State Estimation for Robotics
Assignment 2

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1 Variances of the Noise

1. Base on the data, is the assumption of zero-mean Gaussian noise reasonable? What values of the variances σ_q^2 and σ_r^2, should we use?

Solution:
It is reasonable to assume that the noises are zero-mean Gaussian distributions, because when we use a Gaussian distribution to approximate the true distribution of the noise, we find that the mean of these 2 noises are 0. Hence:

$$\sigma_q^2 = 0.0023 \tag{1}$$

$$\sigma_r^2 = 3.6692 \times 10^{-4} \tag{2}$$

2 Batch objective function

Write out expressions for the batch linear-Gaussian objective function that we will seek to minimize:

$$J(x_{1:K}|u_{1:K}, y_{1:K}) \tag{3}$$

where K is the index of the maximum time.

Solution:
We know it is a 1-D batch estimation problem problem and the system can be written like:

$$motion : x_k = x_{k-1} + Tu_k + w_k \tag{4}$$

$$observation : y_k = x_k + n_k \tag{5}$$

According to the batch solution(using MAP), we can form those matrices as:

$$\mathbf{z} = [\check{x}_0,\ Tu_1,\ ...,\ Tu_k|y_0,\ ...,\ y_k]^T \tag{6}$$

$$\mathbf{x} = [x_0,\ x_1,\ ...,\ x_k]^T \tag{7}$$

$$H = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ -A_0 & 1 & 0 & 0 & 0 \\ 0 & -A_1 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & -A_{K-1} & 1 \\ \hline C_0 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & C_K \end{array}\right] = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{8}$$

$$W = \begin{bmatrix} 0.01 & & & & & & & & & \\ & \sigma_q^2 & & & & & & & & \\ & & \sigma_q^2 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & \sigma_q^2 & & & & & \\ & & & & & \sigma_r^2 & & & & \\ & & & & & & \sigma_r^2 & & & \\ & & & & & & & \sigma_r^2 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \sigma_r^2 \end{bmatrix} \tag{9}$$

Then we can form our cost function as:

$$J(x) = \frac{1}{2}(\mathbf{z}\ -\ \mathbf{Hx})^{\mathbf{T}}\mathbf{W^{-1}}(\mathbf{z}\ -\ \mathbf{Hx}) \tag{10}$$

3 Batch objective function

Derive an expression for the optimal estimates:

$$x_{1:K}^{*} = \underset{x}{argmin}\, J(x_{1:K}|u_{1:K}, y_{1:K}) \tag{11}$$

$$J(x_{1:K}|u_{1:K}, y_{1:K}) \tag{12}$$

where K is the index of the maximum time.

Solution:
$J(x)$ is a exact quadratic function and we can find the minimum by let the first order derivative of $J(x)$ equal to 0.

$$\frac{\partial \mathbf{J(x)}}{\partial \mathbf{x^T}} = -\mathbf{H^T W^{-1}}(\mathbf{z} \ - \ \mathbf{H\hat{x}}) = 0 \tag{13}$$

$$(\mathbf{H^T W^{-1} H})\mathbf{\hat{x}} = \mathbf{H^T W^{-1} z} \tag{14}$$

Hence we get the optimal estimation $\mathbf{\hat{x}}$ by:

$$\mathbf{x_{1:K}^{*}} = \mathbf{\hat{x}} = (\mathbf{H^T W^{-1} H})^{-1}\mathbf{H^T W^{-1} z} \tag{15}$$

Then we can use this function to do Cholesky decomposition or some other tricks to solve these linear equations.

4 estimate only subset of the poses

The dataset has $K = 12709$. It may be computationally intractable to solve for all Kpositions using a batch method. Suppose, however, we only want to solve for a subset pf these poses:

$$x_\delta,\ x_{2\delta},\ x_{3\delta},\ ...,\ x_K \qquad \delta \geq 1$$

we will only use the laser rangefinder reading at corresponding timesteps:

$$r_\delta,\ r_{2\delta},\ r_{3\delta},\ ...,\ r_K \qquad \delta \geq 1$$

But we also want to use all of the odometry speed measurements. Modify the model.

Solution:

We can modify the model as:

$$\mathbf{z}_\delta = [\check{x}_0,\ \Sigma_{i=1}^{\delta} Tu_i,\ \Sigma_{i=\delta+1}^{2\delta} Tu_i,\ ...,\ \Sigma_{i=(n\delta+1)}^{K} Tu_i \mid y_0,\ y_\delta,\ y_{2\delta},\ ...,\ y_k]^T_{\mathbf{2(n+2)}}$$
$$\mathbf{x}_\delta = [x_0,\ x_\delta,\ x_{2\delta},\ ...,\ x_{n\delta},\ x_K]^T_{\mathbf{(n+2)}}$$

where $n = [\frac{K}{\delta}]$.

And the matrices should be redefined as:

$$\mathbf{H}_\delta = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ -A_0 & 1 & 0 & 0 & 0 \\ 0 & -A_\sigma & 1 & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & -A_{n\sigma} & 1 \\ \hline C_0 & 0 & 0 & 0 & 0 \\ 0 & C_\sigma & 0 & 0 & 0 \\ 0 & 0 & C_{2\sigma} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & C_K \end{array}\right] = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]_{\mathbf{2(n+2)\times(n+2)}}$$

$$\mathbf{W}_\delta = \begin{bmatrix} \check{P}_0 & & & & & & & & & \\ & Q_\sigma & & & & & & & & \\ & & Q_{2\sigma} & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & Q_K & & & & & \\ & & & & & R_0 & & & & \\ & & & & & & R_\sigma & & & \\ & & & & & & & R_{2\sigma} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & R_K \end{bmatrix}_{\mathbf{2(n+2)\times(n+2)}}$$

$$
= \begin{bmatrix}
0.01 & & & & & & & & & \\
& \sigma_q^2 & & & & & & & & \\
& & \sigma_q^2 & & & & & & & \\
& & & \ddots & & & & & & \\
& & & & \sigma_q^2 & & & & & \\
& & & & & \sigma_r^2 & & & & \\
& & & & & & \sigma_r^2 & & & \\
& & & & & & & \sigma_r^2 & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & \sigma_r^2
\end{bmatrix}_{\mathbf{2(n+2)\times(n+2)}}
$$

Here we need to point out that: if $\delta = 1$, then the dimension of these two vectors and the matrices would be slightly different from the other δ, since the remainder of $\frac{K}{\delta}$ is equal to 0. i.e. all the $(n + 2)$ should be changed to $(n + 1)$.
Therefore we have:

$$
\mathbf{x}_\delta^* = (\mathbf{H}_\delta^{\mathbf{T}} \mathbf{W}_\delta^{\mathbf{-1}} \mathbf{H}_\delta)^{\mathbf{-1}} \mathbf{H}_\delta^{\mathbf{T}} \mathbf{W}_\delta^{\mathbf{-1}} \mathbf{z} \tag{16}
$$

5 Results

We notice that the error follows the Gaussian distribution and this means that our estimation is only disturbed by white noise (Gaussian noise). Therefore, we can conclude that our estimation is relatively accurate.
Here we also attach the estimated curve and the true curve of the estimation.

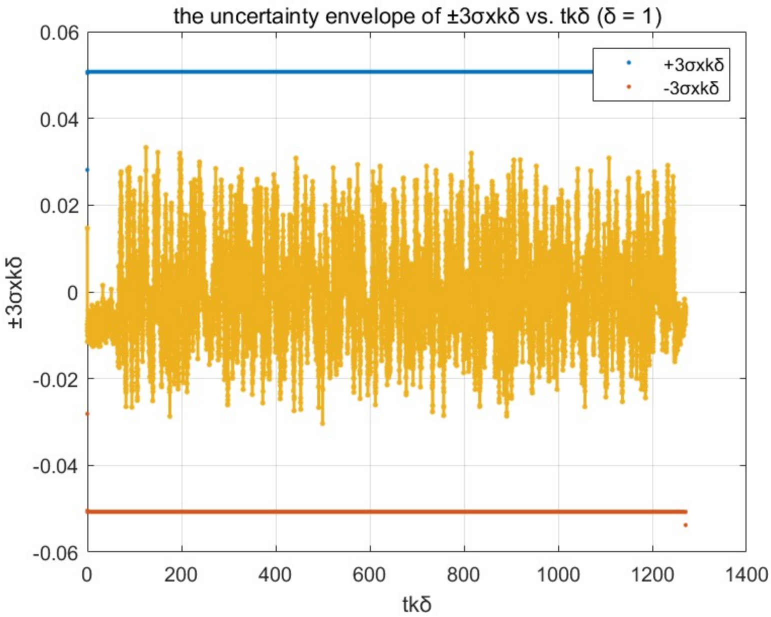

Figure 1: The error between the estimation and the groundtruth ($\delta = 1$)

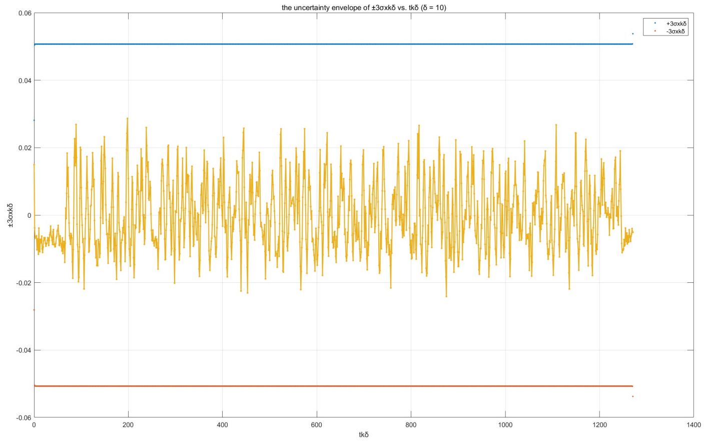

Figure 2: The error between the estimation and the groundtruth ($\delta = 10$)

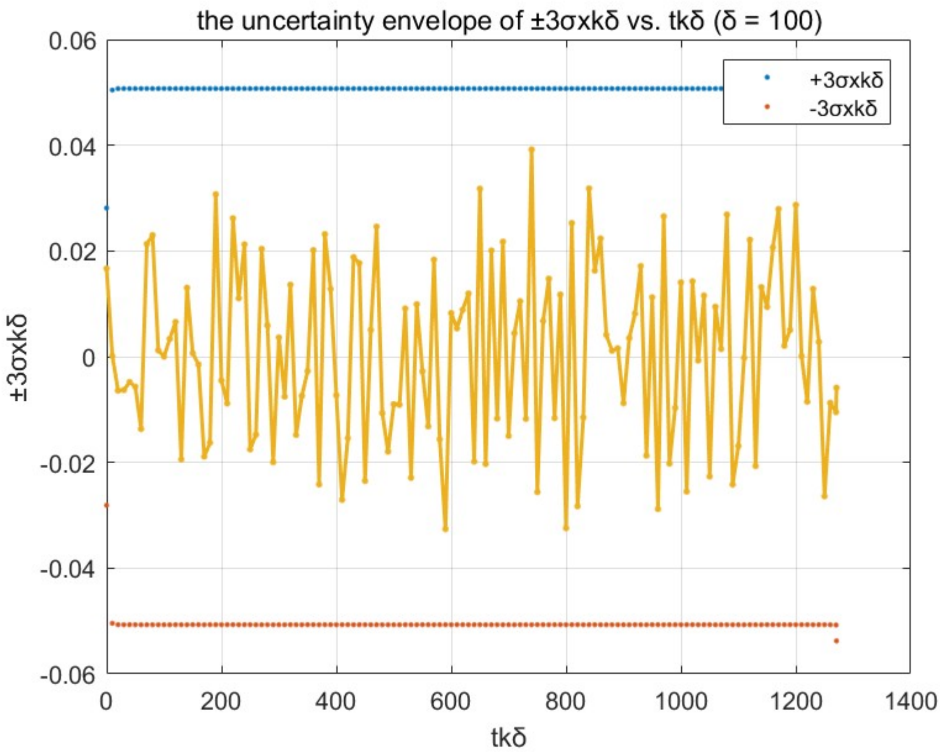

Figure 3: The error between the estimation and the groundtruth ($\delta = 100$)

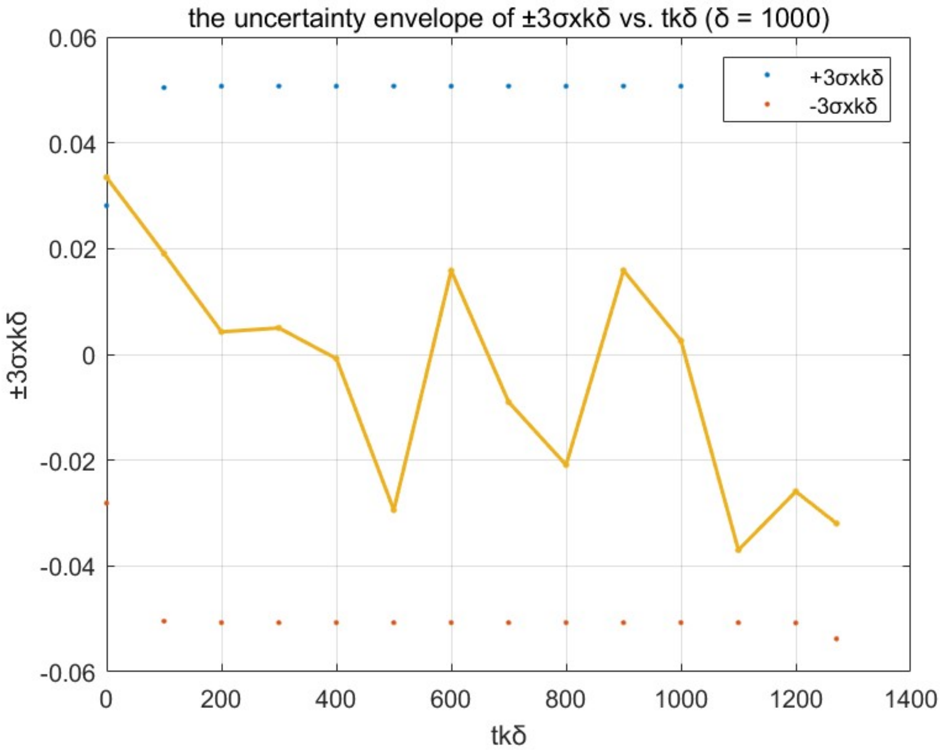

Figure 4: The error between the estimation and the groundtruth ($\delta = 1000$)

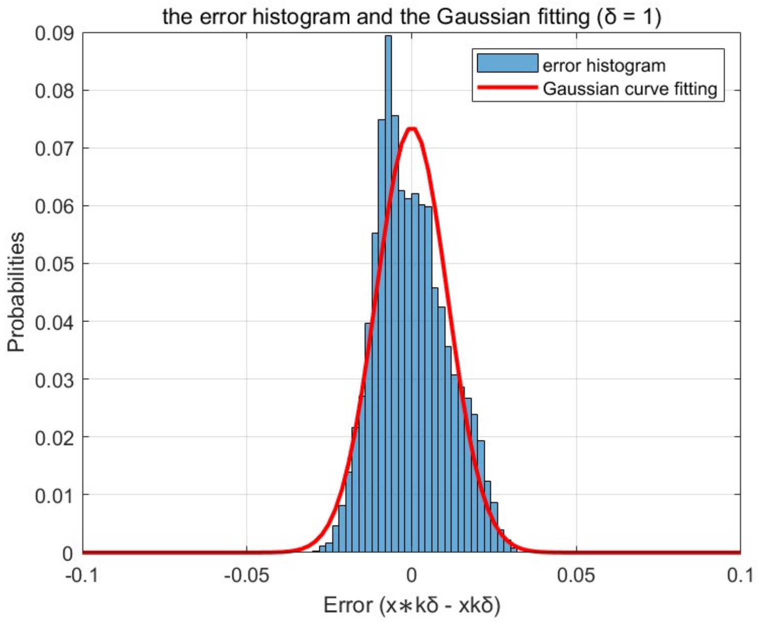

Figure 5: The histogram of error $(x^*_{k\delta} - x_{k\delta}))$ $/(\delta = 1)$

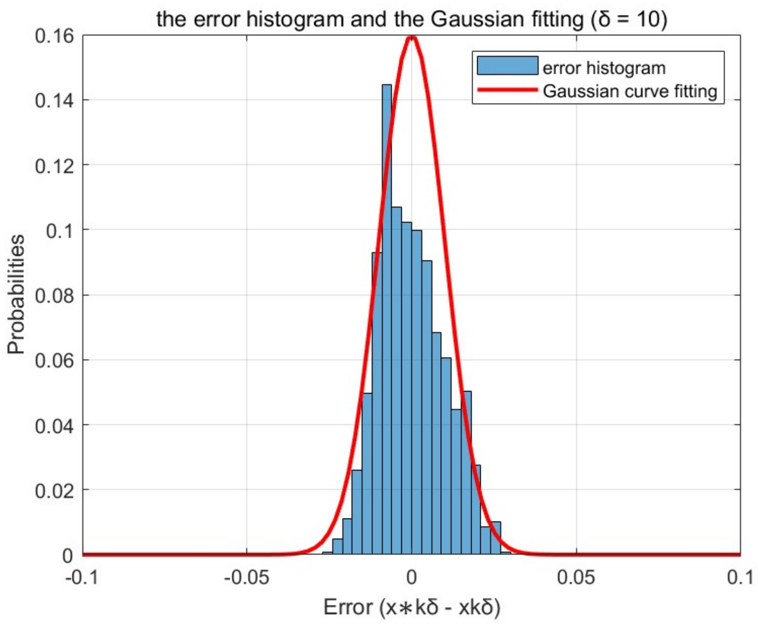

Figure 6: The histogram of error $(x^*_{k\sigma} - x_{k\sigma}))$ $/(\sigma = 10)$

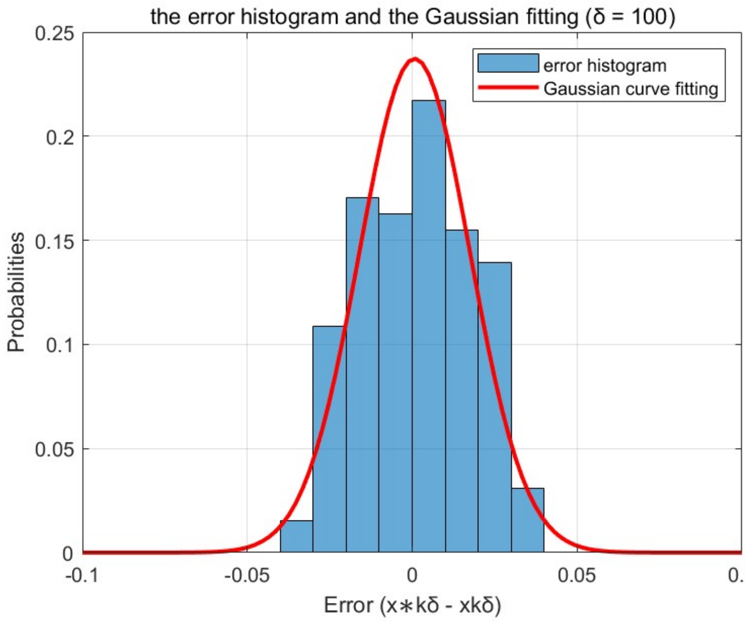

Figure 7: The histogram of error $(x^*_{k\delta} - x_{k\delta}))$ $/(\delta = 100)$

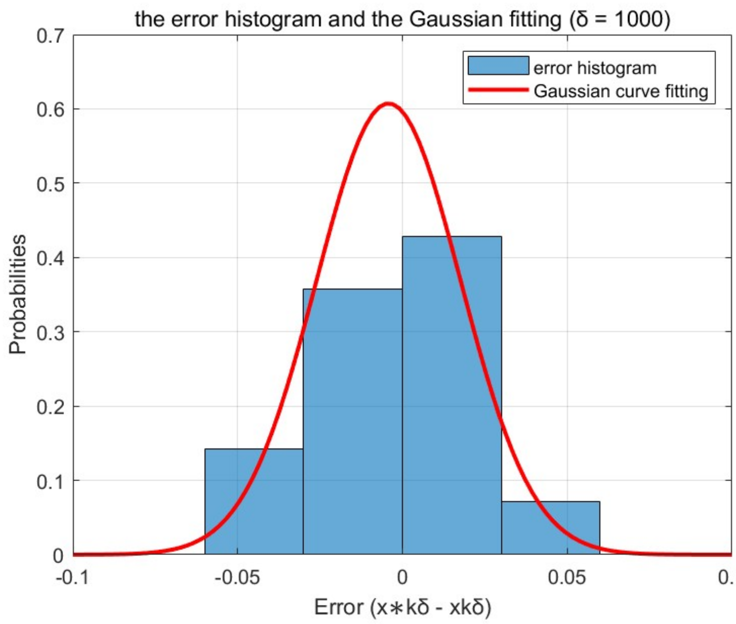

Figure 8: The histogram of error $(x^*_{k\delta} - x_{k\delta}))$ $/(\delta = 1000)$

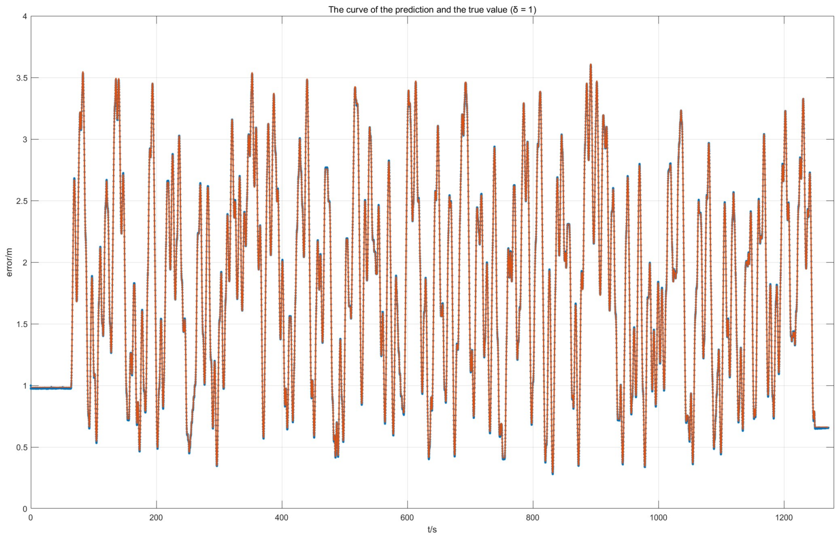

Figure 9: Estimated curve and the true curve ($\delta = 1$)

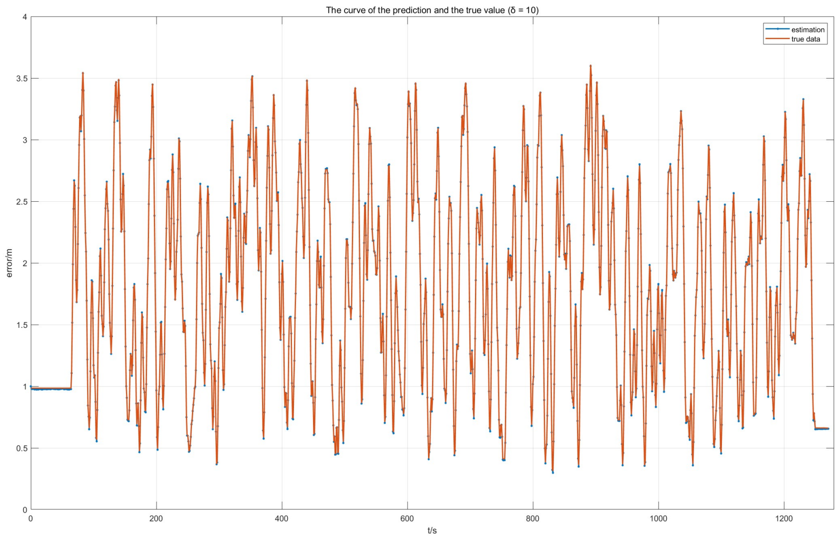

Figure 10: Estimated curve and the true curve ($\delta = 10$)

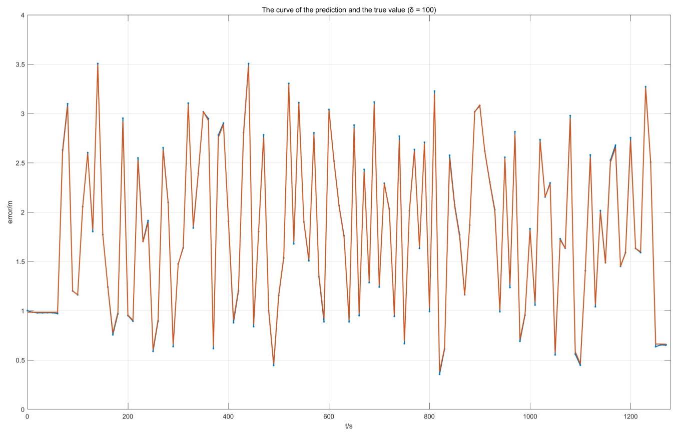

Figure 11: Estimated curve and the true curve ($\delta = 100$)

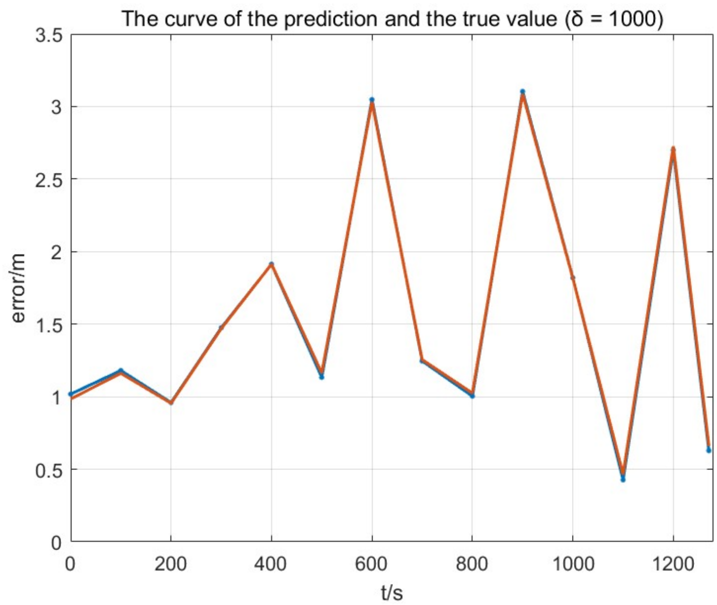

Figure 12: Estimated curve and the true curve ($\delta = 1000$)