

Linear Algebra

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Abstract

The lecture note of 2025 Fall Linear Algebra by professor 李明穗 (Amy Lee) .

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Chapter 0

Introduction

Lecture 1

0.1 Geometry

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- linear
- To study geometry with linearity
- In a different dimension:
 - In 2D: **lines**
 - In 3D: **planes**
 - In n D: **hyperplanes**

0.2 Abstract Algebra

Definition 0.2.1 (Linear Algebra). Here is the definition of Linear algebra.

- Algebra is the study of basic "mathematical structure."
e.g. **Group**, **Ring**, **Field**, ...etc.
- Linear Algebra studies one of the structures called **vector space**.

Note. Followed by logical deduction from the basic definition, we can derive some theorems.

0.3 Applied Science

- **Mathematic:** ODE, PDE.
- **Linear Programming:** developing during World War II
- **Image Processing, Computer Vision, Computer Graphic**, etc.

Chapter 1

Matrices and Gaussian Elimination

1.1 Introduction

The central problem of Linear Algebra is the solution of Linear Equations. The most important and simplest case is when the # of unknowns equals to the # of equations.

Note. There are two ways to solve linear equations:

- The method of elimination (**Gaussian Elimination**)
- Determinants (**Cramer's Rule**)

1.1.1 Four aspects to follow

- (1) The geometry of linear equations.

Note. $n = 2, n = 3 \rightarrow$ higher dimensional space.

- (2) The interpretation of elimination is a factorization of the coefficient matrix.

Definition. Some notation to define:

Definition 1.1.1 (Scalar, Matrix, Vector).

$$Ax = b \quad \begin{cases} \alpha, \beta, \gamma : & \text{scalar} \\ A, B, C : & \text{matrix} \\ a, b, c : & \text{vector} \end{cases}$$

Definition 1.1.2 (Lower/Upper triangular matrix).

$$A = LU \quad \begin{cases} L : & \text{lower triangular matrix} \\ U : & \text{upper triangular matrix} \end{cases}$$

Definition 1.1.3 (Transpose/Inverse).

$$A^T/A^{-1} : \quad \begin{cases} A^T : & \text{Transpose of matrix A} \\ A^{-1} : & \text{Inverse of matrix A} \end{cases}$$

(3) Irregular case and Singular case (**no unique solution**):

Note. no solution or infinitely many solutions

(4) The # of operations to solve the system by elimination

1.2 Geometry of Linear Equation

Example. Consider the linear equation below:

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

- approach 1: row picture \rightarrow two lines in plane

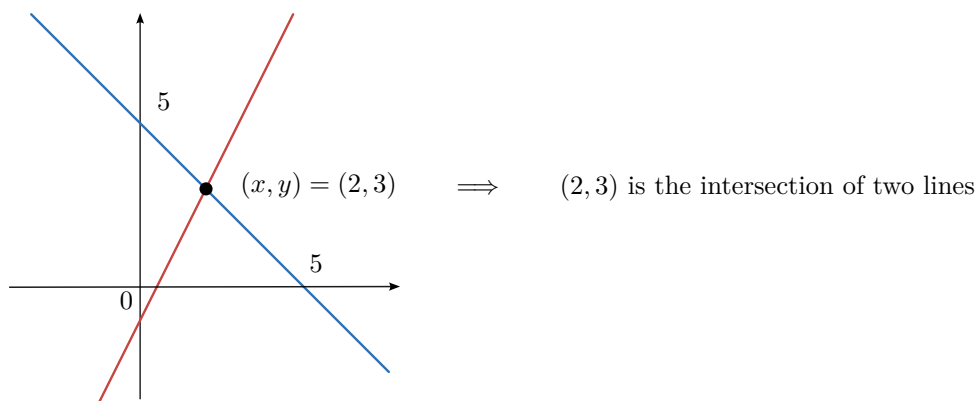


Figure 1.1: Row Picture

- approach 2: column picture

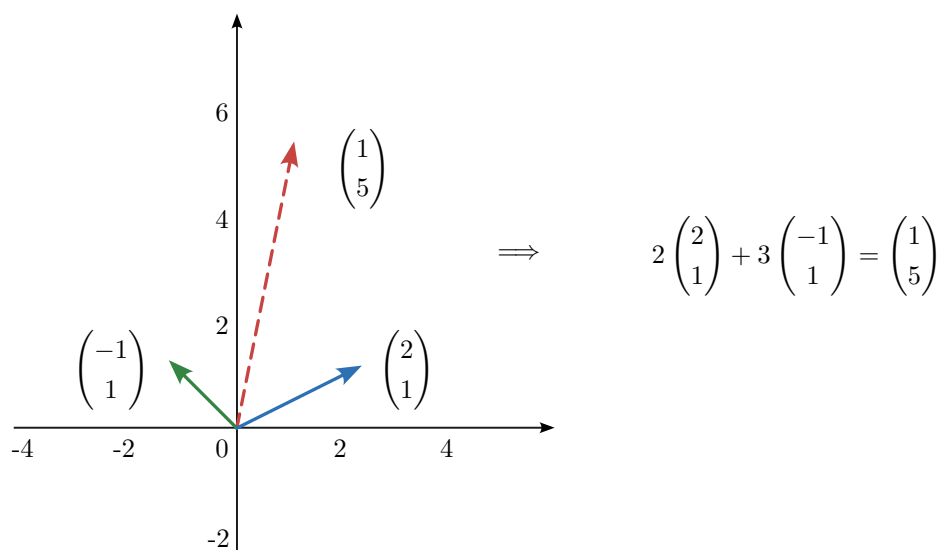


Figure 1.2: Column Picture

Lemma 1.2.1 (Linear Combination).

$$x \begin{pmatrix} 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

To find the **Linear Combination** of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to reach $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

Note. A vector is a $n \times 1$ array with n real numbers, c_n is

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

But in the text, we use

$$(c_1, \dots, c_n)$$

to represent.

Definition. Here are some operations on matrix:

Definition 1.2.1.

$$\alpha \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}_{n \times 1} = \begin{pmatrix} \alpha \cdot c_1 \\ \vdots \\ \alpha \cdot c_n \end{pmatrix}_{n \times 1}, \quad \alpha \in \mathbb{R}$$

Definition 1.2.2.

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{pmatrix}_{n \times 1}$$

Definition 1.2.3.

$$y \in \mathbb{R}$$

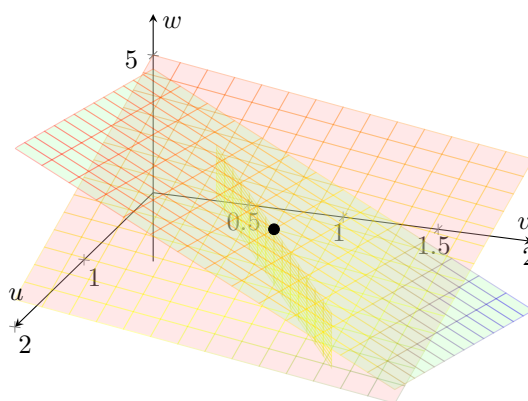
$$y \in \mathbb{R}^2 \implies y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{2 \times 1} \quad y_1, y_2 \in \mathbb{R}$$

$$y \in \mathbb{R}^3 \implies y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3 \times 1} \quad y_1, y_2, y_3 \in \mathbb{R}$$

Example. Consider the linear equation below:

$$\begin{cases} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7u + 2w &= 9 \end{cases}$$

- Row picture



$$(u, v, w) = (1, 1, 2)$$

Lemma 1.2.2. in n -dimension, a line require $(n - 1)$ equation.

Question. How to extend into n -dimensions?

Answer. Consider the following steps:

- Each equation represents a plane or hyperplane.
- The first equation produces a $(n - 1)$ -dimension plane in \mathbb{R}^n
- The second equation produces another $(n - 1)$ -dimension plane in \mathbb{R}^n
- Their intersection in smaller set of $(n - 2)$ -dimension
- $(n - 3) \rightarrow (n - 4) \rightarrow \dots \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow \text{point}$

Then we can find the final intersection. ⊗

- Column picture

$$u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \iff \begin{cases} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7u + 2w &= 9 \end{cases}$$

RHS is a linear combination of 3 column vectors.

Theorem 1.2.1. Solution to a linear equation:

$$\underbrace{(\text{intersection of to points})}_{\text{row pic.}} = \underbrace{(\text{coefficient of linear combination})}_{\text{column pic.}}$$

1.2.1 Singular Case

(1) Row Picture: In 3D case, they didn't intersect at a point.

- **Case 1:** two parallel

$$\begin{cases} 2u + v + w &= 5 \\ 4u + 2v + 2w &= 9 \end{cases}$$

- **Case 2:** three plane perpendicular (\perp)

$$\begin{cases} u + v + w &= 2 \dots (1) \\ 2u + 3w &= 5 \dots (2) \\ 3u + v + 4w &= 6 \dots (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1) + (2) = (3) \quad ; \quad \text{LHS} \Rightarrow (1) + (2) \neq (3)$$

- **Case 2:** three plane have a whole line in common.

$$\begin{cases} u + v + w &= 2 \dots (1) \\ 2u + 3w &= 5 \dots (2) \\ 3u + v + 4w &= 7 \dots (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1) + (2) = (3) \quad ; \quad \text{LHS} \Rightarrow (1) + (2) = (3)$$

- **Case 4:** three parallel

(2) Column Picture:

$$u \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + v \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + w \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = b$$

In the case above, three vectors are linear combination to each other, i.e. three vectors share the same plane.

Lemma 1.2.3 (Singular case). If the three vectors are linear combination to each other (three vector share a common plane), it must be **singular case**.

- If $b = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$, which is on the plane \Rightarrow too many solution to produce b .
- If $b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$, which is not on the plane \Rightarrow no solution.

1.2.2 Fundamental Linear Algebra Theorem (Geometry form)

Theorem 1.2.2 (Fundamental LA Theorem). Consider a linear system

$$Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^m.$$

If the n hyperplanes have no only one intersection or infinitely many points, then the n columns lie in the same plane. (consistency of *row picture* and *column picture*)

Notation. Logic notation:

- If ..., then : \Rightarrow
- If and only if : \Leftrightarrow

Lecture 2

1.3 An Example of Gaussian Elimination

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Example. Here is a linear equation.

$$\begin{cases} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{cases}$$

$$\begin{pmatrix} \boxed{2} & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & \boxed{-8} & -2 & -12 \\ 0 & 8 & 3 & 14 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & \boxed{1} & 2 \end{pmatrix} \quad \text{"pivot"}$$

Then we get $w = 2$, we can plug in the equation i.e.

$$\begin{cases} 2u + v + 1w = 5 \\ -8v - 2w = -12 \\ w = 2 \end{cases} \Rightarrow \text{Forward Elimination}$$

Then we substitute into 2nd, 1st equation to get $v = 1$ and $u = 1 \Rightarrow$ Backend Elimination

Note. By definition, **pivots cannot be zero!**

Question. Under what circumstances could the elimination process break down?

Answer. Here are some situations.

- Something **must** go wrong in the singular case.
- Something **might** go wrong in the nonsingular case.

A zero appears in a pivot position!

If in the process, there are nonzero pivots, then there's only one solution.

⊛

Example.

$$\begin{pmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix}$$

- (1) If $a_{11} = 0 \implies$ nonsingular
- (2) If $a_{22} = 0 \implies$ nonsingular
- (3) If $a_{33} = 1 \implies$ singular

Question. How many separate arithmetical operations does elimination require for n equations in n unknowns?

Answer. For a single operation.

a single operation = each division & each multiplication-subtraction

⊛

• **FE:**

$$\begin{array}{ccccccc} x & x & \cdots & x & = & x \\ \vdots & \vdots & & & & \vdots \\ x & x & \cdots & x & = & x \end{array}$$

$\underbrace{\hspace{10em}}_n$

$$n(n-1) + (n-1)(n-2) + \cdots + (1^2 - 1) = \frac{n^3 - n}{3} \sim \frac{n^3}{3} \text{ steps}$$

• **RHS:**

$$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

• **BF:**

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

1.4 Matrix Notation and Matrix Multiplication

$$\begin{cases} 2u + 4v - 2w = 2 \\ 4u + 9v - 3w = 8 \\ -2u - 3v + 7w = 10 \end{cases} \implies u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + w \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

We can rewrite it in the below form.

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}_{3 \times 3}, \quad x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{3 \times 1}, \quad b = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}_{3 \times 1} \implies x = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}_{3 \times 1}$$

coefficient matrix unknowns RHS solution

$$\boxed{Ax = b}$$

Definition 1.4.1. An $m \times n$ matrix, $A_{m \times n}$ over \mathbb{R} , is an array with m rows and n columns of real numbers, which can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \text{ where } a_{ij} \in \mathbb{R}, \quad \begin{cases} i : \text{index of row} \\ j : \text{index of column} \end{cases}$$

- $\boxed{m \times n}$ is called the **dimensions (size)** of $A \implies$ dimension of a $()_{3 \times 5}$ is 3×5
- $\boxed{a_{ij}}$ is called the **elements/entry/coefficient** of A
- **Addition:** $A = (a_{ij})_{m \times n}, \quad B = (b_{ij})_{m \times n}$

$$A + B = (a_{ij} + b_{ij})_{m \times n}$$

- **Multiplication:** $A = (a_{ij})_{m \times n}, \quad B = (b_{ij})_{n \times r}$

$$AB = (c_{ij})_{m \times r}, \quad \text{where } c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

- **Scalar Multiplication:**

$$\alpha A = (\alpha a_{ij})_{m \times n}$$

•

$$A_{m \times n} x_{n \times 1} = b_{m \times 1}$$

In particular, if

$$A_{1 \times n} B_{n \times 1} = \mathbf{v} \cdot \mathbf{w} = ()_{1 \times 1}.$$

Then it's the **inner product** of vector \mathbf{v} and vector \mathbf{w}

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & 3 & -7 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot (-1) & 4 \cdot (2) & -2 \cdot (2) \\ 4 \cdot (-1) & 9 \cdot (2) & -3 \cdot (2) \\ -2 \cdot (-1) & 3 \cdot (2) & -7 \cdot (2) \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 22 \end{pmatrix}$$

$$(-1) \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 9 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$$

(1) by row: 3 inner product

(2) by column: a linear combination of 3 columns of A **Example (1A).** Ax is a combination of columns of A

$$\begin{aligned} A_{m \times n} x_{n \times 1} &= (A_1 | A_2 | \cdots | A_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1(A_1) + x_2(A_2) + \cdots + x_n(A_n) = \left(\sum_{j=1}^n a_{ij} x_j \right)_{m \times 1} \end{aligned}$$

1.4.1 The Matrix Form of One Elimination Step

Definition (1B). Matrix form**Definition 1.4.2.** zero matrix:

$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Definition 1.4.3. identity matrix:

$$I = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} = I_n = I_{n \times n}; \quad \begin{cases} A_{m \times n} I_n = A_{m \times n} \\ A_{m \times n} = A_{m \times n} I_n \end{cases}$$

Definition 1.4.4. elementary matrix (elimination matrix):

$$E_{ij} = \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & -\ell & \ddots & 0 \\ 0 & \cdots & \text{jth column} & 0 & \cdots & 1 \end{pmatrix} \quad \begin{array}{l} \ell : \text{multiplier} \\ \text{ith row} \end{array}$$

$$E_{ij} \cdot A = \begin{pmatrix} \cdots & -\ell & \cdots & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} \begin{array}{l} \leftarrow \text{i-th} \Rightarrow (\text{i-th row}) + (-\ell)(\text{j-th column}) \\ \leftarrow \text{j-th} \Rightarrow \text{create zero at } (i, j) \text{ position!} \end{array}$$

Example.

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{E_{21}} \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}_A = \begin{pmatrix} 2 & 4 & -2 \\ 0_{21} & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}_{EA}$$

Note. Here is two properties

1. $Ax = b \implies E_{ij}Ax = E_{ij}b$
2. $E_{ij}A \neq AE_{ij}$

1.4.2 Matrix Multiplication

- (1) The (i, j) -th entry of AB is the inner product of the **i-th** of A and the **j-th** of B .
- (2) Each column of AB is the product of a matrix A and **a column of B**

$$\begin{aligned} \implies \text{column } j \text{ of } AB &= A \text{ times } \mathbf{j\text{-th of } B} \\ &= \text{linear combination of } \mathbf{columns \text{ of } A} \\ &= b_{1j}A_{\bullet 1} + b_{2j}A_{\bullet 2} + \cdots + b_{nj}A_{\bullet n} \end{aligned}$$

any numbers

Example.

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}_{A_{2 \times 3}} \begin{pmatrix} 5 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}_{B_{3 \times 3}} = \begin{pmatrix} 16 & 1 & 1 \\ 8 & 0 & -1 \end{pmatrix}_{C_{2 \times 3}}$$

$$\text{1st column of } AB = \begin{pmatrix} 16 \\ 8 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(3) Each row of AB is a product of a row of A and a matrix B .

$$\begin{aligned}\Rightarrow \text{ i-th row of } AB &= \text{ i-th row of } A \text{ times } B \\ &= \text{ linear combination of rows of } B \\ &= a_{i1}B_{1\bullet} + a_{i2}B_{2\bullet} + \cdots + a_{in}B_{n\bullet}.\end{aligned}$$

Theorem 1.4.1. Let A, B and C be matrices (possibly rectangular). Assume that their dimension permit them to be added and multiplied in the following theorem

(1) The matrix multiplication is associative

$$(AB)C = A(BC)$$

(2) Matrix operations are distributive

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

(3) Matrix multiplication is noncommutative

$$AB \neq BA \quad \text{in general}$$

(4) Identity Matrix

$$A_{n \times n} I_n = I_n A_{n \times n} = A_{n \times n}$$

Example.

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

(1)

$$E_{21} F_{31} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \boxed{=} \quad F_{31} E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(2)

$$E_{21} G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \boxed{\neq} \quad G_{32} E_{21}$$

(3)

$$G_{32} F_{31} E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \quad \boxed{\neq} \quad E_{21} F_{31} G_{32} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

"right order"

Note. The product of lower triangular matrices is a lower triangular matrix.

Lecture 3

1.5 Triangular Factors and Row Exchanges

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$$\boxed{Ax = b}$$

$$\implies \textcolor{red}{LU}x = b \implies \begin{cases} Lc = b \\ Ux = c \end{cases}$$

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = b$$

Remark. ℓ : multipliers

$$E_{ij}(\ell) : (\text{i-th row}) + (-\ell)(\text{j-th column})$$

$$\begin{pmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{pmatrix} \xrightarrow[\textcolor{blue}{R_3+(1)R_1}]{\textcolor{blue}{R_2+(-2)R_1}} \begin{pmatrix} 2 & 4 & -2 & 2 \\ \textcolor{blue}{0}_{21} & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{pmatrix} \xrightarrow{\textcolor{blue}{R_3+(-1)R_2}} \begin{pmatrix} \boxed{2} & 4 & -2 & 2 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & \textcolor{blue}{0}_{32} & \boxed{4} & 8 \end{pmatrix} \quad \textcolor{red}{\text{pivot}}$$

$$E_{21}(\textcolor{red}{2}) = E = \begin{pmatrix} 1 & 0 & 0 \\ \textcolor{blue}{-2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{31}(\textcolor{red}{-1}) = F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \textcolor{blue}{1} & 0 & 1 \end{pmatrix}, \quad E_{32}(\textcolor{red}{1}) = G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \textcolor{blue}{-1} & 1 \end{pmatrix}$$

i.e.

$$E_{21}E_{31}E_{32}Ax = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = Ux = c = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} = E_{21}E_{31}E_{32}b$$

Question. How can we undo the steps of Gaussian Elimination?

$$\textcolor{red}{E}^{-1}\textcolor{red}{F}^{-1}\textcolor{red}{G}^{-1}GF EA = A = \underbrace{E^{-1}F^{-1}G^{-1}}_{\text{factors of } A} \boxed{U} = LU \quad \text{i.e.} \quad A = \textcolor{red}{LU}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \textcolor{blue}{-(-2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \textcolor{blue}{-1} & 0 & 1 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \textcolor{blue}{-(-1)} & 1 \end{pmatrix}$$

$$E^{-1}F^{-1}G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{-1} & \boxed{1} & 1 \end{pmatrix} \implies \text{records everything that has been done so far}$$

1.5.1 Triangular Factorization

Theorem 1.5.1. If no exchanges are required, the original matrix A can be written as

$$A = LU$$

- The matrix L is lower triangular with 1's on the diagonal and the multipliers ℓ_{ij} (taken from elimination) below the diagonal.
- The matrix U is the upper triangular matrix which appears after forward elimination and before back-substitution; its diagonal entries are the pivots.

Example.

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 6 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \text{提出2}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 6 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Question.

$$A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 2 & 6 & 5 \\ -1 & 4 & -2 \\ 1 & 2 & 3 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

triangular matrix 有三條對角線

Answer.

⊗

1.5.3 Row Exchange and Permutation Matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \langle \text{Permutation matrix } P_{ij} \rangle$$

Note. Permutation matrix is also an elementary matrix.

Example. Here are some of the example:

1°

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \quad \boxed{R_2 \leftrightarrow R_3}$$

2°

$$PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{pmatrix} \quad \boxed{R_2 \leftrightarrow R_3}$$

3°

$$AP = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 5 & 6 \end{pmatrix} \quad \boxed{C_2 \leftrightarrow C_3}$$

Note. For the permutation matrix:

1° PA : Performing row exchange of A

2° AP : Performing column exchange of A

3° $PAx = Pb$; Should we permute the component of $x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ as well? **NONONONONO!!!**

Example.

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{pmatrix} \quad Ax = b$$

(1) if $d = 0$, the problem is incurable. The matrix is singular.

(2) if $d \neq 0$, $P_{13}A = \begin{pmatrix} d & e & f \\ 0 & 0 & c \\ 0 & a & b \end{pmatrix}$; if $a \neq 0$, $P_{23}P_{13}A = \begin{pmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{pmatrix}$

$$\left| \begin{array}{ccc} P_{23}P_{13} & \neq & P_{13}P_{23} \\ \text{row} & \begin{matrix} 1 & 3 & 3 \\ 2 \rightarrow 2 \rightarrow 1 \\ 3 & 1 & 2 \end{matrix} & \begin{matrix} 1 & 1 & 2 \\ 2 \rightarrow 3 \rightarrow 3 \\ 3 & 2 & 1 \end{matrix} \end{array} \right|$$

Theorem 1.5.3. We separate into two cases:

- In the non singular case, there's a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. In this case,
 - (1) $Ax = b$ has a **unique** solution.
 - (2) It is found by **elimination with row exchange**
 - (3) With the rows reorders in advance, PA can be factored into **LU** ($PA = LU$)
- In singular case, no reordering can produce a full set of pivots.

Example.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix} \xrightarrow[\ell_{21}=2]{\ell_{31}=1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{P_{23}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \langle \text{This is WRONG} \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

To summarize: A good code for Gaussian Elimination keeps a record of L, U and P . They allow the solution ($Ax = b$) from two triangular systems. If the system $Ax = b$ has a unique solution, then we say:

1° The system is nonsingular or

2° The matrix is nonsingular

Otherwise, it is singular.

1.6 Inverse and Transpose

Definition 1.6.1. An $n \times n$ matrix A is **invertible** if \exists an $n \times n$ matrix $B \ni BA = I = AB$

Theorem 1.6.1. If A is invertible, then the matrix B satisfying $AB = BA = I$ is unique!

Proof. Suppose $\exists c \neq B \ni AC = CA = I$

$$B = BI = B(AC) = (BA)C = IC = C \text{ i.e. } B = C$$

we call this matrix B , the **inverse of A** , and denoted as **A^{-1}** ■

Note. Not all $n \times n$ matrices have inverse.

e.g.

1°

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

2° if $Ax = \vec{0}$ has a **nonzero solution**, then A has no inverse!

$$x = A^{-1}(Ax) = A^{-1}\vec{0} = \vec{0} \quad (\rightarrow \leftarrow)$$

Note. The inverse of A^{-1} is A itself. i.e. $(A^{-1})^{-1} = A$.

Note. If $A = (a)_{1 \times 1}$ and $a \neq 0$, then $A^{-1} = (\frac{1}{a})$. The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2 \times 2}$ is

$$\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ if } \det(A) \neq 0$$

Note.

$$A = \begin{pmatrix} d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & d_n \end{pmatrix} \quad d_i \neq 0, \forall i \implies A^{-1} = \begin{pmatrix} 1/d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1/d_n \end{pmatrix}$$

Proposition 1.6.1. If A and B are invertible, then

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A_1A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1}A_1^{-1}$

1.6.1 The Calculation of A^{-1} : Gaussian-Jordan Method

$$A \cdot A^{-1} = I$$

$$A_{n \times n} B_{n \times n} = I_n$$

$$\implies A_{n \times n} (B_1 | B_2 | \cdots | B_n)_{n \times n} = (e_1 | e_2 | \cdots | e_n)_{n \times n}$$

$$\implies (AB_1 | AB_2 | \cdots | AB_n)_{n \times n} = (e_1 | e_2 | \cdots | e_n)_{n \times n}$$

$$\implies AB_1 = e_1; AB_2 = e_2; \cdots; AB_n = e_n \longrightarrow n \text{ linear systems: } Ax = b$$

Definition 1.6.2 (Gaussian-Jordan Method). Instead of stopping at U and switching to back substitution, it continues by subtracting multipliers of a row from the rows above till it reaches a diagonal matrix. Then we divide each row by corresponding pivot.

$$\begin{pmatrix} A & | & I \end{pmatrix} \xrightarrow[\text{LU}]{\times L^{-1}} (U | L^{-1}) \xrightarrow{\times U^{-1}} (I | A^{-1})$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array} \right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$$

1.6.2 Invertible = Nonsingular

Question. What kind of matrices are invertible?

Answer. Here are the example:

1° nonzero pivot [Ch1](#) [Ch4](#)

2° nonzero determinants [Ch4](#)

3° independent columns (rows) [Ch2](#)

4° nonzero eigenvalues [Ch5](#)

which will in the whole course

⊗

Suppose a matrix A has full set of nonzero pivots. By definition, A is nonsingular and the n systems

$$Ax_1 = e_1, Ax_2 = e_2, \dots, Ax_n = e_n$$

can be solved by elimination or Gaussian-Jordan Method.

Row exchanges maybe necessary, but the columns of A^{-1} are uniquely determined.

$$Ax = b \quad PAx = Pb$$

$$PAx_i = Pe_i$$

$$\{Pe_1, Pe_2, \dots, Pe_n\} = \{e_1, e_2, \dots, e_n\}$$

Note. Compute A^{-1} :

$$1^\circ A(x_1 | \dots | x_n) = I = (e_1 | \dots | e_n) \iff Ax_i = e_i, i = 1 \dots n$$

$$2^\circ \text{ Gauss-Jordan Method: } (A | I) \longrightarrow (I | A^{-1})$$

Question. We have found a matrix $A^{-1} \ni AA^{-1} = I$. But is $A^{-1}A = I$

Answer. We can do this by recall.

As previously seen. Recall that every Gauss-Jordan step is a multiplication of matrices on the left. There are three types of elementary matrices:

1° $E_{ij}(\ell)$: to subtract a multiple ℓ of j row from i row.

2° P_{ij} : to exchange row i and j

3° $\boxed{D_i(d)}$: to multiply row i by d i.e. $D_i(d) = \begin{pmatrix} 1 & & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & d & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \end{pmatrix} \rightarrow \text{ith row}$

$$\begin{pmatrix} d_1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & d_n \end{pmatrix} = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & & d_n \end{pmatrix}$$

$$\Rightarrow \text{DEEPEE}A = I \Rightarrow A^{-1}A = I \therefore \text{we have a left inverse!}$$

These are the operation of A^{-1}

⊗

Theorem 1.6.2. For nonsingular and invertible:

- Every nonsingular matrix is invertible.
- Every invertible matrix is nonsingular.

Theorem 1.6.3. A square matrix is invertible \iff it is nonsingular