Introduction to optimization:

Throughout this course (and 173B) our goal will often be to solve problems of the form:

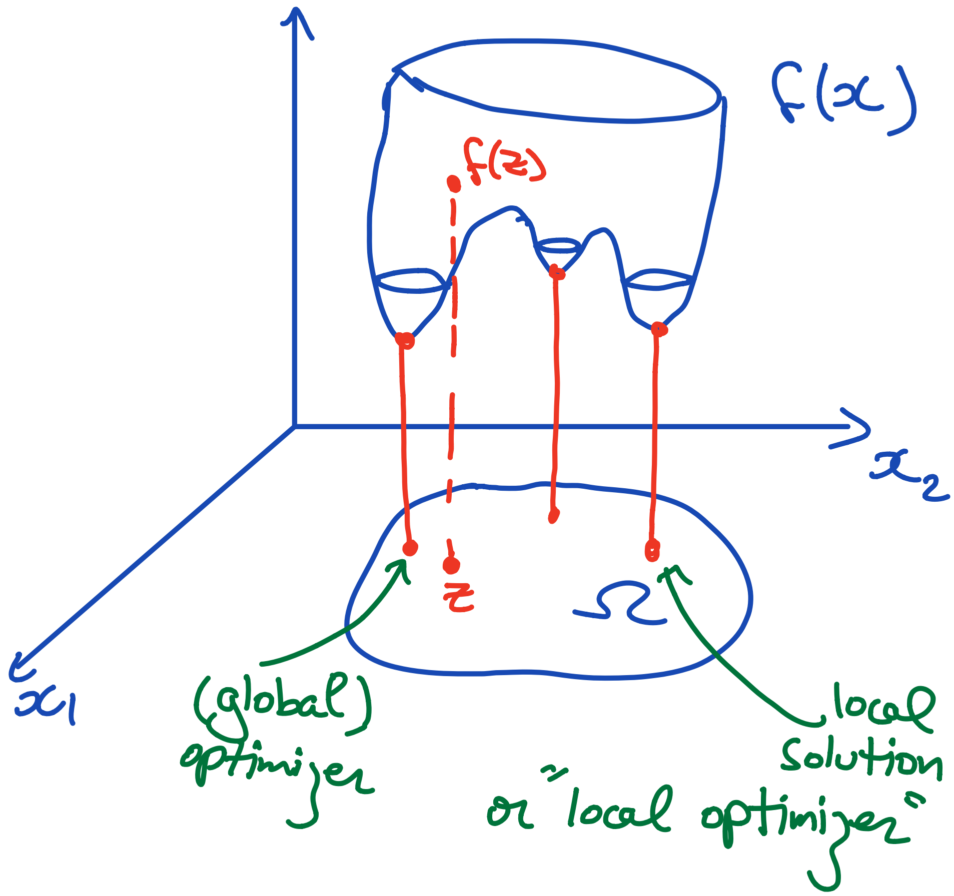
objective function

variable in $\mathbb{R}^n$ (say)

$$\boxed{\min_{x \in \Omega} f(x)} \qquad (1)$$

constraint
($\Omega \subset \mathbb{R}^n$, possible to have $\Omega = \mathbb{R}^n$)

When Ω is $\mathbb{R}^n$, we effectively have no constraint on x, and ~~we~~ call (1) an **unconstrained** opt. problem



Constrained Vs Unconstrained opt.:

When Ω is the whole space, $\Omega = \mathbb{R}^n$, we say the opt. problem is unconstrained

Example:

(1) $$\min_{x_1, x_2} \; 2x_1^2 + 3x_2^2 - 4x_1 x_2 + 7$$

(2) $$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \underbrace{|a_i^T x - \underbrace{b_i}_{\in \mathbb{R}}|^2}$$

each $a_i \in \mathbb{R}^n$

When Ω is a strict subset of $\mathbb{R}^n$
we say the opt. Prob. is constrained

Examples:

(1) $$\min_{\substack{2 \le x_1 \le 5 \\ 3 \le x_2 \le 7}} 2x_1^2 + 3x_2^2 - 4x_1x_2 + 7$$

(2) $$\min_{x \in B_2^n} \sum_{i=1}^{m} |a_i^T x - b_i|^2$$

$b_i \in \mathbb{R}$

each $a_i \in \mathbb{R}^n$

$$B_2^n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \le 1 \right\}$$

Remark: Instead of writing

$$\min_{x \in \Omega} f(x)$$

we may write

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad x \in \Omega$$

Optimal Solution:

We say that x^* is a solution of 1 if:

- $x^* \in \Omega$ (x^* is feasible)
- $\forall x \in \Omega$, $f(x^*) \leq f(x)$

We say that x^* is a local solution of 1 if:

- $x^* \in \Omega$
- There is a neighborhood $\mathcal{N}$ around x^* such that
  $\forall x \in \mathcal{N} \cap \Omega : f(x^*) \leq f(x)$

—x—

Remark: Strict local minimum:
replace $\leq$ by $<$
in the defn's above.

Question: How do you remember checking in Calc?

- local: "$\nabla f = 0$"
- global: plug in all local to f + pick smallest

In practice, very hard to know if even found a local, no way of knowing if found all locals.

So no way of checking.

So an important question is :

How can one check if a certain point $\hat{x}$ is optimal or even locally optimal?

To answer this question (and other questions), we need some def'ns.

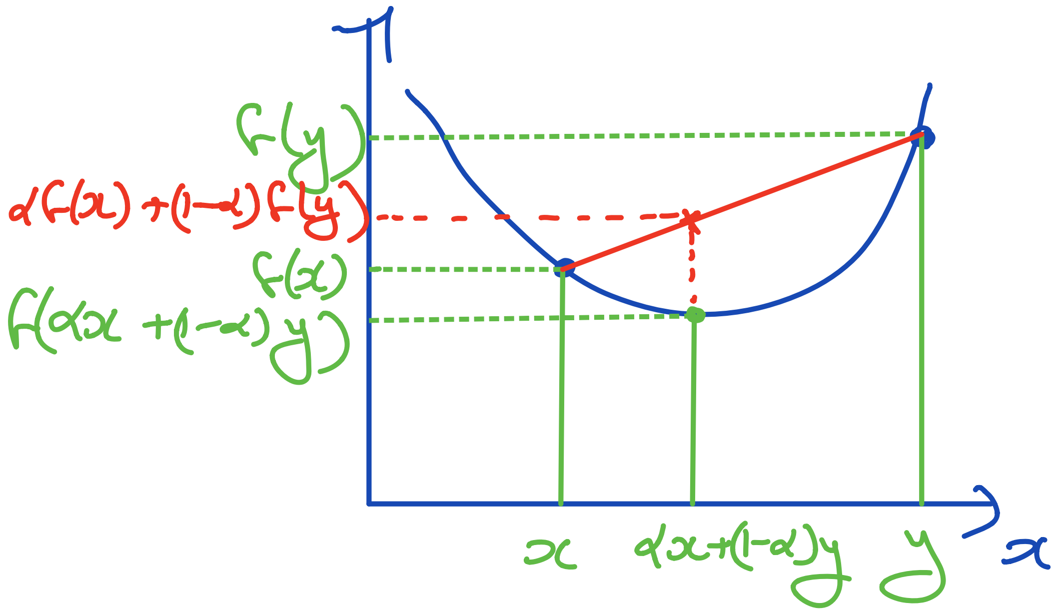
1. Convex functions
2. Convex Sets

Convex Functions

- We say $f : \Omega \to \mathbb{R}$ is convex if $\forall x, y \in \Omega$, and $\forall \alpha \in [0,1]$ we have

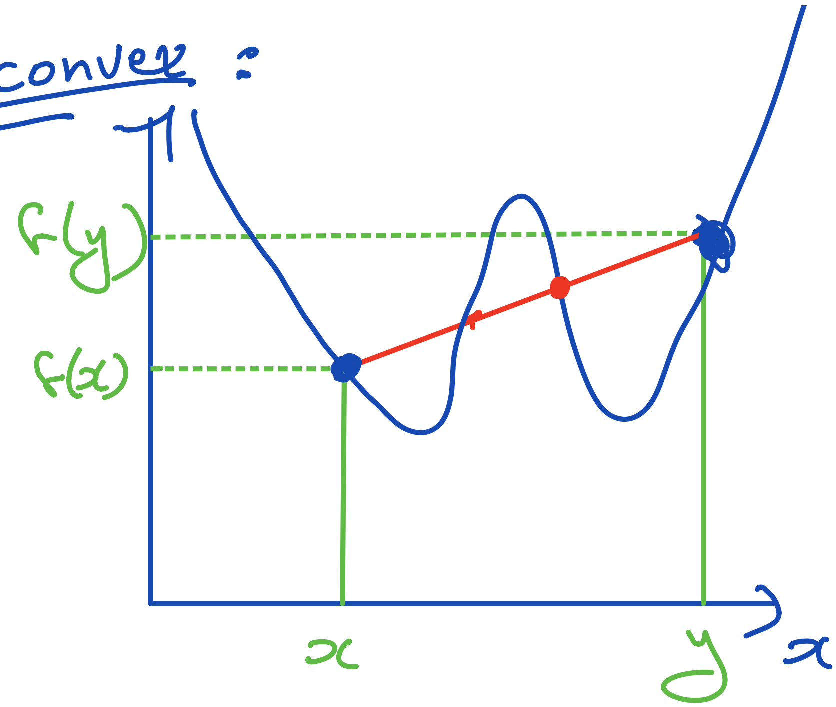
$$\boxed{f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)}$$

- We say f is <u>strictly</u> convex if "$\leq$" is replaced by "$<$" above



"line between $(x, f(x))$ & $(y, f(y))$ is above the graph of the funct."

not convex:



Example: $f(x) = x^2$ is convex

"proof": we need to check the def'n holds so we need

- $f(\alpha x + (1-\alpha) y) = (\alpha x + (1-\alpha) y)^2$

$$= \alpha^2 x^2 + (1-\alpha)^2 y^2 + 2\alpha(1-\alpha) x y$$

- $\alpha f(x) + (1-\alpha) f(y) = \alpha x^2 + (1-\alpha) y^2$

So:

$$\left(\alpha f(x) + (1-\alpha) f(y)\right) - f(\alpha x + (1-\alpha) y)$$

$$= \alpha(1-\alpha)x^2 + \left[(1-\alpha) - (1-\alpha)^2\right] y^2$$

$$- 2\alpha(1-\alpha)xy$$

$$= \alpha(1-\alpha)x^2 + \alpha(1-\alpha)y^2 - 2\alpha(1-\alpha)xy$$

$$= \underbrace{\alpha}_{\geq 0}\underbrace{(1-\alpha)}_{\geq 0} \underbrace{(x-y)^2}_{\geq 0} \geq 0$$

$\forall \alpha \in [0,1]$ & $\forall x, y \in \mathbb{R}$

$\Rightarrow$ convex.

<u>On the other hand:</u>

$f(x) = x^3$ is <u>not convex</u>

<u>bec.</u> $f(\frac{1}{2}) = \frac{1}{8}$, $f(-1) = -1$

$$f\left(\alpha \times \tfrac{1}{2} + (1-\alpha)(-1)\right) = f(1.5\alpha - 1)$$

$$= (1.5\alpha - 1)^3$$

while

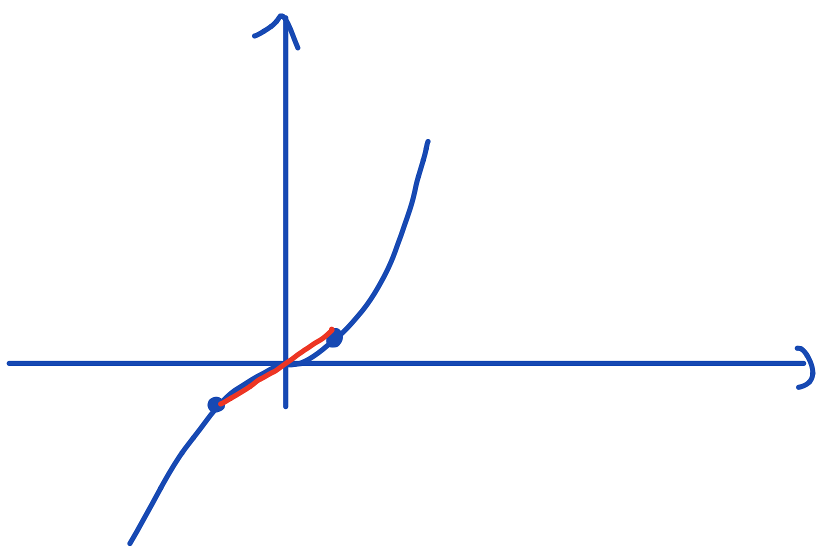
$$\alpha f(1/2) + (1-\alpha) f(-1)$$

$$= \frac{\alpha}{8} + \alpha - 1 = \frac{9}{8}\alpha - 1$$

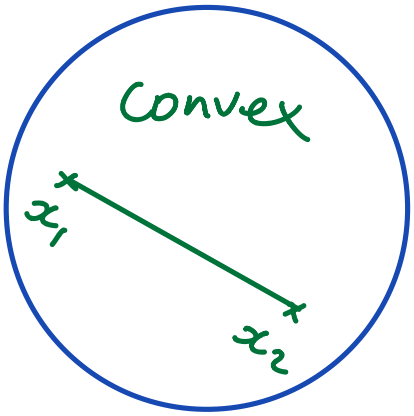
pick $\alpha = 2/3$ then

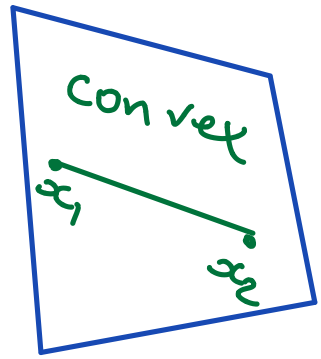
$$f\left(\frac{\alpha}{2} + (1-\alpha)(-1) \right) = 0$$

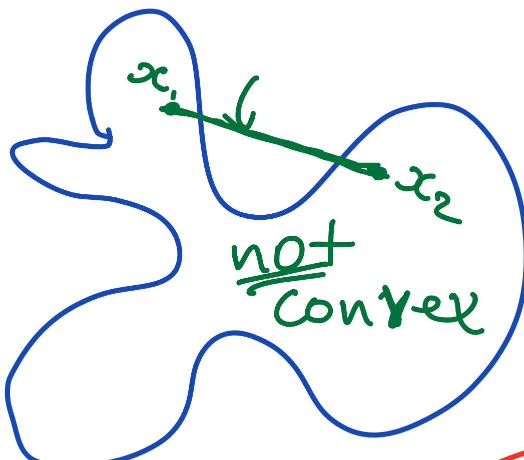
and $\alpha f(1/2) + (1-\alpha) f(-1) = \frac{9}{8} \times \frac{2}{3} - 1 \leq 0$



Convex Sets:



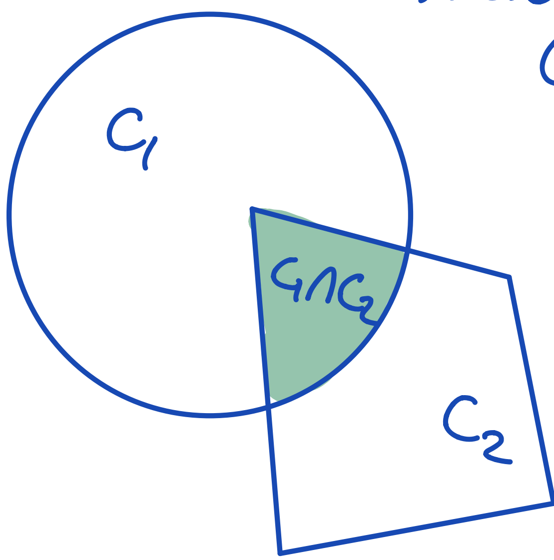




Def'n a set C is convex if $\forall x_1, x_2 \in C$ and $\forall \alpha \in [0,1]$ we have

$$\alpha x_1 + (1-\alpha) x_2 \in C$$

Fun facts: • If C_1 & C_2 are convex sets then $C_1 \cap C_2$ is convex



• The intersection of any collection of convex sets is convex

Less fun fact: $C_1 \cup C_2$ is not necessarily convex

(see the pic. above)

For convex functions defined on convex sets, local minima are global minima

Theorem: Consider the opt. problem

$$\min_{x \in \Omega} f(x)$$

where $f : \Omega \to \mathbb{R}$ is a convex function & Ω is a convex set

then if x^* is a local min it is also a global min.