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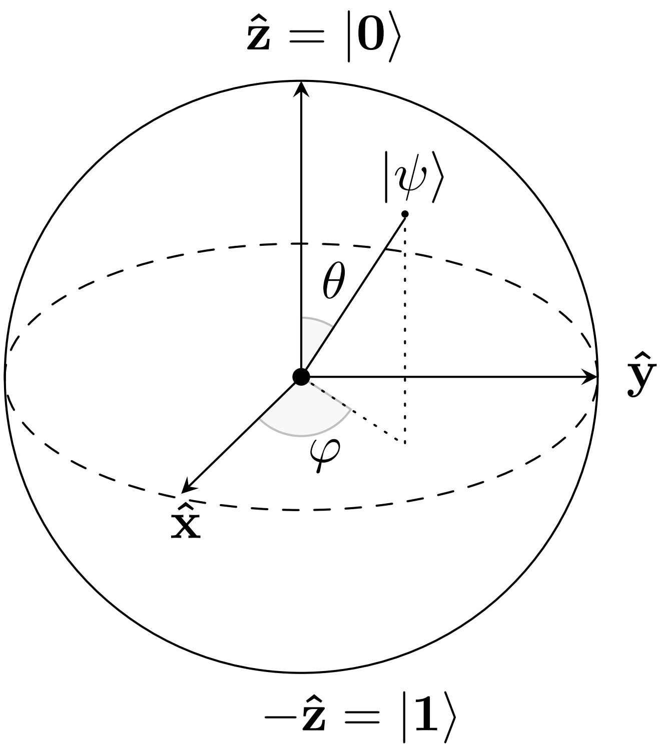
Engineering Physics Department

Entanglement in a spin chain



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Abstract

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Quantum information XY-Γ chain

Keywords :

Acknowledgements

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Glossaries

Introduction

Quantum entanglement, a fundamental feature of quantum mechanics, was first recognized in the early 20th century by Einstein, Podolsky, Rosen, and Schrödinger [5]. This phenomenon describes a situation where the quantum states of two or more particles become intertwined, such that the state of one particle cannot be described independently of the state of the other, no matter the distance separating them.

Entanglement become a practical resource in quantum information science, underpinning technologies such as quantum cryptography [7], quantum teleportation [1], and quantum computing [9]. These applications exploit entanglement to perform tasks that are impossible with classical systems.

Despite its utility, entanglement is a fragile and complex phenomenon, challenging to detect and manipulate. The study of entanglement involves understanding its properties, methods for its detection, and strategies for its quantification and manipulation. These efforts are crucial for advancing our ability to harness entanglement for practical applications, ensuring that it can be effectively used as a resource in quantum communication and computation.

The work presented a quantum simulation of a specific model of 1D lattice of **qubits** (quantum bit) a quantum system capable of existing in two states, such as the spin-up $|\uparrow\rangle$ and spin-down $|\downarrow\rangle$ states of an electron call Heisenberg XY model [8] and with a Dzyaloshinskii-Moriya interaction (DMI) [6, 4]. Finally a protocol to accelerating the entanglement in this lattice.

1. Theoretical fundamentals

In this section, we will discuss how to quantify entanglement in a quantum system. Finally we will express the Heisenberg XY model and the Dzyaloshinskii-Moriya interaction (DMI).

1.1 Quantification of the Entanglement State

Quantifying entanglement is crucial for evaluating the strength and practical applicability of quantum states. Among the various metrics, **concurrence** [10] is a widely used measure, especially in two-qubit systems. This section outlines the calculation of concurrence, which provides a numerical indicator of the degree of entanglement in a two-qubit quantum state. Concurrence $C(\rho)$ is defined as:

$$C(\rho) = \max\left(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\right) \tag{1}$$

Here, λ_i (for $i = 1, 2, 3, 4$) are the square roots of the eigenvalues of the matrix $\rho\tilde{\rho}$, listed in descending order. The density matrix is defined as $\rho = |\psi\rangle\langle\psi|$, where $|\psi(t)\rangle$ represents the state of the two qubits. The matrix $\tilde{\rho}$ is given by:

$$\tilde{\rho} = \left(\sigma^y \otimes \sigma^y\right)\rho^*\left(\sigma^y \otimes \sigma^y\right)$$

where ρ^* is the complex conjugate of ρ, and σ^y is the Pauli-Y matrix:

$$\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Concurrence values range from 0 to 1, where 0 denotes a separable (non-entangled) state, and 1 indicates a maximally entangled state. In two-qubit systems, the most recognized maximally entangled states are the Bell states:

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle\right) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle\right) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\right)
\end{aligned}$$

1.2 Heisenberg XY model with DM interaction

The Heisenberg XY model, can be described by the following Hamiltonian :

$$\hat{\mathcal{H}}_{XY} = J\hbar \sum_{n=1}^{L} \left[\left(\frac{1+\delta}{2} \right) \sigma_n^x \otimes \sigma_{n+1}^x + \left(\frac{1-\delta}{2} \right) \sigma_n^y \otimes \sigma_{n+1}^y \right] + g\hbar \sum_{n=1}^{L+1} \sigma_n^z, \tag{2}$$

here, $\hat{\sigma}_n^x$, $\hat{\sigma}_n^y$, and $\hat{\sigma}_n^z$ are the Pauli matrices of the qubit n in the lattice, J is the exchange constant, δ is the anisotropy parameter, g is the strength of the transverse magnetic field, $\hbar$ is the Planck constant reduce, $\otimes$ is the tensor product between two operator and L is the length of the lattice. The Dzyaloshinskii-Moriya interaction (DMI) adds another term to the Hamiltonian, represented by:

$$\hat{\mathcal{H}}_D = D\hbar \sum_{n=1}^{L} \left(\sigma_n^x \otimes \sigma_{n+1}^y + \gamma \sigma_n^y \otimes \sigma_{n+1}^x \right), \tag{3}$$

where D is the DMI constant and γ is a correction factor associated with the DMI. Therefore, the Hamiltonian for the 1D lattice that we study is:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{XY} + \hat{\mathcal{H}}_D. \tag{4}$$

To get a schematic the system we will study in Figure 1

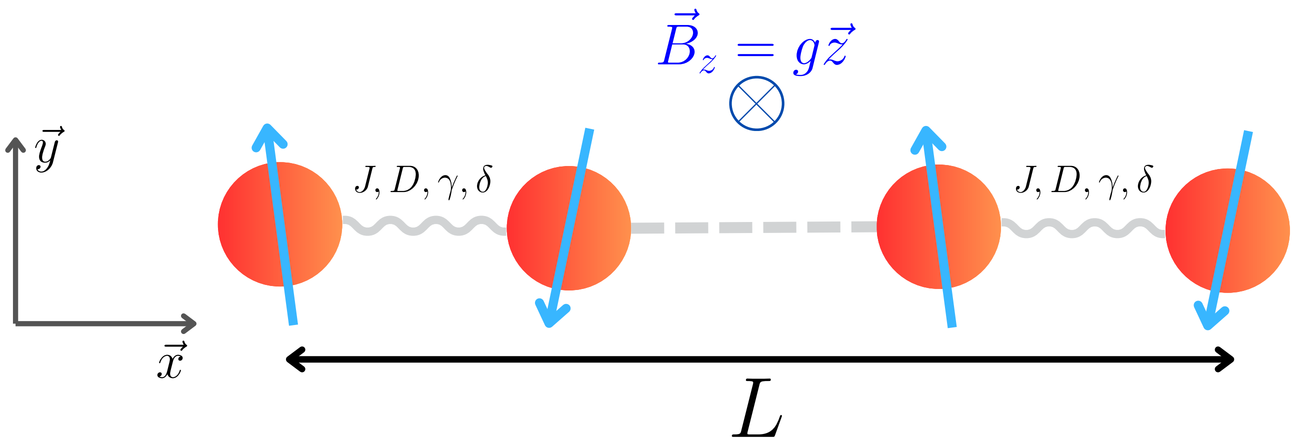


Figure 1: Schematic of the lattice of $L+1$ qubits with the Heisenberg XY model and DMI, where the blue arrows represent the spin states of the qubits, and the gray waves represent the interaction between two neighboring qubits.

2. Methodology

The goal here is to computed the time evolution of the concurrence $C(\rho(t))$ in the lattice Figure 1 with the Hamiltonian of the XY and DMI. For that with have two approch a numerical approch with the language of programmation Python and the analytical approch.

2.1 Numerical approch

To simulation the evolution of the concurrence Definition 1 we have to computed the density matrix but the necessery condition to computed the density matrix $\rho(t)$ in to found the state at the time t the quantum system. For that we use the postulat of the evolution of the quantum system the Schrodïnger equation [3].

$$\forall t \in \mathbb{R}_+,\ i\hbar\partial_t |\psi(t)\rangle = \hat{\mathcal{H}}(t) |\psi(t)\rangle \tag{5}$$

Now let explain a generic algorithm to solve the Schrodïnger equation.

First let introduce the evolution operator $\mathcal{U}(t, t_0)$ which maps $|\psi(t_0)\rangle$ the inital state of the system into $|\psi(t)\rangle$

$$\forall t \in \mathbb{R}_+,\ |\psi(t)\rangle = \mathcal{U}(t, t_0) |\psi(t_0)\rangle \text{ with } \mathcal{U}(t, t_0)\mathcal{U}(t, t_0)^\dagger = \mathcal{U}(t, t_0)^\dagger\mathcal{U}(t, t_0) = \mathrm{id} \tag{6}$$

where id is the identity function and $\mathcal{U}(t, t_0)^\dagger$ is the adjoint of $\mathcal{U}(t, t_0)$. Now we start by considering a Hamiltonian $\hat{\mathcal{H}}(t)$ which is piecewise constant, i.e.,

$$\forall j \in \mathbb{N},\ \hat{\mathcal{H}}(t) = \hat{\mathcal{H}}(t_j) \quad \text{for} \quad t_j < t < t_{j+1}, \tag{7}$$

where t_j are time steps at which the Hamiltonian changes suddenly.

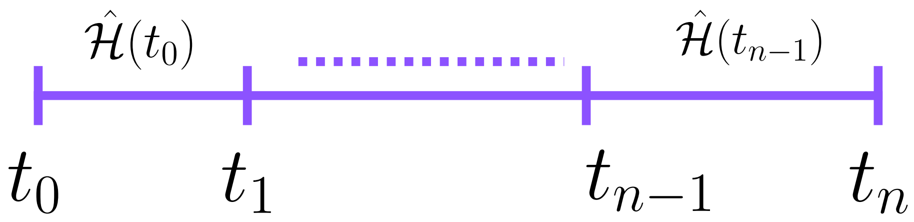


Figure 2: Schematic of the Proposition 7

But now with the Proposition 7, the Definition 6 and the Equation 5 we can express the $\mathcal{U}(t, t_0)$ like this :

$$\mathcal{U}(t, t_0) = e^{-i\hat{\mathcal{H}}(t-t_0)/\hbar} \tag{8}$$

If we know the state of the quantum system at t_0, then we can compute the state quantum system at time t_n with $n > 0$ as:

$$
\begin{aligned}
|\psi(t_1)\rangle &= e^{-i\hat{\mathcal{H}}(t_0)(t_1-t_0)/\hbar}\,|\psi(t_0)\rangle \\
|\psi(t_2)\rangle &= e^{-i\hat{\mathcal{H}}(t_1)(t_2-t_1)/\hbar}\,|\psi(t_1)\rangle = e^{-i\hat{\mathcal{H}}(t_1)(t_2-t_1)/\hbar}e^{-i\hat{\mathcal{H}}(t_0)(t_1-t_0)/\hbar}\,|\psi(t_0)\rangle \\
&\vdots \\
|\psi(t_n)\rangle &= e^{-i\hat{\mathcal{H}}(t_{n-1})(t_n-t_{n-1})/\hbar}\,|\psi(t_{n-1})\rangle \cdots e^{-i\hat{\mathcal{H}}(t_0)(t_2-t_1)/\hbar}e^{-i\hat{\mathcal{H}}(t_1-t_0)/\hbar}\,|\psi(t_0)\rangle
\end{aligned}
$$

Let us consider equally spaced time intervals:

$$t_j = t_0 + j\Delta t.$$

So the state of the quantum system at the time t_n is :

$$\forall n \in \mathbb{N}^*,\ |\psi(t_n)\rangle = \prod_{i=0}^{n-1} e^{-i\hat{\mathcal{H}}(t_i)\Delta t/\hbar}\,|\psi(t_0)\rangle \tag{9}$$

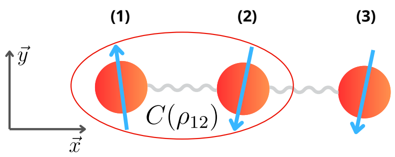
Notice the temporal order of operators $\mathcal{U}(t_j, t_{j-1})$ with $t_j > t_{j-1}$. Operators later in time appear to the left.

So with deduce this algorithm to solve the Schrodïnger equation :

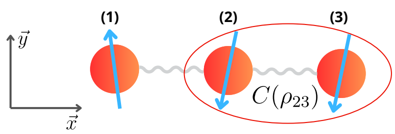
Algorithm 1 Solve Time-Dependent Schrödinger Equation

procedure SOLVE(n : integer, t_f : integer, Hamil: function, $|\psi(t_0)\rangle$: array)
  $\Delta t \leftarrow \frac{t_f}{n}$
  times $\leftarrow$ linspace$(0, t_f, n)$
  state_t $\leftarrow$ empty list
  for $j = 0$ **to** length of times $- 1$ **do**
    $\hat{\mathcal{H}}(t_j) \leftarrow$ Hamil(times$[j]$)
    $\mathcal{U}(t_{j+1} - t_j) \leftarrow$ matrix_exponential $\left(-i \times \hat{\mathcal{H}}(t_j) \times \Delta t\right)$
    $|\psi(t_{j+1})\rangle \leftarrow$ matrix_dot_product$(\mathcal{U}(t_{j+1} - t_j), |\psi(t_0)\rangle)$
    append $|\psi(t_{j+1})\rangle$ to state_t
    $|\psi(t_0)\rangle \leftarrow |\psi(t_{j+1})\rangle$
  end for
  return state_t
end procedure

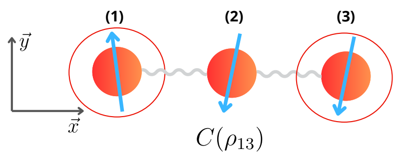
Now we can compute the density matrix at the time t but here if the length of the lattice is $L > 1$ we can not compute the $C(\rho(t))$ according at the Definition 1 the concurrence in between only two qubits but if $L > 1$ we have more of 2 qubits in the lattice so we can't use the of the concurrence. Let see a example with 3 qubits to solve the problem in the figure :



(a) Concurrence between the qubit (1) and (2)



(b) Concurrence between the qubit (2) and (3)



(c) Concurrence between the qubit (1) and (3)

Figure 3: Problem of calculation of the concurrence for many qubits

We understand we the Figure 3 with have to a isolate the pair of the qubit and computed the density matrix of the couple of qubit note ρ_{ij} where $i \neq j$ and $(i, j) \in \mathbb{N}^*$, for that we will use a mathematical tool call the partial trace [2]

Definition 1: Partial trace

Let take a system of $L + 1$ quantum system so

$$(i, j) \in (\mathbb{N}^*)^2 \text{ and } i \neq j,\ \rho_{ij} = \sum_{i=1}^{\dim(A)} \langle e_i| \rho |e_i\rangle \tag{10}$$

Let do a example For example if we want to compute the density matrix of the system of the qubit (1) and (2) schematic in the Figure 3a

3. Results And Discussion

Results And Discussion

3.1 Entanglement in a spin chain with 2 qubits

In this section, we analyze the dynamics of a system consisting of two qubits with the Hamiltonian given by:

$$\hat{H} = \hbar J\left[\left(\frac{1+\delta}{2}\right)\sigma_1^x\otimes\sigma_2^x + \left(\frac{1-\delta}{2}\right)\sigma_1^y\otimes\sigma_2^y\right] + D\hbar\left(\sigma_1^x\otimes\sigma_2^y + \gamma\sigma_1^y\otimes\sigma_2^x\right) + g\left(\sigma_1^z\otimes\hat{\mathbb{I}} + \hat{\mathbb{I}}\otimes\sigma_2^z\right).$$

we focus on the concurrence $C(\rho(t))$, which quantifies the entanglement between the two qubits. The analytical solution for the concurrence, as derived in Appendix B, is given by:

$$C(\rho(t)) = \left|\sin(2\mu t)\right|,$$

where $\mu = \sqrt{J^2 + D^2(\gamma-1)^2}$.

3.1.1 Analysis of Concurrence Dynamics

The concurrence formula reveals a simple yet profound oscillatory behavior of the entanglement between the qubits. The parameter μ plays a crucial role in determining the frequency of these oscillations, which depends on the coupling constants J and D as well as the anisotropy parameter γ.

Impact of Coupling Constants J and D:

The coupling constant J directly contributes to μ, indicating that a stronger XX and YY interaction leads to faster oscillations in concurrence. The term $D^2(\gamma-1)^2$ suggests that the effect of the D coupling on the concurrence is modulated by the anisotropy γ. For $\gamma = 1$, this contribution vanishes, and the oscillation frequency is solely determined by J. However, for $\gamma \neq 1$, the anisotropy introduces additional dynamics through D.

Behavior for Different Regimes of μ:

When μ is large (e.g., large J or significant anisotropy), the concurrence oscillates rapidly, meaning that the system frequently transitions between entangled and separable states. For small μ, the oscillations are slower, indicating more prolonged periods of either high or low entanglement.

Maximal Concurrence: The concurrence achieves its maximum value of 1 when $\sin(2\mu t) = \pm 1$, indicating a fully entangled state. Conversely, $C(|\psi(t)\rangle) = 0$ when $\sin(2\mu t) = 0$, representing separable states where the qubits are not entangled.

3.2 Discussion

The results provide significant insights into the entanglement dynamics in a two-qubit system with anisotropic and cross-coupling terms. The oscillatory nature of the concurrence reflects the intricate interplay between different types of interactions present in the Hamiltonian. This behavior is essential for applications in quantum information processing, where control over entanglement is crucial.

- **Quantum Control:** By tuning the parameters J, D, and γ, one can manipulate the entanglement dynamics, potentially allowing for the design of specific quantum gates or the implementation of quantum error correction protocols that rely on dynamic entanglement.
- **Effect of Anisotropy:** The dependence of μ on γ illustrates how anisotropy can either enhance or diminish the contribution of the D coupling to the entanglement dynamics. This result suggests that systems with tunable anisotropy could be particularly versatile in quantum control schemes.
- **Robustness of Entanglement:** The periodic nature of the concurrence indicates that, under specific conditions, entanglement can be robust over time, recurring predictably as a function of time. This property might be exploited to maintain entanglement over long durations in quantum communication protocols.

In Conclusion concurrence quantifies the entanglement between two qubits. For example, when $t/T_0 = 0.5 + k$, where $k \in \mathbb{N}$, the concurrence is 1. At this time, the state $|\psi(t)\rangle$ is:

$$\left|\psi\left(t = \frac{\pi}{4} + \frac{k\pi}{2}\right)\right\rangle = \frac{1}{\sqrt{2}}\left|\uparrow\downarrow\right\rangle - \frac{i(J+2iD)}{\sqrt{2}\gamma}\left|\downarrow\uparrow\right\rangle$$

This state is evidently entangled according to the definition.

Moreover we can recognize a Bell state we this form:

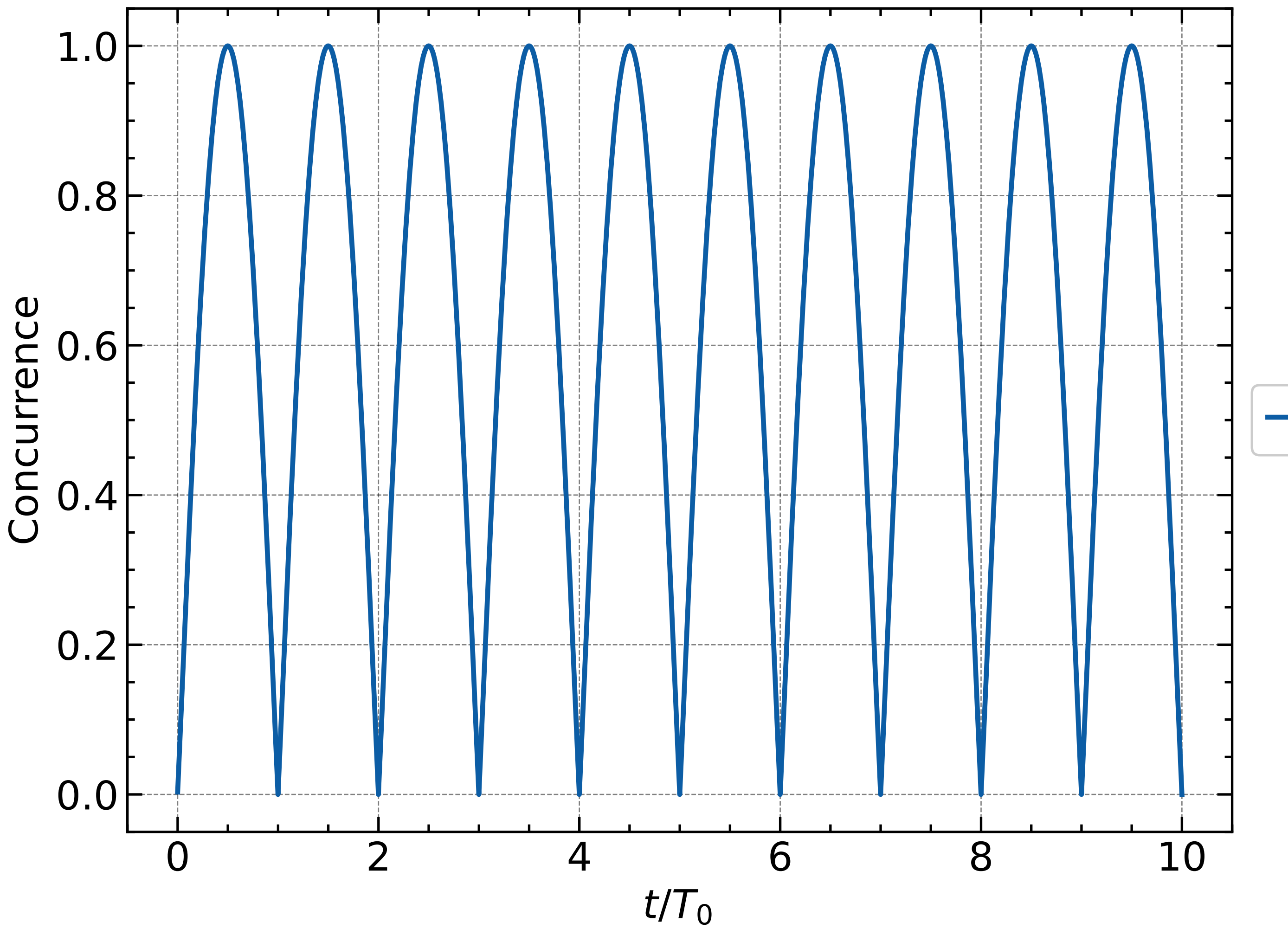
$$\left|\psi\left(t = \frac{\pi}{4} + \frac{k\pi}{2}\right)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - ie^{i\theta}\left|\uparrow\downarrow\right\rangle\right) \tag{11}$$

where $\theta = \arctan\left(\frac{2D}{J}\right)$

And at $t/scale = k$ the concurrence equal to 0,

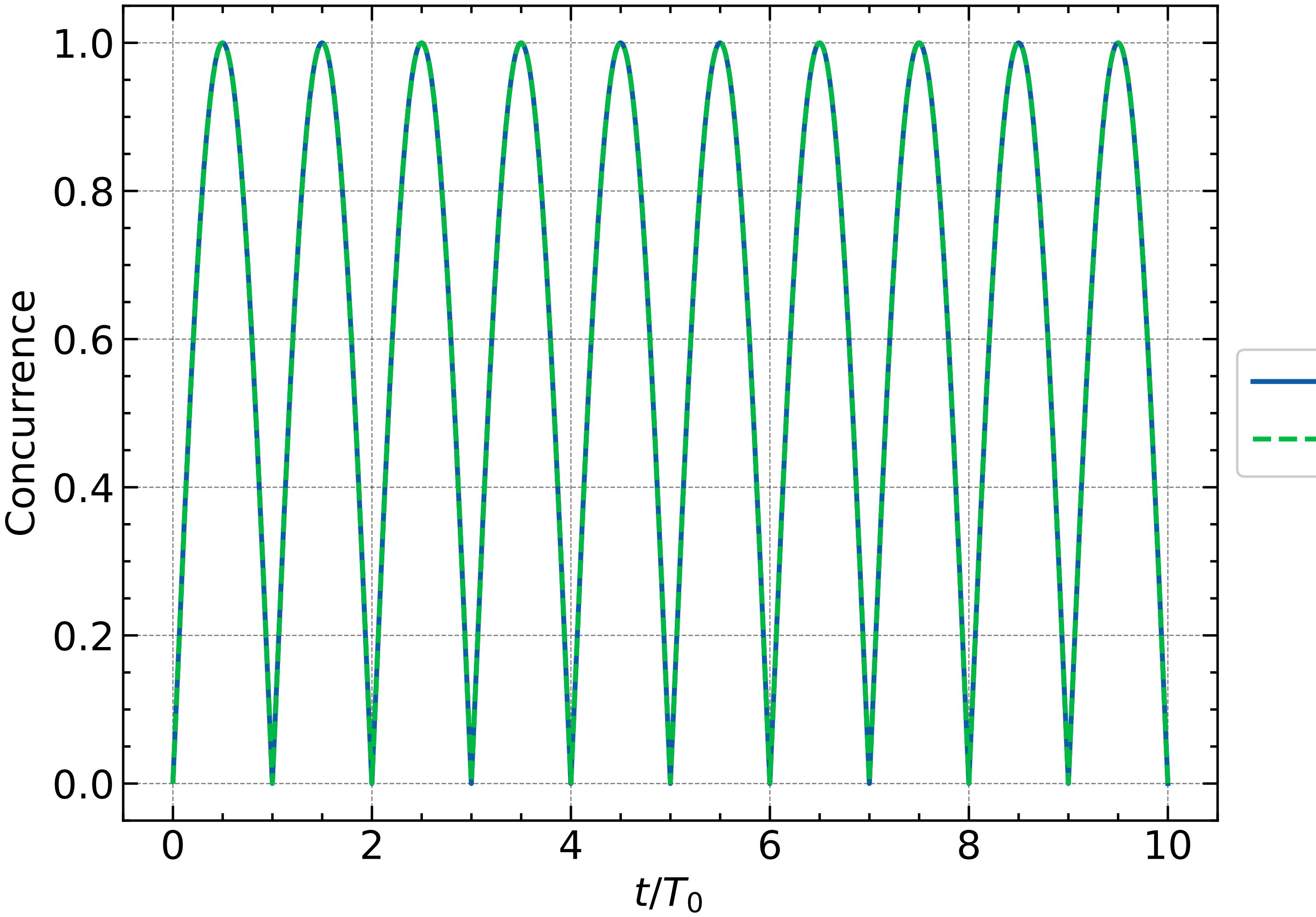
$$\left|\psi\left(t = \frac{k\pi}{2}\right)\right\rangle = \frac{i(J+2iD)}{\sqrt{2}\gamma}\left|\downarrow\uparrow\right\rangle$$

In this case, the state is separable.



Concurrence
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
t/T_0

3.2.1 Analytical approch



3.3 Entanglement in a spin chain with N qubits

3.3.1 Entanglement in a spin chain with 3 qubits

3.3.2 Entanglement in a spin chain with 8 qubits

3.4 Protocol of accelerating of the entanglement

Conclusion

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Appendices

A. Appendix First

A.1 Generalities and Recall on Matrix Series

Definition 2: Norme matrix

For $A = (a_{ij}) \in M_n(\mathbb{C})$, we define

$$\|A\|_\infty = \max\{|a_{ij}|; 1 \leq i, j \leq n\}.$$

$\|\cdot\|_\infty$ is a norm on $M_n(\mathbb{C})$, i.e., it is a function with values in $\mathbb{R}^+$ such that

- $\forall A, \|A\|_\infty = 0 \iff A = 0$,
- $\forall A, \forall \lambda \in \mathbb{C}, \|\lambda A\|_\infty = |\lambda| \cdot \|A\|_\infty$,
- $\forall A, B, \|A + B\|_\infty \leq \|A\|_\infty + \|B\|_\infty$.

In the following, we will denote $\|\cdot\|$ instead of $\|\cdot\|_\infty$.

Definition 3: Converge sequence of matrix

- A sequence (A_k) of $M_n(\mathbb{C})$ is said to be convergent if there exists $A \in M_n(\mathbb{C})$ such that $\forall \epsilon > 0, \exists k_0 \in \mathbb{N}$ such that $k \geq k_0 \Rightarrow \|A_k - A\| < \epsilon$.
- A sequence (A_k) is said to be Cauchy if $\forall \epsilon > 0, \exists k_0 \in \mathbb{N}, \forall k, k_0 \geq k_0, \|A_k - A_{k_0}\| < \epsilon$.

Proposition 1: Characterization of matrix convergence

Let (A_k) be a sequence of $M_n(\mathbb{C})$. Then it is Cauchy if and only if it is convergent.

Proof. The right-to-left direction is classical, and the direct direction relies on the completeness of $\mathbb{C}$ □

Definition 4: Convergent and Absolutely convergent matrix serie

Given a sequence (A_k) of $M_n(\mathbb{C})$, we define the associated series denoted $\sum A_k$ as the sequence (S_k) with general term

$$S_k = \sum_{l=0}^{k} A_l.$$

The series is said to be absolutely convergent if the real series $\sum \|A_k\|$ is convergent.

Proposition 2: Convergent and Absolutely convergent matrix serie

If $\sum A_k$ is absolutely convergent, then it is convergent.

Proof. Let $\epsilon > 0$. For all $k \in \mathbb{N}$, denote $T_k = \sum_{l=0}^{k} \|A_l\|$ and $S_k = \sum_{l=0}^{k} A_l$. By hypothesis, (T_k) converges (in $\mathbb{R}$), so it is Cauchy. Hence, there exists $k_0 \in \mathbb{N}$ such that for $k, k_0 \geq k_0$, we have $|T_k - T_{k_0}| < \epsilon$.

For $k_0 \geq k \geq k_0$, we have

$$\|S_{k_0} - S_k\| = \left\| \sum_{l=k+1}^{k_0} A_l \right\| \leq \sum_{l=k+1}^{k_0} \|A_l\| = T_{k_0} - T_k < \epsilon.$$

Thus, the sequence (S_k) is Cauchy, hence convergent. □

Lemma 1: Matrix inequality

For $A, B \in M_n(\mathbb{C})$,

$$\|AB\| \leq n\|A\|\|B\| \quad \text{and} \quad \|A^k\| \leq n^{k-1}\|A\|^k.$$

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$, and $C = AB = (c_{ij})$. For all i, j,

$$|c_{ij}| = \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right| \leq \sum_{k=1}^{n} |a_{ik}||b_{kj}| \leq \sum_{k=1}^{n} \|A\|\|B\| = n\|A\|\|B\|.$$

Consequently, $\|C\| \leq n\|A\|\|B\|$.

The second inequality is obtained by induction on k using the first. □

A.2 Properties exponential of matrix

Definition 5: Exponential of matrix

Let $A \in M_n(\mathbb{C})$. We define $e^A = \exp(A) \in M_n(\mathbb{C})$ by

$$e^A = \sum_{k=0}^{+\infty} \frac{A^k}{k!}.$$

Theorem 1: Existance of the exponential of matrix

The series $\sum \frac{A^k}{k!}$ converges. Thus, the matrix e^A is well-defined.

Proof. If $A = 0$, it is trivial. We assume A is non-zero. We show that the series is absolutely convergent. Indeed,

$$\left\| \sum_{l=0}^{k} \frac{A^l}{l!} \right\| \leq \sum_{l=0}^{k} \frac{n^{l-1}\|A\|^l}{l!}.$$

Let $u_l = \frac{n^{l-1}\|A\|^l}{l!}$. Then $\frac{u_{l+1}}{u_l} = \frac{n\|A\|}{l+1}$ and this tends to 0 as l tends to $+\infty$. By the D'Alembert criterion, the series $\sum u_l$ is convergent, which implies the absolute convergence of our original series. □

Theorem 2: Matrix exponential and Pauli matrices

Let the matrix $\vec{\sigma} = \hat{\sigma}^x \hat{x} + \hat{\sigma}^y \hat{y} + \sigma_z \hat{z}$ and the vector $\hat{n} = a\hat{x} + b\hat{y} + c\hat{z}$, where $(\hat{x}, \hat{y}, \hat{z})$ is an orthonormal basis. The σ_a are the Pauli matrices for $a \in \{x, y, z\}$.
If $\|\hat{n}\| = 1$ then :

$$e^{i\mu(\hat{n}\cdot\vec{\sigma})} = \hat{\mathbb{I}} \cos(\mu) + i(\hat{n} \cdot \vec{\sigma}) \sin(\mu). \tag{12}$$

where . : is the inner product in $\mathbb{R}^3$

Proof. Exponential of a Pauli vector:

For

$$\vec{\mu} = \mu\hat{n}, \quad |\hat{n}| = 1,$$

one has, for even powers, $2p$, $p = 0, 1, 2, 3, \ldots$

$$(\hat{n} \cdot \vec{\sigma})^{2p} = I,$$

which can be shown first for the $p = 1$ case using the anticommutation relations. For convenience, the case $p = 0$ is taken to be I by convention.

For odd powers, $2q + 1$, $q = 0, 1, 2, 3, \ldots$

$$(\hat{n} \cdot \vec{\sigma})^{2q+1} = \hat{n} \cdot \vec{\sigma}.$$

Matrix exponentiating, and using the Taylor series for sine and cosine,

$$
\begin{aligned}
e^{i\mu(\hat{n}\cdot\vec{\sigma})} &= \sum_{k=0}^{\infty} \frac{i^k[\mu(\hat{n}\cdot\vec{\sigma})]^k}{k!} \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p(\mu\hat{n}\cdot\vec{\sigma})^{2p}}{(2p)!} + i\sum_{q=0}^{\infty} \frac{(-1)^q(\mu\hat{n}\cdot\vec{\sigma})^{2q+1}}{(2q+1)!} \\
&= I\sum_{p=0}^{\infty} \frac{(-1)^p\mu^{2p}}{(2p)!} + i(\hat{n}\cdot\vec{\sigma})\sum_{q=0}^{\infty} \frac{(-1)^q\mu^{2q+1}}{(2q+1)!}.
\end{aligned}
$$

In the last line, the first sum is the cosine, while the second sum is the sine; so, finally,

$$
e^{i\mu(\hat{n}\cdot\vec{\sigma})} = \hat{\mathbb{I}}\cos\mu + i(\hat{n}\cdot\vec{\sigma})\sin\mu.
$$

□

B. Appendix Second

The objective of this section is to solve the Schrödinger equation for a chain of two qubits:

$$\begin{cases} i\hbar\partial_t \left|\psi(t)\right\rangle = \hat{\mathcal{H}} \left|\psi(t)\right\rangle \\ \\ \left|\psi(t=0)\right\rangle = \left|\uparrow\downarrow\right\rangle \end{cases}$$

The Hamiltonian of this system is given by:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{XY} + \hat{\mathcal{H}}_D \tag{13}$$

$$\hat{\mathcal{H}}_{XY} = \hbar J \left[\left(\frac{1+\delta}{2}\right) \sigma_x^{(1)} \otimes \sigma_x^{(2)} + \left(\frac{1-\delta}{2}\right) \sigma_y^{(1)} \otimes \sigma_y^{(2)} \right] + \hbar g \sigma_z^{(1)} \otimes \hat{\mathbb{I}}_2^{(2)} \tag{14}$$

and

$$\hat{\mathcal{H}}_D = D\hbar \left(\sigma_x^{(1)} \otimes \sigma_y^{(2)} + \gamma \sigma_y^{(1)} \otimes \sigma_x^{(2)} \right) \tag{15}$$

Now let define the Hilbert space of the quantum system of two qubits, first of all the Hilbert basis of the first qubit ($\mathcal{H}_1$: Hilbert space of the first qubit) is given by:

$$\left|\uparrow\right\rangle^{(1)} = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \left|\downarrow\right\rangle^{(1)} = \begin{pmatrix}0\\1\end{pmatrix}$$

then the Hilbert basis of the second qubit ($\mathcal{H}_2$: Hilbert space of the second qubit) is given by:

$$\left|\uparrow\right\rangle^{(2)} = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \left|\downarrow\right\rangle^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$$

Therefore the basis for the total Hilbert space ($\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \mathcal{H}_2$) is:

$$\mathcal{B}_{\mathcal{H}_{tot}} = \{ \left|\uparrow\right\rangle^{(1)} \otimes \left|\uparrow\right\rangle^{(2)}, \left|\uparrow\right\rangle^{(1)} \otimes \left|\downarrow\right\rangle^{(2)}, \left|\downarrow\right\rangle^{(1)} \otimes \left|\uparrow\right\rangle^{(2)}, \left|\downarrow\right\rangle^{(1)} \otimes \left|\downarrow\right\rangle^{(2)} \}$$

To simplify the notation in this report, we will use a classical notation:

$$\mathcal{B}_{\mathcal{H}_{tot}} = \{ \left|\uparrow\uparrow\right\rangle, \left|\uparrow\downarrow\right\rangle, \left|\downarrow\uparrow\right\rangle, \left|\downarrow\downarrow\right\rangle \} \tag{16}$$

The Hilbert space, a fundamental construct in quantum mechanics and quantum computing, represents the complete set of possible states for a quantum system. However, the vastness of this space often poses significant computational challenges. The reduction of Hilbert space, therefore, becomes a crucial technique for making quantum computations tractable. This process involves identifying and isolating a subspace of the Hilbert space that captures the essential dynamics and properties of the system under consideration.

B.0.1 Hilbert space reduction

To reduce of the Hilbert space. Let see the action of the $\hat{\mathcal{H}}$ on the initial state

$$\hat{\mathcal{H}}\left|\uparrow\downarrow\right\rangle = \hbar\left[J + iD(\gamma-1)\right]\left|\downarrow\uparrow\right\rangle - +\hbar g\left(\left|\uparrow\downarrow\right\rangle - \left|\uparrow\downarrow\right\rangle\right) \tag{17}$$

With Equation 17 the minimal basis of the system is :

$$\mathcal{B}_{red} = \{\left|\uparrow\downarrow\right\rangle, \left|\downarrow\uparrow\right\rangle\} \tag{18}$$

So let expressed the Hamiltionian in $\mathcal{B}_{red}$, for that let see the action of $\hat{\mathcal{H}}$ on the $\left|\downarrow\uparrow\right\rangle$

$$\hat{\mathcal{H}}\left|\downarrow\uparrow\right\rangle = \hbar\left[J - iD(\gamma-1)\right]\left|\uparrow\downarrow\right\rangle - \hbar g\left(\left|\downarrow\uparrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right) \tag{19}$$

So with the the effectif Hamiltonian in $\mathcal{B}_{red}$ is :

$$\boxed{\hat{\mathcal{H}}_{\text{eff}} = \hbar\begin{bmatrix} 0 & \kappa \\ \kappa^* & 0 \end{bmatrix}} \tag{20}$$

where $\kappa = J + iD(\gamma - 1)$
Let us define $a = \frac{1+\delta}{2}$ and $b = \frac{1-\delta}{2}$, and let us transform the $\hat{\mathcal{H}}_{XY}$ using Proposition **??**:

$$\hat{\mathcal{H}}_{XY} = \hbar J\left[a\left(\hat{\sigma}_1^+ + \hat{\sigma}_1^-\right)\left(\hat{\sigma}_2^+ + \hat{\sigma}_2^-\right) - b\left(\hat{\sigma}_1^+ - \hat{\sigma}_1^-\right)\left(\hat{\sigma}_2^+ - \hat{\sigma}_2^-\right)\right] + \hbar g\sigma_z^{(1)}$$

Now, let us use the bilinearity and associativity of the Kronecker product:

$$\begin{aligned}\hat{\mathcal{H}}_{XY} = \hbar J\Big[&a\left(\hat{\sigma}_1^+\hat{\sigma}_2^+ + \hat{\sigma}_1^+\hat{\sigma}_2^- + \hat{\sigma}_1^-\hat{\sigma}_2^+ + \hat{\sigma}_1^-\hat{\sigma}_2^-\right)\\ &-b\left(\hat{\sigma}_1^+\hat{\sigma}_2^+ - \hat{\sigma}_1^+\hat{\sigma}_2^- - \hat{\sigma}_1^-\hat{\sigma}_2^+ + \hat{\sigma}_1^-\hat{\sigma}_2^-\right)\Big] + \hbar g\left[\hat{\sigma}_1^z + \hat{\sigma}_2^z\right]\end{aligned}$$

Since $a + b = 1$ and $a - b = \delta$, we have:

$$\hat{\mathcal{H}}_{XY} = \hbar J\left[\delta\left(\hat{\sigma}_1^+\hat{\sigma}_2^+ + \hat{\sigma}_1^-\hat{\sigma}_2^-\right) + \left(\hat{\sigma}_1^+\hat{\sigma}_2^- + \hat{\sigma}_1^-\hat{\sigma}_2^+\right)\right] + \hbar g\left[\hat{\sigma}_1^z + \hat{\sigma}_2^z\right]$$

Let us now proceed similarly for $\hat{\mathcal{H}}_D$:

$$\begin{aligned}\hat{\mathcal{H}}_D &= \hbar D\left(\hat{\sigma}_1^x\hat{\sigma}_2^y + \gamma\hat{\sigma}_1^y\hat{\sigma}_2^x\right)\\ &= -i\hbar D\left[\left(\hat{\sigma}_1^+ + \hat{\sigma}_1^-\right)\left(\hat{\sigma}_2^+ - \hat{\sigma}_2^-\right) + \gamma\left(\hat{\sigma}_1^+ - \hat{\sigma}_1^-\right)\left(\hat{\sigma}_2^+ + \hat{\sigma}_2^-\right)\right] \quad \text{by Proposition ??}\\ &= -i\hbar D\left[\left(\hat{\sigma}_1^+\hat{\sigma}_2^+ - \hat{\sigma}_1^+\hat{\sigma}_2^- + \hat{\sigma}_1^-\hat{\sigma}_2^+ - \hat{\sigma}_1^-\hat{\sigma}_2^-\right) + \gamma\left(\hat{\sigma}_1^+\hat{\sigma}_2^+ + \hat{\sigma}_1^+\hat{\sigma}_2^- - \hat{\sigma}_1^-\hat{\sigma}_2^+ - \hat{\sigma}_1^-\hat{\sigma}_2^-\right)\right]\\ &= -i\hbar D\left[(\gamma+1)\hat{\sigma}_1^+\hat{\sigma}_2^+ + (\gamma-1)\hat{\sigma}_1^+\hat{\sigma}_2^- - (\gamma-1)\hat{\sigma}_1^-\hat{\sigma}_2^+ - (\gamma+1)\hat{\sigma}_1^-\hat{\sigma}_2^-\right]\\ &= i\hbar D\left[(\gamma+1)\left(\hat{\sigma}_1^-\hat{\sigma}_2^- - \hat{\sigma}_1^+\hat{\sigma}_2^+\right) + (\gamma-1)\left(\hat{\sigma}_1^-\hat{\sigma}_2^+ - \hat{\sigma}_1^+\hat{\sigma}_2^-\right)\right]\end{aligned}$$

Given the initial state $|\uparrow\downarrow\rangle$, as discussed in the section on reduction of Hilbert space, we can reduce the basis of the Hilbert space to $\mathcal{B}_{red}$. The operators are as follows:

$$\begin{cases} i\left(\hat{\sigma}_1^- \hat{\sigma}_2^+ - \hat{\sigma}_1^+ \hat{\sigma}_2^- |_{\mathcal{B}_{red}}\right) = \sigma_y \\ \hat{\sigma}_1^+ \hat{\sigma}_2^- + \hat{\sigma}_1^- \hat{\sigma}_2^+ |_{\mathcal{B}_{red}} = \sigma_x \\ \hat{\sigma}_1^z |_{\mathcal{B}_{red}} = \sigma_z \\ \hat{\sigma}_2^z |_{\mathcal{B}_{red}} = -\sigma_z \\ \hat{\sigma}_1^+ \hat{\sigma}_2^+ + \hat{\sigma}_1^- \hat{\sigma}_2^- |_{\mathcal{B}_{red}} = \hat{\sigma}_1^- \hat{\sigma}_2^- - \hat{\sigma}_1^+ \hat{\sigma}_2^+ |_{\mathcal{B}_{red}} = 0 \end{cases} \tag{21}$$

$$\hat{\mathcal{H}}_{\text{eff}} = \hbar\left(J\sigma_x + D(\gamma - 1)\sigma_y + g\sigma_z\right)$$

Using Theorem 2 and Equation 20, the evolution operator in the basis $\mathcal{B}_{red}$ is given by:

$$\hat{U}(t, t_0) = e^{-i\hat{\mathcal{H}}_{\text{eff}} t/\hbar} = \cos(\mu t)\hat{\mathbb{I}} - i\sin(\mu t)\left[\frac{J\sigma_x + D(\gamma - 1)\sigma_y}{\mu}\right] \tag{22}$$

where $\mu = \sqrt{J^2 + D^2(\gamma - 1)^2}$.

Thus, using the definition of the time evolution operator and the properties in Proposition **??**, we can compute $|\psi(t)\rangle$:

$$\boxed{|\psi(t)\rangle = \cos(\mu t)\,|\uparrow\downarrow\rangle - i\sin(\mu t)\left[\frac{J + iD(\gamma - 1)}{\mu}\right]|\downarrow\uparrow\rangle} \tag{23}$$

Theorem 3: Concurrence of two qubit

Consider a two-qubit system with the basis states $\mathcal{B} = \{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}$ for its Hilbert space.
For a quantum state $|\psi\rangle$ in $\mathcal{B}$ expressed as

$$|\psi\rangle = C_0\,|\downarrow\downarrow\rangle + C_1\,|\downarrow\uparrow\rangle + C_2\,|\uparrow\downarrow\rangle + C_3\,|\uparrow\uparrow\rangle \tag{24}$$

where $\forall i \in [[0, 3]],\ C_i \in \mathbb{C}$ the concurrence $C(|\psi\rangle)$ is given by

$$C\left(|\psi\rangle\right) = 2\left|C_1 C_2 - C_0 C_3\right| \tag{25}$$

Proof. We start by recalling the definition of concurrence $C(|\psi\rangle)$ for a two-qubit quantum state $|\psi\rangle$ [10]:

$$C\left(|\psi\rangle\right) = \left|\langle\psi|\,\sigma_y \otimes \sigma_y\,|\psi^*\rangle\right| \tag{26}$$

The action of the tensor product $\sigma_y \otimes \sigma_y$ on $|\psi^*\rangle$, considering the transformation properties of σ_y on the basis states, is computed as:

$$\sigma_y \otimes \sigma_y |\psi^*\rangle = C_0^* |\uparrow\uparrow\rangle + C_1^* |\uparrow\downarrow\rangle + C_2^* |\downarrow\uparrow\rangle + C_3^* |\downarrow\downarrow\rangle \tag{27}$$

Next, we evaluate the inner product $\langle\psi| \sigma_y \otimes \sigma_y |\psi^*\rangle$. Recall that the basis states are orthogonal and normalized. Therefore, we only consider terms where the bra and ket vectors match, which gives:

$$\langle\psi| \sigma_y \otimes \sigma_y |\psi\rangle^* = -2C_0C_3 + 2C_1C_2 \tag{28}$$

Thus, the concurrence $C(|\psi\rangle)$ becomes:

$$C(|\psi\rangle) = 2|C_1C_2 - C_0C_3|$$

□

So

$$\begin{aligned} C(|\psi(t)\rangle) &= 2\left| -\cos(\mu t) i \sin(\mu t) \left[\frac{J + iD(\gamma - 1)}{\mu}\right]\right| \\ &= |\sin(2\mu t)| \end{aligned}$$

$$\boxed{C(|\psi(t)\rangle) = |\sin(2\mu t)|}$$