ISDA 609 - Mathematical Modeling Techniques for Data Analytics: Final Project #1

Partha Banerjee, Rohan Fray, Vincent Ying

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Problem Definition

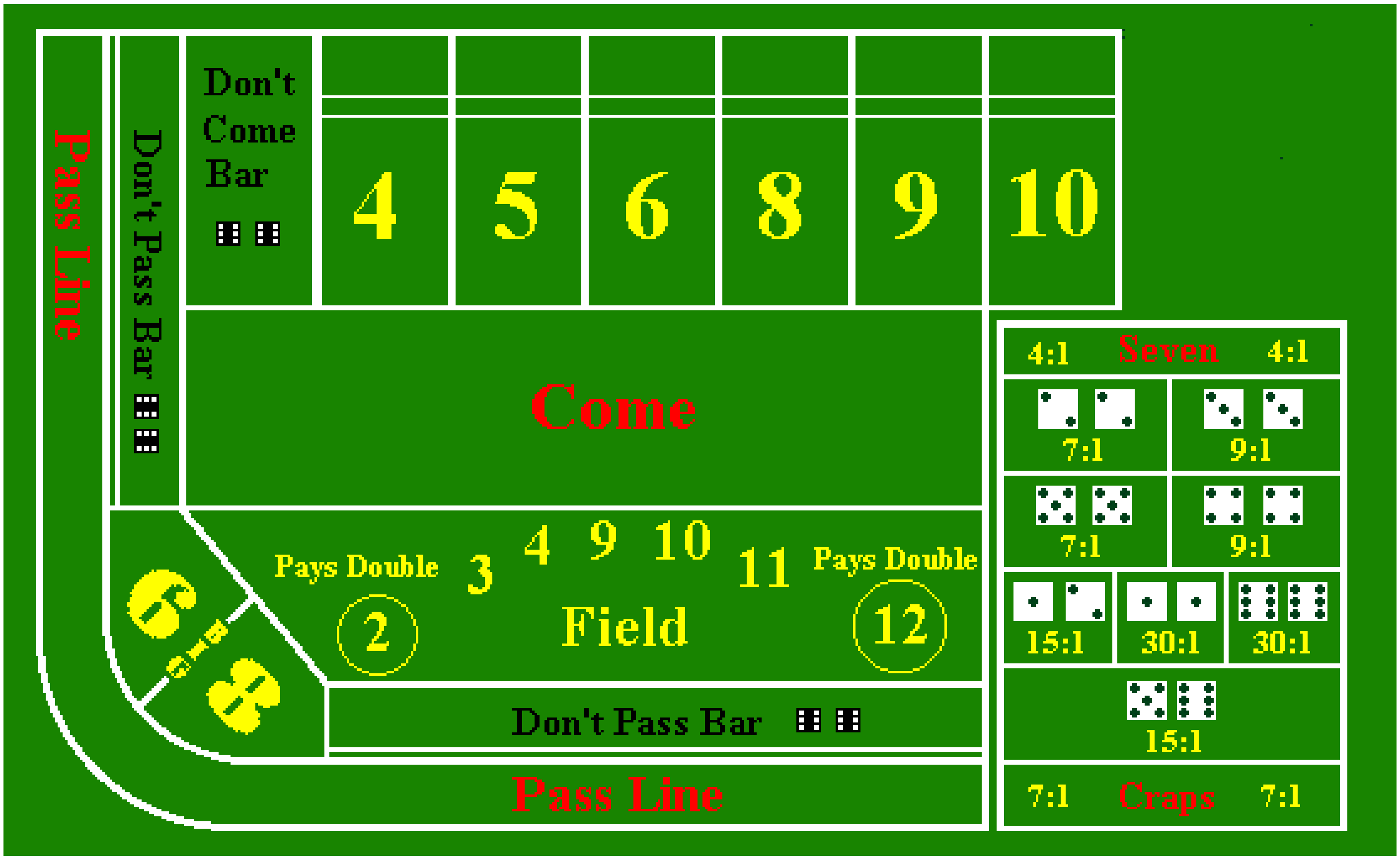
Project 5.3.3 (page 201): Craps Monte Carlo Simulation

Craps - Construct and perform a Monte Carlo simulation of the popular casino game of craps. The rules are as follows:

There are two basic bets in craps, pass and don't pass. In the pass bet, you wager that the shooter (the person throwing the dice) will win; in the don't pass bet, you wager that the shooter will lose. We will play by the rule that on an initial roll of 12 ("boxcars"), both pass and don't pass bets are losers. Both are even-money bets.

Conduct of the game:

- Roll a 7 or 11 on the first roll: Shooter wins (pass bets win and don't pass bets lose).
- Roll a 12 on the first roll: Shooter loses (boxcars; pass and don't pass bets lose).
- Roll a 2 or 3 on the first roll: Shooter loses (pass bets lose, don't pass bets win).
- Roll 4, 5, 6, 8, 9, 10 on the first roll: This becomes the point. The object then becomes to roll the point again before rolling a 7.
- The shooter continues to roll the dice until the point or a 7 appears. Pass bettors win if the shooter rolls the point again before rolling a 7. Don't pass bettors win if the shooter rolls a 7 before rolling the point again.



Write an algorithm and code it in the computer language of your choice. Run the simulation to estimate the probability of winning a pass bet and the probability of winning a don't pass bet. Which is the better bet? As the number of trials increases, to what do the probabilities converge?

Solution - Mathematical Approach

Craps involves the rolling of two dice. The assumption is that the dice are fair and the outcomes of the various rolls are independent.

Simple Mathematics

The possible totals obtained from rolling two dice are as below:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Now let us examine the rules:

- Roll a 7 or 11 on the first roll ("natural"): Shooter wins
  Now the probability of getting 7 or 11 is $\frac{8}{36}$, or about 22.22%.
- Roll a 2 or 3 or 12 on the first roll ("craps"): Shooter loses
  The probability of getting 2 or 3 or 12 is $\frac{4}{36}$, or about 11.11%.
- Roll 4, 5, 6, 8, 9, 10 on the first roll: This becomes the point. The Shooter's goal then becomes to roll the point again before rolling a 7
  The probability of rolling 4 is $\frac{3}{36}$. Once Shooter has rolled the 4, the only cells that matter are the cells containing 4 and 7. All other cells can be ignored. There are 9 cells containing 4 or 7 of which only 3 cells are favorable to Shooter. Hence the probability of Shooter rolling another 4 before a 7 is $\frac{3}{9}$. Therefore, the probability of rolling a 4, and then rolling a 4 before a 7 is $\frac{3}{36} \times \frac{3}{9}$ or about 2.78%.
  Table below summarizes the winning probabilities of Shooter in craps:

Initial Roll	Probability of Winning	Probability in Decimal
4	$\frac{3}{36} \times \frac{3}{9}$	0.0277778
5	$\frac{4}{36} \times \frac{4}{10}$	0.0444444
6	$\frac{5}{36} \times \frac{5}{11}$	0.0631313
7	$\frac{6}{36}$	0.1666667
8	$\frac{5}{36} \times \frac{5}{11}$	0.0631313
9	$\frac{4}{36} \times \frac{4}{10}$	0.0444444
10	$\frac{3}{36} \times \frac{3}{9}$	0.0277778
11	$\frac{2}{36}$	0.0555556
	Total	**0.4929293**

More Mathematical

Let $P(p = n)$ is the probability of rolling a total n. For rolls that are not naturals (7 or 11, say W) or craps (2 or 3 or 12, say L), the probability that the point $p = n$ will be rolled before 7 is found from

$P(win \mid p = n) = \frac{P(p=n)}{P(p=7)+P(p=n))} = \frac{P(p=n)}{\frac{1}{6}+P(p=n)}$

Applying the above, we get the same result as above:

Initial Roll (n)	$P(p = n)$	$P(win \mid p = n)$	$P(win) = P(p = n)P(win \mid p = n)$	Probability in Decimal
2	$\frac{1}{36}$	0	0	0
3	$\frac{2}{36}$	0	0	0
4	$\frac{3}{36}$	$\frac{3}{9}$	$\frac{3}{36} \times \frac{3}{9}$	0.0277778
5	$\frac{4}{36}$	$\frac{4}{10}$	$\frac{4}{36} \times \frac{4}{10}$	0.0444444
6	$\frac{5}{36}$	$\frac{5}{11}$	$\frac{5}{36} \times \frac{5}{11}$	0.0631313
7	$\frac{6}{36}$	1	$\frac{6}{36}$	0.1666667
8	$\frac{5}{36}$	$\frac{5}{11}$	$\frac{5}{36} \times \frac{5}{11}$	0.0631313
9	$\frac{4}{36}$	$\frac{4}{10}$	$\frac{4}{36} \times \frac{4}{10}$	0.0444444
10	$\frac{3}{36}$	$\frac{3}{9}$	$\frac{3}{36} \times \frac{3}{9}$	0.0277778
11	$\frac{2}{36}$	1	$\frac{2}{36}$	0.0555556
12	$\frac{1}{36}$	0	0	0

Hence, $P(win) = \sum_{n=2}^{12} P(p = n)P(win \mid p = n) = 0.4929293$

Hence the probability of the Shooter wins = 49.29%

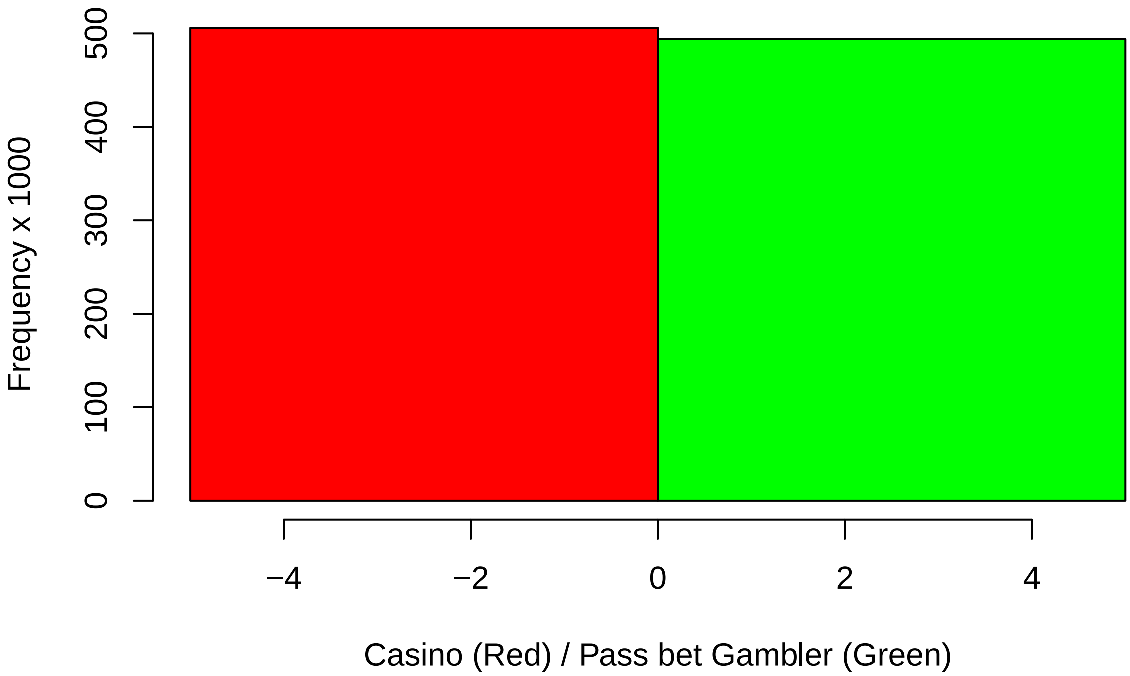
Which implies the probability that the Shooter loses = 50.71%

Note: *We will see the same results in our decision tree model while solve this Craps game.*

Back to Craps Problem

Here we have 2 types of bets - Pass and Don't Pass. In the Pass bet, the gambler wins only when Shooter wins and in Don't Pass bet the gambler wins only when Shooters loses except the Boxcars (Roll a 12 on the first roll).

Hence, for the ***Pass bet gambler***, the winning probability is: 49.29%
And for the Casino (or house), the winning probability is: 50.71%
Thus, *the house has an advantage of about 1.4% on any Pass bet.*



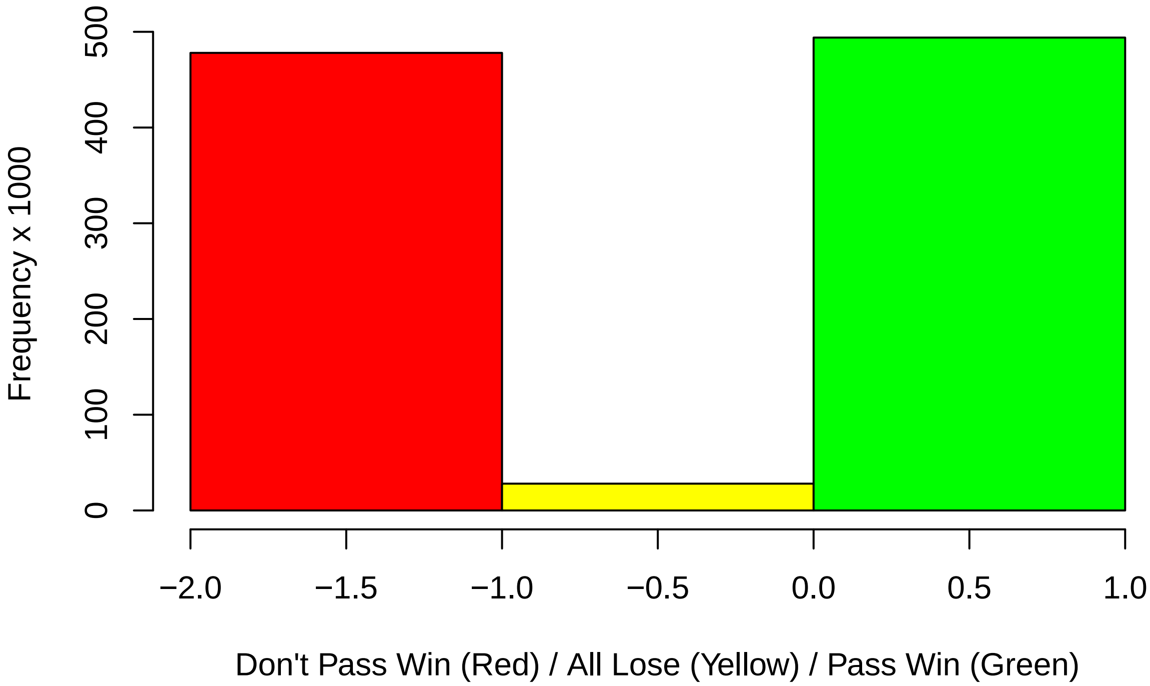
Now the probability of Boxcars = $\frac{1}{36}$ = 2.78%

Hence, for ***Don't Pass bet gambler***,

Event	Probability
TIE on Don't Pass bet	$\frac{1}{36}$ = 0.0277778 = 2.78%
WIN on Don't Pass bet	0.5070707 - $\frac{1}{36}$ = 0.4792929 = 47.93%
LOSE on Don't Pass bet	1 - $\frac{1}{36}$ - 0.4792929 = 0.4929293 = 49.29%

Hence, mathematically we can say that for a particularly game, the winning probability of

- Pass bets: 49.29%
- Don't Pass bets: 47.93%
- Boxcars (both Pass and Don't Pass bets lose): 2.78%



Mathematical Conclusion

The casino has only a slight edge in craps. But, in the long run, the game is a money-maker for the casino since the casino plays on indefinitely. While in Pass bet scenario house is always in advantage, we need to analyze Don't Pass bet bit more.

Since on the tie or Boxcars nobody wins, let us ignore that situation. If we reduce this to a win or lose situation, the probability that a Don't Pass Bet wins is $\frac{0.479293}{(0.479293+0.492929)} = 0.492987$ and the probability that a Don't Pass Bet loses is 1 - 0.492987 = 0.507013. Thus the casino maintains a 1.4% advantage over the player even in Don't Pass bet.

Simulation to verify Mathematical Approach

Monte Carlo Simulation in R

```r
# Number of simulation
N = 10000
cat("Number of simulations = ", N, sep="")
```

```
## Number of simulations = 10000
```

```r
# Define data frame
# First roll: dice1 = fx, dice2 = fy
# Last roll:  dice1 = lx, dice2 = ly
# n = number of rolls
# flag: -1 = Don't Pass, 1 = Pass, 0 = Boxcars
data = data.frame(fx=c(0), fy=c(0), lx=c(0), ly=c(0), n=c(0), flag=c(NA))

for (i in 1:N) {

  # n: number of times dice rolls
  n <- 1

  # First roll
  fx <- sample(1:6,1)
  fy <- sample(1:6,1)
  shooter <- fx + fy

  if (shooter == 12){
    data <- rbind(data, data.frame(fx=c(fx), fy=c(fy), lx=c(""), ly=c(""), n=c(n), flag=c(0)))
  } else if ((shooter == 7)||(shooter==11)){
    data <- rbind(data, data.frame(fx=c(fx), fy=c(fy), lx=c(""), ly=c(""), n=c(n), flag=c(1)))
  } else if ((shooter == 2)||(shooter==3)){
    data <- rbind(data, data.frame(fx=c(fx), fy=c(fy), lx=c(""), ly=c(""), n=c(n), flag=c(-1)))
  } else{
    point <- shooter

    repeat{
      n <- n + 1

      # n-th roll
      lx <- sample(1:6,1)
      ly <- sample(1:6,1)

      if (lx + ly == 7){
        data <- rbind(data, data.frame(fx=c(fx), fy=c(fy), lx=c(lx), ly=c(ly), n=c(n), flag=c(-1)))
        break
      }
      else if (lx + ly == point){
        data <- rbind(data, data.frame(fx=c(fx), fy=c(fy), lx=c(lx), ly=c(ly), n=c(n), flag=c(1)))
        break
```

```
      }
    }
  }
}

# Remove first row inserted at part of initialization
data <- data[!is.na(data$flag),]
```

Data Definition

- fx = Outcome of Dice-1 in first roll
- fy = Outcome of Dice-2 in first roll
- lx = Outcome of Dice-1 in last roll (if subsequent rolls are needed)
- ly = Outcome of Dice-2 in last roll (if subsequent rolls are needed)
- n = number of rolls
- flag:
    - 1 = Pass win
    - 0 = Boxcars (no one wins)
    - -1 = Don't Pass win

Top 25 simulated Craps data:

```
##   fx fy lx ly  n flag
##    5  4  6  3  4    1
##    6  4  4  6  5    1
##    6  2  2  6 12    1
##    4  2  4  2  4    1
##    4  6  3  4  3   -1
##    4  3           1    1
##    6  3  3  4  2   -1
##    2  6  3  5  3    1
##    5  4  3  4 11   -1
##    2  2  5  2  3   -1
##    4  3           1    1
##    2  2  3  4  8   -1
##    5  4  3  6  3    1
##    5  4  3  6  6    1
##    4  4  3  4  5   -1
##    6  5           1    1
##    6  3  5  2  8   -1
##    5  6           1    1
##    5  3  5  3  3    1
##    2  2  2  5  3   -1
##    2  2  4  3  2   -1
##    6  4  5  5  2    1
##    4  3           1    1
##    1  2           1   -1
##    2  1           1   -1
```

Bottom 10 simulated Craps data:

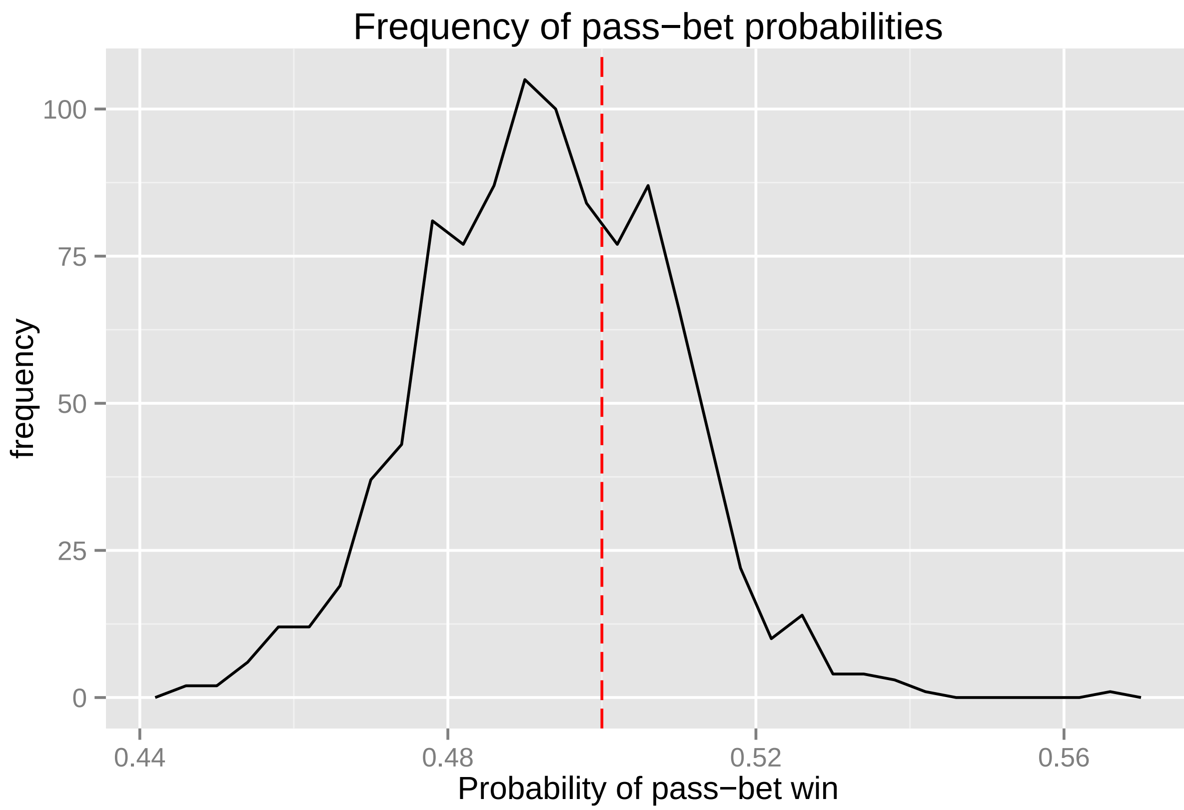
```
##   fx fy lx ly n flag
##    1  2        1   -1
##    2  1        1   -1
##    6  2  5  3 7    1
##    4  2  6  1 3   -1
##    4  4  2  5 2   -1
##    3  1  1  3 5    1
##    2  5        1    1
##    5  1  1  5 4    1
##    3  1  3  1 8    1
##    4  3        1    1
```

Pass wins: 4940 or 49.4%
Don't Pass wins: 4791 or 47.91%
No one wins: 269 or 2.69%

Mean: 0.0149
SD: 0.9863951

If we repeat this simulation of the pass bets 1000 times:

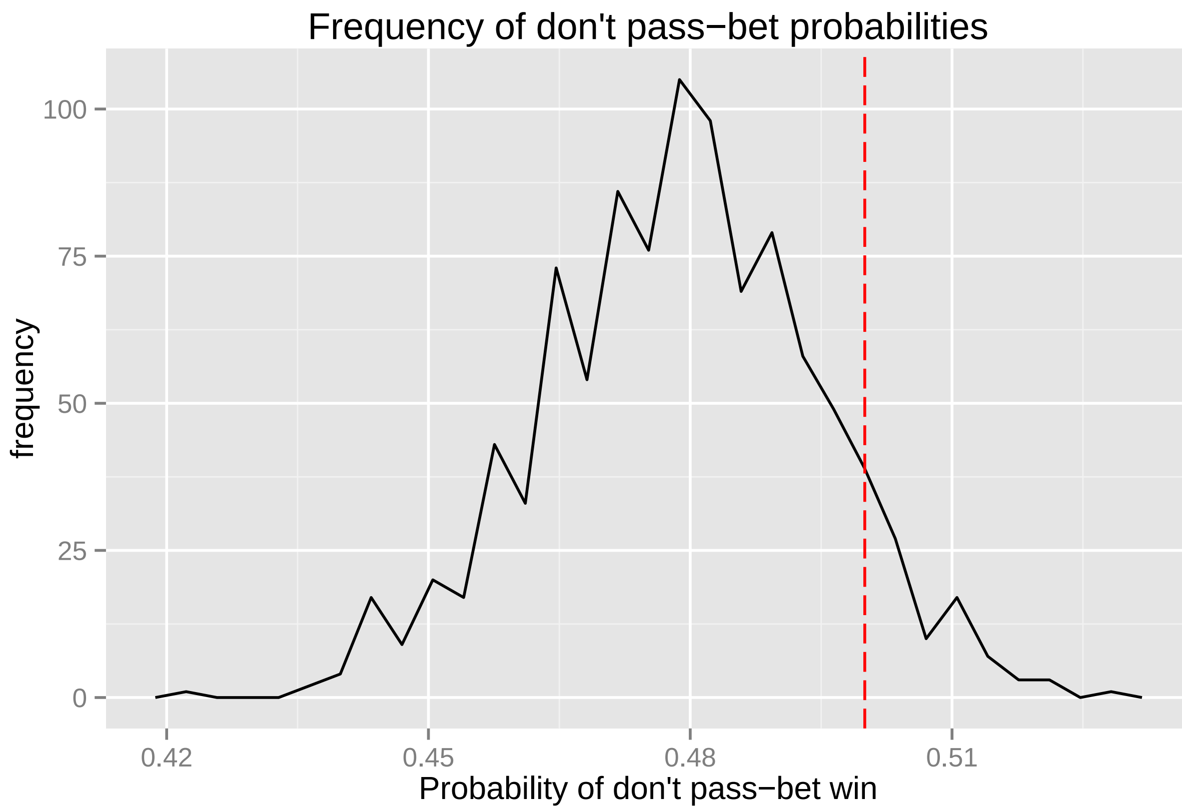
```
## stat_bin: binwidth defaulted to range/30. Use 'binwidth = x' to adjust this.
```



We see that the probabilities are generally below 50%

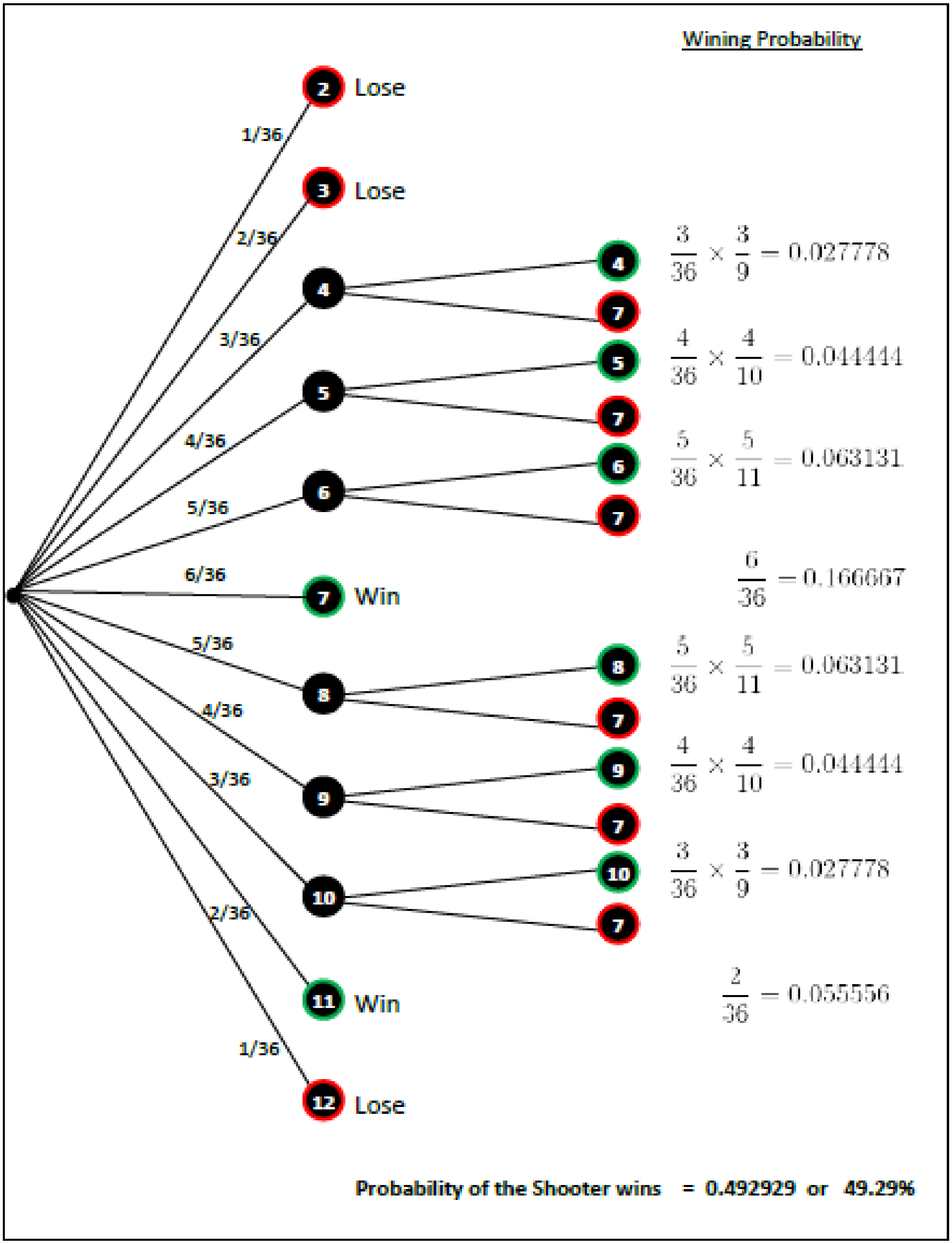
And if we do the same for the Don't Pass bets:

```
## stat_bin: binwidth defaulted to range/30. Use 'binwidth = x' to adjust this.
```



Note that no matter the choice, the odds are usually worse than a coin-flip

Decision tree with expected value of winning



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From the decision tree, we can see that the Probability of Shooter's wining = 49.29%

Note: *This is the same result we received while solveing this game mathematically.*

Conclusion

From both mathematical analysis and simulation game, it is clear that the Craps game is close to a fair gamble where the chance for wining is very close to 50%. However, as it is less than 50% and as the House will play on indefinitely, it will always come out ahead due to this slight imbalance.