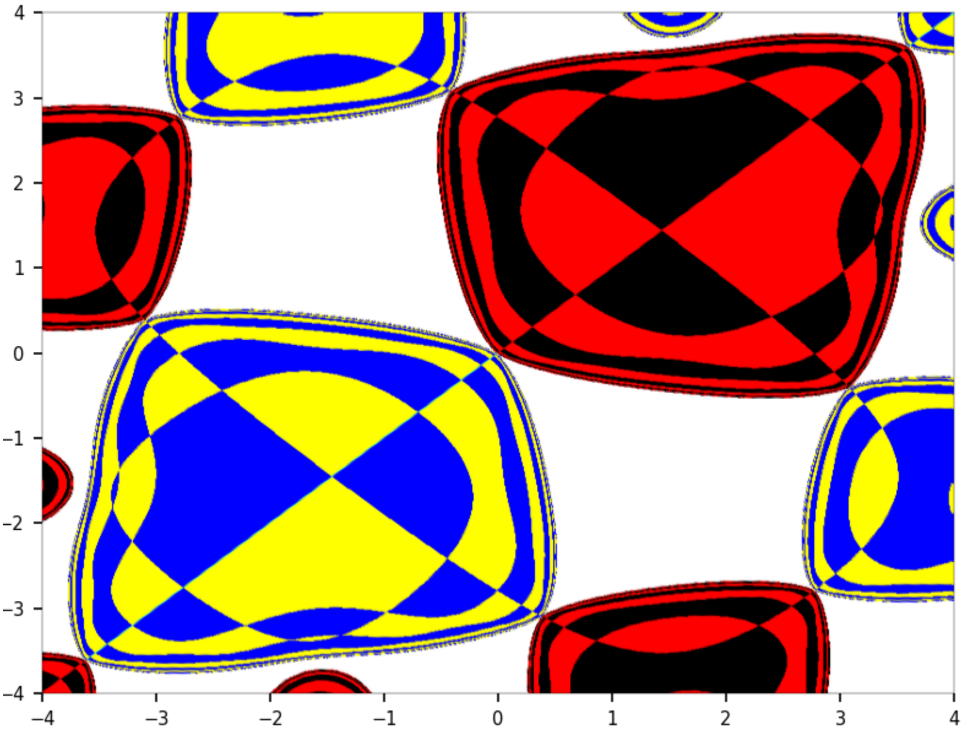
Gentle Introduction To Chaotic Dynamical Systems



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Preface

In about 100 pages, the book covers all important topics about discrete chaotic dynamical systems and related time series and stochastic processes, ranging from introductory to advanced, in one and two dimensions. State-of-the art methods and new results are presented in simple English. Yet, some mathematical proofs appear for the first time in this book: for instance, about the full autocorrelation function of the logistic map, the absence of cross-correlation between digit sequences in a family of irrational numbers, and a very fast algorithm to compute the digits of quadratic irrationals. These are not just new important if not seminal theoretical developments: it leads to better algorithms in random number generation (PRNG), benefiting applications such as data synthetization, security, or heavy simulations. In particular, you will find an implementation of a very fast, simple PRNG based on millions of digits of millions of quadratic irrationals, producing strongly random sequences superior in many respects to those available on the market.

Without using measure theory, the invariant distributions of many systems are discussed in details, with numerous closed-form expressions for classic and new maps, including the logistic, square root logistic, nested radicals, generalized continued fractions (the Gauss map), the ten-fold and dyadic maps, and more. The concept of bad seed, rarely discussed in the literature, is explored in details. It leads to singular fractal distributions with no probability density function, and sets similar to the Cantor set. Rather than avoiding these monsters, you will be able to leverage them as competitive tools for modeling purposes, since many evolutionary processes in economy, fintech, physics, population growth and so on, do not always behave nicely.

A summary table of numeration systems serves as a useful, quick reference on the subject. Equivalence between different maps is also discussed. In a nutshell, this book is dedicated to the study of two numbers: zero and one, with a wealth of applications and results attached to them, as well as some of the toughest mathematical conjectures. It will appeal in particular to busy practitioners in fintech, security, defense, operations research, engineering, computer science, machine learning, AI, as well as consultants and professional mathematicians. For students complaining about how hard this topic is, and deterred by the amount of advanced mathematics, this book will help them get jump-started. While the mathematical level remains high in some sections, it is explained as simply as possible, focusing on what is needed for the applications.

Numerous illustrations including beautiful representations of these systems (generative art), a lot of well documented Python code, and nearly 20 off-the-beaten-path exercises complementing the theory, will help you navigate through this fascinating field. You will see how even the most basic systems offer such an incredible variety of configurations depending on a few parameters, allowing you to model a very large array of phenomena. A surprising application – a synthetic stock exchange and lottery – is described in detail in chapter 5, including full business model and legal aspects.

Finally, chapter 1 also covers time-continuous processes including unusual clustered, reflective, constrained, and integrated Brownian-like processes, random walks and time series, with little math and jargon-free. In the end, my goal is to get you to you use these systems fluently, and see them as gentle, controllable chaos. In short, what real life should be! Quantifying the amount of chaos is also one of the topics discussed in the book.

About the author

Vincent Granville is a pioneering data scientist and machine learning expert, co-founder of Data Science Central (acquired by TechTarget), founder of MLTechniques.com, former VC-funded executive, author and patent owner.

Vincent's past corporate experience includes Visa, Wells Fargo, eBay, NBC, Microsoft, and CNET. Vincent is also a former post-doc at Cambridge University, and the National Institute of Statistical Sciences (NISS). He published in *Journal of Number Theory*, *Journal of the Royal Statistical Society* (Series B), and *IEEE Transactions on Pattern Analysis and Machine Intelligence*. He is also the author of multiple books, available here. He lives in Washington state, and enjoys doing research on stochastic processes, dynamical systems, experimental math and probabilistic number theory.

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