Basic concepts from Graph Theory

Guillaume Obozinski

Ecole des Pontes - ParisTech

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Graphs

Graph defined as a pair $G = (V, E)$

- V is a finite set of nodes
- E is a set of edges

Edges in directed graphs are directed

Directed edges are couples of nodes $(u, v) \in E \subset V \times V$

Edges in undirected graphs are undirected

Directed edges are pairs of nodes $\{u, v\} \in E \subset \binom{V}{2}$

Remarks on graph definitions

Remark 1: A directed graph can always be associated its undirected graph obtained by dropping the orientation.

Remark 2: In this course we will only consider graphs with no self-edges ($\{v, v\}$ or (v, v))

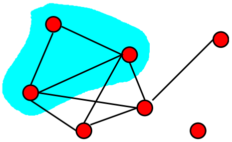
Neighbors and cliques

Neighbors $\mathcal{N}(u)$ of a node u

$$\mathcal{N}(u) = \{ v \in V \mid \{u, v\}\}$$

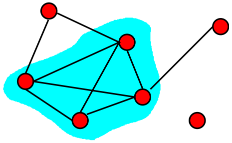
Clique

A totally connected subset of nodes.



Maximal clique

A clique that is not contained in a larger clique.



Path and cycles

Path

A sequence of distinct nodes $(v_0, v_1, \ldots, v_k)$ s.t. $\forall i,\ \{v_{i-1}, v_i\} \in E$.

Cycle

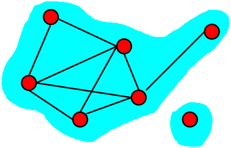
A sequence of nodes $(v_0, v_1, \ldots, v_{k-1}, v_0)$ s.t. $(v_0, v_1, \ldots, v_{k-1})$ is a path and $\{v_{k-1}, v_0\} \in E$.

Connectedness

The relation $a \sim_G b$ defined by "there exists a path between a and b" is an *equivalence relation*[1].

Connected component

The connected components of G are the equivalence classes of the relation $\sim$.



Connected graph

A graph is connected iff it has a single connected component.

[1] A binary relation which is reflexive, symmetric and transitive.

Concepts in undirected graph

Induced graph

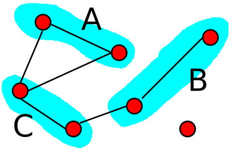
If $G = (V, E)$ is a graph. The *induced graph* on $A \subset V$ is the graph

$$G|_A := (A, E \cap A \times A).$$

Separation

Let A, B, S three disjoint subsets of V.
S separates A from B iff

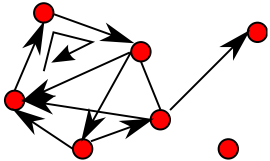
- all paths from $a \in A$ to $b \in B$ go through S
- equivalently: any connected component K the graph induced on $V \setminus S$ is such that either $K \cap A = \varnothing$ or $K \cap B = \varnothing$



Directed Acyclic Graph (DAG)

A directed graph is called *acyclic* if if contains no cycle.

Counterexample:



Some definitions in a DAG

Parent and Child

u is a parent of v iff v is a child of u iff $(u, v) \in E$

Directed path

A sequence of distinct nodes $(v_0, v_1, \ldots, v_k)$ s.t. $\forall i$, $(v_{i-1}, v_i) \in E$.

Directed cycle

A sequence of nodes $(v_0, v_1, \ldots, v_{k-1}, v_0)$ s.t. $(v_0, v_1, \ldots, v_{k-1})$ is a path and $(v_{k-1}, v_0) \in E$.

Ancestor

u is a ancestor of v ($u \preceq_G v$) iff there is a directed path from u to v.
u is strict ancestor if in addition $u \neq v$.

Descendant

v is a (strict) descendant of u iff u is a (strict) ancestor of v.

Topological order for a DAG

A topological order is a *total order* $\prec$ on V compatible with the partial order $\prec_G$ in the sense that

$$u \prec_G v \;\Rightarrow\; u \prec v$$

In other words, if $v_1, v_2, \ldots, v_n$ are in topological order the ancestors of v_i are among $(v_j)_{j \leq i}$.

Proposition

A topological order always exists.

Proof: Induction by removing a maximal element.

Trees

Undirected tree

An undirected tree is a (connected) undirected graph without cycle

Directed tree

An directed tree is a (connected) DAG in which each node has a single parent.

Remarks:

- This definition is the one used in graphical model theory. (Not universally used in CS)
- A directed tree is *not* an undirected tree with any orientation of the edges
- Sometimes called a *rooted tree* because edges must be oriented away from a root.
- A DAG whose underlying undirected graph is a tree is called a *polytree* (or an *oriented tree*).

Forest vs trees, etc

- In graph theory, a forest is a disjoint union of trees (both in the directed and undirected case)
- In this course, we will often not make the distinction and use the word tree even for a graph which is a forest, because the same theory applies to both.
- More generally, when a graph has several connected components, we will be able to treat one component at a time for all relevant task of graphical model theory (i.e. learning, inference and decoding).