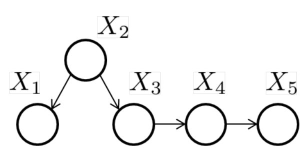
Course on probabilistic graphical models
Master MVA
Practice exercises 2

These exercises are not meant to provide an exhaustive coverage of the material to review for the final exam. To some extend they focus more specifically on material that is not covered in the homeworks. Also, all these exercises should not be taken as representative of the difficulty of the questions posed at the exam, although several questions of the exam are likely to have a similar style. Some exercises are easy, and a few can be much harder so don't be discouraged if you find some of them difficult. They are primarily designed to help you review and consolidate your understanding of the course.

Sum-product in a directed tree

Consider running the sum-product algorithm on the following directed graphical model (DGM), where we suppose that each random variable takes value in $\{1, \ldots, K\}$:



Express all answers as functions of the conditional probabilities $p(x_i|x_{\pi_i})$ for all i.

1. What is the message $m_{1\to 2}(x_2)$ sent from node X_1 to node X_2 during sum-product? Give its simplest form.

2. Suppose that we observe the value of $X_3 = \bar{x}_3$ and that we want to compute $p(x_2|\bar{x}_3)$. Give the message $m_{3\to 2}(x_2)$ sent from node X_3 to node X_2.

3. Give the expression in terms of messages during sum-product for $p(x_2|\bar{x}_3)$, as well as its simplified form.

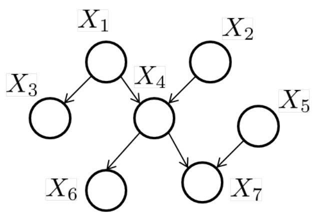
Short problems

1. **Stationary distribution.** Consider a real-valued Gaussian homogeneous Markov chain specified by the recurrence $X_{t+1} = \rho X_t + \epsilon_t$ with $\epsilon_t \perp\!\!\!\perp X_t$ and $(\epsilon_t)_t$ an i.i.d. sequence following the normal distribution $\mathcal{N}(0, \sigma^2)$. Assuming that it is a Gaussian distribution, find the stationary distribution of this Markov chain, that is the distribution P such that if $\mathbb{P}(X_1 \in A) = P(A)$ then $\mathbb{P}(X_t \in A) = P(A)$.

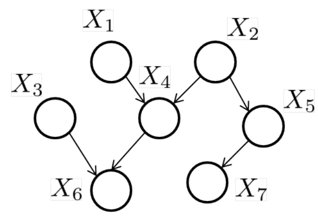
2. **Max entropy.** What is the family of distribution $(p_\alpha)_{\alpha\in\mathbb{R}_+}$ with p_α the distribution of maximal entropy on $\mathbb{N}$, the set of natural integers $\{0, 1, 2, \ldots\}$, such that $\mathbb{E}_p[X] = \alpha$?

3. **Sampling.** Propose a sampling scheme to sample exactly from the distribution $\mathbb{P}(X \in \cdot \mid \|X - y\|_2 \le 1)$ where $y \in \mathbb{R}^d$ and X is a multivariate Gaussian random variable $\mathcal{N}(0, I_d)$. Prove that the proposed sampling scheme indeed yields a variable that has exactly the desired distribution.

4. **Factorization.** Consider the two directed graphical models below. Is it possible to have a distribution p on $X_1, \ldots, X_7$ such that $p \in \mathcal{L}(G_1)$ and $p \in \mathcal{L}(G_2)$? Justify your answer.



G_1



G_2

Message passing for a Gaussian graphical model

Consider the directed Gaussian graphical models with the graph $G = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \big\{(1, 2), (1, 3), (2, 4)\big\}$ with the conditional probability densities $p(x_1) \propto \exp\big(-\frac{1}{2\sigma_1^2}(x_1 - \mu_1)^2\big)$ and for all $(i, j) \in E$, $p(x_j|x_i) \propto \exp\big(-\frac{1}{2\sigma_j^2}(x_j - \rho_j x_i - \mu_j)^2\big)$.

We consider the problem of computing the distributions $p(x_1|x_3, x_4)$ and $p(x_2|x_3, x_4)$ using the sum-product algorithm in the graph in which the variables x_3 and x_4 are fixed to some given values.

1. Show that the sum-product algorithm consists in exchanging messages $\mu_{i\to j}(x_j)$ that are functions of the variable x_j and computed as integrals.

2. Show furthermore that each message is a Gaussian with a given mean and variance.

3. Show that instead of passing messages that are functions, it is sufficient to exchange messages that consist of the corresponding mean and variances.

4. Express the update formulas for these mean-and-variance messages in terms of the means and variances propagated at all nodes in all directions.

5. Similarly express the computation needed to compute the mean and variance of $p(x_1|x_3, x_4)$ and $p(x_2|x_3, x_4)$ from the messages arriving at the nodes 1 and 2 respectively.