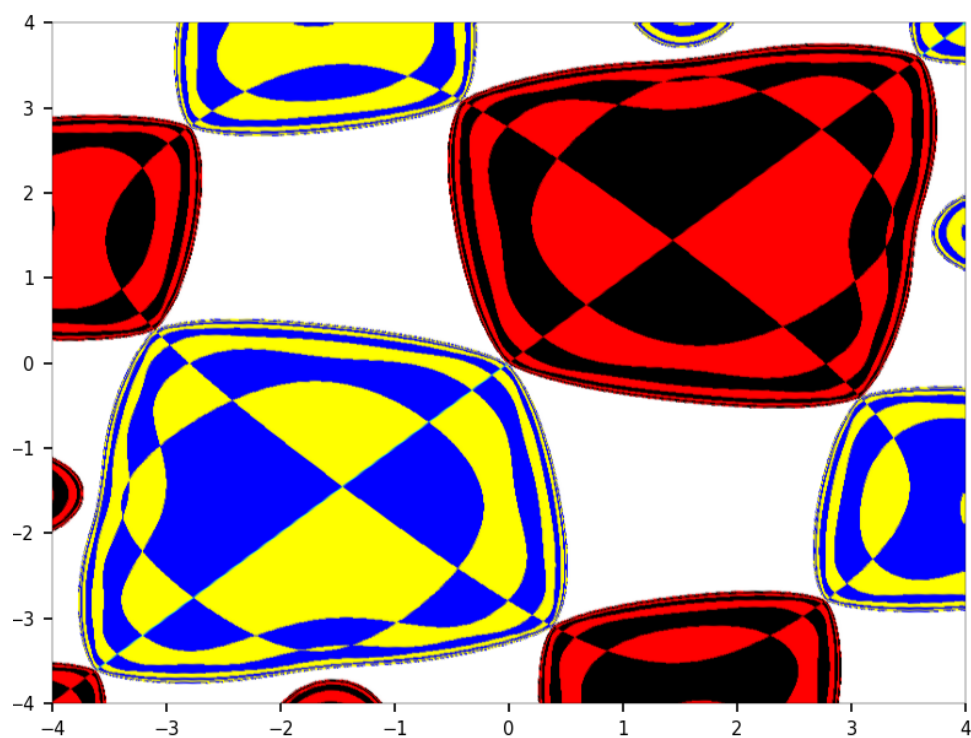

Gentle Introduction To Chaotic Dynamical Systems



Preface

In less than 100 pages, the book covers all important topics about discrete chaotic dynamical systems and related time series and stochastic processes, ranging from introductory to advanced, in one and two dimensions. State-of-the-art methods and new results are presented in simple English. Yet, some mathematical proofs appear for the first time in this book: for instance, about the full autocorrelation function of the logistic map, the absence of cross-correlation between digit sequences in a family of irrational numbers, and a very fast algorithm to compute the digits of quadratic irrationals. These are not just new important if not seminal theoretical developments: it leads to better algorithms in random number generation (PRNG), benefiting applications such as data synthetization, security, or heavy simulations. In particular, you will find an implementation of a very fast, simple PRNG based on millions of digits of millions of quadratic irrationals, producing strongly random sequences superior in many respects to those available on the market.

Without using measure theory, the invariant distributions of many systems are discussed in details, with numerous closed-form expressions for classic and new maps, including the logistic, square root logistic, nested radicals, generalized continued fractions (the Gauss map), the ten-fold and dyadic maps, and more. The concept of bad seed, rarely discussed in the literature, is explored in details. It leads to singular fractal distributions with no probability density function, and sets similar to the Cantor set. Rather than avoiding these monsters, you will be able to leverage them as competitive tools for modeling purposes, since many evolutionary processes in economy, fintech, physics, population growth and so on, do not always behave nicely.

A summary table of numeration systems serves as a useful, quick reference on the subject. Equivalence between different maps is also discussed. In a nutshell, this book is dedicated to the study of two numbers: zero and one, with a wealth of applications and results attached to them, as well as some of the toughest mathematical conjectures. It will appeal in particular to busy practitioners in fintech, security, defense, operations research, engineering, computer science, machine learning, AI, as well as consultants and professional mathematicians. For students complaining about how hard this topic is, and deterred by the amount of advanced mathematics, this book will help them get jump-started. While the mathematical level remains high in some sections, it is explained as simply as possible, focusing on what is needed for the applications.

Numerous illustrations including beautiful representations of these systems (generative art), a lot of well documented Python code, and nearly 20 off-the-beaten-path exercises complementing the theory, will help you navigate through this fascinating field. You will see how even the most basic systems offer such an incredible variety of configurations depending on a few parameters, allowing you to model a very large array of phenomena. A surprising application – a synthetic stock exchange and lottery – is described in detail in chapter 5, including full business model and legal aspects.

Finally, chapter 1 also covers time-continuous processes including unusual clustered, reflective, constrained, and integrated Brownian-like processes, random walks and time series, with little math and jargon-free. In the end, my goal is to get you to use these systems fluently, and see them as gentle, controllable chaos. In short, what real life should be! Quantifying the amount of chaos is also one of the topics discussed in the book.

About the author

Vincent Granville is a pioneering data scientist and machine learning expert, co-founder of Data Science Central (acquired by TechTarget), founder of [MLTechniques.com](https://mltechniques.com), former VC-funded executive, author and patent owner.



Vincent's past corporate experience includes Visa, Wells Fargo, eBay, NBC, Microsoft, and CNET. Vincent is also a former post-doc at Cambridge University, and the National Institute of Statistical Sciences (NISS). He published in *Journal of Number Theory*, *Journal of the Royal Statistical Society* (Series B), and *IEEE Transactions on Pattern Analysis and Machine Intelligence*. He is also the author of multiple books, available [here](#). He lives in Washington state, and enjoys doing research on stochastic processes, dynamical systems, experimental math and probabilistic number theory.

List of Figures

1.1	Brownian motion (green), integrated (orange) and moving average (red)	9
1.2	Integrated Brownian (top left), Brownian (top right) and nowhere continuous (bottom)	10
1.3	Reflected random walk with $a = b = \frac{1}{2}$	12
1.4	Invariant measure (density function) of reflected random walk with $a = b = \frac{1}{2}$	12
1.5	Typical path S_n with $0 \leq n \leq 50,000$ for four types of random walks	15
1.6	$\delta_n = 1 - \text{Var}[S_{n+1}] + \text{Var}[S_n]$ for four types of random walks, with $0 \leq n \leq 5000$	16
1.7	Same as Figure 1.6, using a more aesthetic but less meaningful chart type	17
1.8	Clustered Brownian process	20
2.1	First 20,000 coefficients (their logarithm) in the continued fraction of π	27
2.2	Base $(-5, 4)$ leads to better randomness [blue fit] than $(3, -2)$ [orange fit]	29
2.3	Convergence of iterated functions towards the invariant density in red	31
2.4	Series A, C, D (left) with corresponding chaos measurements (right)	34
2.5	Series E (left) with corresponding chaos measurements (right)	35
2.6	Non-periodic orbit with fractal dimension and basin of repulsion	35
2.7	Basins of attraction of the 2D sine map ($\lambda = 2, \rho = -0.25$)	37
3.1	Logistic map: raw (Y_k, Y_{k+20}) on the left, versus rescaled on the right	43
3.2	Error due to using an approximation for the invariant distribution	47
3.3	$F_Z(z)$ vs $\log_2 z$ on the left; approximation error on the right	51
3.4	Invariant distribution when $a = 0.4, b = 0.6, p = 0.8$	54
3.5	Basins of attraction of the 2D sine map ($\lambda = 0.5, \rho = -1$)	55
3.6	Six spectacular orbits of the 2D sine map	56
3.7	Chaotic convergence path of the η series in the complex plane	58
4.1	Orbit of $L(z, \chi)$ at $\sigma = \frac{1}{2}$, with $0 < t < 200$ and $\chi = \chi_4$ (left) versus pseudo-random χ (right) . .	64
4.2	$L_3^*(n)$ test statistic for four sequences: Python[200] and SQRT[90,91] fail	66
4.3	$ L_3(n) $ test statistic for four sequences: Python[200] and SQRT[90,91] fail	67
4.4	Correlations are computed on sequences consisting of 300 binary digits	78
5.1	Smooth neutral ROI table with moderate risk / reward	91
5.2	Smooth neutral ROI table with high risk / high reward	91
5.3	Smooth neutral ROI table with low risk / low reward	92
5.4	Bracketed neutral ROI table with about 20 brackets	92
5.5	Cumulative profits (left) and distribution of daily profits (right), with a 0.025% fee	94
5.6	Cumulative profits (left) and distribution of daily profits (right), with no fee	94

Contents

List of Figures	3
1 Random Walks, Brownian Motions, and Related Stochastic Processes	7
1.1 From random walks to Brownian motions	7
1.2 General Properties	8
1.3 Integration, differentiation, moving averages	9
1.4 Reflected random walks	11
1.4.1 Exercises	13
1.4.2 Python code	13
1.5 Constrained random walks	14
1.5.1 Three fundamental properties of pure random walks	14
1.5.2 Random walks with more entropy than pure random signal	15
1.5.2.1 Applications	15
1.5.2.2 Algorithm to generate quasi-random sequences	16
1.5.2.3 Variance of the modified random walk	17
1.5.3 Random walks with less entropy than pure random signal	17
1.5.4 Python code: computing probabilities and variances attached to S_n	18
1.5.5 Python code: path simulations	19
1.6 Two-dimensional Brownian motions	20
2 Introduction to Discrete Chaotic Dynamical Systems	22
2.1 Definitions, properties, and examples	22
2.1.1 Properties of discrete chaotic dynamical systems	23
2.1.2 Shortlist of dynamical systems with known solution	24
2.1.2.1 Digits and numeration system attached to a dynamical system	24
2.1.2.2 Examples with exact solution in closed form	25
2.1.2.3 Gauss map: expectation, variance and autocorrelation	26
2.1.3 Exercises: Gauss map and continued fractions	26
2.1.4 Random numbers based on a bivariate numeration system	28
2.1.4.1 Properties of the bivariate numeration system	28
2.1.4.2 Increasing point spread and randomness	28
2.1.4.3 Python implementation of distance to randomness	30
2.2 Iterative method to find the invariant distribution	31
2.2.1 Results and discussion about convergence	31
2.2.2 Python code to numerically solve the functional equation	32
2.2.3 Curious, very accurate approximation of π	33
2.3 Measuring the amount of chaos	33
2.3.1 Hurst exponent and related metrics	33
2.3.1.1 Detrending moving averages and spreadsheet illustration	34
2.3.2 Lyapunov exponent and related metrics	35
2.4 The pillow map: a fascinating bivariate system	36
2.4.1 The one-dimensional version: Arnold's tongues	36
2.4.2 Two-dimensional case	36
2.4.3 Python code to generate the basins of attraction	37
3 Special Probability Distributions Associated to Chaos	40
3.1 Equivalence between logistic and dyadic map	40
3.1.1 Homomorphism between two dynamical systems	41
3.1.2 Autocorrelation function: exact computation	42
3.1.3 Random numbers based on digits of irrational numbers	44
3.1.4 Inverting a dynamical system: cryptography application	45
3.1.4.1 Systems harder to reverse-engineer	45

3.2	Probabilistic properties of numeration systems	45
3.2.1	Square root logistic map	45
3.2.2	Nested radicals	46
3.2.3	Summary table for common systems	47
3.3	Special probability distributions	48
3.3.1	A different version of the binary numeration system	48
3.3.1.1	Geometric invariant distribution	49
3.3.1.2	Gaussian-like invariant distribution	49
3.3.1.3	Connection to the binary numeration system	49
3.3.2	Singular distributions with no density function	50
3.3.2.1	Bad seeds in the stochastic dyadic map	50
3.3.2.2	Nested radicals and fractal invariant distributions	50
3.3.2.3	Generalizations and characteristic function	52
3.3.3	Exercises	52
3.4	Cool synthetic images and generated art	55
3.4.1	Revisiting the 2D sine map	55
3.4.2	The Riemann zeta function	57
4	Random Numbers Based on Quadratic Irrationals	61
4.1	Introduction	61
4.2	Pseudo-random numbers	62
4.2.1	Strong pseudo-random numbers	62
4.2.1.1	New test of randomness for PRNGs	63
4.2.1.2	Theoretical background: the law of the iterated logarithm	63
4.2.1.3	Connection to the Generalized Riemann Hypothesis	63
4.2.2	Testing well-known sequences	65
4.2.2.1	Reverse-engineering a pseudo-random sequence	66
4.2.2.2	Illustrations	66
4.3	Python code	67
4.3.1	Fixes to the faulty random function in Python	68
4.3.2	Prime test implementation to detect subtle flaws in PRNG's	68
4.3.3	Special formula to compute 10 million digits of $\sqrt{2}$	71
4.4	Military-grade PRNG Based on Quadratic Irrationals	74
4.4.1	Fast algorithm rooted in advanced analytic number theory	74
4.4.2	Fast PRNG: explanations	75
4.4.3	Python code	76
4.4.4	Computing a digit without generating the previous ones	77
4.4.5	Security and comparison with other PRNGs	77
4.4.5.1	Important comments	78
5	Application: Synthetic Stock Exchange	79
5.1	Introduction	79
5.2	Winning, sequences and customized ROI tables	80
5.3	Implementation details with example	81
5.3.1	Seeds, public and private algorithms	81
5.3.2	Winning numbers and public data	81
5.3.2.1	Using the public algorithm to win	82
5.3.2.2	Python code	82
5.3.3	Optimizing computations	83
5.3.4	Seeds with billions of digits and enhanced system	85
5.3.4.1	Towards maximum security	85
5.3.4.2	Real example	85
5.3.5	Collision risks	87
5.3.6	Exercises	87
5.4	Customized ROI tables and business model	90
5.4.1	Bracketed and parametric ROI tables: examples	90
5.4.1.1	Types of ROI tables, with examples	91
5.4.1.2	The 32-bit system	92
5.4.1.3	Free simulator, time to positive return	93
5.4.2	Daily payouts for the operator: Brownian motion	93
5.4.2.1	Example with truncated geometric distribution	94
5.4.2.2	Python code and algorithm to find optimum transaction fee	95
5.4.3	Legal engineering	97

Bibliography

- [1] David Bailey, Jonathan Borwein, and Neil Calkin. *Experimental Mathematics in Action*. A K Peters, 2007. [49](#)
- [2] David Bailey and Richard Crandall. Random generators and normal numbers. *Experimental Mathematics*, 11, 2002. Project Euclid [\[Link\]](#). [78](#)
- [3] Rabi Bhattacharya and Edward Waymire. *Random Walk, Brownian Motion, and Martingales*. Springer, 2021. [14](#)
- [4] Luis Báez-Duarte et al. Étude de l'autocorrélation multiplicative de la fonction ‘partie fractionnaire’. *The Ramanujan Journal*, 78:215–240, 2005. [\[Link\]](#). [26](#)
- [5] Keith Conrad. *L-functions and the Riemann Hypothesis*. 2018. 2018 CTNT Summer School [\[Link\]](#). [65](#)
- [6] D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Springer, second edition, 2002. Volume 1 – Elementary Theory and Methods. [20](#)
- [7] D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Springer, second edition, 2014. Volume 2 – General Theory and Structure. [20](#)
- [8] Harold G. Diamond and Wen-Bin Zhang. *Beurling Generalized Numbers*. American Mathematical Society, 2016. Mathematical Surveys and Monographs, Volume 213 [\[Link\]](#). [65](#)
- [9] Benjamin Epstein. Some applications of the Mellin transform in statistics. *Annals of Mathematical Statistics*, 19:370–370, 1948. [\[Link\]](#). [53](#)
- [10] P. A. Van Der Geest. The binomial distribution with dependent Bernoulli trials. *Journal of Statistical Computation and Simulation*, pages 141–154, 2004. [\[Link\]](#). [15](#)
- [11] B.V. Gnedenko and A. N. Kolmogorov. *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, 1954. [21](#)
- [12] Manuel González-Navarrete and Rodrigo Lambert. Non-markovian random walks with memory lapses. *Preprint*, pages 1–14, 2018. arXiv [\[Link\]](#). [14](#)
- [13] Vincent Granville. *Stochastic Processes and Simulations: A Machine Learning Perspective*. MLTechniques.com, 2022. [\[Link\]](#). [21](#), [36](#)
- [14] Vincent Granville. *Synthetic Data and Generative AI*. MLTechniques.com, 2022. [\[Link\]](#). [8](#), [10](#), [15](#), [16](#), [29](#), [57](#), [65](#)
- [15] Charles M. Grinstead and Laurie Snell. *Introduction to Probability*. American Mathematical Society, second edition, 1997. [\[Link\]](#). [93](#)
- [16] Nasr-Eddine Hamri and Yamina Soula. Basins and critical curves generated by a family of two-dimensional sine maps. *Electronic J. of Theoretical Physics*, 24:139–150, 2010. [\[Link\]](#). [36](#)
- [17] Adam J. Harper. Moments of random multiplicative functions, II: High moments. *Algebra and Number Theory*, 13(10):2277–2321, 2019. [\[Link\]](#). [62](#)
- [18] Adam J. Harper. Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos. *Forum of Mathematics, Pi*, 8:1–95, 2020. [\[Link\]](#). [62](#), [63](#)
- [19] Adam J. Harper. Almost sure large fluctuations of random multiplicative functions. *Preprint*, pages 1–38, 2021. arXiv [\[Link\]](#). [64](#)
- [20] T. W. Hilberdink and M. L. Lapidus. Beurling Zeta functions, generalised primes, and fractal membranes. *Preprint*, pages 1–31, 2004. arXiv [\[Link\]](#). [65](#)
- [21] Dixon J. Jones. A chronology of continued square roots and other continued compositions, through the year 2016. *Preprint*, pages 1–98, 2017. arXiv:1707.06139 [\[Link\]](#). [47](#)
- [22] R. Kannan, A. K. Lenstra, and L. Lovász. Polynomial factorization and nonrandomness of bits of algebraic and some transcendental numbers. *Mathematics of Computation*, 50:335–250, 1988. [\[Link\]](#). [85](#)

- [23] Yuk-Kam Lau, Gerald Tenenbaum, and Jie Wu. On mean values of random multiplicative functions. *Proceedings of the American Mathematical Society*, 142(2):409–420, 2013. [\[Link\]](#). 62, 63
- [24] D. Lenz. Spectral theory of dynamical systems as diffraction theory of sampling functions. *Monatshefte für Mathematik*, 192:625–649, 2020. [\[Link\]](#). 36
- [25] Corina Macovei et al. The autocorrelation function of the logistic map chaotic signal in relation with the statistical independence issue. *IEEE 13th International Conference on Communications*, pages 25–30, 2020. [\[Link\]](#). 43
- [26] Peter Mörters and Yuval Peres. *Brownian Motion*. Cambridge University Press, 2010. Cambridge Series in Statistical and Probabilistic Mathematics, Volume 30 [\[Link\]](#). 14, 20
- [27] John Noonan and Doron Zeilberger. The Goulden-Jackson cluster method: extensions, applications and implementations. *Journal of Difference Equations and Applications*, 5:355–377, 1999. [\[Link\]](#). 87
- [28] Ying-Hui Shao et al. Comparing the performance of FA, DFA and DMA using different synthetic long-range correlated time series. *Preprint*, pages 1–9, 2018. arXiv:1208.4158 [\[Link\]](#). 34
- [29] Grzegorz Sikora. Statistical test for fractional Brownian motion based on detrending moving average algorithm. *Chaos, Solitons & Fractals*, 116:54–62, 2018. [\[Link\]](#). 34
- [30] E.C. Titchmarsh and D.R. Heath-Brown. *The Theory of the Riemann Zeta-Function*. Oxford Science Publications, second edition, 1987. 65
- [31] Maistrenko V et al. Chaotic synchronization and antisynchronization in coupled sine maps. *International Journal of Bifurcation and Chaos*, 15:2161–2177, 2005. [\[Link\]](#). 36
- [32] Luyao Wang and Hai Cheng. Pseudo-random number generator based on logistic chaotic system. *Entropy*, 21, 2019. [\[Link\]](#). 78
- [33] David Williams. *Probability with Martingales*. Cambridge University Press, 1991. 93
- [34] Lan Wua, Yongcheng Qi, and Jingping Yang. Asymptotics for dependent Bernoulli random variables. *Statistics and Probability Letters*, pages 455–463, 2012. [\[Link\]](#). 14
- [35] Gordan Žitkovic. *Introduction to Stochastic Processes*. University of Texas, 2019. Lecture notes [\[Link\]](#). 93

Index

- algebraic number, 66
- analytic function, 64
- arcsine law, 8
- Arnold's tongues, 36
- attactor, 37
- attractor distribution, 12, 21, 23, 52
- autocorrelation function, 36, 66
- autoregressive models, 10

- Bailey–Borwein–Plouffe formulas, 66
- base (numeration systems)
 - bivariate, 28
 - golden ratio base, 34
 - irrational, 28
- basin of attraction, 36, 55
- basin of repulsion, 36, 56
- Berry-Esseen inequality, 62
- beta distribution, 32, 42
- Beurling primes, 65
- bifurcation, 33, 37
- Brown noise, 10
- Brownian motion, 7, 14, 20, 33, 93, 95
 - fractional, 34
 - Lévy flight, 21

- Cantor set, 50, 51
- Cauchy distribution, 21
- Cauchy-Riemann equations, 64
- central limit theorem, 21
- characteristic function, 52
- characteristic polynomial, 10
- Chebyshev's bias (prime numbers), 64
- circle map, 36
- code (dynamical systems), 24
- Collatz conjecture, 72
- completeness (numeration systems), 41, 46
- complex random variable, 62
- computational complexity, 75
- continued fractions, 23
 - generalized, 25
- convergence
 - conditional, 64

- Dedekind zeta function, 65
- detrending moving average, 34
- Diehard tests of randomness, 63
- differentiated process, 9
- digit, 24, 40, 47, 79
- Dirichlet character, 64, 65
- Dirichlet eta function, 58
- Dirichlet functional equation, 64
- Dirichlet series, 61
- Dirichlet's theorem, 64
- Dirichlet- L function, 64
- distribution
 - beta, 32, 42
 - Cauchy, 21
 - Fréchet, 21
 - Gauss-Kuzmin, 32
 - Lévy, 21
 - Rademacher, 62
 - Weibull, 21
- dyadic map, 34, 46, 50, 52, 65, 79, 83
- dynamical systems, 65
 - bivariate, 28
 - discrete, 22
 - dyadic map, 65
 - ergodicity, 65
 - Gauss map, 23
 - logistic map, 65
 - shift map, 65

- Egyptian fractions, 41
- empirical distribution, 23, 29, 54, 62
 - multivariate, 62
- entropy, 36
 - approximate entropy, 35
- equidistribution modulo 1, 67
- ergodicity, 8, 23, 42, 65
- Euler product, 61
- evolution parameter, 22
- exception set, 50
- extreme value theory, 8, 21

- Feigenbaum's constant, 33
- fixed point algorithm, 12
- fixed-point algorithm, 31, 37
- fractal dimension, 10, 36
- fractional Brownian motion, 34
- fractional part function, 67
- Fréchet distribution, 21
- functional equation, 12, 46, 48, 50

- Gamma function, 21
- Gauss map, 23, 24, 45, 52
- Gauss-Kuzmin distribution, 24, 32
- Gaussian primes, 65
- geometric distribution, 49
- gmpy2 (Python library), 84
- golden ratio base, 34
- greedy algorithm, 41, 87

- Hartman–Wintner theorem, 14
- hash table, 75
- Hoeffding inequality, 17
- homomorphism, 41, 49
- Hurst exponent, 10, 35
- Hurwitz function, 26
- Hurwitz map, 26
- integer square root, 84
- integrated process, 9
- interarrival times, 20
- invariant distribution, 23, 41, 47, 49, 52
 - joint distribution, 43
 - see invariant measure, 23
- invariant measure, 12
- irrational base, 24
- iterated logarithm, 14, 62, 63
- Khinchin’s constant, 27
- Kolmogorov–Smirnov statistic, 29
- Kolmogorov–Smirnov test, 62
- law of the iterated logarithm, 14, 62, 63, 88
- Lebesgue measure, 23
- logistic map, 22, 32, 33, 65
- Lyapunov exponent, 35
- Lévy distribution, 21
- Lévy flight, 21
- map, 22
 - Arnold’s tongues, 36
 - circle, 36
 - dyadic, 24, 46
 - Gauss, 23, 24
 - logistic, 22, 32, 33
 - sine, 36
 - ten-fold, 24
- Markov chain
 - MCMC, 61
- Markov property, 8
- martingale
 - Wald martingale, 93
- Mellin transform, 53
- memoryless property, 8
- Mersenne twister, 16, 65, 68, 95
- Monte Carlo simulations, 61
- moving average process, 9
- mpmath (Python library), 84
- multiplicative function
 - completely multiplicative, 61, 64
 - Rademacher, 62
- nested radicals, 32, 45, 46, 52
- normal number, 26, 62
 - strongly normal, 63, 83, 89
- numeration system, 24
- numerical stability, 27, 44
- orbit (dynamical systems), 36, 56
- Perron–Frobenius theorem, 23
- pink noise, 10
- Poisson point process, 20
- prime test (of randomness), 15, 63, 74, 88
- probability generating function, 15
- probability integral transform, 45
- pseudo-random numbers, 15, 16, 43, 44, 62
 - combined generators, 78
 - congruential PRGN, 29, 68
 - Diehard tests, 63, 75
 - Mersenne twister, 16, 68
 - prime test, 15, 63
 - strongly random, 63, 66
 - TestU01, 63
- Pólya’s theorem, 8
- quadratic irrational, 65, 68, 74
- R-squared, 95
- Rademacher distribution, 53, 62
- Rademacher function, 62
 - random, 64
- random multiplicative function, 62
 - Rademacher, 64
- random variable
 - complex, 62
- random walk, 7, 14, 33
 - first hitting time, 15, 18, 93
 - symmetric, 93
 - zero crossing, 14
- redundancy (numeration systems), 41
- reflected random walk, 11
- Riemann Hypothesis
 - Generalized, 63
- Riemann zeta function, 13, 26, 36, 61, 64
- scale-invariant, 8
- seed (dynamical systems), 22, 47, 48, 75, 81
 - bad seed, 23, 42, 50, 52
 - bivariate, 28
 - good seed, 23, 42, 44, 45, 79, 83
- shift map, 65
- sine map, 36
- spectral theory, 36
- square-free integer, 63, 75
- stable distribution, 21
- state space, 22
- stationarity, 8, 33
- stochastic integral equation, 12, 31
- synthetic data, 61, 74
- ten-fold map, 24, 35
- transcendental number, 66
- transfer operator, 23
- Weibull distribution, 21
- white noise, 7, 10
- Wiener process, 7
- XOR operator, 68