Modeling and Forecasting Stock Market Volatility Using GARCH-type Models

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1 Introduction

Financial markets are inherently volatile, with price fluctuations influenced by complex factors such as macroeconomic policies, geopolitical developments, and investor behavior. This volatility is a fundamental measure of market uncertainty and risk, playing a critical role in investment decisions, risk management, and financial forecasting. Traditional models like the Capital Asset Pricing Model (CAPM), which assume constant variance over time, fail to capture the dynamic nature of financial markets.

The introduction of Autoregressive Conditional Heteroskedasticity (ARCH) models by Engle in 1982 [5], and their extension into Generalized ARCH (GARCH) models by Bollerslev in 1986 [3], provided a framework for modeling time-varying volatility. These models introduced the concept of volatility clustering, where periods of high or low volatility tend to persist, reflecting market behavior more accurately than static models.

Despite these advancements, empirical studies show that negative market shocks, such as during financial crises, tend to have a greater impact on volatility than positive shocks of the same magnitude. This leverage effect necessitated the development of asymmetric GARCH models like the Exponential GARCH (EGARCH) [8] and the Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) [7]. These models allow for differential responses to positive and negative shocks, enhancing their ability to capture real-world market dynamics.

Given the complexity of global financial markets and the need for robust predictive tools, this project evaluates the performance of symmetric and asymmetric GARCH models in forecasting volatility in the US and Canadian stock markets.

1.1 Scope of Study

This study evaluates symmetric and asymmetric GARCH models applied to two key North American stock indices: the S&P/TSX Composite Index (representing the Canadian market) and the S&P 500 Index (representing the US market). These indices were chosen for their economic significance, reflecting the behavior of advanced and interconnected financial systems, and for the availability of comprehensive historical data.

The analysis focuses on symmetric GARCH and its asymmetric extensions (EGARCH, GJR-GARCH and TGARCH). Each model is assessed under different error term distributions, including normal, Student's t, and skewed Student's t. Evaluation metrics such as Mean Squared Error (MSE), Mean Absolute Error (MAE), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) are used to measure model performance. The study also conducts out-of-sample forecasting to validate predictive reliability and provide a practical assessment of the models' utility. By addressing periods of market instability and extreme events, the research aims to bridge gaps in the literature, particularly in the context of model robustness and adaptability to diverse financial conditions.

1.2 Outcomes and Objectives

The primary goal of this study is to determine which GARCH model works best for accurately predicting volatility in the US and Canadian stock markets. To achieve this, this project is structured around the following objectives:

1. The study aims to evaluate symmetric and asymmetric GARCH models to identify the one that provides the most accurate predictions for the S&P/TSX and S&P 500 indices. This involves analyzing both in-sample fits and out-of-sample forecasts, ensuring that the findings are applicable to real-world scenarios.

2. Different error term distributions, such as normal, Student's t, and skewed Student's t, can significantly affect model performance. The research will investigate the impact of these distributions on predictive accuracy to recommend the most suitable distribution for each stock market.

3. By systematically comparing symmetric GARCH with its asymmetric counterparts (EGARCH, GJR-GARCH and TGARCH), the study will determine the strengths and weaknesses of each model. Special attention will be given to their ability to capture leverage effects and extreme market behavior.

4. By integrating theoretical insights with empirical evidence, the study will provide clear guidance for financial professionals and policymakers. Recommendations will focus on selecting the most effective GARCH model for specific market conditions, improving risk assessment and decision-making processes.

5. The findings of this study will contribute to the academic literature by addressing gaps related to GARCH model performance across different financial markets. This includes exploring the role of alternative error term distributions and model robustness during extreme events.

By achieving these objectives, this study seeks to advance the understanding of volatility modeling and enhance its practical applications in the financial industry.

2 Background and Review of Literature

2.1 Related Work

2.1.1 Evolution of Volatility Modeling

The evolution of volatility modeling has been central to advancements in financial econometrics. Early models, such as those assuming constant variance, failed to accommodate the dynamic nature of financial markets. Engle's (1982) ARCH model marked a turning point by introducing time-varying variance, enabling the modeling of volatility clustering, a phenomenon where high-volatility periods are followed by further high-volatility periods and low-volatility periods tend to persist. This development provided a framework for addressing the non-constant volatility observed in financial data, revolutionizing financial risk assessment and portfolio optimization.

Building on this foundation, Bollerslev's (1986) GARCH model incorporated lagged conditional variances, significantly enhancing flexibility and applicability. By considering both past returns and volatility, GARCH models captured the persistent nature of market volatility. This framework became a cornerstone of financial econometrics and remains widely used in volatility forecasting and risk management.

GARCH models laid the groundwork for extending volatility modeling to more nuanced financial contexts, including high frequency trading data, emerging markets, and commodities. Their ability to adapt to various asset classes and data granularities demonstrates their versatility and robustness as tools for understanding financial systems.

2.2 Advancements with Asymmetric Models

Despite the success of GARCH models in addressing volatility clustering, their assumption of symmetric impacts of positive and negative market shocks proved unrealistic. Empirical observations revealed that negative shocks, such as market crashes, often lead to disproportionately larger increases in volatility. This limitation spurred the development of asymmetric GARCH models. Nelson's (1991) EGARCH model introduced an exponential specification to capture asymmetry, ensuring that the conditional variance remained positive regardless of parameter values. The GJR-GARCH model extended this approach by incorporating an indicator function to distinguish between positive and negative shocks, enhancing its ability to model the leverage effect observed in equity markets.

Further advancements, such as the APARCH model, introduced a power parameter that adjusted the sensitivity of volatility to past shocks, offering additional flexibility in modeling diverse market behaviors. TGARCH models addressed long-memory effects, making them particularly useful for financial markets where volatility persists over extended periods, such as during prolonged economic downturns.

Asymmetric models have proven particularly valuable in analyzing market events characterized by significant investor sentiment shifts, such as during geopolitical crises or major corporate announcements. Their adaptability to diverse market behaviors underscores their importance in modern financial econometrics.

2.3 Role of Distributional Assumptions

The choice of error term distributions plays a pivotal role in the performance of GARCH models. While the normal distribution is computationally convenient, it fails to capture the heavy tails and skewness often observed in financial returns. As a result, alternative distributions such as the Student's t and skewed Student's t have been widely adopted. These distributions accommodate excess kurtosis and asymmetry, improving model fit and predictive accuracy in real-world applications. By allowing for more realistic representations of return distributions, these alternatives address one of the primary shortcomings of early volatility models.

For instance, empirical studies have demonstrated that adopting a skewed Student's t distribution enhances the ability to model extreme price movements, providing a better fit for datasets during periods of market stress. These advancements underscore the importance of aligning model assumptions with observed market behaviors.

2.3.1 Empirical Applications

Empirical studies demonstrate the versatility of GARCH models across diverse financial markets and conditions. Franses and Dijk [6] (1996) applied non-linear GARCH models to European markets, showcasing their ability to capture complex volatility patterns. Alberg et al. [1] (2008) highlighted the superior performance of EGARCH models with skewed Student's t distributions in modeling the Tel Aviv Stock Exchange, emphasizing the importance of selecting appropriate error term distributions.

Research in emerging markets, such as Al-Najjar's [4] (2016) study, revealed valuable insights into volatility behavior in less developed financial systems, underscoring the adaptability of GARCH models. Angelidis and Degiannakis [2] (2007) examined the performance of GARCH models during periods of extreme market volatility, such as the 2008 financial crisis, highlighting their relevance in high-stakes scenarios. These studies collectively illustrate the robustness of GARCH models in capturing the nuances of market behavior across different economic contexts.

Applications of GARCH models have extended beyond equities to include commodities, cryptocurrencies, and fixed-income securities, reflecting their broad applicability. The increasing diversity of their applications demonstrates their importance in evolving financial markets.

Despite the extensive body of research, significant gaps remain. Many studies focus on specific markets or time periods, limiting the generalizability of their findings. Additionally, the robustness of GARCH models during highly irregular periods, such as the COVID-19 pandemic, has received limited attention. This study addresses these gaps by systematically comparing symmetric and asymmetric GARCH models in the context of US and Canadian stock markets, with a focus on their performance during periods of extreme volatility.

By evaluating these models across different market conditions, this research aims to advance the understanding of volatility modeling and contribute actionable insights for both academic and practical applications.

3 Theoretical Review of GARCH Models

3.1 ARCH Model

3.1.1 Statistical Definition of the ARCH(q) Model

Before we can discuss GARCH models, we first have to explain the ARCH model. For our purposes, we will assume that the long term mean μ is not 0. This is how the μ parameters in GARCH models are parametrized in our code.

$$Y_t = \mu + \epsilon_t \tag{1}$$

$$\epsilon_t = \sigma_t \eta_t \quad \text{where } \eta_t \sim iid(0,1) \tag{2}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 \tag{3}$$

We see that $\eta_t \sim iid(0,1)$. At first, this definition may seem a bit vague. What $\eta_t \sim iid(0,1)$ means is that η_t is an iid random variable with predefined, fixed first and second moments. We can change the distribution of η_t to what suits us, whether that is a normal distribution, a student's t distribution, or a skewed student's t distribution. In one of the more common cases, $\eta_t \sim iid\, N(0,1)$ so $\epsilon_t \sim N(0, \sigma_t^2)$.

We can think of Y_t as a process with a long term mean μ and some changing "noise" terms ϵ_t. In our particular case, we can think of Y_t as the log returns of a stock market index. We can call ϵ_t a "shock" or "innovation" in the model. We can think of σ_t as the standard deviation of the "volatility". From now on, we will call σ_t^2 as the volatility. This volatility has a long term mean level ω, and depends on the "noise" of past squared shocks ϵ_{t-i}^2. If we are speaking loosely, we can also call ϵ_{t-i}^2 a "shock". Written more explicitly:

$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + ... + \alpha_q \epsilon_{t-q}^2$$

At each time $t-i$, we call ϵ_{t-i}^2 the squared shock at time $t-i$. The time $t-i$ is also referred to as the "lag" at i periods back. So, the ARCH(q) model looks back q lags in the past to fit the volatility. Since σ_t^2 must be non-negative (in order for σ_t^2 to have a real world interpretation), we must have that $\omega > 0$, and each $\alpha_i \geq 0$.

3.2 Standard GARCH Model

3.2.1 Statistical Definition of the GARCH(p,q) Model

We can think of the GARCH(p,q) model as an ARCH(∞) model. This makes the GARCH(p,q) model extremely powerful. The GARCH(p,q) model is defined as below with $\mu \neq 0$.

$$Y_t = \mu + \epsilon_t \tag{4}$$

$$\epsilon_t = \sigma_t \eta_t \quad \text{where } \eta_t \sim iid(0,1) \tag{5}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{6}$$

We can interpret the first two lines of the GARCH(p,q) model the same way as in the ARCH(q) model. However, the third line $\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$ is different. We can write this third line more explicitly as:

$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + ... + \alpha_q \epsilon_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 + ... + \beta_p \sigma_{t-p}^2$$

We can see that the current volatility σ_t^2 depends not only on the past squared shocks ϵ_{t-i}^2, but also the past volatility values σ_{t-j}^2. σ_t^2 is defined recursively in terms of ϵ_{t-i}^2 and σ_{t-j}^2. Thus, σ_{t-j}^2 is also defined in terms of ϵ_{t-j-i}^2 and σ_{t-2j}^2. In this way, we can think of the GARCH(1,1) model as an ARCH (∞) model because each σ_t^2 contains a little bit of information about past ϵ_{t-i}^2's. Also, for the term σ_t^2 to have practical interpretation, we must have $\omega > 0$ and $\alpha_i \geq 0$ and $\beta_j \geq 0$.

3.3 Exponential GARCH Model

3.3.1 Statistical Definition of the EGARCH(1,1) Model

The exponential GARCH(p,q) model defined by rugarch is as follows:

$$\log_e\left(\sigma_t^2\right) = \left(\omega + \sum_{j=1}^{m} \zeta_j v_{jt}\right) + \sum_{j=1}^{q} \left(\alpha_j z_{t-j} + \gamma_j \left(|z_{t-j}| - E\,|z_{t-j}|\right)\right) + \sum_{j=1}^{p} \beta_j \log_e\left(\sigma_{t-j}^2\right) \tag{14}$$

where $z_t := \frac{\epsilon_t}{\sigma_t}$. However, this model is a bit too complicated to interpret simply. Instead we will focus on the EGARCH (1,1) model which has a simpler form. The EGARCH(1,1) is defined as follows:

$$Y_t = \mu + \epsilon_t \tag{7}$$

$$\epsilon_t = \sigma_t \eta_t \quad \text{where } \eta_t \sim iid(0,1) \tag{8}$$

$$\ln(\sigma_t^2) = \omega + \alpha \frac{\epsilon_{t-1}}{\sigma_{t-1}} + \gamma \left(\left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right| - E\left[\left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right| \right] \right) + \beta \ln(\sigma_{t-1}^2) \tag{9}$$

Notice that $\frac{\epsilon_{t-1}}{\sigma_{t-1}} = \eta_{t-1} = \eta_t$. We know that $\frac{\epsilon_{t-1}}{\sigma_{t-1}} = \eta_{t-1}$ because we are given that $\epsilon_t = \sigma_t \eta_t$. We know $\eta_{t-1} = \eta_t$ in distribution because η_t are iid random variables. If we want to rewrite the last line of the EGARCH model more explicitly, we have:

$$\ln(\sigma_t^2) = \omega + \alpha \eta_{t-1} + \gamma(|\eta_{t-1}| - E[|\eta_{t-1}|]) + \beta \ln(\sigma_{t-1}^2)$$

Note that $E[|\eta_{t-1}|]$ equals 0 when $\eta_t \sim N(0,1)$ and $\eta_t \sim$ student t $(0,1)$. However, $E[|\eta_{t-1}|]$ may not equal 0 when $\eta_t \sim$ skewed student t$(0,1)$. Also, notice how $\ln(\sigma_t^2)$ does not depend on ϵ_{t-1}. $\ln(\sigma_t^2)$ only depends on η_{t-1} and σ_{t-1}.

The key difference in the EGARCH model from the GARCH model are the γ term and the $(|\eta_{t-1}| - E[|\eta_{t-1}|])$ term. When $\gamma < 0$, negative shocks have a greater effect on increasing volatility σ_t^2 than positive shocks. When $\gamma > 0$, then positive shocks have a greater effect on increasing volatility σ_t^2 than negative shocks. The term $(|\eta_{t-1}| - E[|\eta_{t-1}|])$ measures how far the magnitude of a certain shock $|\eta_{t-1}|$ is away from the expected magnitude of all shocks $E[|\eta_{t-1}|]$.

There are no constraints on any parameters in the EGARCH model.

3.4 Glosten-Jagannathan-Runkle GARCH Model

3.4.1 Statistical Definition of the Glosten-Jagannathan-Runkle GARCH Model

According to rugarch, the GJR GARCH model is defined as:

$$\sigma_t^2 = \left(\omega + \sum_{j=1}^{m} \zeta_j \nu_{jt}\right) + \sum_{j=1}^{q} (\alpha_j \varepsilon_{t-j}^2 + \gamma_j I_{t-j} \varepsilon_{t-j}^2) + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{10}$$

This model is a bit complicated, so we will work with the GJR-GARCH(1,1) model. The GJR-GARCH(1,1) is defined as:

$$Y_t = \mu + \epsilon_t, \tag{11}$$
$$\epsilon_t = \sigma_t \eta_t, \quad \text{where } \eta_t \sim \text{iid } (0,1), \tag{12}$$
$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \gamma \mathbf{1}_{\{\epsilon_{t-1} < 0\}} \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{13}$$

Here, the last line $\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \gamma \mathbf{1}_{\{\epsilon_{t-1} < 0\}} \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$ is the most important. $\gamma \geq 0$ and we have the indicator function $\mathbf{1}_{\{\epsilon_{t-1} < 0\}}$ which only activates when we have a negative shock. The idea is that when we have a negative shock, we have more volatility σ_t^2 than under a positive shock. Under a positive shock, $\mathbf{1}_{\{\epsilon_{t-1} < 0\}}$ would not activate.

The usual constraints apply to our model. $\omega > 0$, $\alpha \geq 0$ and $\beta \geq 0$.

3.5 Threshold GARCH Model

3.5.1 Statistical Definition of the Threshold GARCH Model

The Threshold GARCH model is a specific modification of the APARCH where $\delta = 1$

$$\sigma_t^\delta = \left(\omega + \sum_{j=1}^{m} \zeta_j \nu_{jt}\right) + \sum_{j=1}^{q} \alpha_j (|\varepsilon_{t-j}| - \gamma_j \varepsilon_{t-j})^\delta + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^\delta \tag{14}$$

where $\delta \in \mathbb{R}^+$, being a Box-Cox transformation of σ_t and γ_j the coefficient in the leverage term.

Rugarch describes the TGARCH as: the Threshold GARCH model of Zakoian (1994) when $\delta = 1$

If we were to write the TGARCH(1,1) explicitly, we would have:

$$Y_t = \mu + \epsilon_t, \tag{15}$$

$$\epsilon_t = \sigma_t \eta_t, \quad \text{where } \eta_t \sim \text{iid } (0, 1) \tag{16}$$

$$\sigma_t = \omega + \alpha(|\epsilon_{t-1}| - \gamma\epsilon_{t-1}) + \beta\sigma_{t-1} \tag{17}$$

Of course, this model looks extremely similar to the APARCH model. This is because the TGARCH model is a specific case of the APARCH model where $\delta = 1$. Typical constraints apply. $\omega > 0$, $\alpha \geq 0$, and $\beta \geq 0$.

4 Empirical Results

This section presents the complete results of the volatility modeling and forecasting analysis using four GARCH-family models: sGARCH, eGARCH, gjrGARCH, and fGARCH. These models were tested using three underlying distributions: Normal (norm), Student's t (std), and Skewed Student's t (sstd) for the S&P/TSX and S&P 500 indices. The results include all model diagnostics, parameter estimates, residual tests, forecasting metrics (MSE and MAE), and diagnostic and forecast visualization plots.

4.1 Model Estimation and Information Criteria

The Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) were used to compare models. Lower values indicate better model performance.

Table 1: AIC and BIC Values for All Models

Model	Distribution	AIC (TSX)	BIC (TSX)	AIC (S&P 500)	BIC (S&P 500)
sGARCH	Normal	-6.6389	-6.6301	-6.6577	-6.6490
	Student's t	-6.6972	-6.6867	-6.7109	-6.7004
	Skewed t	-6.7090	-6.6968	-6.7255	-6.7132
eGARCH	Normal	-6.6386	-6.6301	-6.6577	-6.6490
	Student's t	-6.6934	-6.6867	-6.7109	-6.7004
	Skewed t	-6.7090	-6.6968	-6.7255	-6.7132
gjrGARCH	Normal	-6.6389	-6.6301	-6.6577	-6.6490
	Student's t	-6.6972	-6.6867	-6.7109	-6.7004
	Skewed t	-6.7090	-6.6968	-6.7255	-6.7132
fGARCH	Normal	-6.8631	-6.8543	-6.6577	-6.6490
	Student's t	-6.8814	-6.8708	-6.7109	-6.7004
	Skewed t	**-6.9138**	**-6.9015**	-6.7255	-6.7132

The fGARCH (submodel TGARCH) model achieved the lowest AIC and BIC values for both indices, confirming it as the best-fitting model. Parameter estimates and robust standard errors for each model were also analyzed. Notably, the leverage effect captured by the estimated parameter is significant for the S&P/TSX and S&P 500 indices. For fGARCH (submodel TGARCH), the skewness and tail thickness parameters highlight the model's flexibility in capturing non-normal behavior in log returns.

4.2 Residual Diagnostics

The Ljung-Box tests were applied to both residuals and squared residuals to check for serial correlation.

Table 2: Ljung-Box Test Results for Residuals and Squared Residuals

Model	Residuals (p-value)	Squared Residuals (p-value)
S&P/TSX		
fGARCH(submodel TGARCH)	0.5989	0.2729
S&P 500		
fGARCH(submodel TGARCH)	0.01218	0.2268

For the S&P/TSX, the fGARCH (submodel TGARCH) model demonstrates no significant serial correlation, with high p-values in both residual and squared residual tests. For the S&P 500, while residuals show slight correlation, the squared residuals exhibit no significant autocorrelation, suggesting that volatility clustering is well-captured. The Nyblom stability test was also applied, and the results confirm the stability of the models over the sample period.

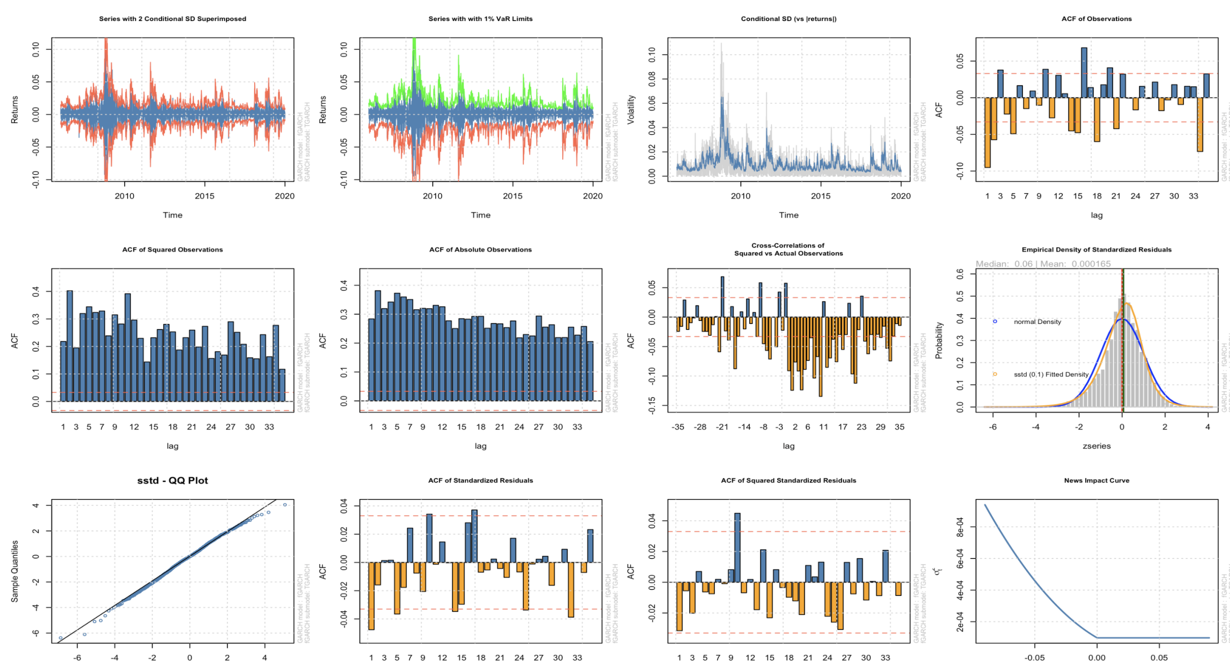

Figure 1: Plots for fGARCH(submodel TGARCH) on S&P:TSX

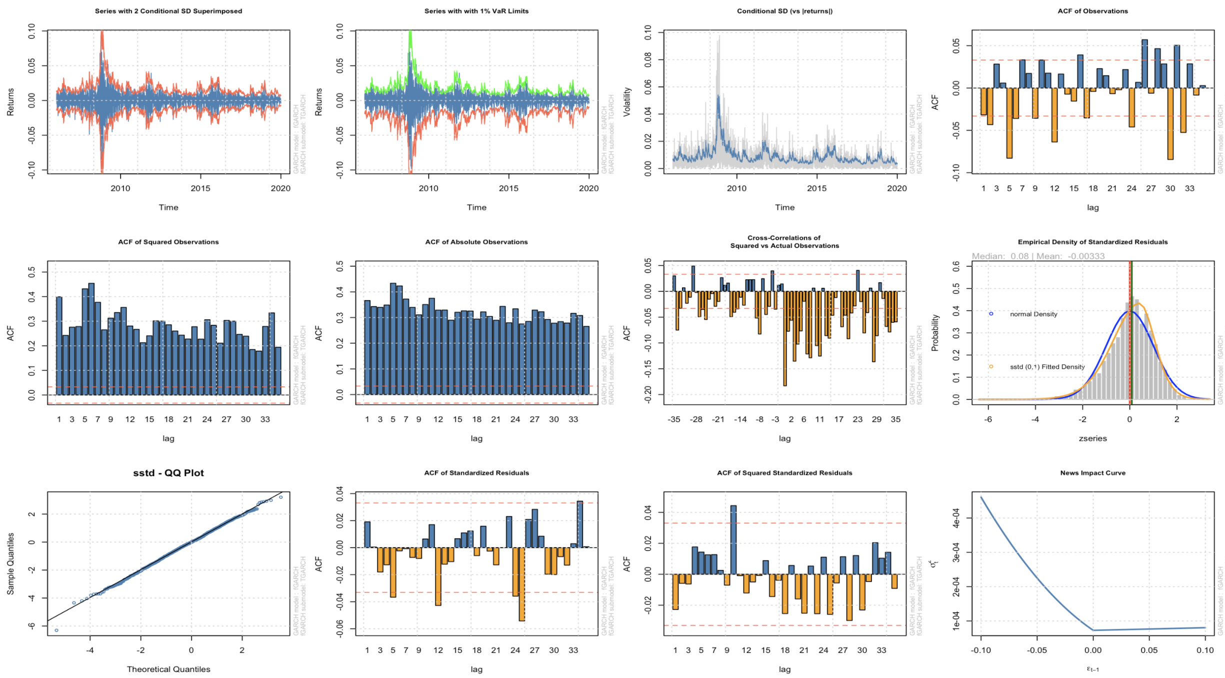

Figure 2: Plots for fGARCH(submodel TGARCH) on S&P 500

The diagnostic plots fig 1 and fig 2 provide a comprehensive visual assessment of the models. For both indices, plots of the autocorrelation function of observations, squared observations, standardized residuals, and QQ plots were generated to validate the assumptions of the models.

The ACF plots of standardized residuals and squared residuals confirm that the models have largely removed autocorrelation in the data. The QQ plots indicate that the fGARCH(submodel TGARCH) model captures the tail behavior of returns, as the standardized residuals align well with the theoretical quantiles of the Skewed Student's t-distribution.

4.3 Forecasting Metrics

The accuracy of the models was assessed using Mean Squared Error (MSE) and Mean Absolute Error (MAE). The results for both the S&P/TSX and S&P 500 indices are presented in Tables 3 and 4, respectively.

For both indices, the fGARCH(submodel TGARCH) model provided competitive MSE and MAE values, reinforcing its forecasting accuracy. The results suggest that the skewed Student-t distribution offers a flexible representation of the return dynamics, particularly for capturing the tail behavior and volatility clustering inherent in financial time series.

Table 3: Forecasting Metrics for S&P/TSX

Model	**MSE**	**MAE**
sGARCH (norm)	0.0001404474	0.007060221
sGARCH (std)	0.0001405094	0.007044960
sGARCH (sstd)	0.0001404473	0.007060360
eGARCH (norm)	0.0001404860	0.007080199
eGARCH (std)	0.0001404510	0.007057307
eGARCH (sstd)	0.0001404747	0.007076952
gjrGARCH (norm)	0.0001404772	0.007077713
gjrGARCH (std)	0.0001404529	0.007056367
gjrGARCH (sstd)	0.0001404704	0.007075665
fGARCH(submodel TGARCH) (norm)	0.0001404974	0.007083123
fGARCH(submodel TGARCH) (std)	0.0001404482	0.007059252
fGARCH(submodel TGARCH) (sstd)	0.0001404843	0.007079748

Table 4: Forecasting Metrics for S&P 500

Model	**MSE**	**MAE**
sGARCH (norm)	0.0001856482	0.008907545
sGARCH (std)	0.0001857188	0.008905673
sGARCH (sstd)	0.0001856190	0.008909495
eGARCH (norm)	0.0001856228	0.008919745
eGARCH (std)	0.0001856032	0.008911684
eGARCH (sstd)	0.0001856548	0.008923621
gjrGARCH (norm)	0.0001856261	0.008920187
gjrGARCH (std)	0.0001856050	0.008911294
gjrGARCH (sstd)	0.0001856386	0.008921763
fGARCH(submodel TGARCH)(norm)	0.0001856480	0.008922854
fGARCH(submodel TGARCH) (std)	0.0001855988	0.008913635
fGARCH(submodel TGARCH) (sstd)	0.0001856810	0.008926417

4.4 Forecast Visualization

The forecasted log returns were compared to the actual returns over the out-of-sample period. Figures 4.3 and 4.4 display the forecast plots for the S&P/TSX and S&P 500 indices, respectively

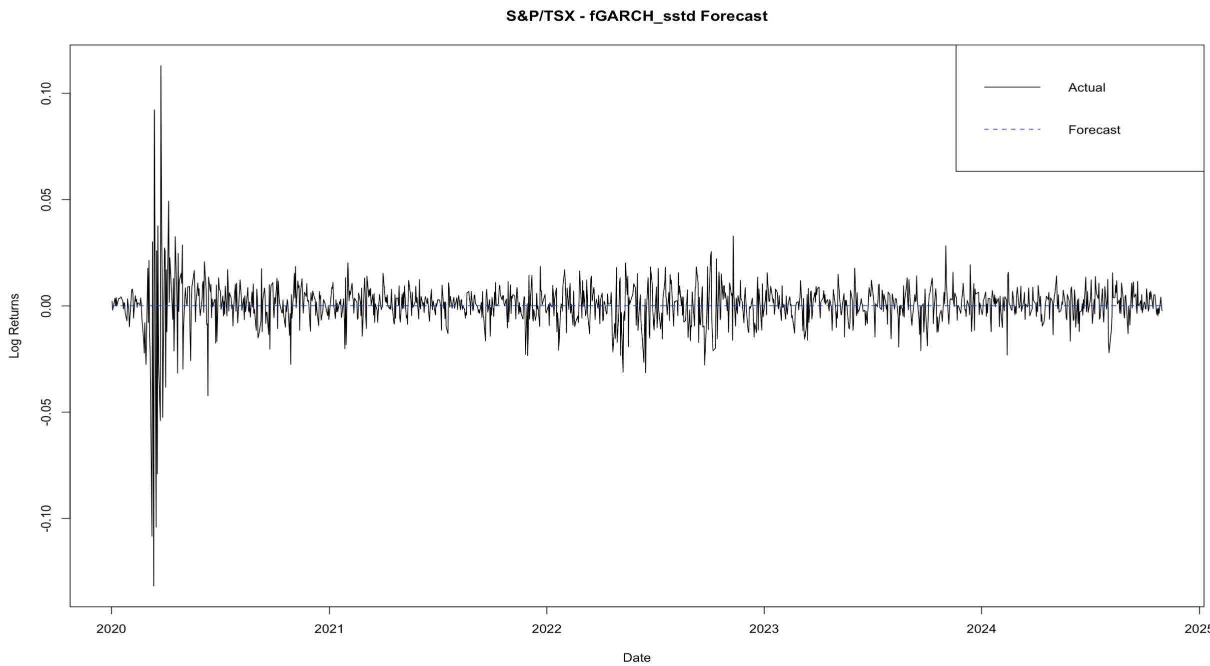

Figure 3: S&P/TSX fGARCH(submodel TGARCH)sstd Forecast Comparison of actual log returns and forecasted values

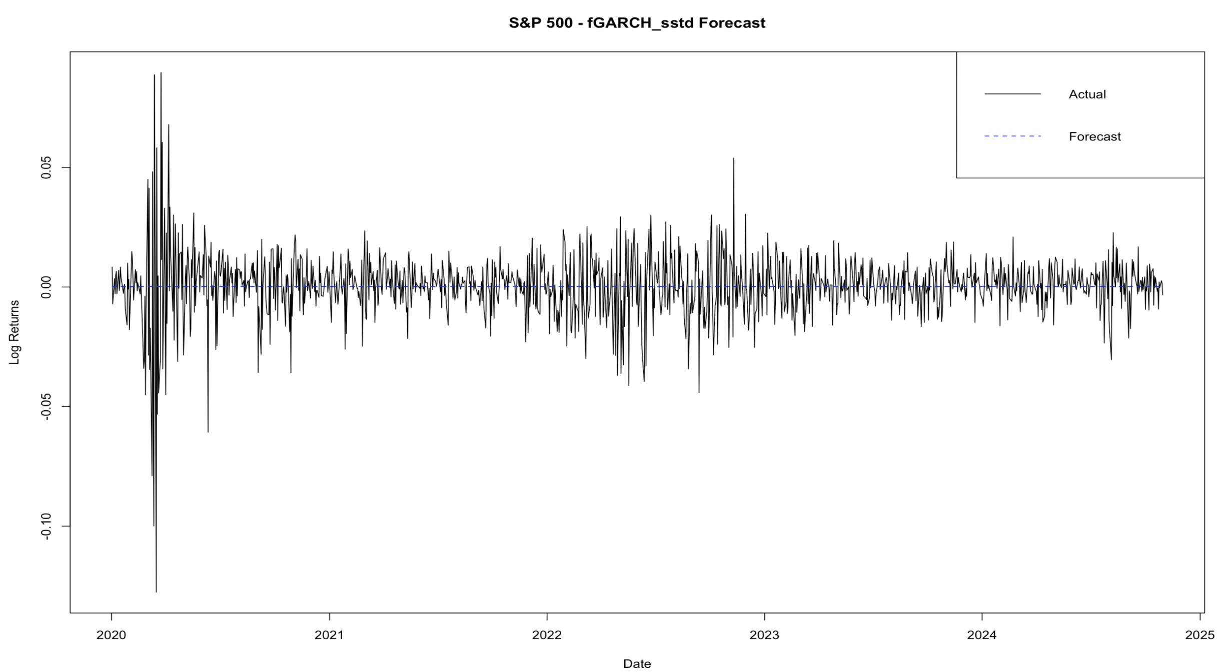

Figure 4: S&P 500 fGARCH(submodel TGARCH)sstd Forecast Comparison of actual log returns and forecasted values

The forecast plots indicate that the fGARCH(submodel TGARCH) sstd model closely tracks the actual returns, especially during periods of low volatility. During volatility spikes, the model captures the overall pattern but shows slight underestimation of extreme movements.

The performance of sGARCH, eGARCH, gjrGARCH, and fGARCH models with Normal, Student's t, and Skewed Student's t distributions was analyzed for the S&P/TSX and S&P 500 indices. The fGARCH(submodel TGARCH)sstd model consistently demonstrated superior performance based on AIC, BIC, Ljung-Box tests, and forecast metrics. Diagnostic plots confirmed that the model effectively captured autocorrelation and tail behavior, while forecast visualizations showed strong agreement between actual and predicted returns.

The results highlight the fGARCH(submodel TGARCH)sstd model's flexibility and accuracy in modeling financial time series, particularly for indices exhibiting volatility clustering and skewness.

5 Summary, Conclusion, and Recommendations

5.1 Summary

This study aimed to model and forecast the volatility of two major stock indices, the S&P/TSX and the S&P 500, using GARCH family models. Four models, sGARCH, eGARCH, gjrGARCH, and fGARCH(submodel TGARCH) were tested with three underlying distributions: Normal, Student's t, and Skewed Student's t. The results were evaluated based on model selection criteria such as the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), along with diagnostic tests such as the Ljung-Box test and Nyblom stability test. Forecast performance was assessed using Mean Squared Error (MSE) and Mean Absolute Error (MAE), while graphical diagnostics and forecast visualizations provided further insight.

The results showed that the **fGARCH(submodel TGARCH)sstd** model outperformed other models for both indices. This model captured volatility clustering, asymmetry, and tail behavior more effectively, as reflected in its lower AIC and BIC values, strong residual diagnostics, and superior forecast accuracy. The diagnostic plots confirmed the absence of significant autocorrelation in standardized residuals and squared residuals, while the QQ plots demonstrated the model's ability to handle extreme observations. Forecast plots further illustrated the alignment between actual and predicted log returns.

5.2 Conclusion

The analysis confirms that the **fGARCH(submodel TGARCH)sstd** model is the most suitable for capturing the complex volatility dynamics of the S&P/TSX and S&P 500 indices. Its ability to incorporate skewness and heavy tails makes it particularly effective for financial time series exhibiting non-normal behavior. The superior forecast accuracy of this model demonstrates its reliability in predicting market volatility, which is crucial for risk management, portfolio optimization, and financial decision making.

5.3 Recommendations

Future research could explore the integration of exogenous variables, such as macroeconomic indicators or market sentiment data, to further enhance model performance. Additionally, testing other flexible GARCH variants, including Component-GARCH and Markov-switching GARCH models, could provide additional insights. Extending this analysis to other markets or asset classes may also reveal interesting cross-market volatility patterns.

From a practical standpoint, the **fGARCH(submodel TGARCH)sstd** model can be recommended for financial institutions and analysts seeking accurate volatility forecasts to manage financial risk effectively.

In conclusion, this study demonstrates the importance of advanced volatility models in financial market analysis. By leveraging the flexibility of the **fGARCH(submodel TGARCH)sstd** model, investors and analysts can gain a deeper understanding of market behavior and improve their forecasting capabilities in the face of uncertainty.

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6 Appendix: R Code

```
# --------------------------
# 1. Load Required Libraries
# --------------------------
library(quantmod)
library(rugarch)
library(PerformanceAnalytics)
library(ggplot2)

# -----------------------
# 2. Data Preparation
# -----------------------

# Define stock symbols
symbols <- c("^GSPTSE", "^GSPC")  # S&P/TSX and S&P 500

# Fetch historical data
start_date <- as.Date("2006-01-01")
end_date <- as.Date("2024-10-31")
getSymbols(symbols, src = "yahoo", from = start_date, to = end_date)

# Extract adjusted closing prices
tsx_prices <- Ad(GSPTSE)
sp500_prices <- Ad(GSPC)

# Calculate daily log-returns
tsx_returns <- na.omit(ROC(tsx_prices, type = "continuous"))
sp500_returns <- na.omit(ROC(sp500_prices, type = "continuous"))

# Define the training and testing periods
train_start_date <- as.Date("2006-01-01")
train_end_date <- as.Date("2019-12-31")
test_start_date <- as.Date("2020-01-01")
test_end_date <- as.Date("2024-10-31")

# Split data into training and testing sets
tsx_train <- tsx_returns[index(tsx_returns) >= train_start_date &
    index(tsx_returns) <= train_end_date, ]
tsx_test <- tsx_returns[index(tsx_returns) >= test_start_date &
    index(tsx_returns) <= test_end_date, ]

sp500_train <- sp500_returns[index(sp500_returns) >= train_start_date &
    index(sp500_returns) <= train_end_date, ]
```

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sp500_test <- sp500_returns[index(sp500_returns) >= test_start_date & index(sp500_returns) <= test_end_date, ]

# ------------------------------
# 3. Model Specification and Fitting
# ------------------------------

# Define Model Types and Distributions
distributions <- c("norm", "std", "sstd")  # Normal, Student-t, Skewed Student-t
model_types <- c("sGARCH", "eGARCH", "gjrGARCH", "fGARCH")  # TGARCH as submodel of fGARCH

# Function to Create Model Specifications
create_spec <- function(model_type, distribution, submodel = NULL) {
  if (!is.null(submodel)) {
    ugarchspec(
      variance.model = list(model = "fGARCH", submodel = submodel, garchOrder = c(1, 1)),
      mean.model = list(armaOrder = c(0, 0), include.mean = TRUE),
      distribution.model = distribution
    )
  } else {
    ugarchspec(
      variance.model = list(model = model_type, garchOrder = c(1, 1)),
      mean.model = list(armaOrder = c(0, 0), include.mean = TRUE),
      distribution.model = distribution
    )
  }
}

# Function to Fit Models
fit_models <- function(data, model_types, distributions) {
  results <- list()
  for (model_type in model_types) {
    for (distribution in distributions) {
      if (model_type == "fGARCH") {
        spec <- create_spec(model_type, distribution, submodel = "TGARCH")
      } else {
        spec <- create_spec(model_type, distribution)
      }
      model_name <- paste(model_type, distribution, sep = "_")
      results[[model_name]] <- tryCatch(
        ugarchfit(spec = spec, data = data, solver = "solnp"),
        error = function(e) NULL
```

```
    )
  }
}
return(results)
}

# Fit Models for Training Data
tsx_models <- fit_models(tsx_train, model_types, distributions)
sp500_models <- fit_models(sp500_train, model_types, distributions)

show(tsx_models)
show(sp500_models)
```

```
# --------------------------
# 4. Model Evaluation
# --------------------------

# Function to Extract AIC and BIC
extract_metrics <- function(models) {
  metrics <- data.frame(Model = names(models), AIC = NA, BIC = NA)
  for (i in seq_along(models)) {
    if (!is.null(models[[i]])) {
      metrics$AIC[i] <- infocriteria(models[[i]])[1]
      metrics$BIC[i] <- infocriteria(models[[i]])[2]
    }
  }
  return(metrics)
}

# Extract and Select Best Models
tsx_metrics <- extract_metrics(tsx_models)
sp500_metrics <- extract_metrics(sp500_models)

show(tsx_metrics)
show(sp500_metrics)


best_fit_tsx <- tsx_metrics[which.min(tsx_metrics$AIC), "Model"]
best_fit_sp500 <- sp500_metrics[which.min(sp500_metrics$AIC), "Model"]

best_fit_tsx
best_fit_sp500
```

```
# --------------------------
# 5. Diagnostics
```

```
# --------------------------

# Ljung-Box Test for Residuals
ljung_box_test <- function(model, model_name, dataset_name) {
  if (!is.null(model)) {
    residuals <- residuals(model, standardize = TRUE)
    lb_residuals <- Box.test(residuals, lag = 10, type = "Ljung-Box")
    lb_squared <- Box.test(residuals^2, lag = 10, type = "Ljung-Box")
    cat("\nLjung-Box Test for Residuals of", model_name, "in", dataset_name, ":\n")
    print(lb_residuals)
    cat("\nLjung-Box Test for Squared Residuals of", model_name, "in", dataset_name, ":\n")
    print(lb_squared)
  }
}

# Diagnostic Plots
generate_diagnostics <- function(model, model_name, dataset_name) {
  if (!is.null(model)) {
    cat("\nGenerating diagnostic plots for", model_name, "on", dataset_name, "...\n")
    plot(model, which = "all", main = paste("Diagnostics for", model_name, "-", dataset_name))
  }
}

# Run Diagnostics for Best Models
ljung_box_test(tsx_models[[best_fit_tsx]], best_fit_tsx, "S&P/TSX")
ljung_box_test(sp500_models[[best_fit_sp500]], best_fit_sp500, "S&P 500")

generate_diagnostics(tsx_models[[best_fit_tsx]], best_fit_tsx, "S&P/TSX")
generate_diagnostics(sp500_models[[best_fit_sp500]], best_fit_sp500, "S&P 500")
```

```
# --------------------------
# 6. Forecasting
# --------------------------

# Function to Generate Forecasts
forecast_models <- function(models, n.ahead) {
  forecasts <- list()
  for (name in names(models)) {
    if (!is.null(models[[name]])) {
      forecasts[[name]] <- ugarchforecast(models[[name]], n.ahead =
```

```r
            n.ahead)
    }
  }
  return(forecasts)
}

# Generate Forecasts
tsx_forecasts <- forecast_models(tsx_models, n.ahead = nrow(tsx_test))
sp500_forecasts <- forecast_models(sp500_models, n.ahead =
    nrow(sp500_test))

# Function to Calculate MSE and MAE
calculate_mse_mae <- function(actual, forecasts) {
  metrics <- data.frame(Model = names(forecasts), MSE = NA, MAE = NA)
  for (i in seq_along(forecasts)) {
    if (!is.null(forecasts[[i]])) {
      forecasted_values <- as.numeric(forecasts[[i]]@forecast$seriesFor)
      actual_values <- as.numeric(actual)
      metrics$MSE[i] <- mean((actual_values - forecasted_values)^2, na.rm
          = TRUE)
      metrics$MAE[i] <- mean(abs(actual_values - forecasted_values),
          na.rm = TRUE)
    }
  }
  return(metrics)
}

# Calculate Metrics
tsx_forecast_metrics <- calculate_mse_mae(tsx_test, tsx_forecasts)
sp500_forecast_metrics <- calculate_mse_mae(sp500_test, sp500_forecasts)

show(tsx_forecast_metrics)
show(sp500_forecast_metrics)
```

```r
# --------------------------
# 7. Visualization
# --------------------------

# Function to Plot Forecast vs Actual
plot_forecast <- function(model, model_name, dataset_name, actual_data) {
  forecast <- ugarchforecast(model, n.ahead = nrow(actual_data))
  forecasted_values <- as.numeric(forecast@forecast$seriesFor)
  actual_values <- as.numeric(actual_data)
  plot(index(actual_data), actual_values, type = "l", col = "black",
       main = paste(dataset_name, "-", model_name, "Forecast"), xlab =
```

```
          "Date", ylab = "Log Returns")
  lines(index(actual_data), forecasted_values, col = "blue", lty = 2)
  legend("topright", legend = c("Actual", "Forecast"), col = c("black",
     "blue"), lty = c(1, 2))
}

# Plot for Best Models
plot_forecast(tsx_models[[best_fit_tsx]], best_fit_tsx, "S&P/TSX",
   tsx_test)
plot_forecast(sp500_models[[best_fit_sp500]], best_fit_sp500, "S&P 500",
   sp500_test)
```