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Ex 2.16.

(i) Suppose we have a prior, $p(x) = N(\mu, \Sigma)$, for a random variable, $x \in R^N$. We measure the quantity according to the following measurement model.

$$y = Cx + n$$

where $n \sim N(0, R)$ is measurement noise that is statiscally independent from x. Provide an expression for $p(y|x)$

Solution:

$\because$ y is a linear transformation of x. and $n \sim N(0,R)$ is statiscally independent from x.

$\therefore P(y|x) = P(Cx+n|x) = P(Cx|x)\, P(n)$

$\therefore P(Cx|x) \sim N(Cx, 0)$ (since x is fixed)

$\therefore P(y|x) \sim N(Cx, R)$

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(ii) Form the joint likelihood for x and y by filling in the following:

$$p(x,y) = p(y|x)\,p(x) = \mathcal{N}\left(\begin{bmatrix} ? \\ ? \end{bmatrix}, \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\right)$$

Solution:

$\because$ $P(y|x) \sim \mathcal{N}(C\mu, C\Sigma C^T + R)$, $p(x) \sim \mathcal{N}(\mu, \Sigma)$

$\therefore$ We can easily write that the mean of the joint distribution is $\begin{bmatrix} \mu \\ C\mu \end{bmatrix}$

$\therefore$ also we can conclude that the joint covariance matrix has the form of:

$$\Sigma_{x,y} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^T & \Sigma_{yy} \end{bmatrix} = \begin{bmatrix} \Sigma & \Sigma_{xy} \\ \Sigma_{xy}^T & C\Sigma C^T + R \end{bmatrix}$$

$\therefore$ We need to compute Σ_{xy}

$\because$ $\Sigma_{xy} = E[(x-\mu)(y-C\mu)^T] = E[(x-\mu)(x-\mu)^T C^T]$

$$= E[(x-\mu)(x-\mu)^T]\,C^T$$

$$= \Sigma C^T$$

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$\therefore \Sigma_{xy} = \Sigma C^T \qquad \Sigma_{yx} = C\Sigma^T$

$\because \Sigma$ is s.p.d

$\therefore \Sigma_{xy} = \Sigma C^T \qquad \Sigma_{yx} = C\Sigma$

$\therefore$ insert Σ_{xy}, Σ_{yx} back to the joint covariance matrix

$$\therefore p(x,y) = p(y|x)p(x) \sim \mathcal{N}\left(\begin{bmatrix} \mu \\ C\mu \end{bmatrix}, \begin{bmatrix} \Sigma & \Sigma C^T \\ C\Sigma & C\Sigma C^T + R \end{bmatrix} \right)$$

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(iii) Now factor the joint likelihood the other way

$$p(x,y) = p(x|y)\,p(y)$$

Provide expressions for $p(x|y)$ and $p(y)$

Solution:

We have the joint distribution:

$$p(x,y) \sim \mathcal{N}\left(\begin{bmatrix} \mu \\ C\mu \end{bmatrix}, \begin{bmatrix} \Sigma & \Sigma C^T \\ C\Sigma & C\Sigma C^T + R \end{bmatrix} \right)$$

here we can use the schur complement

$$\begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} = \begin{bmatrix} 1 & \Sigma_{xy}\Sigma_{yy}^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} & 0 \\ 0 & \Sigma_{yy} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Sigma_{yy}^{-1}\Sigma_{yx} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \Sigma C^T (C\Sigma C^T + R)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Sigma - \Sigma C^T (C\Sigma C^T + R)^{-1} C\Sigma & 0 \\ 0 & \Sigma_{yy} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (C\Sigma C^T + R)^{-1} C\Sigma & 1 \end{bmatrix}$$

take the inverse on both sides

$$\begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -(C\Sigma C^T + R)^{-1} C\Sigma & 1 \end{bmatrix} \begin{bmatrix} (\Sigma - \Sigma C^T (C\Sigma C^T + R)^{-1} C\Sigma)^{-1} & 0 \\ 0 & (C\Sigma C^T + R)^{-1} \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & -\Sigma C^T (C\Sigma C^T + R)^{-1} \\ 0 & 1 \end{bmatrix}$$

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$\because$ we have

$$p(x,y) = \frac{1}{\sqrt{(2\pi)^N \det(\Sigma)}} e^{-\frac{1}{2}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \mu \\ C\mu \end{bmatrix}\right)^T \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix}^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \mu \\ C\mu \end{bmatrix}\right)}$$

$\therefore$ insert the inverse we got in the previous step. and we only focus on the exponential part.

$$\therefore \begin{bmatrix} x-\mu \\ y-C\mu \end{bmatrix}^T \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix}^{-1} \begin{bmatrix} x-\mu \\ y-C\mu \end{bmatrix} = \begin{bmatrix} x-\mu \\ y-C\mu \end{bmatrix}^T \begin{bmatrix} \Sigma & \Sigma C^T \\ C\Sigma & C\Sigma C^T + R \end{bmatrix} \begin{bmatrix} x-\mu \\ y-C\mu \end{bmatrix}$$

$$\begin{bmatrix} x-\mu \\ y-C\mu \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ -(C\Sigma C^T+R)^{-1}C\Sigma & 1 \end{bmatrix} \begin{bmatrix} (\Sigma - \Sigma C^T(C\Sigma C^T+R)^{-1}C\Sigma)^{-1} & 0 \\ 0 & (C\Sigma C^T+R)^{-1} \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & -\Sigma C^T(C\Sigma C^T+R)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x-\mu \\ y-C\mu \end{bmatrix}$$

$$= \begin{bmatrix} x-\mu-(y-C\mu)(C\Sigma C^T+R)^{-1}C\Sigma \\ y-C\mu \end{bmatrix}^T \begin{bmatrix} (\Sigma - \Sigma C^T(C\Sigma C^T+R)^{-1}C\Sigma)^{-1} & 0 \\ 0 & (C\Sigma C^T+R)^{-1} \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & -\Sigma C^T(C\Sigma C^T+R)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x-\mu \\ y-C\mu \end{bmatrix}$$

$$= (x-\mu-\Sigma C^T(C\Sigma C^T+R)^{-1}(y-C\mu))^T (\Sigma - \Sigma C^T(C\Sigma C^T+R)^{-1}C\Sigma)^{-1}$$
$$(x-\mu-\Sigma C^T(C\Sigma C^T+R)^{-1}(y-C\mu)) + (y-C\mu)^T(C\Sigma C^T+R)^{-1}(y-C\mu)$$

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$$\boxed{\left(x-\mu-\Sigma C^T(C\Sigma C^T+R)^{-1}(y-C\mu)\right)^T\left(\Sigma-\Sigma C^T(C\Sigma C^T+R)^{-1}C\Sigma\right)^{-1} \times \left(x-\mu-\Sigma C^T(C\Sigma C^T+R)^{-1}(y-C\mu)\right)} +$$

$$\boxed{(y-C\mu)^T(C\Sigma C^T+R)^{-1}(y-C\mu)}$$

$\therefore$ the first term is the expression for $p(x|y)$

the second term is the expression for $p(y)$

$\therefore$ $p(x|y) \sim \mathcal{N}\left(\mu+\Sigma C^T(C\Sigma C^T+R)^{-1}(y-C\mu),\ \Sigma-\Sigma C^T(C\Sigma C^T+R)^{-1}C\Sigma\right)$

$p(y) \sim \mathcal{N}\left(C\mu,\ C\Sigma C^T+R\right)$

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(iv) The Bayesian posterior is $p(x|y)$. Looking at this expression, will the covariance be larger or smaller that of $p(x)$? Explain briefly.

Solution:

$$p(x|y) \sim \mathcal{N}\left(\mu + \Sigma C^T (C\Sigma C^T + R)^{-1}(y - C\mu),\ \Sigma - \Sigma C^T (C\Sigma C^T + R)^{-1} C\Sigma\right)$$

since Σ is s.p.d, R is also s.p.d.
we can write the distribution in this way

$$p(x|y) \sim \mathcal{N}\left(\mu + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y),\ \Sigma - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}\right)$$

we can know that $\Sigma_{xy}, \Sigma_{yx}, \Sigma_{yy}, \Sigma$ are all s.p.d

$\therefore\ \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} > 0$

$\therefore$ the covariance decreases after the measure, which means that the uncertainty of x after Bayesian inference is smaller than the uncertainty of $p(x)$.

$\downarrow$

(the posterior knowledge)

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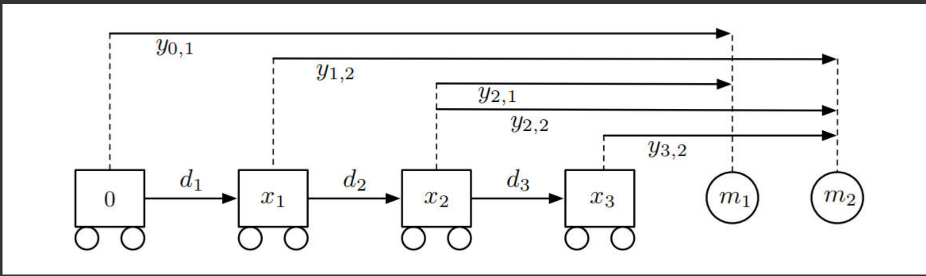
Ex 3.8

Consider the simple one-dimensional problem in the diagram; all quantities live on the x-axis.



The x_k are robot poses, the d_k are odometry measurements, the m_l are landmark positions, and the $y_{k,l}$ are landmark measurements. It is an example of SLAM (simultaneous localization and mapping). Since we are going to estimate the robot and the landmark positions.

(i) Define (linear) error terms for the odometric and landmark measurements.

Solution.

Since we should define a linear error:

$$\begin{cases} e_k = [d_1-(x_1-x_0),\ d_2-(x_2-x_1),\ d_3-(x_3-x_2)] \\ e_{k,l} = [y_{0,1}-(m_1-x_0)\ \ y_{2,1}-(m_1-x_2)\ \ y_{1,2}-(m_2-x_1)\ \ y_{2,2}-(m_2-x_2)\ \ y_{3,2}-(m_2-x_3)]^T \end{cases}$$

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(ii) Organize these error into a stacked error for all the measurements,

$$e = \begin{bmatrix} e_1 & e_2 & e_3 & e_{0,1} & e_{2,1} & e_{1,2} & e_{2,2} & e_{3,2} \end{bmatrix}^T$$

in terms of the measurements and the state,

$$y = \begin{bmatrix} d_1 & d_2 & d_3 & y_{0,1} & y_{2,1} & y_{1,2} & y_{2,2} & y_{3,2} \end{bmatrix}^T$$

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & m_1 & m_2 \end{bmatrix}^T$$

Solution:

here we can define a observation Matrix C

$\therefore$ $y = Cx + e$ and $e = y - Cx$

the original errors are written as follow:

$$\begin{cases} e_k = \left[d_1 - (x_1 - x_0),\ d_2 - (x_2 - x_1),\ d_3 - (x_3 - x_2) \right]^T \\ e_{k,l} = \left[y_{0,1} - (m_1 - x_0) \quad y_{2,1} - (m_1 - x_2) \quad y_{1,2} - (m_2 - x_1) \quad y_{2,2} - (m_2 - x_2) \quad y_{3,2} - (m_2 - x_3) \right]^T \end{cases}$$

$$e = \big[d_1 - (x_1 - x_0),\ d_2 - (x_2 - x_1),\ d_3 - (x_3 - x_2),$$
$$y_{0,1} - (m_1 - x_0),\ y_{2,1} - (m_1 - x_2),\ y_{1,2} - (m_2 - x_1),\ y_{2,2} - (m_2 - x_2),\ y_{3,2} - (m_2 - x_3) \big]^T$$

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We can conclude that

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \Rightarrow y = Cx + e$$

$$\therefore e = (y - Cx)$$

(iii) Assuming both the odometric and landmark measurements are corrupted by zero-mean Gaussian noise with covariance 1, define a cost function that we can minimize to find the state that is most likely given the measurements.

Solution:

here we can define a cost function in this way:

$$J(x) = \sum_{k=1}^{K} J_y(x)$$

$$J_y(x) = \frac{1}{2}(y - Cx)^T \mathbf{1}^{-1} (y - Cx)$$

C is the matrix we wrote in the previous page.

the next step we need to do is to minimize this cost function.

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(iv) Take the derivative of the cost function, J, with respect to the state, x, and set this to zero to find an equation for the optimal state, x^*.

Solution.

$$J(x) = \frac{1}{2}(y - Cx)^T 1 (y - Cx)$$

$$\therefore \left.\frac{\partial J(x)}{\partial x^T}\right|_{\hat{x}} = -C^T 1^{-1}(y - Cx)\Big|_{\hat{x}} = -C^T(y - C\hat{x})$$

$\because$ we want the first order derivative to be 0

$$\therefore -C^T(y - C\hat{x}) = 0$$

$$-C^T y + C^T C\hat{x} = 0 \Rightarrow C^T C\hat{x} = C^T y$$

$\therefore$ Since our cost function is exactly quadratic and so there is only one minimum and it is also the global minimum, where the derivative is equal to zero.

$\therefore$ $x^* = (C^T C)^{-1} C^T y$ is the optimal state for equation.

(v) Will x^* be unique in this problem? Why / why not?

Solution.

here we need to check whether the C^TC is full-rank, in other words, the determinant of C^TC is not equal to 0 (not singular)

$$C^TC = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{bmatrix}$$

$\det(A) = 29 > 0$

$\Longrightarrow \operatorname{rank}(C^TC) = 5$

it is full-rank

$\therefore$ the solution is unique.

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Ex 3.9

Consider the one-dimensional simultaneous localization and mapping (SLAM) setup below with K robot positions and one landmark. There are odometry measurements, d_k, between each pair of positions as well as landmark range measurements y_k, from the first and last robot positions.

The motion and observation models are as follows:

$$\begin{bmatrix} p_k \\ m_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ m_{k-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (d_k + w_k) \qquad w_k \sim N(0, \tfrac{2}{K})$$

$$y_k = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} p_k \\ m_k \end{bmatrix} + n_k \qquad n_k \sim N(0,1)$$

where p_k is the position of the robot and m_k is the position of the (stationary) landmark, both at time K. We will use the Kalman filter approach to carry out the simultaneous estimation of the robot and landmark positions.

(i) The Kalman filter mean and covariance will be initialized as:

$$\hat{x}_0 = \begin{bmatrix} \hat{p}_0 \\ \hat{m}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ ? \end{bmatrix}, \quad \hat{P}_0 = \begin{bmatrix} 0 & ? \\ ? & ? \end{bmatrix}$$

where we have set $\hat{p}_0 = 0$ to make the problem observable and also set the robot position covariance to zero. Use the first landmark range measurement y_0 to help initialize the ? entries.

Solution.

① Since we have no other information of the initial position of the landmark, we can only approximately set it as y_0

$$\therefore \hat{x}_0 = \begin{bmatrix} \hat{p}_0 \\ \hat{m}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ y_0 \end{bmatrix} \Leftarrow \hat{m}_0 = y_0 + \hat{p}_0 = y_0$$

② since we apply the measurement y_0 as the initial value, we also include a uncertainty (y_0 is an observation)

$$\therefore y_0 = m_0 + n_k \leftarrow n_k \sim N(0,1)$$

$\because$ we assumed at time 0 the position of the robot is totally certain, i.e. the covariance of the $\hat{p}_0$ is 0.

$\therefore$ the cross-covariance between y_0 and $\hat{p}_0$ should be 0, and the covariance of y_0 is 1.

$\therefore$ $\hat{x}_0 = \begin{bmatrix} \hat{p}_0 \\ \hat{m}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ y_0 \end{bmatrix}$ $\quad \hat{P}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

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(ii) Apply the prediction step of the kalman filter K times in a row (with no correction) to determine expressions for the mean and covariance at time K, just before applying the correction step associated with y_K:

$$\check{x}_K = ? \qquad \check{P}_K = ?$$

Solution:

we assume that $\hat{x}_{k-1} = \begin{bmatrix} \hat{p}_{k-1} \\ \hat{m}_{k-1} \end{bmatrix}$, $\hat{P}_{k-1}$ at time K

$$\therefore \begin{bmatrix} \check{p}_k \\ \check{m}_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{p}_{k-1} \\ \hat{m}_{k-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (d_k) = \begin{bmatrix} \sum_{k=1}^{K} d_k \\ y_0 \end{bmatrix}$$

$$\check{P}_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \hat{P}_{k-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T + \begin{bmatrix} \frac{2}{K} & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{k-1} \hat{P}_{k-1} A_{k-1}^T + Q_k = \begin{bmatrix} \sum_{k=1}^{K} \frac{2}{K} & 0 \\ 0 & 1 \end{bmatrix}$$

the reason why $Q = \begin{bmatrix} \frac{2}{K} & 0 \\ 0 & 0 \end{bmatrix}$ is that the process noise w has nothing to do with $\hat{m}_{k-1}$ so only the $\check{p}_k$ will be affected.

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(iii) Apply the final correction step of the Kalman filter (associated with y_k) to determine final expressions for the mean and covariance at time k:

$$\hat{x}_k = ? \qquad \hat{P}_k = ?$$

Solution:

According to the form of a standard Kalman filter, we take $C_k = [-1 \quad 1]$, $\check{P}_k = A_{k-1}\hat{P}_{k-1}A_{k-1}^T + Q_k$, $R_k = 1$ and the measurement at step k is y_k

$\therefore$ the Kalman gain $K_k = \check{P}_k C_k^T (C_k \check{P}_k C_k^T + R_k)^{-1}$

$$\therefore \check{P}_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{k-1} \frac{2}{k} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{k} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{k} \frac{2}{k} & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_k = \begin{bmatrix} \sum_{k=1}^{k} \frac{2}{k} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \left([-1, 1] \begin{bmatrix} \sum_{k=1}^{k} \frac{2}{k} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1 \right)^{-1}$$

$$= \frac{1}{\sum_{k=1}^{k} \frac{2}{k} + 2} \left[-\sum_{k=1}^{k} \frac{2}{k}, \; 1 \right]^T$$

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① . $\hat{x}_k = \check{x}_k + K_k(y_k - C_k\check{x}_k)$

$$\hat{x}_k = \check{x}_k + \check{P}_k C_k^T (C_k \check{P}_k C_k^T + R_k)^{-1} (y_k - C_k \check{x}_k)$$

$$\hat{x}_k = \begin{bmatrix} \sum_{k=1}^{K} d_k \\ y_0 \end{bmatrix} + \left(\frac{1}{\sum_{k=1}^{K} \frac{2}{K} + 2} \begin{bmatrix} -\sum_{k=1}^{K} \frac{2}{K} \\ 1 \end{bmatrix} \right) \left(y_k + \sum_{k=1}^{K} d_k - y_0 \right)$$

② for $\hat{P}_k$:

$$\hat{P}_k = (1 - K_k C_k)\check{P}_k = \left(1 - \check{P}_k C_k^T (C_k \check{P}_k C_k^T + R_k)^{-1} C_k\right)\check{P}_k$$

$$= \check{P}_k - \check{P}_k C_k^T (C_k \check{P}_k C_k^T + R_k)^{-1} C_k \check{P}_k$$

$$\hat{P}_k = \begin{bmatrix} \sum_{k=1}^{K} \frac{2}{K} & 0 \\ 0 & 1 \end{bmatrix} -$$

$$\left(\frac{1}{\sum_{k=1}^{K} \frac{2}{K} + 2} \begin{bmatrix} -\sum_{k=1}^{K} \frac{2}{K} \\ 1 \end{bmatrix} \right) \begin{bmatrix} -1, 1 \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{K} \frac{2}{K} & 0 \\ 0 & 1 \end{bmatrix}$$

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(iv) if we had used the batch approach (see previous question) would our estimate for the position of the robot and landmark at time K be different or the same? Explain.

Solution:

the positions of the robot and the landmark would be the same.

firstly we know that from the recursive method we finally get a kalman filter (forward RTS Smoother), the form is like:

$$
KF \begin{cases} \check{P}_k = A_{k-1}\hat{P}_{k-1}A_{k-1}^T + Q_k \\ \check{x}_k = A_k \hat{x}_{k-1} + v_k \end{cases} \text{prediction}
$$

$$
\quad K_k = \check{P}_k C_k^T (C_k \check{P}_k C_k^T + R_k)^{-1} \quad \text{kalman gain}
$$

$$
\quad \begin{cases} \hat{P}_k = (1 - K_k C_k)\check{P}_k \\ \hat{x}_k = \check{x}_k + K_k(y_k - C_k \check{x}_k) \end{cases} \text{update}
$$

Since our model is the same, using batch solution involving cholesky decomposition, which allows us to transform the batch solution into the recursive solution So batch solution depends not only on the previous state, but also other

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states before k-1, while recursive solution uses the Information at k-1 only. (k-1 contains all the info in the past)

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Ex. 4

Consider the uncertainty ellipsoid inside the set of points x given by equation.

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = M^2$$

(a) (dim=2) Show that the area inside the uncertainty ellipse define is $A = M^2 \pi \sqrt{\det \Sigma}$.

Solution:

here we suppose that:

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = M^2 \quad \text{(dimension is 2)} \qquad ①$$

since Σ^{-1} (the covariance) is positive definite, so we can do a Eigendecomposition on Σ.

$$\Sigma = U \Lambda U^T \qquad ②$$

where $U^T U = U U^T = \mathbf{1}$; $\Lambda = \mathrm{diag}(\lambda_i), i=1,2$, $U = (u_1, u_2)$

λ_1, λ_2 are the eigenvalues of Σ.

$\therefore$ we have:

$$\Sigma = U \Lambda U^T = (u_1 \;\; u_2) \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1^T \\ u_2^T \end{pmatrix} = \sum_{i=1}^{2} u_i \lambda_i u_i^T \qquad ③$$

$$\Sigma^{-1} = (U \Lambda U^T)^{-1} = U \Lambda^{-1} U^T = \sum_{i=1}^{2} u_i \frac{1}{\lambda_i} u_i^T \qquad ④$$

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We put ④ back to ① and we get.

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = (x-\mu)^T \sum_{i=1}^{2} u_i \frac{1}{\lambda_i} u_i^T (x-\mu) = M^2$$

$$\therefore M^2 = \sum_{i=1}^{2} \frac{1}{\lambda_i} \left[(x-\mu)^T \mu_i\right]\left[\mu_i^T (x-\mu)\right] \quad ⑤$$

then we suppose $y = \begin{pmatrix} y_1^T \\ y_2^T \end{pmatrix} = (x-\mu)^T u_i$

$\therefore \quad M^2 = \frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2}$ (it is a ellipse function) ⑥

$\therefore$ we normalize this function and we will get

$$\frac{y_1^2}{\lambda_1 M^2} + \frac{y_2^2}{\lambda_2 M^2} = 1 \quad \left(\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} = 1 \right) \quad ⑦$$

$\therefore \quad a = \sqrt{\lambda_1 M^2} \qquad b = \sqrt{\lambda_2 M^2}$

$$\therefore S = \pi ab = M^2 \pi \sqrt{\lambda_1 \lambda_2} \quad ⑧$$

$\because$ Eigendecomposition won't change the determinant of the original Matrix

so $|\det \Sigma| = |\det \Lambda| = |\lambda_1 \lambda_2|$

$$\therefore S = \pi ab = M^2 \pi \sqrt{\lambda_1 \lambda_2} = M^2 \pi \sqrt{\det \Sigma} \quad ⑨$$

end.

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(b) (dim=N) Show that the volume inside the uncertainty hyper-ellipsoid is

$$V = \frac{M^N \pi^{\frac{N}{2}} \sqrt{\det \Sigma}}{\Gamma(\frac{N}{2}+1)}$$

where Γ is the Gamma function (the generalization of the factorial function for real numbers).

Observe that $N=2$ gives the above formula for A.

Solution:

first we rewrite the function in this way:

$$V = \frac{M^N \pi^{\frac{N}{2}} \sqrt{\det \Sigma}}{(\frac{N}{2}+1-1)!} = \frac{M^N \pi^{\frac{N}{2}} \sqrt{\det \Sigma}}{(\frac{N}{2})!} \quad ①$$

we know that the volume of the hyper-ellipsoid can be express like.

$$V_n = \frac{2}{n} \frac{\pi^{\frac{n}{2}} (a_1 a_2 a_3 \cdots a_n)}{\Gamma(\frac{N}{2})} = \frac{\pi^{\frac{n}{2}} (a_1 a_2 \cdots a_n)}{\Gamma(\frac{N}{2}+1)} \quad ②$$

where $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1$

$\therefore$ we do the trick again in n-dimension.

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suppose $(x-\mu)^T\Sigma^{-1}(x-\mu)=M^2 \quad x\in R^N$

$$\therefore \underset{N\times N}{\Sigma} = U\Lambda U^T = (u_1\ u_2\ u_3 \cdots u_N)\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \ \lambda_N\end{pmatrix}\begin{pmatrix}u_1^T\\u_2^T\\ \vdots \\ u_N^T\end{pmatrix} \quad ③$$

$$\therefore \Sigma^{-1} = U\Lambda^{-1}U^T = \sum_{i=1}^{N} u_i \frac{1}{\lambda_i} u_i^T \quad ④$$

$\therefore$ put ④ back to $(x-\mu)^T\Sigma^{-1}(x-\mu)$

$$\therefore (x-\mu)^T \sum_{i=1}^{N} u_i \frac{1}{\lambda_i} u_i^T (x-\mu) = M^2$$

$$\therefore \sum_{i=1}^{N} \frac{1}{\lambda_i}\left[(x-\mu)^T u_i\right]\left[u_i^T(x-\mu)\right] = M^2$$

$\therefore$ take $y=(x-\mu)^T u_i$

$$\therefore \sum_{i=1}^{N}\frac{1}{\lambda_i} y_i y_i^T = \sum_{i=1}^{N}\frac{1}{\lambda_i} y_i^2 = M^2$$

$$\therefore \frac{y_1^2}{\lambda_1}+\frac{y_2^2}{\lambda_2}+\frac{y_3^2}{\lambda_3}+\cdots\frac{y_N^2}{\lambda_N} = M^2$$

$$\therefore \frac{y_1^2}{\lambda_1 M^2}+\frac{y_2^2}{\lambda_2 M^2}+\cdots+\frac{y_N^2}{\lambda_N M^2} = 1 \quad ⑤$$

$$\therefore \begin{cases} a_1=\sqrt{\lambda_1 M^2} \\ a_2=\sqrt{\lambda_2 M^2} \\ \vdots \\ a_N=\sqrt{\lambda_N M^2}\end{cases} \quad ⑥$$

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put ⑥ back to ②

we get:

$$V_N = \frac{M^N \pi^{\frac{N}{2}} \sqrt{\lambda_1 \lambda_2 \cdots \lambda_N}}{\Gamma\left(\frac{N}{2}+1\right)} = \frac{M^N \pi^{\frac{N}{2}} \sqrt{\det \Sigma}}{\left(\frac{N}{2}\right)!}$$

when $D=2$

$$V_N = \frac{M^2 \pi^{\frac{2}{2}} \sqrt{\det \Sigma}}{\left(\frac{2}{2}\right)!} = M^2 \pi \sqrt{\det \Sigma}$$

end.