Question 1: Answer Sheet

To show any Language decided by a two-dimensional TM can also be decided by a standard TM, we must simulate any transition in a two-dimensional TM in a standard TM.

Clearly, if the two dimensional TM moves left or right, a standard TM can do the same.

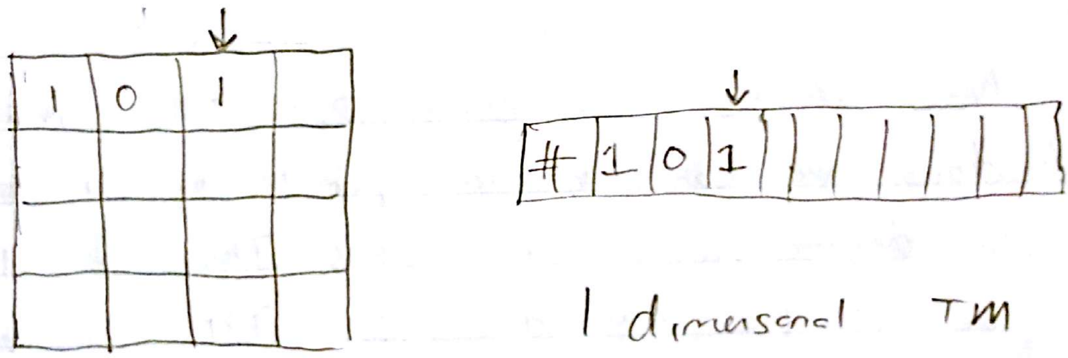
Now, if the two dimensional TM moves up or down, we can use a special symbol, say, "#" to denote a new level. The general idea will be if the two dimensional TM moves down, the Standard TM will search towards the right for a "#" symbol and move its head to the "appropriate position" If there is no "#" symbol, that means it is the first time the two-dimensional TM moved to that level, and the standard TM will write a # in the right-most position.

If the two-dimensional TM moves up, the standard TM searches left for a "#" symbol and moves to the appropriate position.

Question 1: Answer Sheet

To formalize, we will describe exactly how the Standard TM will get to the "appropriate" position after the Two-dimensional TM moves up or down.

WLOG, we illustrate with an example. Suppose the input string is 101. We start by adding a "#" symbol in the left-most position of the tape in the standard TM. Then copy the input string as normal.



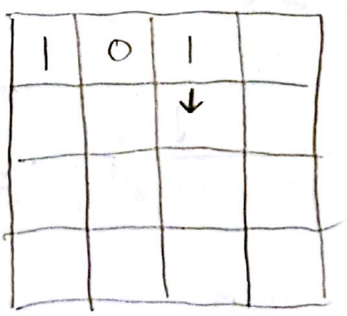
2 dimensional TM

1 dimensional TM

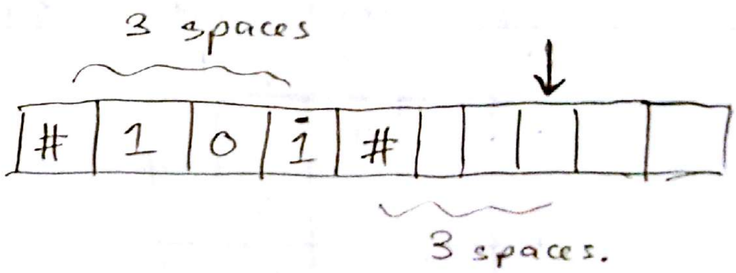
Note, if the 2-dimensional TM moves left or right, the 1 dimensional TM can do the same. The only minor exception is if the 2-dimensional TM in its upper and left most state moves left, it will stay in the same position. Similarly, in the standard TM, if after moving to the left and encountering a "#" symbol, the next move is to always move 1 step back to the right. By doing so, the standard TM also effectively "stays" in the same position.

Question 1: Answer Sheet.

Now, let's say the 2-dimensional TM moves down one step. The standard TM then "dots" the current space and counts how many spaces in-between the closest "#" symbol to the left. This will be used to remember the relative "column" position. The standard TM then searches to the right for a "#" symbol. If it does not find one, the standard TM moves one step to the right and writes a "#" symbol to denote a new level. It then moves to the right the same number of spaces between the "dotted" symbol and the previous "#" symbol. Again, to illustrate with the same example



2 dimensional TM moving "down"



Standard TM: After placing "#" symbol moves 3 spaces to right since that is the number of spaces from previous "#" symbol to dotted symbol

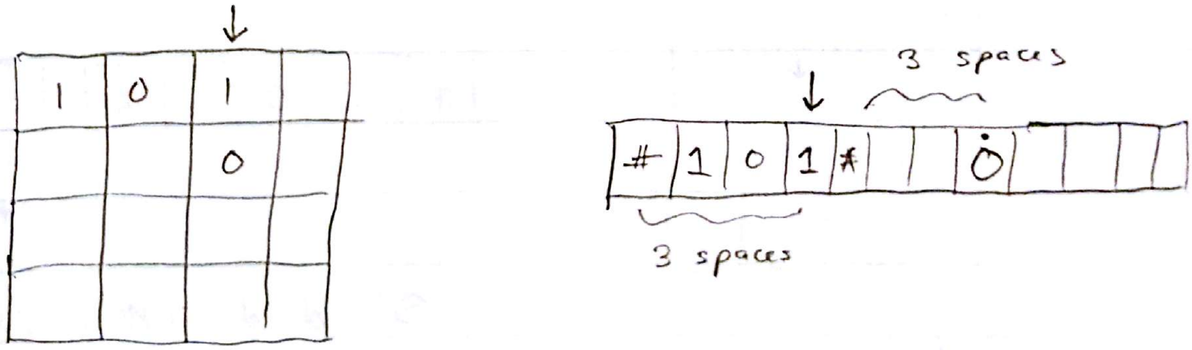
Question 1: Answer Sheet

Again, notice that writing to the tape or moving left and right is the same with 2-dimensional tape as standard TM.

Now, let's simulate when the 2-dimensional TM moves "up" with a standard TM:

Again, mark the current space with a dot and count number of spaces between closest "#" symbol on left and "dotted" space. (Call this "x").
Then move to the left until the head reaches the 2nd "#" symbol on the left. Then move the head back to the right by "x" spaces

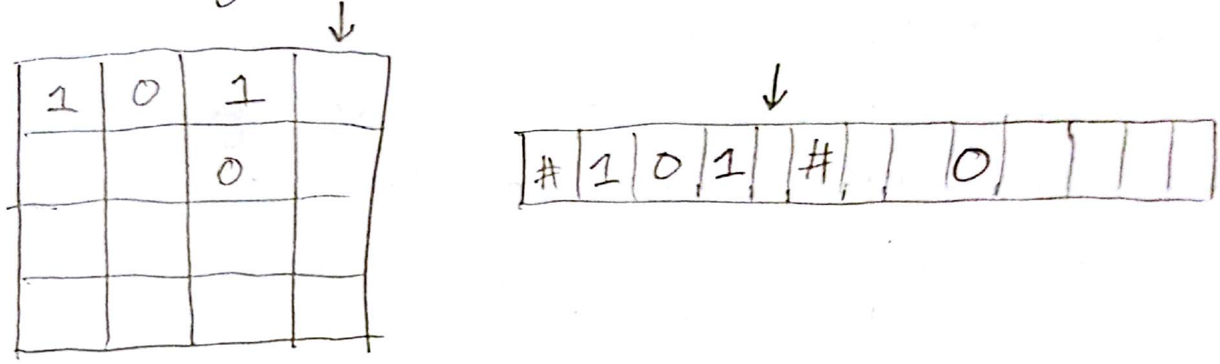
Illustrate with same example



Now, notice that in the standard TM if we want to move to the right, but we encounter a "#" symbol, we can just right shift everything over by 1.

Question 1: Answer Sheet

Illustrating with example



We've now simulated both up and down in the 2-dimensional TM and dealt with certain edge cases moving left and right with a standard TM.

Thus, we have shown any language decided by a 2-dimensional TM can also be decided by a standard TM.

Question 2: Answer Sheet

We will show A is uncountably infinite using the following logic.

$$\mathcal{L} = A \cup \bar{A}$$

We will show $\bar{A}$ is countable. Notice that if $\bar{A}$ is countable, then A must be uncountable. If not, then $A \cup \bar{A}$ would be the union of two countable sets which would be countable. This would imply $\mathcal{L}$ is countable, but we know $\mathcal{L}$ is uncountable.

WTS: $\bar{A}$ is countable.

$$\bar{A} = \{ L \in \mathcal{L} : |L| \neq \infty \}$$

(that is the set of all languages over Σ with finitely many words).

We will show $\bar{A}$ is countable by showing there is a 1:1 correspondence between the elements of $\bar{A}$ and the rational numbers. Since the rational numbers are countably infinite, this would show $\bar{A}$ is countable.

Let $x_1, x_2, x_3, \ldots,$ be words over the alphabet Σ.

Question 2: Answer Sheet

For any given $\alpha \in \bar{A}$, where α is some Language with finitely many words, let $\alpha = \{x_i, x_j, \ldots, x_n\}$

Define $f(\alpha) = i/j/\ldots/n \in$ Set of Rational numbers.

Notice that $i/j/\ldots/n$ is a rational number

Also notice that for any $\alpha_1, \alpha_2 \in \bar{A}$ s.t $\alpha_1 \neq \alpha_2 \Rightarrow f(\alpha_1) \neq f(\alpha_2)$.

If $f(\alpha_1) = f(\alpha_2)$ that would imply $\alpha_1 = \alpha_2$.

Thus, we have shown $\bar{A}$ corresponds to the set of rational numbes and thus $\bar{A}$ is comtable.

As stated previously, this also shows A is Un comtably infinite.

Question 3: Answer sheet

a). We use Rice's Theorem to show L is undecidable. Clearly L is nontrivial. Let M_1 be a TM that accepts all inputs, and let M_2 be a TM that accepts no inputs. Then $\langle M_1 \rangle \in L$ and $\langle M_2 \rangle \notin L$, which shows L is nontrivial.

Now, let $L(M_1) = L(M_2)$. This implies
$|L(M_1)| > 1 \iff |L(M_2)| > 1$
Thus, if $\langle M_1 \rangle \in L \Rightarrow \langle M_2 \rangle \in L$
if $\langle M_1 \rangle \notin L \Rightarrow \langle M_2 \rangle \notin L$.

By Rice's theorem, L is undecidable.

b) L is recognizable. This is similar to a problem 9 on Problem Set 12. We can run M on all inputs in parallel (using dovetailing). If M accepts at least 2 inputs, then accept.

c). Since in part (a) we've shown L is undecidable, and in part (b), L is recognizable, $\overline{L}$ must be not recognizable. Otherwise that would imply L is decidable, which we've already shown to be false.

eorem 5.22: If $A \le_m B$ and B is decidable, then A is decidable.

f: Let M be a decider for B. Construct N: On input w
1. Compute $f(w)$
2. Run M on input $f(w)$ and output $M(w)$

Since f reduces A to B, If $w \in A$; $f(w) \in B \Rightarrow$ accept
If $w \notin A$; $f(w) \notin B \Rightarrow$ reject.

orollary 5.23: If $A \le_m B$ and A is undecidable, then B is undecidable.

heorem 4.22: A is decidable iff A and $\overline{A}$ are both recognizable.

he compliment of a decidable language is also decidable.: If A is decidable then $\overline{A}$ Complement of undecidable

heorem 5.28: If $A \le_m B$ and B is recognizable, then A is recognizable.

rollary 5.29: If $A \le_m B$ and A is not Recognizable, then B is not Recognizable

ample of proof of undecidability by reduction

$_{TM} = \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$; $HALT_{TM} = \{\langle M, w\rangle \mid M$ is a TM and M halts on input $w\}$

f by Contradiction: Assume TM R decides $HALT_{TM}$. We construct TM S to decide A_{TM}, with S operating

"On input $\langle M, w\rangle$, an encoding of a TM M and a string w:
1. Run TM R on input $\langle M, w\rangle$.
2. If R rejects, reject
3. If R accepts, Simulate M on w until halts
4. If M accepted, accept; If M rejects, reject.

This shows $A_{TM} \le_m HALT_{TM}$

us, if R decides $HALT_{TM}$ then S decides A_{TM}, contradiction since A_{TM} is undecidable.

Practice Exam 4: #12
a) Show intersection of finitely many decidable
Let L_1, L_2 be decidable. $\exists M_1, M_2$ th
Consider $L_1 \cap L_2$. On any input, run M_1, M_2
accept, accept. Else reject.
b) False Let L be an undecidable language.
$\overline{L} = \bigcap_{w \in L} \overline{\{w\}}$. But $\{w\}$ is decidable, and $\bigcap_{w \in L} \{w\}$ is th countably infinite de
b) Is intersection of many decid

ce's Theorem

t L be a Language consisting of TM descriptions where L fulfills two conditions
1) L is nontrivial; contains, some, but not all TM descriptions. ($\exists M_1, M_2$ s.t. $\langle M_1\rangle \in L$ & $\langle M_2$
2) Whenever $L(M_1) = L(M_2)$ we have $\langle M_1\rangle \in L$ iff $\langle M_2\rangle \in L$
(either $\langle M_1\rangle, \langle M_2\rangle \in L$ or $\langle M_1\rangle, \langle M_2\rangle \notin L$)

Then L is undecidable.

xample of Rice's Theorem And showing Reductions

$L = \{\langle M\rangle : |L(M) \cap \{0,1\}| = 1\}$ $H_{TM} = \{\langle M\rangle : H$ halts on input $\varepsilon\}$

ce's Theorem to show L is undecidable. L is clearly nontrivial Now, let $L(M_1) = L(M_2)$ then if $\langle M_1\rangle \in L$, $|L(M_1)$
$\Rightarrow |L(M_2) \cap \{0,1\}| = 1$. So $M_2 \in L$. We could use a similar argument to show $\Leftarrow$ so L is undecidable

w, we show $H_{TM} \le_m L$.
input $\langle M\rangle$:
On input x':
1. Simulate M on ε
2. If $x' = 1$; accept
3. reject.

show if $\langle M\rangle \in H_{TM} \Rightarrow \langle M'\rangle \in L$
$\langle M\rangle \in H_{TM}$, simulating M on ε will halt,
d M' only accepts 1 so $\langle M\rangle \in L$.
, we show if $\langle M\rangle \notin H_{TM} \Rightarrow \langle M'\rangle \notin L$.
$\langle M\rangle \notin H_{TM}$, then simulating M on ε will loop
M' accepts nothing so $\langle M'\rangle \notin L$.

Now, we show $H_{TM} \le_m \overline{L}$:
On input $\langle M\rangle$:
On input x':
1. If $x' = 1$, accept
2. Simulate M on ε
3 accept.

If $\langle M\rangle \in H_{TM}$, M' accepts everything so $\langle M'\rangle \in \overline{L}$
If $\langle M\rangle \notin H_{TM}$, M' only accepts 1, so $\langle M'\rangle \notin \overline{L}$

Is L recognizable, what about $\overline{L}$.
Both are unrecognizable. Assume L is recognizable.
$H_{TM} \le_m L \Rightarrow H_{TM}$ recognizable.
$H_{TM} \le_m \overline{L} \Rightarrow \overline{H}_{TM} \le_m L \Rightarrow \overline{H}_{TM}$ recognizable
$\Rightarrow H_{TM}$ decidable, contradiction
Same for $\overline{L}$.

Practice Exam 4: #3
$L = \{\langle M\rangle : L(M) = \Sigma^*\}$. L accept all inputs.
a) Clearly, L is non-trivial. Let
If $\langle M_1\rangle \in L$ and $L(M_1) = L(M_2)$
$L(M_1) = L(M_2) = \Sigma^* \Rightarrow M_2 \in \Sigma^*$.
L is undecidable.
b) L is not recognizable. We show $\overline{H}$
Construct M' on input M
On input x':
1) Simulate M on ε $|x'|$ times
2) If halt, reject
3) Accept
Since $\overline{H}_{TM}$ is unrecognizable, L is not Recognizable.
c) $\overline{L}$ is also not recogni
$\overline{H}_{TM} \le_m \overline{L}$. On inpu
on inpu
1) Simu
2) If
Since $\overline{H}$

To show if L is recognizable /or not, usually
$\overline{H}_{TM} \le_m L$
L not recognizable
which shows L is not recognizable.
Construct M' on input M
On input x':
1) Simulate M on ε

Problem Set 12 #7:
$L = \{\langle M\rangle : |L(M)|$
Is L recognizable?
$\overline{H}_{TM} \le_m L$.
Is $\overline{L}$ recognizable? No, we sh
On input x':

mmon Languages in Class:

	Recognizable	Corecognizable	Decidable
$= \{\langle M, w\rangle \mid M$ is a TM and accepts $w\}$	✓	×	×
$= \{\langle M, w\rangle \mid M$ is a TM & halts on $w\}$	✓	×	×
$= \{\langle M\rangle : M$ halts on input $\varepsilon\}$	✓	×	×

...ring : A finite sequence of symbols

...nguage : A set of strings.

...et M be a TM

...(M) : Language recognized by M : Collection of strings that M accepts.

A Language is Turing Recognizable if some Turing Machine Recognizes it.

Deciders: A TM that always accepts or rejects.

A Language is Turing - decidable if some Turing machine decides it.

...ovetailing :

Show Two stacks equivalent to TM.

Simulate TM with Two stacks, Left / Right

1. Start by pushing the input string onto the Left stack.
Think of the first stack as the contents of the tape to the left of the current position, ...
second as the contents to the right. Start by pushing the normal "bottom of stack" ...
stacks, then we can simulate TM by popping from the right stack and pushing to the le...
right and vice versa to move left. If we hit the bottom of the left stack ... stay, if we hit bottom of the right stack, push blank symbol on the left

Problem Set 12 #9

a)