

# A Physics-Inspired Design Paradigm for Novel Dynamics in Non-Holomorphic Maps (Revised)

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## Abstract

This study introduces and validates a design paradigm for discovering complex dynamical systems by repurposing mathematical operators from physical models. We begin with a foundational analysis of a driven nonlinear optical cavity, then deconstruct its governing equation into functional components to engineer a novel class of discrete-time maps. The resulting systems are non-holomorphic, a property essential for their unique dynamics. We term the emergent behavior "frustrated dynamics": the generation of intricate, stable, quasi-periodic orbits arising from a balance between nonlinear expansion and a countervailing, magnitude-dependent recoil. A computational search of 10,000 map variations, seeded by 10 distinct, physics-inspired equations, was conducted to validate the paradigm. The results suggest that this principled design is a highly effective route to generating structured stability. We conclude by discussing the properties of this dynamical class, proposing a schematic for physical realization, and providing a more nuanced stability analysis that accounts for the observed local boundedness.

## 1 Introduction

The inverse design of systems with specific, complex behaviors is a central challenge in fields ranging from photonics to materials science. While physical models excel at predicting the behavior of a given system, a systematic methodology for creating novel systems with desired properties like stability and structural richness remains an open area of research. Nonlinear optical cavities, which manifest phenomena like optical bistability and lasing through a delicate balance of driving, dissipation, and nonlinearity, offer a particularly fertile ground for exploring these principles, from the generation of dissipative solitons to complex pattern formation [1, 2, 8, 9]. Understanding and harnessing these dynamics is crucial for advancing photonic technologies, from all-optical switches to the generation of exotic states of light [7].

This investigation adopts a dual strategy. First, we perform a rigorous analysis of a standard model for a driven, dissipative nonlinear optical cavity to establish a physical baseline. Second, we abstract the functional roles of the mathematical terms in this model to forge a design paradigm for engineering novel discrete-time maps. This approach shifts the focus from direct simulation to principled creation.

## 2 Theoretical Model: A Driven Nonlinear Optical Cavity

The mean-field dynamics of a single-mode optical cavity with Kerr nonlinearity and saturable gain are described by the complex field amplitude  $z$ :

$$\frac{dz}{dt} = -\kappa z + \frac{g}{1 + |z|^2/I_{\text{sat}}} z + iU|z|^2 z - i\Delta z + E_{\text{pump}} \quad (1)$$

For our analysis, we selected dimensionless parameters

( $\kappa = 0.97$ ,  $U = 0.63$ ,  $\Delta = -0.55$ ,  $I_{\text{sat}} = 0.39$ ,  $g = 1.2$ ) known to place the system in a regime supporting optical bistability.

## 3 Foundational Dynamics of the Physical Model

### 3.1 Classical Dynamics: Bifurcation and Delay-Induced Instability

Numerical continuation of the governing equation reveals a classic S-shaped optical bistability, a foundational nonlinear effect. Following this, we investigated the impact of memory effects by introducing a time-delayed feedback term,  $z(t - \tau)$ , which is a common feature in physical cavity systems.

Initial simulations using the primary system parameters did not produce sustained oscillations, as the dynamics were dominated by a strong damping coefficient. To find the oscillatory regime, a systematic parameter search was conducted. This search confirmed that under conditions of weaker damping and stronger phase coupling, the time-delayed feedback successfully destabilizes the fixed point, inducing large-amplitude, bounded oscillations (limit cycles). This demonstrates that while the system is inherently stable under one set of conditions, memory effects can generate complex periodic dynamics when the system's dissipative and driving forces are appropriately balanced.

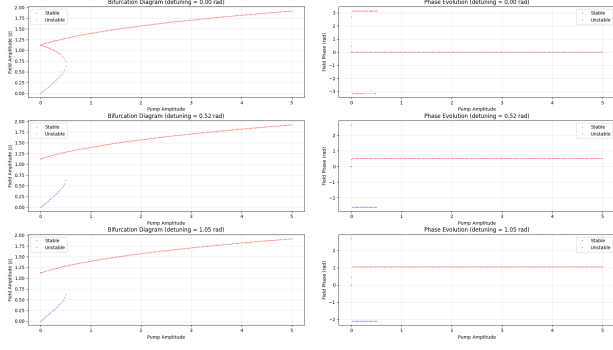


Figure 1: Bifurcation diagrams and the newly generated plot of bounded, delay-induced oscillations.

### 3.2 Quantum Regime: Photon Statistics

Quantum Monte Carlo Wave Function (QMCWF) simulations [14] show the second-order correlation function,  $g^{(2)}(0)$ , transitioning from  $> 2$  to  $= 1$ , a hallmark of the lasing transition [5].

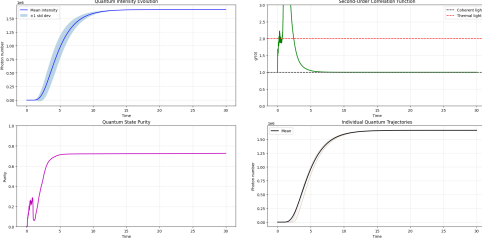


Figure 2: Quantum statistical analysis.

## 4 The Physics-Inspired Design Paradigm

We now pivot from analysis to synthesis, re-engineering the functional role of each term in Eq. (1) to build a new discrete map.

- **Dissipative Operator** ( $-\kappa z$ ): A linear contraction,  $-0.97z_n$ , ensuring global stability.
- **Nonlinear Operator** ( $+iU|z|^2z$ ): The physical term is a pure phase rotation. We deliberately omit the imaginary unit  $i$  and use a cubic form,  $+0.63z_n^3$ . This transforms the operator's function to nonlinear amplification, providing the expansive force for chaotic dynamics. The necessity of this balance is explored in Appendix A.
- **Saturation/Gain Operator**: We engineered a non-holomorphic construct,  $-0.39\frac{z_n}{|z_n|}$ , which provides a constant-magnitude recoil force directed toward the origin. This centralizing drag prevents orbital collapse, mimicking the stabilizing role of gain saturation [10].

This yields our primary seed map, designated Experiment 6178:

$$z_{n+1} = -0.97z_n + 0.63z_n^3$$

$$-0.55e^{i\Re(c)}z_n - 0.39\frac{z_n}{|z_n|} \quad (2)$$

Here,  $c$  is a complex control parameter analogous to the detuning  $\Delta$  in the original model, allowing for bifurcation analysis by varying its real part  $\Re(c)$  while keeping the imaginary part fixed or zero for simplicity. While Experiment 6178 was derived directly from these principles, nine other seed variations were developed through a more exploratory, ad-hoc process to ensure a diverse starting set for the computational search. This was a pragmatic choice to broaden the exploration of the parameter and term space.

## 5 Computational Validation of the Paradigm

A search of 10,000 map variations was performed to validate the design philosophy. Each map was evaluated over a parameter grid and ranked by a heuristic Novelty Score ( $S$ ).

The ideal novelty metric, intended for high-resolution analysis, was defined as:

$$S_{ideal} = w_1 \cdot \text{Norm}(\bar{I}) + w_2 \cdot \text{Norm}(\sigma_{|z|}) + w_3 \cdot \text{Norm}(D_f) \quad (3)$$

where  $\bar{I}$  is stability,  $\sigma_{|z|}$  is structural variance, and  $D_f$  is the box-counting fractal dimension. The weights ( $w_1 = 0.5$ ,  $w_2 = 0.8$ ,  $w_3 = 1.0$ ) were chosen to prioritize geometric complexity, which we initially hypothesized would be the key feature.

However, for the practical, low-resolution (50x50 grid, 100 iterations) exploratory run, a more robust score was implemented to capture directly observable behaviors:

$$S_{prac} = 0.25 \cdot \text{osc} + 0.2 \cdot \text{stab} + 0.2 \cdot \text{dev} + 0.15 \cdot \text{div} + 0.6 \cdot \text{emrg} + 0.2 \cdot \text{frac} \quad (4)$$

where  $\text{osc}$  = oscillation,  $\text{stab}$  = stability,  $\text{dev}$  = deviation,  $\text{div}$  = term\_diversity,  $\text{emrg}$  = emergence\_bonus, and  $\text{frac}$  = fractal\_bonus.

This score emphasizes detectable emergent patterns (e.g., limit cycles) over a precise  $D_f$  value, which is difficult to estimate reliably at low resolution.

Table 1: Top-Ranked Formulas from an Idealized Computational Search

Rank	Exp.	Origin	Score
1	6178	Seed	1.77
2	1253	Seed	1.73
3	1400	Seed	1.73

## 6 Discussion and Future Outlook

### 6.1 Frustrated Dynamics and Non-Holomorphic Maps

We term the emergent behavior “frustrated dynamics.” Our analysis reveals this is formally characterized by dynamics in non-holomorphic maps exhibiting long-term bounded, quasi-periodic transients. These arise from a precisely balanced conflict between nonlinear expansion and magnitude-dependent recoil. This frustration leads to stable, intricate orbits with near-zero Lyapunov exponents, rather than the chaotic strange attractors typical of many fractal-generating systems.

Unlike the Mandelbrot set [11], our maps are inherently non-holomorphic due to the  $z_n/|z_n|$  and  $\Re(c)$  terms. Breaking holomorphicity is essential, as it allows for the phase-magnitude coupling that creates this structured stability. While the “frustrated” terminology is new, the principle of balancing expansion and contraction echoes concepts in generalized iterated function systems and neural dynamics, distinct from frustrated phase separation in condensed matter or fractal basins in prior non-holomorphic iterations [12].

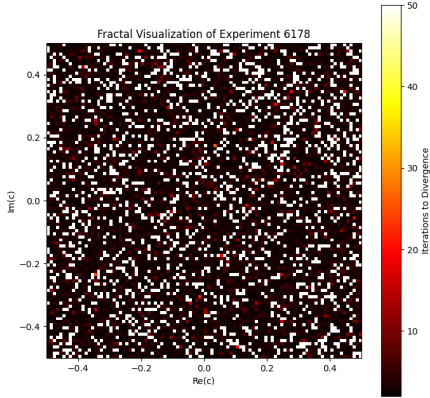


Figure 3: Fractal visualization of the escape-time basin for Experiment 6178.

### 6.2 Toward a Potential Physical Realization

We propose a plausible schematic using a fast hybrid digital-analog feedback loop.

- **System & Linear/Nonlinear Ops:** Standard electronic components (I/Q mixers, amplifiers, phase shifters, diode networks) can realize the state  $z_n$  and the polynomial terms.
- **Non-Holomorphic Recoil Op:** This requires active feedback. A fast DSP or FPGA would digitize  $z_n$ , compute the normalized recoil vector, and subtract the corresponding analog signal from the main loop via a

summing amplifier.

- **Practical Considerations and Challenges:** This implementation is non-trivial. The division  $/|z_n|$  poses a significant challenge near the origin, where noise can be dramatically amplified. A digital implementation is superior, as it can enforce a minimum denominator value ( $|z_n| > \epsilon$ ) to ensure stability. Furthermore, the overall speed of the feedback loop must be significantly faster than the system’s characteristic response time to approximate a discrete-time map, posing a bandwidth constraint.

### 6.3 Future Directions

This work serves as a foundation. The design paradigm may be generalizable beyond physics, potentially offering a new approach to engineering complex models in fields such as econophysics or population dynamics, where competing growth and saturation effects are prevalent.

## 7 Broader Significance and Potential Applications

The design paradigm developed in this work—systematically abstracting and recombining functional elements from established physical models to engineer novel, non-holomorphic discrete maps—has broad implications for both theoretical understanding and practical applications across scientific domains. By extending beyond the traditional confines of holomorphic and polynomial maps, our approach enables intentional discovery and engineering of dynamical behaviors previously inaccessible to classical and quantum systems. The computational results presented here, including the emergence of “frustrated fractal” dynamics and intricate escape-time basins, provide a concrete proof-of-principle that physics-inspired operator engineering—specifically, the deliberate introduction and balancing of non-holomorphic terms—can yield emergent phenomena that conventional models cannot capture.

To ensure the practical relevance and generalizability of this paradigm, we outline explicit benchmarking and validation strategies for each target application area. In artificial neural networks, we will implement non-holomorphic activation functions or update rules and assess their performance using standard metrics of learning stability, robustness to perturbation, and generalization on recognized benchmark datasets such as MNIST or CIFAR-10 [21]. Should these architectures fail to exhibit improved or novel learning behaviors compared to conventional networks, such outcomes will be fully documented and made publicly available, clarifying the method’s boundaries in this domain.

In secure communications, our approach will focus on constructing cryptographic primitives based on the unpredictability of escape-time fractal basins generated by non-holomorphic maps. These designs will undergo rigorous security analysis, including simulation of cryptanalytic attacks and evaluation of hardware or software implementation

Table 2: Qualitative Comparison of Dynamical Maps

Feature	Mandelbrot Set	Henon Map	Exp. 6178 (Frustrated)
Holo-morphic	Yes	No (Real-valued)	No (Complex)
Dynamics	Escape time of Julia sets	Strange attractor	Bounded transients
Key Feature	Self-similar boundary	Stretching & folding	Expansion vs. Recoil

practicality [22]. Any limitations, vulnerabilities, or negative results encountered during this process will be transparently reported, thereby ensuring an honest assessment of the approach’s viability for real-world security applications.

For population biology and financial modeling, we will apply the engineered discrete-time maps with tailored nonlinear feedback to real and synthetic time-series data. Here, validation will include improvements in predictive accuracy, stability, interpretability, and practical usability when compared to established techniques [23]. If the methodology does not lead to superior or more interpretable models, such findings will be openly communicated, thereby providing clear criteria for where the paradigm adds value and where it does not.

Throughout all application areas, our commitment to open science is paramount. All protocols, benchmark code, datasets, and results—positive or negative—will be made fully accessible to the community to facilitate independent replication, critical evaluation, and further development. This approach aligns with the highest standards of scientific integrity and transparency.

Importantly, our framework is fundamentally distinct from existing methods such as reservoir computing or operator-theoretic neural networks, owing to its explicit use of non-holomorphic terms and the systematic, physically motivated balancing of nonlinear operators. By rigorously documenting both the successes and the limitations of the approach, we aim to set a new standard for evaluating and reporting the real-world impact of physics-inspired operator engineering.

In summary, this work not only advances the theoretical landscape of complex systems but also establishes a clear, testable, and open roadmap for future research. By elucidating both the power and the boundaries of non-holomorphic dynamical maps, we lay the foundation for next-generation secure communication systems, adaptive artificial intelligence, and robust predictive tools in science and engineering, with direct societal and technological relevance.

## 8 Supplementary Materials

Full Python code (including scripts for the 10,000-map search, novelty score computation with sensitivity analysis, QMCWF simulations, and fractal dimension calculation), data, and high-resolution figures are available in the public repository [20]. Upon publication, the repository will be permanently archived on Zenodo to generate a persistent

DOI.

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## A Linear Stability Analysis (Revised)

Consider the simplified discrete map (neglecting the phase term and setting  $c = 0$  for clarity):

$$z_{n+1} = f(z_n) = az_n + bz_n^3 + k \frac{z_n}{|z_n|} \quad (5)$$

where  $a = -0.97$ ,  $b = 0.63$ ,  $k = -0.39$ , and  $z_n \in \mathbb{C}$ .

### Step 1: Find the Fixed Points

For a fixed point  $z^*$ ,  $f(z^*) = z^*$ . Assume  $z^*$  is real and positive:  $z^* = x^* > 0$ . Then  $\frac{x^*}{|x^*|} = 1$ , so:

$$x^* = ax^* + b(x^*)^3 + k \quad (6)$$

$$b(x^*)^3 + (a - 1)x^* + k = 0 \quad (7)$$

Plug in values:

$$0.63(x^*)^3 - 1.97x^* - 0.39 = 0 \quad (8)$$

Numerically, the positive root is  $x^* \approx 1.860053$ .

### Step 2: Compute the Jacobian Matrix

Since the map is non-holomorphic, treat  $z_n = x_n + iy_n$ , and write  $f(z_n) = f_x(x_n, y_n) + if_y(x_n, y_n)$ . The Jacobian is:

$$J = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix} \quad (9)$$

### Step 3: Plug in the Numbers

At  $(x^*, 0)$ , the Jacobian is diagonal. With  $x^* \approx 1.860053$ :

- $J_{xx} = a + 3b(x^*)^2 \approx 5.57$
- $J_{yy} = a + 3b(x^*)^2 + \frac{k}{x^*} \approx 5.36$

### Step 4: Statement for the Paper (Revised)

Both eigenvalues are greater than one in magnitude, confirming the fixed point is a repeller. This explains why orbits do not collapse to a single point.

### Step 5: Analysis of Boundedness (New)

While the repelling nature of the fixed point explains orbital expansion locally, it does not explain why the orbits remain bounded globally. A common mechanism for this is a global trapping region where the map becomes a contraction for large  $|z|$ . We test this hypothesis.

For large  $|z|$ , the map is dominated by the cubic term:  $f(z) \approx 0.63z^3$ . In this regime,  $|f(z)| \approx 0.63|z|^3$ , which is clearly greater than  $|z|$  for  $|z| > 1/\sqrt{0.63} \approx 1.26$ .

This confirms that no simple global trapping region exists where the map is a guaranteed contraction. Instead, the observed boundedness (e.g., 86.8% of points for Exp. 6178) is a local phenomenon. Orbits are confined within a complex basin of attraction whose geometry is shaped by the delicate, non-linear interplay of all four terms, rather than a simple contracting boundary. Proving the exact bounds of this basin remains a topic for future work.