

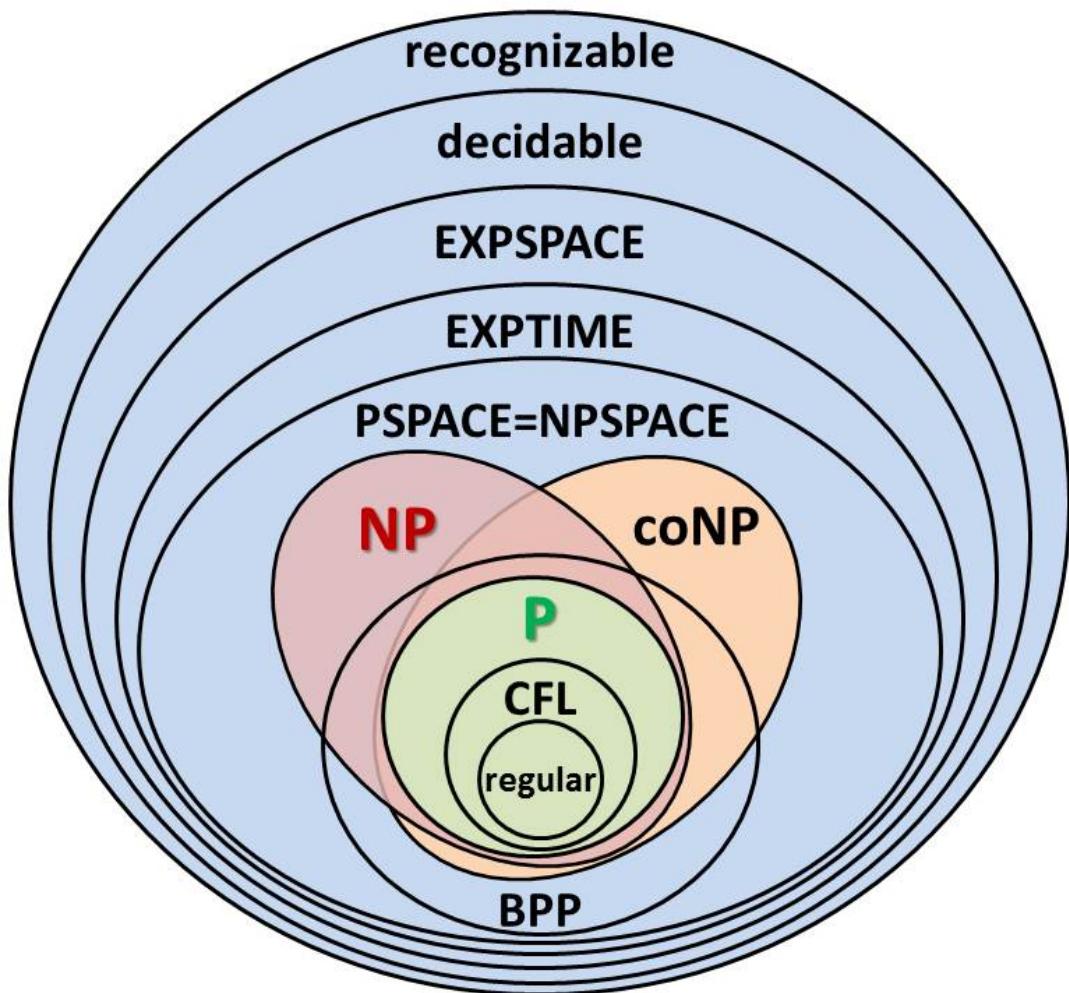
Introduction to Computation Theory

Vinsong

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Abstract

The lecture note of 2025 Fall Introduction to Computation Theory by professor 林智仁.



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Chapter 0

Basic Knowledge

Lecture 1

0.1 Mathematical Notions

2025-09-01

0.1.1 Set & its operation

Definition 0.1.1 (Set). Omitted

Definition (Sequence & Tuple). Here are some definitions of basic containers

Definition 0.1.2 (Sequence). Sequence is the objects in order, which have two properties:

- Order:

$$(1, 2, 3) \neq (2, 1, 3)$$

- Repetition:

$$\text{Sequence : } (1, 2, 3) \neq (1, 1, 2, 3)$$

$$\text{Set : } \{1, 2, 3\} = \{1, 1, 2, 3\}$$

Definition 0.1.3 (Tuple). Finite sequence, $(1, 2, 3)$ is a 3-tuple

Definition 0.1.4 (Cartesian Product). Here is the Cartesian Product between two sets. We define

$$A = \{1, 2\}, B = \{x, y\}$$

then,

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

0.1.2 Function & Relation

Definition 0.1.5 (Function). Function is a machine with single output.

Definition (Equivalence Relations). Here are the properties of Equivalence Relations.

Definition 0.1.6 (reflexive).

$$\forall x, xRx$$

Definition 0.1.7 (symmetric).

$$\forall x, y, xRy \iff yRx$$

Definition 0.1.8 (transitive).

$$xRy, yRz \implies xRz$$

Example.

$$i \equiv_7 j, \text{ if } 0 = i - j \pmod{7}$$

- Reflexive

$$i - i = 0 \pmod{7}$$

- Symmetric

$$i - j = 7a, j - i = -7a$$

- Transitive

$$i - j = 7a, j - k = 7b \implies i - k = 7(a + b)$$

0.1.3 String & Languages

Definition (String & Languages). Here is the definition of Language.

Example (Alphabet).

$$\{0, 1\}$$

Example (String).

$$01000$$

Definition 0.1.9 (Language). Set of Strings

$$L(A)$$

is the language of A

0.2 Definitions, Theorems, and Proofs

- **Definition:** Introduce new concept.
- **Statement:** A sentence that is either true or false.
- **Theorem:** A statement that is true.
 - **Lemma:** A “helping” theorem.
 - **Corollary:** A theorem that follows easily from another theorem.

0.2.1 Proof by Construction

Proposition 0.2.1. Sum of degrees of every graph is even

Proof. Each edge contributes 2 nodes, so

$$\sum_{v \in V} \deg(v) = 2 \times |E|$$

Hence, the sum of degrees of every graph is even. ■

Note. The implication is the definition of graphs.

0.2.2 Proof by Contradiction

Assume the statement is false, then deduce a contradiction.

0.2.3 Proof by Induction

- Basis: Prove for $n = 0$ or $n = 1$ or some trivial case.
- Inductive Step: Assume true for $n = k$ (Induction Hypothesis), prove for $n = k + 1$.

Chapter 1

Regular Languages

1.1 Deterministic Finite Automata (DFA)

- Automaton: single
- Automata: plural

Definition 1.1.1 (Deterministic Finite Automata (DFA)). We define a DFA as a 5-tuple

$$(Q, \Sigma, \delta, q_0, F)$$

where

- Q : Set of states (**Finite**)
- Σ : Alphabet (i.e. set of input characters) (**Finite**)
- $\delta: Q \times \Sigma \rightarrow Q$: Transition Function
- $q_0 \in Q$: Start state
- $F \subset Q$: Set of accept states

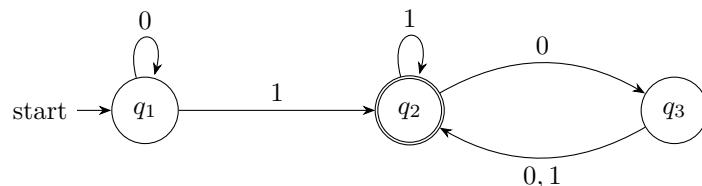


Figure 1.1: A state diagram

If we call this machine M , then we have.

$$M = (Q, \Sigma, \delta, q_0, F)$$

For the example given above,

$$Q = \{q_1, q_2, q_3\}$$

$$\Sigma = \{0, 1\}$$

$$q_0 = q_1$$

$$F = \{q_2\}$$

The δ function:

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

Definition 1.1.2. The language that recognize by a Machine M is denoted as

$$L(M) = A$$

We say A is recognized (accepted) by M .

1.1.1 Definition of Computation

Let,

- $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton.
- $w = w_1, \dots, w_n$ be a string over Σ .

Theorem 1.1.1. M accepts w if \exists states $r_0 \dots r_n$ such that

- (1) $r_0 = q_0$
- (2) $r_{i+1} = \delta(r_i, w_{i+1}), \quad i = [0, n - 1]$
- (3) $r_n \in F$

Definition 1.1.3 (Regular Language). A language is regular if recognized by some automata.

1.1.2 Regular Operations

Definition. Assume A, B are given languages,

Definition 1.1.4 (Union).

$$A \cup B = \{w \mid w \in A \vee w \in B\}$$

Definition 1.1.5 (Concatenation).

$$A \circ B = \{w_1 w_2 \mid w_1 \in A, w_2 \in B\}$$

Definition 1.1.6 (Kleene Star).

$$A^* = \{w_1 \cdots w_k \mid k \geq 0, w_i \in A\}$$

which can also be defined as

$$\bigcup_{i=1}^{\infty} A_i = \{\varepsilon\} \cup A \cup A^2 \cup A^3 \cup \dots, \quad A^0 = \{\varepsilon\}, \quad A^n = \{wv \mid w \in A^{n-1}, v \in A\}$$

Definition 1.1.7 (closed). We say an operation R is closed if the following property holds if

$$x \in A, y \in A, \text{ then } xRy \in A$$

Theorem 1.1.2. Regular languages are closed under the union, concatenation, and Kleene star.

Proof. We define two machines as follows

$$M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$$

$$M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$$

if we union them, we can define a new machine

$$M_1 \cup M_2 = \begin{cases} M = (Q, \Sigma, \delta, q_0, F) \\ Q = \{(r_1, r_2) \mid r_1 \in Q_1, r_2 \in Q_2\} \\ \delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a)) \\ q_0 = (q_1, q_2) \\ F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\} \end{cases}$$

Hence, regular languages are closed under union. ■

Lecture 2

1.2 Nondeterministic Finite Automata (NFA)

2025-09-08

First, we see a NFA that accept strings with 1 in 3rd position from the end,

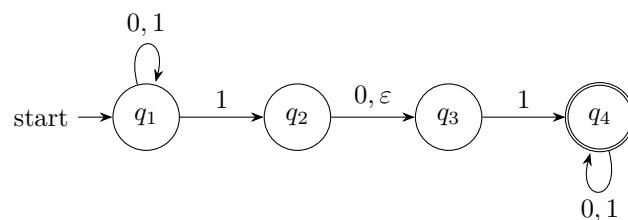


Figure 1.2: NFA machine

- δ is not a function, i.e. $\delta(q_1, 1) = q_1$ or q_2
- ε between q_2, q_3 means q_2 can move to q_3 without any input

We can transport NFA to DFA by some method, for example, for the above NFA we can have:

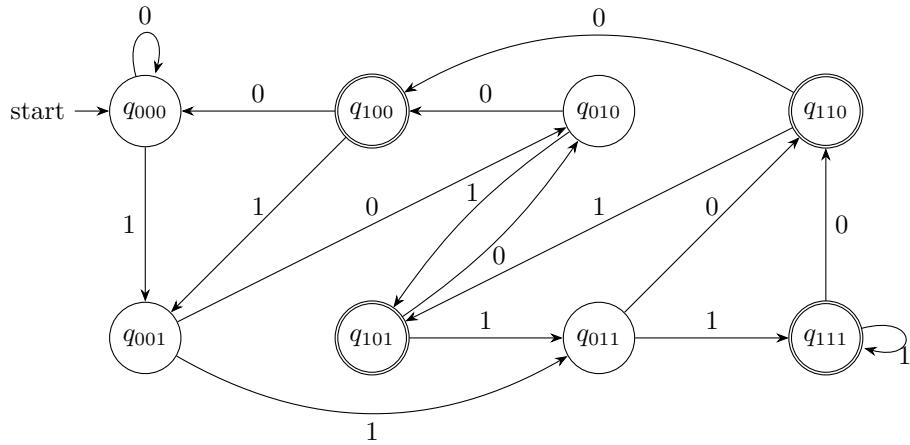


Figure 1.3: NFA machine transport to DFA

We can record it in three bits, it will be complicated.

Definition 1.2.1 (power set).

$$P(Q) = \{X | X \in Q\}$$

which contain all the $2^{|Q|}$ combinations.

Definition 1.2.2 (Nondeterministic Finite Automata (NFA)). We define a NFA as a 5-tuple

$$M = (Q, \Sigma_\varepsilon, \delta, q_0, F)$$

where

- Q : Set of states (**Finite**)
- $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$
- $\delta: Q \times \Sigma_\varepsilon \rightarrow P(Q)$
- $q_0 \in Q$
- $F \subset Q$

Theorem 1.2.1. We have w

$$w = y_1 \cdots y_m \quad \text{where } y_i \in \Sigma_\varepsilon$$

A sequence $r_0 \cdots r_m$ such that

- (1) $r_0 = q_0$
- (2) $r_{i+1} = \delta(r_i, y_{i+1}), \quad i = [0, n - 1]$
- (3) $r_n \in F$

Note. So m may not be the original length (as y_i may be ε)

1.2.1 Equivalence of DFA and NFA

From DFA \Rightarrow NFA. Formally DFA is not an NFA due to Σ and Σ_ε . but we can easily handle this by adding

$$q_i, \varepsilon \rightarrow \emptyset$$

For NFA \Rightarrow DFA, we have the example on the slides on a graph.

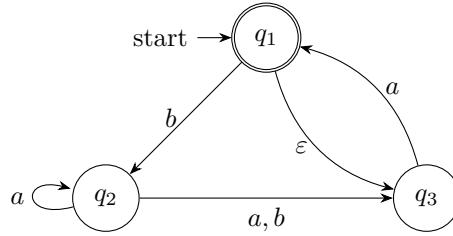


Figure 1.4: NFA example

\Downarrow

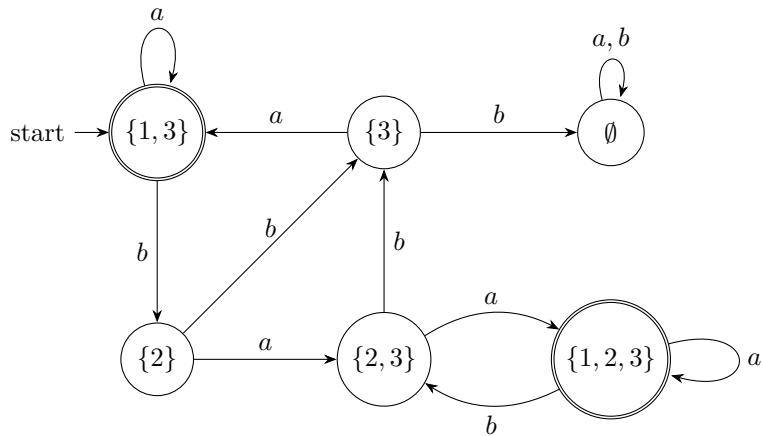


Figure 1.5: DFA conversion example

- Remove the states that are not reachable.
- Remove the states that not handle the ε transition. For example, the start state

$$\{q_1\} \text{ wrong} \rightarrow \{q_1, q_3\} \text{ correct}$$

Definition 1.2.3.

$$E(\{q_0\}) = \{q_0\} \cup \{\text{states reached by } \varepsilon \text{ from } q_0\}$$

Then we can redefine the procedure formally.

Theorem 1.2.2. Given a NFA

$$M = (Q, \Sigma, \delta, q_0, F)$$

We can convert it to a DFA

$$M' = (Q', \Sigma, \delta', q'_0, F')$$

where

- $Q' = P(Q)$
- $q'_0 \in P(Q) = E(\{q_0\})$
- $F' = \{R \mid R \in Q', R \cap F \neq \emptyset\}$
- δ' :

$$\delta'(R, a) = \bigcup_{r \in R} E(\delta(r, a))$$

1.2.2 Closure under regular operations

We give two NFAs N_1, N_2 ,

$$N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$$

$$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$$

note that $\varepsilon \notin \Sigma$, and the graph of them are:

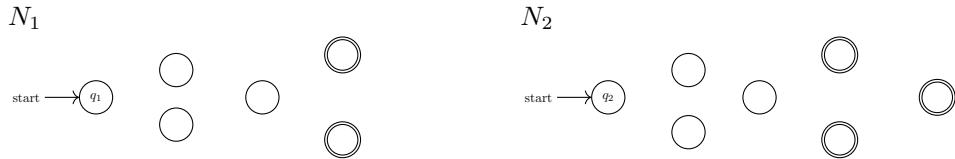


Figure 1.6: N_1, N_2

- **Union:** We can construct the $N_1 \cup N_2$ in

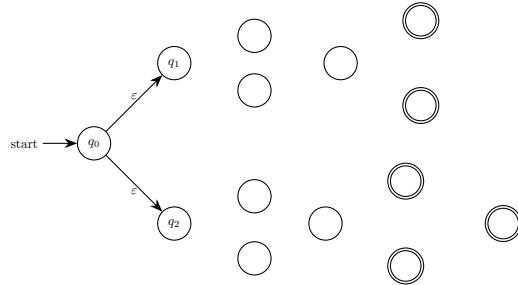


Figure 1.7: $N_1 \cup N_2$

Proposition 1.2.1 (Construction of Union). New NFA is

$$N_1 \cup N_2 = (Q, \Sigma, \delta, q_0, F)$$

where

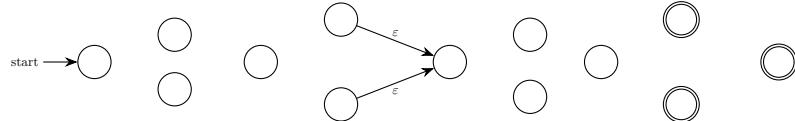
- $Q = Q_1 \cup Q_2 \cup \{q_0\}$

- δ :

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \\ \delta_2(q, a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0, a = \varepsilon \\ \emptyset & q = q_0, a \neq \varepsilon \end{cases}$$

- $F = F_1 \cup F_2$

- **Concatenation:** We can construct the $N_1 \circ N_2$ in

Figure 1.8: $N_1 \circ N_2$

Proposition 1.2.2 (Construction of Concatenation). New NFA is

$$N_1 \circ N_2 = (Q, \Sigma, \delta, q_0, F)$$

where

- $Q = Q_1 \cup Q_2$

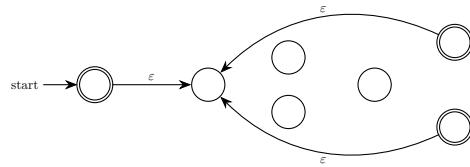
- δ :

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1, F_1 \\ \delta_2(q, a) & q \in Q_2 \\ \delta_1(q, \varepsilon) \cup \{q_2\} & q \in F_1, a = \varepsilon \\ \delta_1(q, \varepsilon) & q \in F_1, a \neq \varepsilon \end{cases}$$

- $q_0 = q_1$

- $F = F_2$

- **Kleene star:** N_1^* can also accept $\{\emptyset\}$, then we can construct the N_1^* in

Figure 1.9: N_1^*

Proposition 1.2.3 (Construction of Kleene Star). New NFA is

$$N_1^* = (Q_1, \Sigma, \delta_1, q_0, F_1)$$

where

- $Q = Q_1 \cup \{q_0\}$

o δ :

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1, F_1 \\ \delta_1(q, a) \cup \{q_1\} & q \in F_1, a = \varepsilon \\ \delta_1(q, \varepsilon) & q \in F_1, a \neq \varepsilon \\ \{q_1\} & q = q_0, a = \varepsilon \\ \emptyset & q = q_0, a \neq \varepsilon \end{cases}$$

o $F = F_1 \cup \{q_0\}$

Note. Some operations are also closed under regular languages,

o **Intersection:**

$$A_1 \cap A_2$$

Use the product automaton (the same construction as for Union). A string is accepted if and only if the state is in the accept states of both N_1 and N_2 at the same time.

o **Set Difference:**

$$A_1 - A_2$$

Use the product automaton as well. A string is accepted if the state is in the accept states of N_1 but *not* in the accept states of N_2 .

o **Complement:**

$$A_1^c = \Sigma^* - A_1$$

Since Σ^* is regular and the class of regular languages is closed under set difference, A_1^c is also regular.

Lecture 3

1.3 Regular expressions

2025-09-15

A regular expression is a tool to describe a language.

Definition 1.3.1 (Regular expressions). R is a regular expression if it is one of the following expressions:

- (1) a , where $a \in \Sigma$
- (2) ε ($\varepsilon \notin \Sigma$)
- (3) \emptyset
- (4) $R_1 \cup R_2$, where R_1, R_2 are regular expressions
- (5) $R_1 \circ R_2$, where R_1, R_2 are regular expressions
- (6) R_1^* , where R_1 is a regular expression

If there is no parentheses, we follow the order of:

Kleene star → Concatenation → Union

Remark.

$$R^+ = RR^*, \quad R^+ \cup \{\varepsilon\} = R^*$$

For \emptyset and ε , we have

- ε : empty string
- \emptyset : empty language (language without any string)

$$(0 \cup \varepsilon)1^* = 01^* \cup 1^*$$

$$(0 \cup \emptyset)1^* = 01^*$$

$$\emptyset 1^* = 1^* \emptyset = \emptyset$$

Example. Here are some examples,

- Strings that start and end with the same symbol:

$$0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$$

- $(\Sigma\Sigma)^*$: strings with even length
- $R \cup \emptyset = R$
- $R \circ \varepsilon = R$
- $\emptyset^* = \{\varepsilon\}$

Floating point numbers can also be represented by regular expressions. For example,

$$(+ \cup - \cup \varepsilon)(DD^* \cup DD^*.D^* \cup D^*.DD^*), \text{ where } D = \{0, \dots, 9\}$$

Example.

$$\begin{aligned} 72 &\in DD^* \\ 2.1 &\in DD^*.D^* \\ 7. &\in DD^*.D^* \\ .01 &\in D^*.DD^* \end{aligned}$$

Lemma 1.3.1. Language by a regular expression \implies Regular (described by an automaton)

Proof. The proof is by induction,

- $R = a \in \Sigma$ can be recognized by

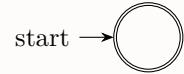


$$N = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\})$$

$$\delta(q_1, a) = \{q_2\}$$

$$\delta(r, b) = \emptyset, r \neq q_1 \text{ or } b \neq a$$

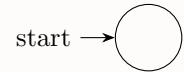
- $R = \epsilon$



$$N = (\{q_1\}, \Sigma, \delta, q_1, \{q_1\})$$

$$\delta(q_1, a) = \emptyset, \forall a$$

- $R = \emptyset$



$$N = (\{q\}, \Sigma, \delta, q, \emptyset)$$

$$\delta(r, a) = \emptyset, \forall r, a$$

- $R = R_1 \cup R_2, R = R_1 \circ R_2, R = R_1^*$ have proof by NFA.

■

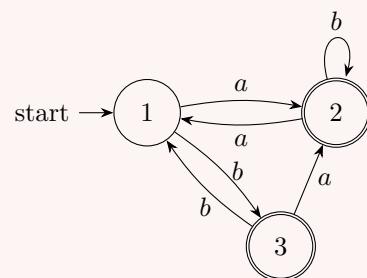
1.3.1 Convert a DFA to a regular expression

The idea is:

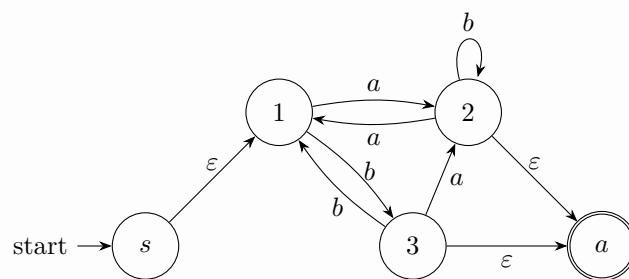
1° DFA \rightarrow GNFA

2° Remove states from GNFA until only the start and accept states.

Question. Convert the following DFA into regular expression.



Answer. First, convert to GNFA:



Next, is to remove the states one by one. We skip, so we can get the answer:

$$(a(aa \cup b)^*ab \cup b)((ba \cup a)(aa \cup b)^*ab \cup bb)^*((ba \cup a)(aa \cup b)^* \cup \varepsilon) \cup a(aa \cup b)^*$$

which is very complicated. ⊗

Definition 1.3.2 (Generalized NFA(GNFA)). We define a GNFA as a 5-tuple

$$G = (Q, \Sigma, \delta, q_{start}, q_{accept})$$

where

- F is not a se, but a single accept state q_{accept}
- δ function is:

$$(Q - \{q_{accept}\}) \times (Q - \{q_{start}\}) \rightarrow R$$

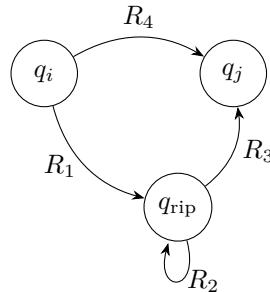
where R is all regular expressions over Σ .

- Two new states:

$$q_{start} \rightarrow q_0 \text{ with } \varepsilon$$

$$\text{any } q \in F \rightarrow q_{accept} \text{ with } \varepsilon$$

Consider q_{rip} is the state being removed



The new regular expression between q_i and q_j is

$$q_i \xrightarrow{(R_1)(R_2)^*(R_3) \cup (R_4)} q_j$$

We can write the whole process into an algorithm.

Algorithm 1.1: CONVERT(G) —State-Elimination from GNFA to RE

Input: $G = (Q, \Sigma, \delta, q_s, q_a)$ a GNFA
Output: A regular expression R for the language of G

```

1  $k \leftarrow |Q|;$ 
2 ;                                     // number of states
3 if  $k = 2$  then
4   return  $\delta(q_s, q_a)$ ;           // the (single) edge label from  $q_s$  to  $q_a$ 
5 Choose any  $q_{rip} \in Q \setminus \{q_s, q_a\}$ ;
6  $Q' \leftarrow Q \setminus \{q_{rip}\}$ ;
7 Initialize  $\delta'$  as the restriction of  $\delta$  to  $Q' \times Q'$ ;
8 foreach  $q_i \in Q' \setminus \{q_a\}$  do
9   foreach  $q_j \in Q' \setminus \{q_s\}$  do
10     $R_1 \leftarrow \delta(q_i, q_{rip})$ ;
11     $R_2 \leftarrow \delta(q_{rip}, q_{rip})$ ;
12     $R_3 \leftarrow \delta(q_{rip}, q_j)$ ;
13     $R_4 \leftarrow \delta(q_i, q_j)$ ;
14     $\delta'(q_i, q_j) \leftarrow R_4 \cup (R_1 R_2^* R_3)$ ;
15  $G' \leftarrow (Q', \Sigma, \delta', q_s, q_a)$ ;
16 return CONVERT( $G'$ );

```

Lecture 4

1.4 Pumping lemma

2025-09-22

1.4.1 Non regular language

Some languages cannot be recognized by DFA such as,

$$\{0^n 1^n \mid n \geq 0\}$$

We might remember #0 first, but # of possible n 's is ∞ , so we have some method to prove that the language is non-regular.

Theorem 1.4.1 (pumping lemma). If A is regular, $\exists p$ such that $\forall s \in A, |s| \geq p$,

$\exists x, y, z$, such that $s = xyz$ and

$$1^\circ \quad \forall i \geq 0, xy^i z \in A$$

$$2^\circ \quad |y| > 0$$

$$3^\circ \quad |xy| \leq p$$

Proof. Skip, which is on the slides. ■

1.4.2 Example for Pumping Lemma

Question. Show that the language $L = \{0^n 1^n \mid n \geq 0\}$ is not regular using the pumping lemma.

Answer. Now consider the string

$$s = 0^p 1^p$$

We know that $|s| \geq p$. By the lemma, s can be split into xyz such that

$$xy^i z \in B, \forall i \geq 0, |y| > 0, \text{ and } |xy| \leq p$$

1° If $y = 0 \cdots 0$, then

$$xy = 0 \cdots 0 \quad \text{and} \quad z = 0 \cdots 0 1 \cdots 1.$$

Thus,

$$xy^2 z : \#0 > \#1.$$

Hence $xy^2 z \notin B$, a contradiction.

2° If $y = 1 \cdots 1$, then similarly

$$xy^2 z \notin B \quad \text{as} \quad \#0 < \#1.$$

3° If $y = 0 \cdots 0 1 \cdots 1$, then

$$xy^2 z \notin B \quad \text{since it is not of the form } 0^* 1^*.$$

Note. Just pick one is sufficient to show the answer.

(*)

Question. Show that the language $C = \{w \mid \#0 = \#1\}$ is not regular using the pumping lemma.

Answer. We can use the situation in the previous example, consider

$$s = 0^p 1^p$$

We can't proof the third condition due to $C = \{w \mid \#0 = \#1\}$ which just require the $\#0 = \#1$.

Then we can use the third condition

$$|xy| \leq p$$

which means y are strict into the first 0^p we can only consider the first case.

$$|xy| \leq p \Rightarrow y = 0 \cdots 0 \text{ in } s = 0^p 1^p$$

Then,

$$xy^2 z \notin C$$

(*)

Lemma 1.4.1. When using pumping lemma, we usually use contradiction, so we use

$$\forall p \exists s \in A, |s| \geq p, \left[\forall x, y, z \left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \exists i \geq 0, xy^i z \notin A \right) \right].$$

Use the claim and the first, second condition to get the negation of the third condition.

Question. $D = \{1^{n^2} \mid n \geq 0\}$ is not regular

Answer. We pick

$$s = 1^{p^2} \in D$$

Then, if $s = xyz, |xy| \leq p, |y| > 0$, we can get

$$p^2 < |xy^2z| \leq p^2 + p \leq (p+1)^2$$

hence, $xy^2z \notin D$.

⊗

Chapter 2

Context-Free Languages

Lecture 5

2.1 Context-Free Grammars (CFG)

2025-10-20

Which is more powerful, and can be used in compilers. A **Grammar** is a collection of substitution rules that describe the structure of a language.

Example. Consider a grammar G_1 :

$$\begin{aligned}A &\rightarrow 0A1 \\A &\rightarrow B \\B &\rightarrow \#\end{aligned}$$

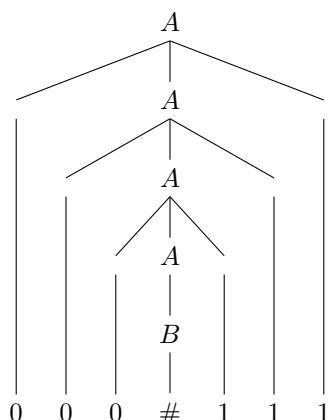
Here are the jargon terms:

- Each of one is called a **substitution rule**.
- **Variables** (non-terminals): A, B (Capital letters)
- **Terminals**: $0, 1, \#$ (Lowercase letters, numbers, symbols)
- **Start variable**: A (the variable we start with)

The process of generating strings is called **derivation**. G_1 generates $000\#\underline{111}$ by

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#\underline{111}$$

We can show the derivation using a **parse tree**:



2.1.1 Definition of CFG

The language of grammar G is denoted by $L(G)$, for the language we discuss here,

$$L(G_1) = \{0^n \# 1^n \mid n \geq 0\}$$

Now we give the formal definition of CFG.

Definition 2.1.1 (Context-Free Grammar). We defined a CFG as a 4-tuple

$$G = (V, \Sigma, R, S)$$

where

- V : Variables (Finite)
- Σ : Terminals (Finite)
- R : Rules:

$$\text{Variables} \rightarrow \text{Strings of Variables and Terminals (including } \varepsilon\text{)}$$
- $S \in V$: Start variable

For instance, for G_1 ,

$$G_1 = (\{A, B\}, \{0, 1, \#\}, R, A)$$

where R is:

$$A \rightarrow 0A1 \mid B, \quad B \rightarrow \#$$

Notation. If u, v, w are strings and rule $A \rightarrow w$ is applied, then we say

$$uAv \text{ yields } uwv$$

denoted as

$$uAv \Rightarrow uwv$$

Notation. If

$$u = v \text{ or } u \Rightarrow u_1 \Rightarrow \dots \Rightarrow u_k \Rightarrow v$$

then we write

$$v \xrightarrow{*} u$$

Definition 2.1.2 (Language of a CFG). The language generated by a CFG G with start variable S is

$$L(G) = \{w \in \Sigma^* \mid S \xrightarrow{*} w\}$$

2.1.2 Examples of CFGs

Question. Consider the grammar $G_2 = (\{S\}, \{a, b\}, R, S)$:

$$S \rightarrow aSb \mid SS \mid \varepsilon$$

What is $L(G_2)$?

Answer. If we let a, b be the left and right parentheses respectively, then $L(G_2)$ is the set of all balanced parentheses. (*)

Example. Consider the grammar $G_3 = (V, \Sigma, R, S)$ where

- $V = \{\langle \text{expr} \rangle, \langle \text{term} \rangle, \langle \text{factor} \rangle\}$
- $\Sigma = \{+, \times, (,), a\}$
- $R:$

$$\begin{aligned}\langle \text{expr} \rangle &\rightarrow \langle \text{term} \rangle + \langle \text{expr} \rangle \mid \langle \text{term} \rangle \\ \langle \text{term} \rangle &\rightarrow \langle \text{factor} \rangle \times \langle \text{term} \rangle \mid \langle \text{factor} \rangle \\ \langle \text{factor} \rangle &\rightarrow (\langle \text{expr} \rangle) \mid a\end{aligned}$$

Consider the string $a + a \times a$:

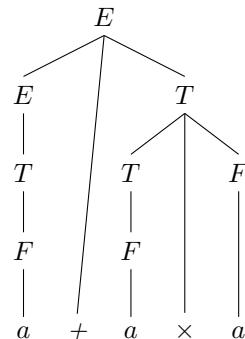


Figure 2.1: Parse tree of $a + a \times a$

Consider the string $(a + a) \times a$:

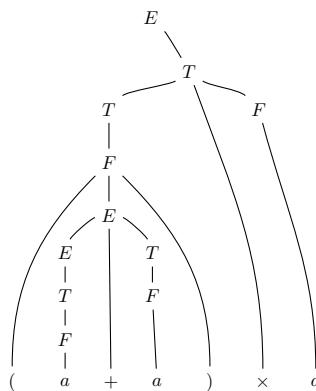


Figure 2.2: Parse tree of $(a + a) \times a$

Note. The example above shows that CFGs can express operator precedence and associativity.

2.1.3 Design of CFGs

We can design CFGs in many methods. Here are some common patterns:

- Combining smaller parts:

Example. $L(G) = \{a^n b^n \mid n \geq 0\} \cup \{b^n a^n \mid n \geq 0\}$

We can let the rule R be:

$$\begin{aligned} S_1 &\rightarrow aS_1b \mid \varepsilon \\ S_2 &\rightarrow bS_2a \mid \varepsilon \\ S &\rightarrow S_1 \mid S_2 \end{aligned}$$

- From DFA:

Lemma 2.1.1. For any regular language A , there exists a CFG G such that $L(G) = A$. The rules of CFG can be

$$\begin{aligned} R_i &\rightarrow aR_j \quad \text{for each transition } \delta(q_i, a) = q_j \\ R_i &\rightarrow \varepsilon \quad \text{if } q_i \in F \end{aligned}$$

The difference is that CFG allows the format

$$R_i \rightarrow aR_jb$$

But DFA only allows

$$R_i \rightarrow aR_j$$

where we treat R_i as the state and let $\delta(R_i, a) = R_j$.

2.1.4 Parse Trees and Ambiguity

If we let the rules of G_3 be

$$\langle \text{expr} \rangle \rightarrow \langle \text{expr} \rangle + \langle \text{expr} \rangle \mid \langle \text{expr} \rangle \times \langle \text{expr} \rangle \mid (\langle \text{expr} \rangle) \mid a$$

We can see the following two parse trees for $a + a \times a$:

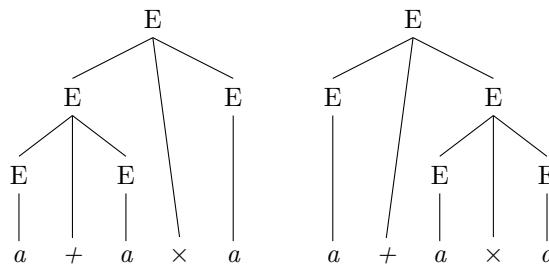


Figure 2.3: Two different parse trees for $a + a \times a$ under ambiguous grammar

This is called **ambiguity**. A CFG is **ambiguous** if there exists some string with two or more different parse trees. The above G_3 is **unambiguous**, G'_3 with new rules is **ambiguous**.

However, an unambiguous grammar may also generate same parse tree but different derivations. Consider G_3 :

- We can do derivation

$$\begin{aligned}\langle \text{expr} \rangle &\Rightarrow \langle \text{expr} \rangle + \langle \text{term} \rangle \\ &\Rightarrow \langle \text{expr} \rangle + \langle \text{term} \rangle \times \langle \text{factor} \rangle\end{aligned}$$

- We can also do derivation

$$\begin{aligned}\langle \text{expr} \rangle &\Rightarrow \langle \text{expr} \rangle + \langle \text{term} \rangle \\ &\Rightarrow \langle \text{term} \rangle + \langle \text{term} \rangle\end{aligned}$$

which is not considered ambiguous. So we have the following definition:

Definition 2.1.3 (leftmost derivation). A **leftmost derivation** is a derivation where at each step, the leftmost variable is replaced.

Then we can have the formal definition of ambiguity:

Definition 2.1.4 (Ambiguous). A is **ambiguous** if $w \in A$ and there exists two or more different leftmost derivations for w .

Definition 2.1.5 (Inherent Ambiguity). A language is **inherently ambiguous** if it only has ambiguous grammars.

Example. Consider the language

$$L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$$

We can consider the string $a^2b^2c^2$. It can be generated by two different leftmost derivations. First we consider

$$S \Rightarrow S_1 \mid S_2$$

- Using $i = j$:

$$\begin{aligned}S_1 &\rightarrow AC \\ A &\rightarrow aAb \mid \epsilon \\ C &\rightarrow cC \mid \epsilon\end{aligned}$$

the derivation is

$$S_1 \Rightarrow AC \Rightarrow aAbC \Rightarrow aaAbbC \Rightarrow aabbC \Rightarrow aabbcC \Rightarrow aabbcc$$

- Using $j = k$:

$$\begin{aligned}S_2 &\rightarrow A'C' \\ A' &\rightarrow aA' \mid \epsilon \\ C' &\rightarrow bC'c \mid \epsilon\end{aligned}$$

the derivation is

$$S_2 \Rightarrow A'C' \Rightarrow aA'C' \Rightarrow aaA'bC'c \Rightarrow aabbC'cc \Rightarrow aabbcc$$

Lecture 6

2.2 Chomsky Normal Form

2025-10-27

We want to simplify the structure of context-free grammars. One useful normal form is the Chomsky Normal Form (CNF).

Definition 2.2.1 (Chomsky Normal Form). A context-free grammar is in **Chomsky Normal Form** if all its production rules are of the form:

- $A \rightarrow BC$, where A, B, C are non-terminal symbols and B, C are not the start symbol.
- $A \rightarrow a$, where $a \in \Sigma (\varepsilon \notin \Sigma)$
- $S \rightarrow \varepsilon$ is allowed, where S is the start symbol.

Example. Convert the following CFG to CNF:

$$\begin{aligned} S &\rightarrow ASA \mid aB \\ A &\rightarrow B \mid S \\ B &\rightarrow b \mid \varepsilon \end{aligned}$$

First, we add S_0 as the new start symbol:

$$S_0 \rightarrow S \quad S \rightarrow ASA \mid aB \quad A \rightarrow B \mid S \quad B \rightarrow b \mid \varepsilon$$

Next, we remove the ε -productions $B \rightarrow \varepsilon$:

$$S_0 \rightarrow S \quad S \rightarrow ASA \mid aB \mid \textcolor{red}{a} \quad A \rightarrow B \mid \textcolor{red}{\varepsilon} \mid S \quad B \rightarrow b$$

Next, we remove the ε -productions $A \rightarrow \varepsilon$:

$$S_0 \rightarrow S \quad S \rightarrow ASA \mid aB \mid a \mid \textcolor{red}{AS} \mid \textcolor{red}{SA} \mid \textcolor{red}{S} \quad A \rightarrow B \mid S \quad B \rightarrow b$$

Next, we remove single production $S \rightarrow S$:

$$S_0 \rightarrow S \quad S \rightarrow ASA \mid aB \mid a \mid AS \mid SA \quad A \rightarrow B \mid S \quad B \rightarrow b$$

Next, we remove single production $S_0 \rightarrow S$:

$$S_0 \rightarrow \textcolor{red}{ASA} \mid \textcolor{red}{aB} \mid \textcolor{red}{a} \mid \textcolor{red}{AS} \mid \textcolor{red}{SA} \quad S \rightarrow ASA \mid aB \mid a \mid AS \mid SA \quad A \rightarrow B \mid S \quad B \rightarrow b$$

Next, we remove single production $A \rightarrow B, A \rightarrow S$:

$$S_0 \rightarrow ASA \mid aB \mid a \mid AS \mid SA \quad S \rightarrow ASA \mid aB \mid a \mid AS \mid SA \quad A \rightarrow \textcolor{red}{b} \mid \textcolor{red}{ASA} \mid \textcolor{red}{aB} \mid \textcolor{red}{a} \mid \textcolor{red}{AS} \mid \textcolor{red}{SA} \quad B \rightarrow b$$

Finally, we convert to CNF by introducing new variables for terminals and breaking down long productions:

$$\begin{aligned} S_0 &\rightarrow AA_1 \mid UB \mid a \mid AS \mid SA \\ S &\rightarrow AA_1 \mid UB \mid a \mid AS \mid SA \\ A &\rightarrow b \mid AA_1 \mid UB \mid a \mid AS \mid SA \\ A_1 &\rightarrow SA \\ B &\rightarrow b \\ U &\rightarrow a \end{aligned}$$

2.2.1 Procedure of Converting CFG to CNF

To convert any CFG to CNF, we can follow these steps:

1° **Add** a new start symbol S_0 with the production

$$S_0 \rightarrow S$$

2° **Remove** all ε -productions, except for the start symbol, i.e. $A \rightarrow \varepsilon$ ($A \neq S_0$), for any

$$\dots \rightarrow uAv$$

add the production

$$\dots \rightarrow uv$$

3° **Remove** single productions of $A \rightarrow B$ where $A, B \in V/\{S\}$.

$$A \rightarrow B, B \rightarrow \gamma \Rightarrow A \rightarrow \gamma$$

Remark. $A \rightarrow \gamma$ can't be a unit rule previously removed.

4° **Convert** remaining productions to CNF:

$$A \rightarrow u_1 u_2 \dots u_k \quad u_i \in V \cup \Sigma$$

and

$$\text{if } k = 1, \text{ then } u_i \in \Sigma$$

Convert as follows:

$$A \rightarrow u_1 A_1$$

$$A_1 \rightarrow u_2 A_2$$

⋮

Replaced every terminal $u_i \in \Sigma$ with a new variable U_i :

$$U_i \rightarrow u_i \quad u_i \in \Sigma$$

2.2.2 Infinite Loop in Converting

Example. Consider the grammar:

$$S \rightarrow B \mid \varepsilon$$

$$B \rightarrow S \mid \varepsilon$$

We first add a new start symbol:

$$S_0 \rightarrow S \quad S \rightarrow B \mid \varepsilon \quad B \rightarrow S \mid \varepsilon$$

Next, we remove the ε -productions:

$$S_0 \rightarrow S \mid \varepsilon \quad S \rightarrow B \quad B \rightarrow S \mid \varepsilon$$

Next, we remove the ε -productions again:

$$S_0 \rightarrow S \mid \varepsilon \quad S \rightarrow B \mid \varepsilon \quad B \rightarrow S$$

This process will continue indefinitely. The reason is $S \rightarrow \varepsilon$ has been handled. So there is no need to add $S \rightarrow \varepsilon$.

2.3 Pushdown Automata

We now introduce the machine that recognizes context-free languages (CFL), called Pushdown Automata (PDA). PDA is a machine with a **stack**, which is a way to store previous states.

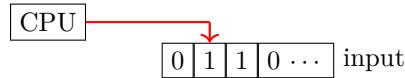


Figure 2.4: DFA or NFA

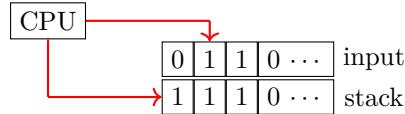


Figure 2.5: Pushdown Automata (PDA)

Example. Consider the language $A = \{0^n 1^n \mid n \geq 0\}$. We can design a PDA to recognize A :

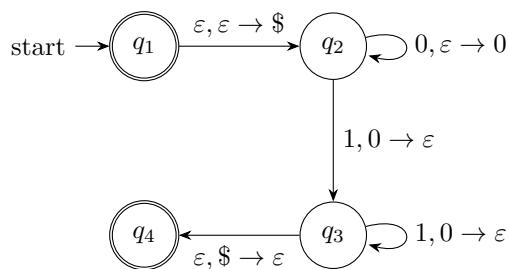


Figure 2.6: PDA for $A = \{0^n 1^n \mid n \geq 0\}$

$\$$ is a special bottom stack symbol to indicate the initial state of the stack. The PDA works as follows:

- $q_2 \rightarrow q_2$, put 0 into stack
- $q_2 \rightarrow q_3$ and $q_3 \rightarrow q_3$, read 1 and pop 0 up

If the input is 0011 which is same as $\epsilon 0011 \epsilon$, the process is as follows:

q_1, \emptyset, ϵ
 $q_2, \{\$\}, 0$
 $q_2, \{0, \$\}, 0$
 $q_2, \{0, 0, \$\}, 1$
 $q_3, \{0, \$\}, 1$
 $q_3, \{\$\}, \epsilon$
 $q_4, \{\}$

Notation. $\{\}$: contents of the stack before processing the input character.

2.3.1 Formal definition of PDA

Definition 2.3.1 (Pushdown Automata). A **pushdown automaton** (PDA) is a 6-tuple

$$(Q, \Sigma, \Gamma, \delta, q_0, F)$$

, where

- Q : States
- Σ : Input alphabet
- Γ : Stack alphabet
- δ : Transition function

$$Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$$

- $q_0 \in Q$: Start state
- $F \subset Q$: Set of accepting states

The definition of the above PDA for $A = \{0^n 1^n \mid n \geq 0\}$ is as follows:

- $Q = \{q_1, q_2, q_3, q_4\}$
- $\Sigma = \{0, 1\}$
- $\Gamma = \{0, \$\}$
- $q_0 = q_1$
- $F = \{q_1, q_4\}$

For the transition function, we care about three things:

- Current state
- Current input
- **Top of the stack**

The transition function δ works as follows:

	0	1	ε
0	0 \$ ε	0 \$ ε	0 \$ ε
q_1			$\{(q_2, \$)\}$
q_2	$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$	
q_3		$\{(q_3, \varepsilon)\}$	
q_4			$\{(q_4, \varepsilon)\}$

For example, we say the transition of $q_2 \rightarrow q_3$ to be

$$\delta(q_2, 1, 0) = \{(q_3, \varepsilon)\}$$

2.3.2 Nondeterministic situation

Example. Design a PDA for the language $B = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } j = k\}$.

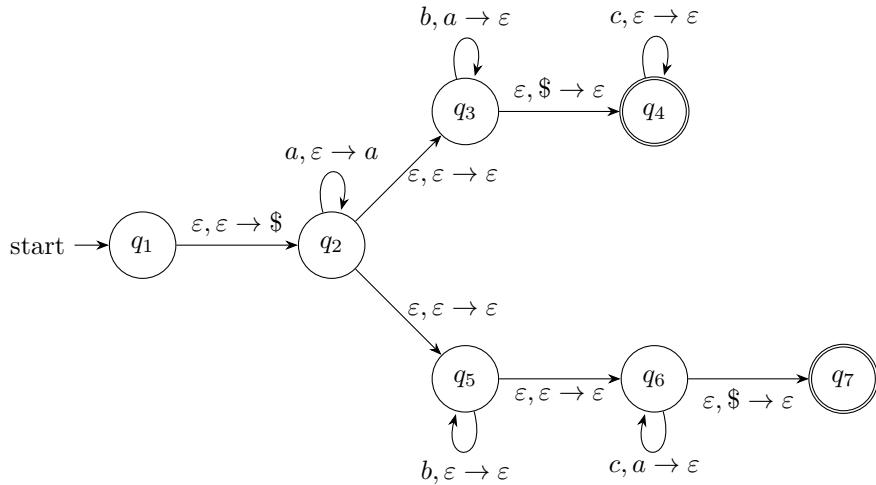
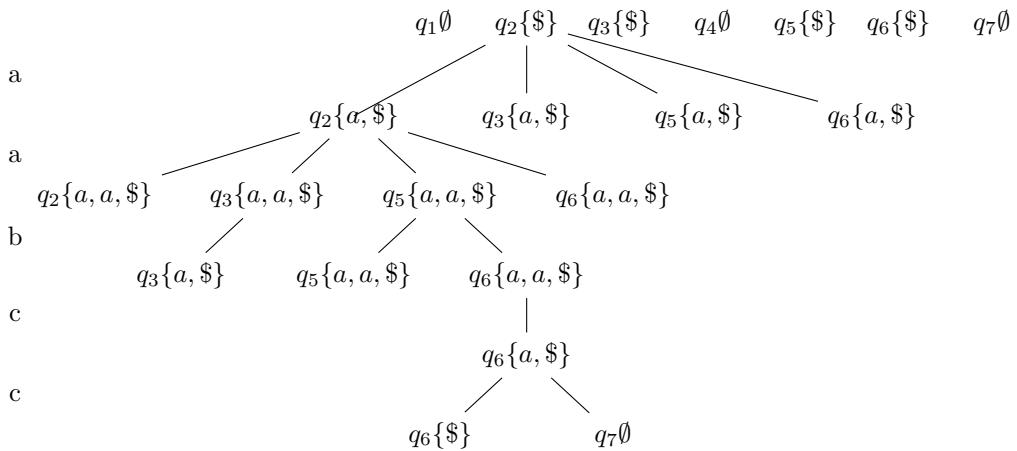


Figure 2.7: Nondeterministic PDA

We input a^2bc^2 , to illustrate the process, we can build the following computation tree:



Example. Design a PDA for the language $C = \{ww^R \mid w \in \{0,1\}^*\}$.

Idea. Symbols pushed to stack, nondeterministically guess middle is reached

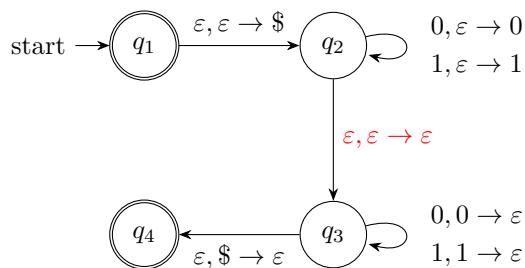


Figure 2.8: PDA for $C = \{ww^R \mid w \in \{0,1\}^*\}$

2.3.3 Converting CFL to PDA

Example. Convert the CFG G to PDA that recognizes $L(G)$:

$$S \rightarrow aTb \mid b$$

$$T \rightarrow Ta \mid \varepsilon$$

Idea. For rule substitution, we replace the left-hand side variable with the right-hand side string i.e.

$$A \rightarrow \gamma \Rightarrow \text{pop } A \text{ from stack, push } \gamma \text{ to stack}$$

if there are multiple productions for A , we push them **in a reversed way**.

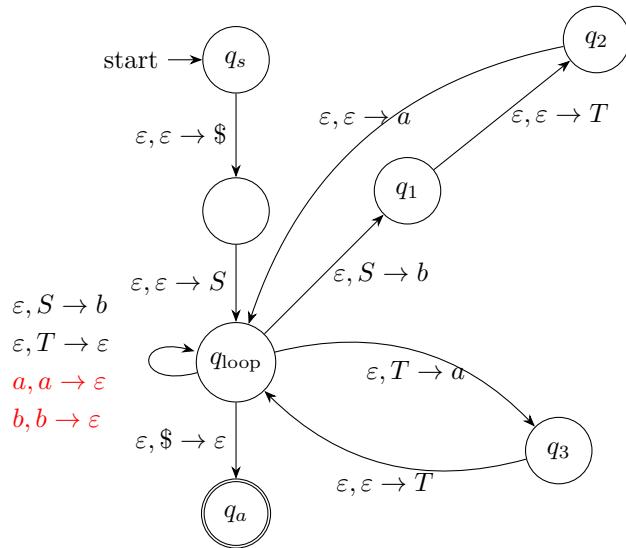


Figure 2.9: PDA for CFG G

Remark. There are two transitions we must add to process the "input":

$$a, a \rightarrow \varepsilon$$

$$b, b \rightarrow \varepsilon$$

The procedure of converting CFG to PDA is as follows:

$$\begin{aligned} q_{\text{start}} &\xrightarrow{\varepsilon} q_{\text{loop}}, \{S, \$\} \xrightarrow{\varepsilon} q_1, \{b, \$\} \xrightarrow{\varepsilon} q_2, \{T, b, \$\} \\ &\xrightarrow{\varepsilon} q_{\text{loop}}, \{a, T, b, \$\} \xrightarrow{a} q_{\text{loop}}, \{T, b, \$\} \\ &\xrightarrow{\varepsilon} q_3, \{a, b, \$\} \xrightarrow{\varepsilon} q_{\text{loop}}, \{T, a, b, \$\} \\ &\xrightarrow{\varepsilon} q_3, \{a, a, b, \$\} \xrightarrow{\varepsilon} q_{\text{loop}}, \{T, a, a, b, \$\} \\ &\xrightarrow{\varepsilon} q_3, \{a, a, a, b, \$\} \xrightarrow{\varepsilon} q_{\text{loop}}, \{T, a, a, a, b, \$\} \\ &\xrightarrow{\varepsilon} q_{\text{loop}}, \{a, a, a, b, \$\} \xrightarrow{a} q_{\text{loop}}, \{a, a, b, \$\} \\ &\xrightarrow{a} q_{\text{loop}}, \{a, b, \$\} \xrightarrow{a} q_{\text{loop}}, \{b, \$\} \\ &\xrightarrow{b} q_{\text{loop}}, \{\$\} \xrightarrow{\varepsilon} q_{\text{accept}} \end{aligned}$$

Proposition 2.3.1. Even with a non-deterministic setting, we ensure that only strings generated by this CFG can be accepted by the PDA

- A string is accepted only if all characters are processed (this is part of the PDA definition!)
- We have $\$$ to ensure that the stack is empty in the end

2.3.4 Converting PDA to CFL

Lemma 2.3.1. Language recognized by PDA \Rightarrow context free

Note. We need PDA to satisfy

- 1° Single start state
- 2° Stack empty before accepting
- 3° Each transition push or pop, but not both

Idea. For each pair of states $p, q \in Q$ of a PDA P , we have A_{pq} and

A_{pq} generates $x \Rightarrow P$ from p with empty stack to q with empty stack, reading x

First, we discuss how to handle transitions

$$\forall p, q, r \in Q, A_{pq} \rightarrow A_{pr}A_{rq}$$

We let the

- x -axis: input string
- y -axis: stack height

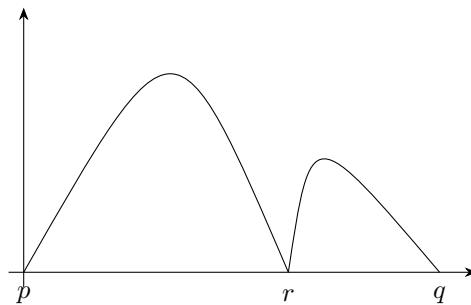


Figure 2.10: PDA transition $A_{pq} \rightarrow A_{pr}A_{rq}$

If we can go

from p to r without changing stack

and

from r to q without changing stack

then we can do

from p to q without changing stack

Next, we have

$$\forall p, q, r, s \in Q, a, b \in \Sigma_\varepsilon, t \in \Gamma$$

If,

$$(r, t) \in \delta(p, a, \varepsilon) \text{ and } (q, \varepsilon) \in \delta(s, b, t)$$

we discuss how to handle transitions

$$A_{pq} \rightarrow a A_{rs} b$$

Then we have

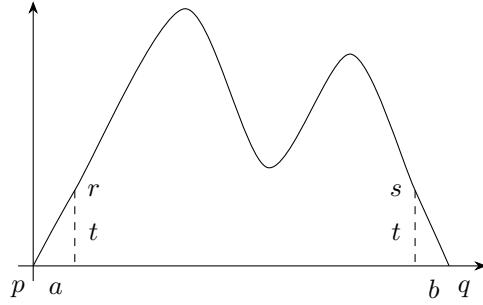


Figure 2.11: PDA transition $A_{pq} \rightarrow a A_{rs} b$

Finally, we have the following base case:

$$\forall p \in Q, A_{pp} \rightarrow \varepsilon$$

To follow the condition (1°), we give a new example

Example. Consider the language $L = \{0^n 1^n \mid n \geq 1\}$.

Now q_1 is not an accept state

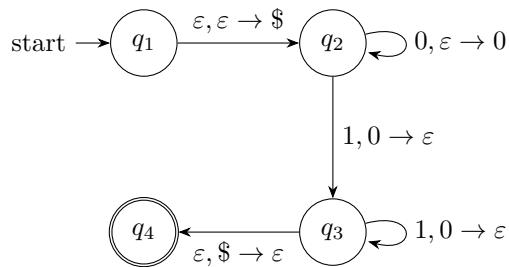


Figure 2.12: PDA for $A = \{0^n 1^n \mid n \geq 1\}$

Consider two elements in Γ

$$t_0 = \$, \quad t_1 = 0$$

- $t = \$$

p	r	s	q	t	a	b
1	2	3	4	\$	ε	ε

then we can get the rule

$$A_{14} \rightarrow A_{23}$$

- $t = 0$

p	r	s	q	t	a	b
2	2	2	3	0	0	1
2	2	3	3	0	0	1

then we can get the rules

$$\begin{aligned} A_{23} &\rightarrow 0A_{22}1 \\ A_{23} &\rightarrow 0A_{23}1 \end{aligned}$$

Other rules: 64 rules

$$\begin{aligned} A_{11} &\rightarrow A_{11}A_{11} \\ A_{11} &\rightarrow A_{12}A_{21} \\ A_{11} &\rightarrow A_{13}A_{31} \\ A_{11} &\rightarrow A_{14}A_{41} \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} A_{11} &\rightarrow \varepsilon \\ A_{22} &\rightarrow \varepsilon \\ A_{33} &\rightarrow \varepsilon \\ A_{44} &\rightarrow \varepsilon \end{aligned}$$

2.3.5 Procedure of converting PDA to CFL

Proposition 2.3.2. Given a PDA

$$P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{accept}\})$$

We construct a CFG with variables

$$\text{var}(G) = \{A_{pq} \mid p, q \in Q\}$$

and start variable

$$S = A_{q_0 q_{accept}}$$

With rules

- 1° Single start state
- 2° Stack empty before accepting
- 3° Each transition push or pop, but not both

A new start $q_s \rightarrow q_{s'}$ with $\varepsilon, \varepsilon \rightarrow \$$, and for any $q \in F$, we have $\varepsilon, a \rightarrow \varepsilon$ back to q , $\forall a \in \Sigma$. Then from any $q \in F$, we do $\varepsilon, \$ \rightarrow \varepsilon$ to q_a

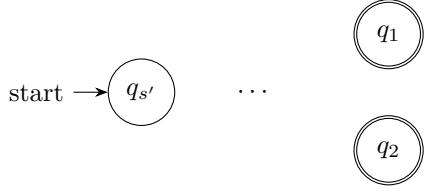
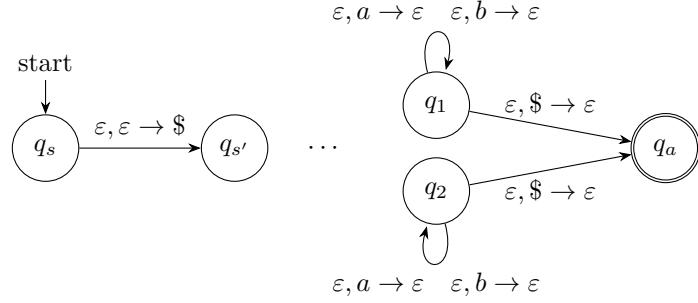


Figure 2.13: PDA with single accept state and empty stack before accepting

The new one will become



These is not enough to ensure condition (3°), we can do some modifications:

- To have each transition either push or pop (but not both), replace

$$q_1 \xrightarrow{a, a \rightarrow b} q_2$$

with the pair

$$q_1 \xrightarrow{a, a \rightarrow \varepsilon} q_3, \quad q_3 \xrightarrow{\varepsilon, \varepsilon \rightarrow b} q_2.$$

- Likewise, replace

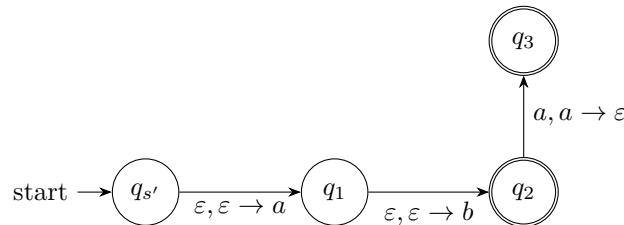
$$q_1 \xrightarrow{a, \varepsilon \rightarrow \varepsilon} q_2$$

with

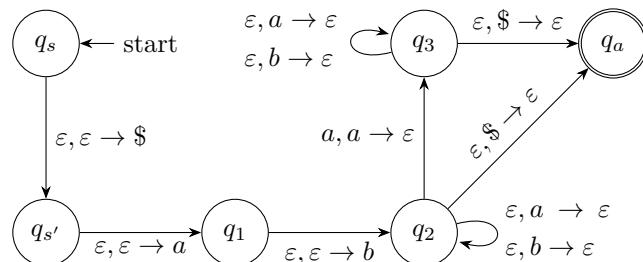
$$q_1 \xrightarrow{a, \varepsilon \rightarrow X} q_3, \quad q_3 \xrightarrow{\varepsilon, X \rightarrow \varepsilon} q_2,$$

where X is a fresh stack marker introduced for this simulation.

For another example, consider the PDA



After the modification, we have



The new PDA will accept the string a but the original PDA rejects it. Hence, we need to modify something else:

- A new start $q_s \rightarrow q_{s'}$ with $\varepsilon, \varepsilon \rightarrow \$$
- A **new state q_{pop}** that have $\varepsilon, a \rightarrow \varepsilon$ back to q_{pop} , $\forall a$.
- For $q \in F$, add a transition $\varepsilon, \varepsilon \rightarrow \varepsilon$ from q to q_{pop}
- Add a new accept state q_a and a transition $\varepsilon, \$ \rightarrow \varepsilon$ from q_{pop} to q_a

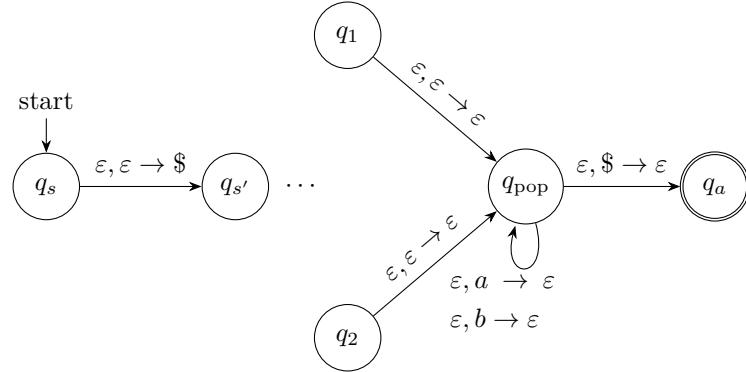


Figure 2.14: PDA with single accept state and empty stack before accepting

2.4 Deterministic Pushdown Automata

Lecture 7

PDA is non-deterministic in general. However, there is a special class of PDA called **Deterministic** 2025-11-03 Pushdown Automata (DPDA). From Ch.1 we know

$$\text{DFA} \equiv \text{NFA}$$

but

$$\text{DPDA} \neq \text{PDA} \implies \text{CFL} \neq \text{DCFL}$$

Definition 2.4.1 (Deterministic Pushdown Automaton (DPDA)). A deterministic pushdown automaton (DPDA) is a 6-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, F)$$

where

- Q : States
- Σ : Input alphabet
- Γ : Stack alphabet
- δ : Transition function

$$Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow (Q \times \Gamma_\varepsilon) \cup \{\emptyset\}$$

- $q_0 \in Q$: Start state
- $F \subset Q$: Set of accepting states

To build a DPDA, we first look at the difference between PDA and DPDA.

As previously seen. For PDA,

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$$

Note. In DPDA, for $\forall q \in Q, a \in \Sigma, x, \gamma \in \Gamma$, at most and at least one of the following is true:

$$\delta(q, a, x) = (p, \gamma), \quad \delta(q, a, \varepsilon) = (p, \gamma), \quad \delta(q, \varepsilon, x) = (p, \gamma), \quad \delta(q, \varepsilon, \varepsilon) = (p, \gamma)$$

the rest must be \emptyset .

2.4.1 Acceptance, Rejection of DPDA

The Rejection of DPDA is similar to PDA, which should only happen when

- Not end at an accept state after the last symbol.
- DPDA fails to read the input
 1. pop an empty stack
 2. Endless ε -transition

Example. $L = \{0^n 1^n \mid n \geq 0\}$

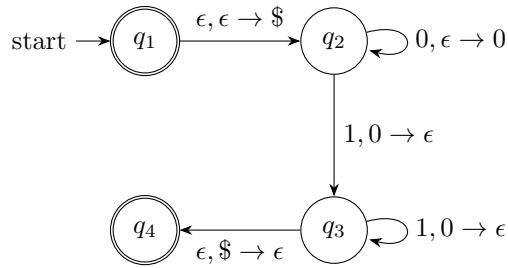


Figure 2.15: DPDA for $L = \{0^n 1^n \mid n \geq 0\}$

The Transition function is defined as follows:

	0			1			ϵ		
	0	\$	ϵ	0	\$	ϵ	0	\$	ϵ
q_1	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	$(q_2, \$)$
q_2	\emptyset	\emptyset	$(q_2, 0)$	(q_3, ϵ)	q_r	\emptyset	\emptyset	\emptyset	\emptyset
q_3	q_r	\emptyset	\emptyset	(q_3, ϵ)	\emptyset	\emptyset	\emptyset	(q_4, ϵ)	\emptyset
q_4	q_r	q_r	\emptyset	q_r	q_r	\emptyset	\emptyset	\emptyset	\emptyset
q_r	q_r	q_r	\emptyset	q_r	q_r	\emptyset	\emptyset	\emptyset	\emptyset

To find this transition table, for instance,

- consider the state q_1 :

$$\delta(q_1, \epsilon, \epsilon) = (q_2, \$)$$

then we can implies that

$$\delta(q_1, a, \gamma) = \delta(q_1, a, \epsilon) = \delta(q_1, \epsilon, \gamma) = \emptyset, \quad \forall a \in \Sigma = \{0, 1\}, \gamma \in \Gamma = \{0, \$\}$$

- consider the state q_2 :

$$\delta(q_2, 1, 0) = (q_3, \epsilon)$$

then we can implies that

	0			1			ϵ		
	0	\$	ϵ	0	\$	ϵ	0	\$	ϵ
q_2	\emptyset	\emptyset	$(q_2, 0)$	(q_3, ϵ)	$\neq \emptyset$	\emptyset	\emptyset	\emptyset	\emptyset

due to

$$\delta(q_2, 1, \epsilon) = \delta(q_2, \epsilon, \$) = \delta(q_2, \epsilon, \epsilon) = \emptyset$$

Formally we have

$$\delta(q_2, 1, \$) = (q_r, \epsilon)$$

For the string 011, the computation of the DPDA is as follows:

$$q_1 \xrightarrow{\epsilon} q_2, \{\$\} \quad q_1 \xrightarrow{0} q_2, \{0, \$\} \quad q_1 \xrightarrow{1} q_3, \{\$\} \quad q_1 \xrightarrow{\epsilon} q_4, \emptyset$$

Follow the graph,

$$\delta(q_4, 1, \epsilon) \text{ and } \delta(q_4, \epsilon, \epsilon) = \emptyset$$

hence, the DPDA rejects 011.

Chapter 3

The Church-Turing Thesis

Lecture 8

3.1 Turing Machines

2025-11-10

As previously seen. To discuss the **computability** of problems. We need a more powerful model.
We already have seen

- Finite Automata (FA): with limited memory (states)
- Pushdown Automata (PDA): unlimited memory with LIFO structure (stack)

We now introduce a new computational model called **Turing Machine** (TM), which has an unlimited tape as memory.

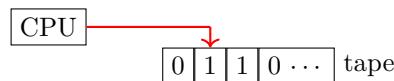


Figure 3.1: Illustration of a Turing Machine

A Turing Machine consists of these properties different from FA and PDA:

- write/read tape
- head that can move left/right on the tape
- unlimited tape length
- reject/accept take immediate effect
- machine can **never** halt

Example.

$$B = \{w\#w \mid w \in \{0,1\}^*\}$$

Remark. We can proof that this language is not CFL by pumping lemma for CFL.

Notation. \square is the blank symbol on the tape.

Idea. Zig-zag to the corresponding places on the two sides of the $\#$ and determine whether they match.

- Scan to check if there is a $\#$.
- Check w and w if they match.

Figure 3.2: Illustration of algorithm

Definition 3.1.1 (Turing Machine (TM)). A Turing Machine is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

where

- Q : States
- Σ : Input alphabet, where $\sqcup \notin \Sigma$
- Γ : Tape alphabet, where $\Sigma \subset \Gamma$ and $\sqcup \in \Gamma$
- δ : Transition function

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

- $q_0 \in Q$: Start state

- $q_{\text{accept}} \in Q$

- $q_{\text{reject}} \in Q$, $q_{\text{reject}} \neq q_{\text{accept}}$

The input

$$w = w_1 w_2 \cdots w_n \in \Sigma^*$$

will be put in the position $1, 2, \dots, n$ of the tape, and the rest of the tape is filled with \sqcup .

Example.

$$L = \{0^{2^n} \mid n \geq 0\}$$

Idea. Cross off every second, and check if the remaining is even (except the last one).

The procedure should be:

- 1° left → right, make remark on every second 0
- 2° if step 1° left with only one unmarked 0, accept
- 3° if step 1° left with odd #0 left, reject
- 4° move head to the leftmost
- 5° go to step 1°

The definition of the machine is

$$\begin{aligned} Q &= \{q_0, q_1, q_2, q_3, q_4, q_5, q_{\text{accept}}, q_{\text{reject}}\} \\ \Sigma &= \{0\} \\ \Gamma &= \{0, x, \sqcup\} \end{aligned}$$

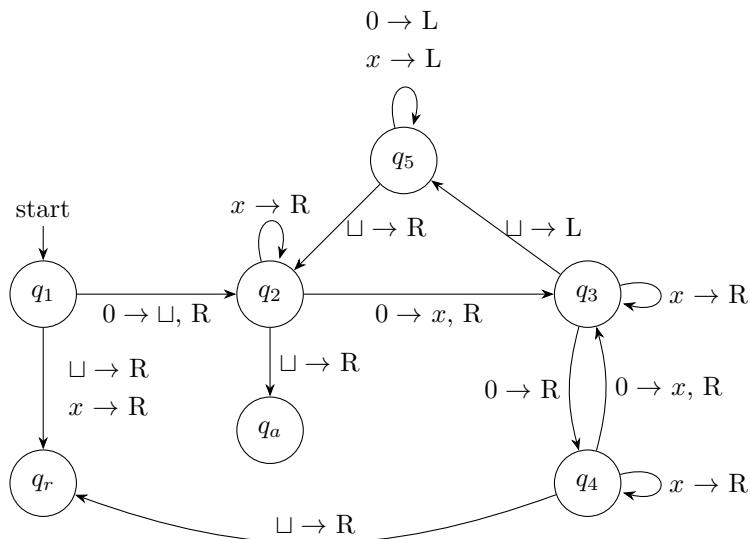


Figure 3.3: TM for $L = \{0^{2^n} \mid n \geq 0\}$

Notation.

$$0 \rightarrow R \equiv 0 \rightarrow 0, R$$

Consider the input 0000:

$q_1 0000$	$\sqcup q_2 000$	$\sqcup x q_3 00$	$\sqcup x 0 q_4 0$	$\sqcup x 0 x q_3$
$\sqcup x 0 q_5 x$	$\sqcup x q_5 0 x$	$\sqcup q_5 x 0 x$	$q_5 \sqcup x 0 x$	$\sqcup q_2 x 0 x$
$\sqcup x q_2 0 x$	$\sqcup x x q_3 x$	$\sqcup x x x q_3 \sqcup$	$\sqcup x x q_5 x$	$\sqcup x q_5 x x$
$\sqcup q_5 x x x$	$q_5 \sqcup x x x$	$\sqcup q_2 x x x$	$\sqcup x q_2 x x$	$\sqcup x x q_2 x$
$\sqcup x x x q_2$	$\sqcup x x x \sqcup q_a$			

The transition function table is as follows:

	0	x	\sqcup
q_1	q_2, \sqcup, R	q_{reject}, x, R	$q_{\text{reject}}, \sqcup, R$
q_2	q_3, x, R	q_2, x, R	$q_{\text{accept}}, \sqcup, R$
\vdots			

Note. There is no need for transition for q_{accept} and q_{reject} since the machine halts when it enters these states.

Idea. We can get the design idea of Turing Machine

- q_1 : mark the start by \sqcup
 - first element must be 0, otherwise, reject
 - Using \sqcup , so the start is known
- $q_2 \rightarrow q_3$: handle initial 00
- $q_3 \rightarrow q_4 \rightarrow q_3$: sequentially $00 \rightarrow 0x$
 - If not pairs (e.g., $0x0x0x$), fails
 - This is the place of checking if # of remained zeros is even
- $q_3 \rightarrow q_5 \rightarrow q_2$ back to beginning
- first First 0 (or \sqcup) is considered the single final 0

$$q_2 \rightarrow \dots \rightarrow q_2 \rightarrow \dots \rightarrow q_{\text{accept}}$$

check if a single 0 is left in the string.

3.1.1 Configuration of Turing Machine

Definition 3.1.2 (current configuration). The current configuration of a Turing Machine is represented as

$$uqv$$

where

- $u \in \Gamma^*$: the string on the left of the head
- $q \in Q$: the current state
- $v \in \Gamma^*$: the string on the right of the head

The head is reading the first symbol of v . If $v = \epsilon$, then the head is reading a blank symbol \sqcup .

Definition 3.1.3. $a, b, c \in \Gamma$, $u, v \in \Gamma^*$, $q_i, q_j \in Q$ then the transition from configuration

- If $\delta(q_i, b) = (q_j, c, L)$, then

$$uaq_i bv \vdash uq_j acv$$

- If $\delta(q_i, a) = (q_j, b, L)$, then

$$uaq_i bv \vdash uacq_j v$$

3.1.2 Turing Recognizable and Turing Decidable Languages

Definition 3.1.4 (Turing Recognizable). A language L is Turing recognizable if some Turing Machine M recognizes it.

For a Turing Machine there are three possible outcomes:

- Accept the input by entering q_{accept}
- Reject the input by entering q_{reject}
- Loop forever without halting

A language is very difficult to decide if the TM loops forever on some inputs. We now define a more restricted type of model, called **Decider**.

Definition 3.1.5 (Turing Decidable). A language L is Turing decidable if some Turing Machine M decides it.

We will discuss more about Decidability in later chapters (Ch.4).

3.1.3 Example of Turing Machine

Example. $L = \{w\#w \mid w \in \{0, 1\}^*\}$

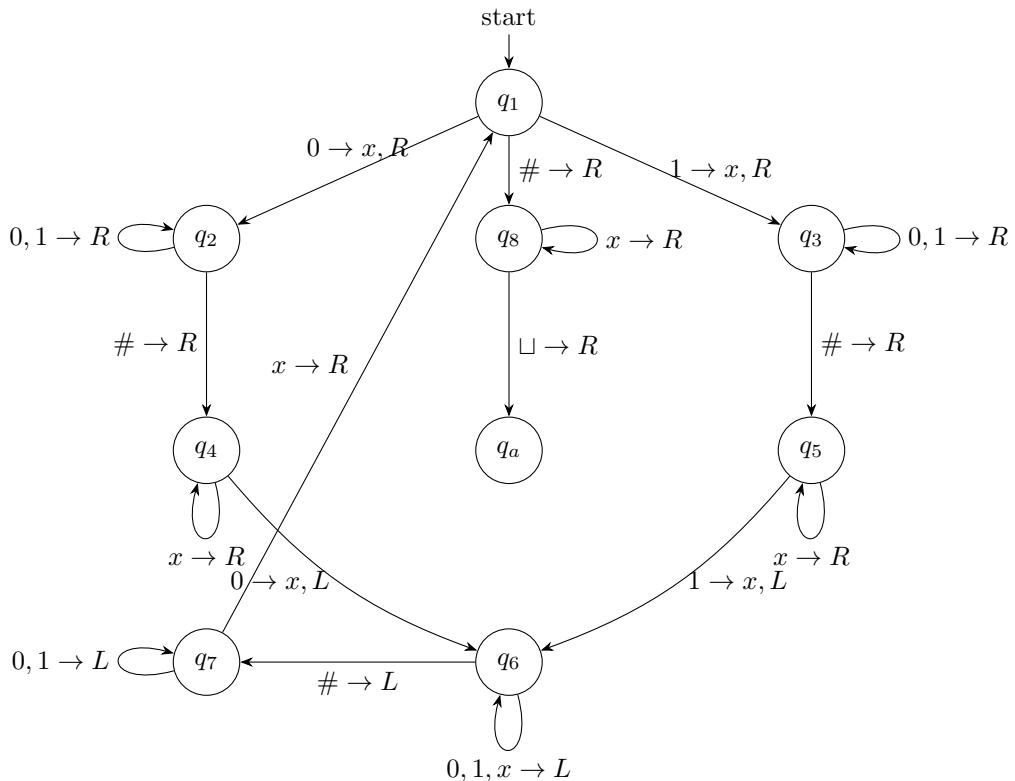


Figure 3.4: Turing Machine of $L = \{w\#w \mid w \in \{0, 1\}^*\}$

Remark. Links to q_r are not shown

Simulate 01#01

$$\begin{array}{llll}
 q_101\#01 & xq_21\#01 & x1q_2\#01 & x1\#q_401 \\
 x1q_6\#x1 & xq_71\#x1 & q_7x1\#x1 & xq_11\#x1 \\
 xxq_3\#x1 & xx\#q_5x1 & xx\#xq_51 & xx\#q_6xx \\
 xxq_6\#xx & xq_7x\#xx & xxq_1\#xx & xx\#q_8xx \\
 xx\#xxq_8\sqcup & xx\#xx \sqcup q_a & &
 \end{array}$$

Idea. The diagram:

$$q_1 \rightarrow q_2 \rightarrow q_4 \rightarrow q_6$$

check 0 at the same position of the two strings

$$q_1 \rightarrow q_3 \rightarrow q_5 \rightarrow q_6$$

check 1 at the same position of the two strings

Example. $C = \{a^i b^j c^k \mid i \times j = k, i, j, k \geq 1\}$

Idea. The procedure should be:

- 1° check if the input is $a^+ b^+ c^+$
- 2° back to the leftmost a
- 3° fix an a , for each b , cross off a c
- 4° store b back, cancel one a , repeat step 3

- Step 1 can be done by a DFA (as DFA is a special case of TM).
- Step 2 can be done by moving left until \sqcup is reached.
- Step 3 is similar to previous examples.

Example. $E = \{\#x_1\#x_2 \dots \#x_l \mid x_i \in \{0, 1\}^*, x_i \neq x_j\}$

Idea. Sequentially compare every pairs

$$\begin{aligned}
 &x_1x_2, x_1x_3, \dots, x_1x_l \\
 &x_2x_3, \dots, x_2x_l \\
 &\vdots \\
 &x_{l-1}x_l
 \end{aligned}$$

For x_i, x_j , mark #'s strings by $\dot{\#}$ i.e.

$$\dot{\#}x_1\#x_2\dot{\#}x_3 : x_1 \text{ and } x_3 \text{ are compared}$$

We can copy x_i, x_j to the right end of the tape and compare them there with the pattern of $w\#w$.

Lecture 9

3.2 Multi-tape Turing machines

2025-11-24

3.2.1 Variants of Turing machines

Example. The transition function may be defined as

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

Notation. S stands for “stay”, meaning the head does not move.

This kind of Turing machine can be simulated by a standard Turing machine.

$$\begin{aligned} q_1, a \rightarrow q_2, b, S &\equiv q_1, a \rightarrow q_{\text{temp}}, b, R \\ q_{\text{temp}}, \gamma \rightarrow q_2, \gamma, L &\quad \forall \gamma \in \Gamma \end{aligned}$$

3.2.2 Multi-tape Turing machines

In this variant, there are multiple tapes, each with its own head.

- Input: In the first tape.
- Other tapes: Blank initially.

Definition 3.2.1 (Multi-tape Turing machine). Transition function:

$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, S\}^k \quad (\text{for } k \text{ tapes})$$

Example. Given $w = 0^{2n}$, $n \geq 1 \Rightarrow$ Generate ww .

Idea. We have the following simulation:

- Copy w into the second tape.
- Check if $|w|$ is even.
- copy w from the second tape to the first tape (append).

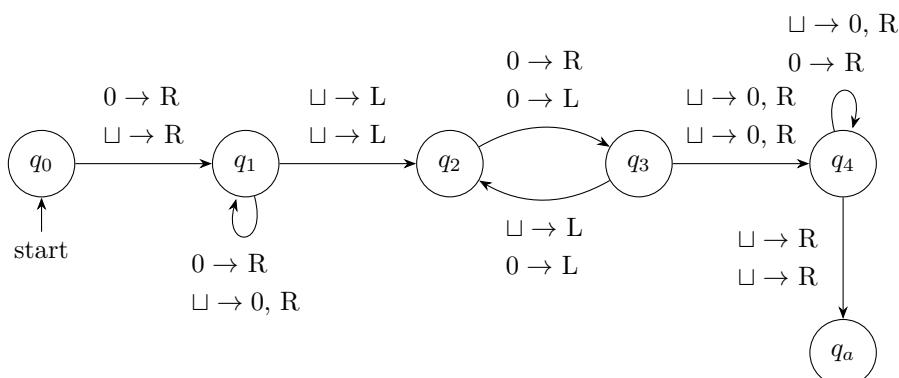


Figure 3.5: Multi-tape Turing machine to compute ww from $w = 0^{2n}$

For the simulation details,

- $q_0 \rightarrow q_1$: Use \sqcup to indicate the beginning of the second tape.
- loop in q_1 : Copy w from the first tape to the second tape.
- $q_2 \rightarrow q_3 \rightarrow q_2$:
 - 1° Check if length of w is even.
 - 2° Head of the first tape zig-zag between last 0 and then \sqcup after.
 - 3° Head of the second tape moves to the beginning.

Remark. If length of w is even, we will at q_3 when reaching the beginning of the second tape.

- q_4 : copy w from the second tape to the first tape (append).

Below is the illustration of the simulation 0000:

$$\begin{array}{cccc}
 q_0 \sqcup 0 0 0 0 & 0 q_1 \sqcup 0 0 0 & \dots & 0 0 0 0 q_1 \sqcup \\
 \sqcup 0 0 q_2 0 & \sqcup 0 q_3 0 0 & 0 0 0 q_2 0 & 0 0 0 0 q_3 \\
 0 0 0 0 q_4 & \dots & 0 0 0 0 0 0 q_4 & 0 0 0 0 q_4
 \end{array}$$

Below is the illustration of the simulation 000:

$$\begin{array}{cccc}
 q_0 \sqcup 0 0 0 & 0 q_1 \sqcup 0 0 & \dots & 0 0 0 q_1 \sqcup \\
 \sqcup 0 q_2 0 & \sqcup q_3 0 0 & 0 0 q_2 0 &
 \end{array}$$

Note. Due to the properties of “deterministic”,

$$\delta(q_0, 0, 0), \delta(q_0, 0, \sqcup), \delta(q_0, \sqcup, 0), \delta(q_0, \sqcup, \sqcup)$$

those not specified transitions go to q_r .

3.2.3 Multi-tape TM \equiv Single-tape TM

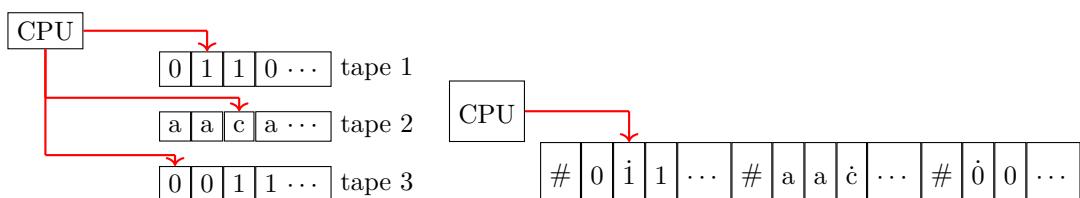


Figure 3.6: Simulation of multi-tape Turing machine by single-tape Turing machine

- $\#$ can be used to separate different tapes.
- \dot{c} can be used to indicate the head position of each tape.
- $\Gamma' = \{\Gamma, \dot{\Gamma}\}$

Example. Right shifting a sequence w .

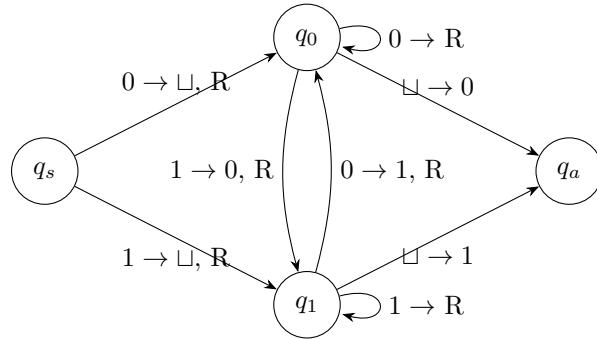


Figure 3.7: Single-tape simulation Turing machine to right shift a sequence

Note. Because Γ is finite, the simulation must succeed by add state for each $\gamma \in \Gamma$.

3.3 Nondeterministic Turing Machines

Definition 3.3.1 (Nondeterministic Turing machine). Transition function:

$$\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

In NTM, by definition w is accepted if any branch works, which means unless all branches are finite,

$$\text{NTM} \longrightarrow \text{accept or loop}$$

Thus, NTM is an “acceptor”.

Example. $A = \{w \text{ contain } aab\}$

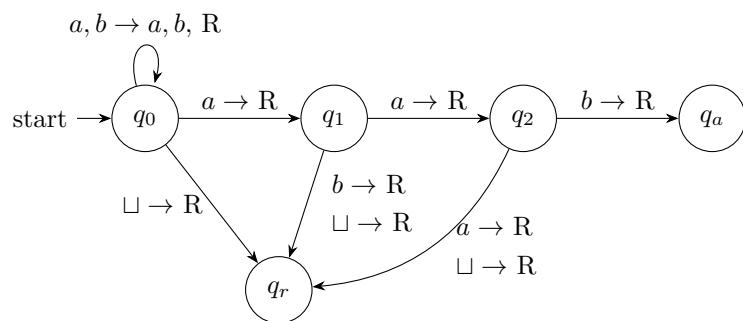


Figure 3.8: Nondeterministic Turing machine to accept $A = \{w \text{ contain } aab\}$

Note. Determine where aab starts nondeterministically.

Example. $L = \{0^n \mid n\text{-composite number}\}$

- Nondeterministically guess p, q , sequentially try from 2 to $n - 1$.
- Check if $n = p \times q$

3.3.1 NTM \equiv TM

A language is Turing-recognizable \Rightarrow it is recognized by a NTM, due to TM being a special case of NTM.
Proof done for \Leftarrow direction.

For the other direction, we need to simulate NTM by TM. Like NFA we use a tree structure to represent the computation finite # branches. To traverse the tree, we can use BFS or DFS.

Remark. BFS is preferred, because DFS may get stuck in an infinite branch.

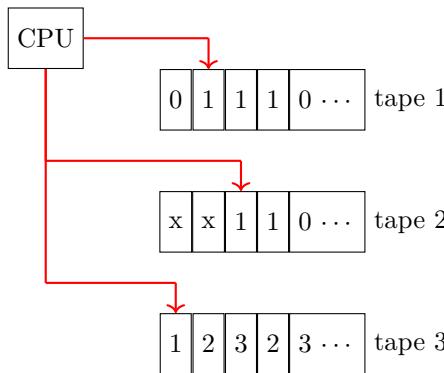


Figure 3.9: Simulation of NTM by TM using multi-tape Turing machine

- Tape 1: Store the input, never alter.
- Tape 2: Simulate the current branch up to certain layer by copying Tape 1.
- Tape 3: Store the path to a node.

Definition 3.3.2. Suppose max # branches is 3 at each node. If the content of the 3rd tape is 231 that means

$$\text{root} \rightarrow \text{2nd child} \rightarrow \text{3rd child} \rightarrow \text{1st child}$$

Hence, NTM can be simulated by 3-tape TM, and we have shown that multi-tape TM can be simulated by single-tape TM. Thus, NTM \equiv TM. ■

Corollary 3.3.1. NTM is a decider if all branches halt on all inputs.

Language decidable \Leftrightarrow some NTM decides it

Proof. We separate into two directions.

- \Rightarrow One TM decides it and a TM is an NTM. This TM halts on all inputs (one branch)
- \Leftarrow Now NTM terminates on all branches. We can construct a TM to decide the language
 - each branch is finite every input halts \exists a finite max length.
 - # branches finite. The tree to process this input is finite.
 - Thus the three-tape TM used earlier can accept/reject the input in a finite number of steps.

3.4 Hilbert's problems

Informally, an algorithm is a collection of instructions. Not until 1900 did Hilbert propose 23 unsolved problems in mathematics. The 10th problem is:

Is there a general method to determine whether a given polynomial equation with integer coefficients has an integer solution?

Hilbert didn't use the word "algorithm" but "general method". However, Hilbert explicitly asked the algorithm be "devised".

3.4.1 Church-Turing thesis

This is proposed by Alonzo Church and Alan Turing in 1936.

Any function which would naturally be regarded as computable can be computed by a Turing machine.

Definition 3.4.1.

Intuitive Algorithm \equiv Turing machine algorithm

3.4.2 Hilbert's 10th problem

Using the Church-Turing thesis, we have a question:

Question. Define

$$D = \{P \mid P : \text{polynomial with integer root}\}$$

is D decidable?

We first simplify the problem to one variable case.

$$D_1 = \{P \mid P : \text{polynomial of } x \text{ with integer root}\}$$

If we try all integers one by one, it may not halt if no integer root exists. Thus, D_1 is Turing-recognizable but not decidable.

However, it can be proved that roots of a 1-variable polynomial is within the range

$$-M \leq x \leq M, \quad M = \pm k \frac{\max |c_i|}{|c_1|}$$

where

- k : # of terms
- c_i : coefficients of the polynomial
- c_1 : leading coefficient

For instance, for $4x^3 - 2x^2 + x - 7$, we have

$$M = \pm 4 \cdot \frac{7}{4} = \pm 7$$

The multi-variable case is much more complicated. In 1970, Matiyasevich proved that no such algorithm exists. Thus, we have the conclusion:

Theorem 3.4.1 (Matiyasevich, 1970). D is not decidable.

3.4.3 Description of Turing machines

A Turing machine can be described in 3 levels:

- **High-level description:** Describe the operations of the Turing machine without managing the tape and head.
- **Implementation-level description:** Describe how the Turing machine moves the head.
- **Formal description:** Specify the states, input alphabet, tape alphabet, transition function, start state, accept state, and reject states of the Turing machine.

Example. Describe a Turing machine that decides the language

$$A = \{\langle G \rangle \mid G : \text{a connected undirected graph}\}$$

- **High-level description:** We separate into three steps.
 - 1° Mark node in G .
 - 2° Repeat until no new nodes marked:
 - For each node in G , if it is marked, mark all its neighbors.
 - 3° If all nodes marked: accept, otherwise: reject.
- **Implementation-level description:**

$$\langle G \rangle = (1, 2, 3, 4)((1, 2), (2, 3), (3, 1), (1, 4))$$

is the input format.

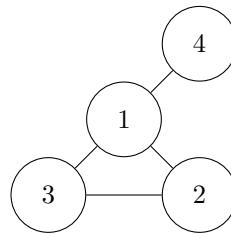


Figure 3.10: Graph representation on tape

- The first step is to check if the input is in the correct format.
- In the first step we begin with seeing if the first part of the input $\langle G \rangle$ includes distinct numbers (as node IDs should be different)
- This is similar to an example before.

$$\{\#x_1\#x_2\#\cdots x_n\# \mid x_i \in \{0, 1\}^*, x_i \neq x_j\}$$

- Then we can talk about how the head is moved.

Chapter 4

Decidability

Lecture 10