Explanation to Wikipedia's Variation of Parameter

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Wikipedia's explanation is too concise. [wiki12] First, you have

$$y^{(n)}(x) + \sum_{k=0}^{n-1} a_k(x)y^{(k)}(x) = b(x).$$
 (i)

Here, y means the general solution of (i).

I just use the notations in Wikipedia: For i between 1 and n, inclusive, y_i is the solution of the homogenous equation

$$y_i^{(n)}(x) + \sum_{k=0}^{n-1} a_k(x) y_i^{(k)}(x) = 0.$$
 (ii)

Unlike Wikipedia, in equation (ii), the i under the summation sign is changed to k so as to avoid confusion of symbols. In other words, we use i to represent $c_i(x)$ and $y_i(x)$, while using k to represent the coefficient of the terms in (i) (i.e. $a_k(x)$) and the derivatives of $y_i(x)$ of different orders. (i.e. $y_i^{(k)}(x)$ Let y_p be the particular solution for the non-homogeneous ODE.

We make an educated guess like this:

$$y_p(x) = \sum_{i=1}^n c_i(x)y_i(x)$$
 (iii)

where $c_i(x)$ is a differentiable function for each i.

By doing so, we have created n unknowns (that is, $c_i(x)$, i = 1, ... n), and they are also known as *freedom*. We may make use of the freedom like this:

$$y_p'(x) = \left[\sum_{i=1}^n c_i'(x) y_i(x) \right] + \left[\sum_{i=1}^n c_i(x) y_i'(x) \right]$$
 (iv)

Continuing the differentiation gives complicated expression, so we use our freedom to give some restrictions on $c_i(x)$, i = 1, ..., n. i.e. We set

$$\sum_{i=1}^{n} c_i'(x)y_i(x) = 0$$
 (v)

Why is there such a condition? It is because we want to simplify (iv) by ruling out *something*. (i.e. setting something to zero) What is *something*?

- 1. $y_i(x) = 0$ or $y'_i(x) = 0$ or $c_i(x) = 0$ for each i = 1, ... nThey are clearly unreasonable restrictions.
- 2. $c'_i(x) = 0$ for each i = 1, ...nThen $c_i(x)$ is a constant function for each i = 1, ...n. In other words, $y_p(x) = \sum_{i=1}^n c_i(x)y_i(x)$ is just linear combinations of $y_i(x)$. Remembering that $y_i(x)$ are denoted as the solution of (ii), we conclude that this restriction can't be accepted.

Being too "strong", the above guesses fails. Thus, what we need to rule out is *either* one of the middle bracket in (iv). Why *must* it be the first one?

Otherwise assume that

$$\sum_{i=1}^{n} c_i(x) y_i'(x) = 0,$$
 (vi)

which leads us to

$$y'_p(x) = \sum_{i=1}^n c'_i(x)y_i(x)$$
 (vii)

Continue the above process of differentiating $y_p(x)$.

$$y_p''(x) = \left[\sum_{i=1}^n c_i''(x)y_i(x)\right] + \left[\sum_{i=1}^n c_i'(x)y_i'(x)\right]$$
 (viii)

Making assumptions similar to (vi), we have

$$\sum_{i=1}^{n} c'_{i}(x)y'_{i}(x) = 0; \text{ and}$$
 (ix)

$$y_p''(x) = \sum_{i=1}^n c_i''(x)y_i(x)$$
 (x)

From (vi) and (ix), we know that if we carry out the process repeatedly, we get a system of "linear" equations with "unknowns" $y'_i(x)$ (which is actually known

since $y_i(x)$ have been found out from (ii).) The most insensible thing of this "system" is the "coefficient matrix" with entries $c_i^{(k)}$ on the k-th row and the i-th column. For the sake of clarity, the "system" is illustrated below.

$$\begin{cases}
c_1(x)y_1'(x) + c_2(x)y_2'(x) + \dots + c_n(x)y_n'(x) &= 0 \\
c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + \dots + c_n'(x)y_n'(x) &= 0 \\
\vdots && & (*) \\
c_1^{(k)}(x)y_1'(x) + c_2^{(k)}(x)y_2'(x) + \dots + c_n^{(k)}(x)y_n'(x) &= 0
\end{cases}$$

We don't care about the range of k such that the R.H.S. of the last equation in the above system is zero, as well as the missing equation which can be yielded after substituting $y_p^{(k)}(x)$ (for example, equations (vii) and (x)) into (i)

Therefore, equations (vi) to (*) are all wrong. Instead, we should have

$$y_p'(x) = \sum_{i=1}^n c_i(x)y_i'(x)$$
 (xi)

$$y_p''(x) = \left[\sum_{i=1}^n c_i'(x)y_i'(x)\right] + \left[\sum_{i=1}^n c_i(x)y_i''(x)\right]$$
(xii)

Again, we set

$$\left| \sum_{i=1}^{n} c_i'(x) y_i'(x) = 0 \right|$$
 (xiii)

so that

$$y_p''(x) = \sum_{i=1}^n c_i(x)y_i''(x).$$
 (xiv)

For each $1 \le k \le n-1$, we have

$$\sum_{i=1}^{n} c_i'(x)y_i^{(k-1)}(x) = 0$$
 (xv)

and

$$y_p^{(k)}(x) = \sum_{i=1}^n c_i(x) y_i^{(k)}(x).$$
 (xvi)

Why can't we apply (xv) and (xvi) to the case of k = n? Differentiating (xvi) with respect to x, we get

$$y_p^{(n)}(x) = \left[\sum_{i=1}^n c_i'(x)y_i^{(n-1)}(x)\right] + \left[\sum_{i=1}^n c_i(x)y_i^{(n)}(x)\right]$$
 (xvii)

Recall from (i) that we have

$$y_p^{(n)}(x) + \sum_{k=0}^{n-1} a_k(x) y_p^{(k)}(x) = b(x).$$
 (xviii)

Substitute (xvi) and (xvii) into (xviii),

$$\begin{split} &\left[\sum_{i=1}^{n}c_{i}'(x)y_{i}^{(n-1)}(x)\right] + \left[\sum_{i=1}^{n}c_{i}(x)y_{i}^{(n)}(x)\right] + \\ &\left\{\sum_{k=0}^{n-1}a_{k}(x)\left[\sum_{i=1}^{n}c_{i}(x)y_{i}^{(k)}(x)\right]\right\} = b(x). \\ &\left[\sum_{i=1}^{n}c_{i}'(x)y_{i}^{(n-1)}(x)\right] + \left[\sum_{i=1}^{n}c_{i}(x)y_{i}^{(n)}(x)\right] + \\ &\left\{\sum_{i=1}^{n}c_{i}(x)\left[\sum_{k=0}^{n-1}a_{k}(x)y_{i}^{(k)}(x)\right]\right\} = b(x). \\ &\left[\sum_{i=1}^{n}c_{i}'(x)y_{i}^{(n-1)}(x)\right] + \left\{\sum_{i=1}^{n}c_{i}(x)\left[y_{i}^{(n)} + \sum_{k=0}^{n-1}a_{k}(x)y_{i}^{(k)}(x)\right]\right\} = b(x) \\ &\left[\sum_{i=1}^{n}c_{i}'(x)y_{i}^{(n-1)}(x)\right] + \left[\sum_{i=1}^{n}c_{i}(x)(0)\right] = b(x) \end{split}$$

Thus, we have

$$\sum_{i=1}^{n} c_i'(x) y_i^{(n-1)}(x) = b(x)$$
 (xix)

From equations (xv) and (xix),

By Cramer's Rule,

$$c_i'(x) = \frac{W_i(x)}{W(x)},$$

where W(x) is the Wronskian of $y_i(x)$ and $W_i(x)$ is the determinant of replacing the *i*-th column of W(x) by $(0,0,\ldots,0,b(x))$. Hence,

$$c_i(x) = \int \frac{W_i(x)}{W(x)} \, \mathrm{d}x. \tag{xx}$$

Substitute (xx) into (iii),

$$y_p(x) = \sum_{i=1}^n y_i(x) \int \frac{W_i(x)}{W(x)} dx$$

References

[wiki12] Variation of Parameters. (n.d.). In *Wikipedia*. Retrieved October 26, 2013, from https://en.wikipedia.org/wiki/Variation_of_parameters