Formula of Verdermonde Determinant

Common Knowledge

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Approach Induction

Target "Kill" the last row and column.

1 First Attempt

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} = \begin{vmatrix} 0 & \lambda_1 - \lambda_n & \cdots & \lambda_1^{n-1} - \lambda_n^{n-1} \\ 0 & \lambda_2 - \lambda_n & \cdots & \lambda_2^{n-1} - \lambda_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} \quad (R_k \to R_k - R_n, k \neq n)$$

Question: Use $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$?

$$\begin{vmatrix} 1 & \lambda_{1} & \cdots & \lambda_{1}^{n-1} \\ 1 & \lambda_{2} & \cdots & \lambda_{2}^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{n} & \cdots & \lambda_{n}^{n-1} \end{vmatrix}$$

$$= (-1)^{n+1} \begin{vmatrix} \lambda_{1} - \lambda_{n} & \lambda_{1}^{2} - \lambda_{n}^{2} & \cdots & \lambda_{1}^{n-1} - \lambda_{n}^{n-1} \\ \lambda_{2} - \lambda_{n} & \lambda_{2}^{2} - \lambda_{n}^{2} & \cdots & \lambda_{2}^{n-1} - \lambda_{n}^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_{n} & \lambda_{n-1}^{2} - \lambda_{n}^{2} & \cdots & \lambda_{n-1}^{n-1} - \lambda_{n}^{n-1} \end{vmatrix}$$

$$= (-1)^{n+1} (\lambda_{1} - \lambda_{n})(\lambda_{2} - \lambda_{n}) \cdots (\lambda_{n-1} - \lambda_{n}) \begin{vmatrix} 1 & \lambda_{1} + \lambda_{n} & \cdots & \lambda_{1}^{n-2} + \lambda_{1}^{n-3} \lambda_{n} + \cdots + \lambda_{1} \lambda_{n}^{n-3} + \lambda_{n}^{n-2} \\ 1 & \lambda_{2} + \lambda_{n} & \cdots & \lambda_{2}^{n-2} + \lambda_{2}^{n-3} \lambda_{n} + \cdots + \lambda_{2} \lambda_{n}^{n-3} + \lambda_{n}^{n-2} \end{vmatrix}$$

$$\vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{n-1} + \lambda_{n} & \cdots & \lambda_{n-1}^{n-2} + \lambda_{n-1}^{n-3} \lambda_{n} + \cdots + \lambda_{n-1} \lambda_{n}^{n-3} + \lambda_{n}^{n-2} \end{vmatrix}$$

It doesn't seem good. Let's try another way.

2 Second Attempt

We "kill" the last row by making the entries zero, except the first one.

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

$$(1)$$

$$\begin{vmatrix} 1 & \lambda_{n} & \cdots & \lambda_{n}^{n-1} \\ 1 & \lambda_{1} - \lambda_{n} & \cdots & \lambda_{1}^{n-1} - \lambda_{n} \lambda_{1}^{n-2} \\ 1 & \lambda_{2} - \lambda_{n} & \cdots & \lambda_{2}^{n-1} - \lambda_{n} \lambda_{2}^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{n-1} - \lambda_{n} & \cdots & \lambda_{n-1}^{n-1} - \lambda_{n} \lambda_{2} n - 1 n - 2 \\ 1 & 0 & \cdots & 0 \end{vmatrix}$$

$$= (-1)^{n+1} \begin{vmatrix} \lambda_{1} - \lambda_{n} & \cdots & (\lambda_{1} - \lambda_{n}) \lambda_{1}^{n-2} \\ \lambda_{2} - \lambda_{n} & \cdots & (\lambda_{2} - \lambda_{n}) \lambda_{2}^{n-2} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \end{vmatrix}$$

$$= (1)^{n+1} \begin{vmatrix} \lambda_{1} - \lambda_{n} & \cdots & (\lambda_{1} - \lambda_{n}) \lambda_{1}^{n-2} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \end{vmatrix}$$

$$= (1)^{n+1} \begin{vmatrix} \lambda_{1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \end{vmatrix}$$

$$= (1)^{n+1} \begin{vmatrix} \lambda_{1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \end{vmatrix}$$

$$= (1)^{n+1} \begin{vmatrix} \lambda_{1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \end{vmatrix}$$

$$= (1)^{n+1} \begin{vmatrix} \lambda_{1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_{n} & \cdots & (\lambda_{n-1} - \lambda_{n}) \lambda_{n-1}^{n-2} \end{vmatrix}$$

$$= (-1)^{n+1} \begin{vmatrix} \lambda_1 - \lambda_n & \cdots & (\lambda_1 - \lambda_n)\lambda_1^{n-2} \\ \lambda_2 - \lambda_n & \cdots & (\lambda_2 - \lambda_n)\lambda_2^{n-2} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1} - \lambda_n & \cdots & (\lambda_{n-1} - \lambda_n)\lambda_{n-1}^{n-2} \end{vmatrix}$$

$$(3)$$

$$= (-1)^{n+1} (\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n) \cdots (\lambda_{n-1} - \lambda_n) \begin{vmatrix} 1 & \cdots & \lambda_1^{n-2} \\ 1 & \cdots & \lambda_2^{n-2} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \lambda_{n-1}^{n-2} \end{vmatrix}$$
(4)

$$= (\lambda_n - \lambda_1)(\lambda_n - \lambda_2) \cdots (\lambda_n - \lambda_{n-1}) \begin{vmatrix} 1 & \cdots & \lambda_1^{n-2} \\ 1 & \cdots & \lambda_2^{n-2} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \lambda_{n-1}^{n-2} \end{vmatrix}$$

$$(5)$$

- In (2), $k = n 2, n 3, \dots, 3, 2$. Start with C_{n-2} first.
- In the last step, I changed the factors from $\lambda_k \lambda_n$ to $\lambda_n \lambda_k$, where $k = 1, 2, \dots, n-1$, so that it is in the form of what you see in Wikipedia.

OK! I believe that you know what to do next to get the formula for Verdermonde determinant.

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)$$