Linear Algebra

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Abstract

The lecture note of 2025 Fall Linear Algebra by professor 李明穗（Amy Lee）.

Contents

Chapter 0

Introduction

Lecture 1

2 Sep. 13:20

0.1 Geometry

- linear
- To study geometry with linearity
- In a different dimension:
  - In 2D: **lines**
  - In 3D: **planes**
  - In nD: **hyperplanes**

0.2 Abstract Algebra

Definition 0.2.1 (Linear Algebra). Here is the definition of Linear algebra.

- Algebra is the study of basic "mathematical structure."
  e.g. **Group**, **Ring**, **Field**, ...etc.
- Linear Algebra studies one of the structures called **vector space**.

Note. Followed by logical deduction from the basic definition, we can derive some theorems.

0.3 Applied Science

- **Mathematic**: ODE, PDE.
- **Linear Programming**: developing during World War II
- **Imange Processing**, **Computer Vision**, **Computer Graphic**, etc.

Chapter 1

Matrices and Gaussian Elimination

1.1 Introduction

The central problem of Linear Algebra is the solution of Linear Equations. The most important and simplest case is when the # of unknowns equals to the # of equations.

Note. There are two ways to solve linear equations:

- The method of elimination **(Gaussian Elimination)**
- Determinants **(Crammer's Rule)**

1.1.1 Four aspects to follow

(1) The geometry of linear equations.

Note. $n = 2, n = 3 \quad \rightarrow \quad$ higher dimensional space.

(2) The interpretation of elimination is a factorization of the coefficient matrix.

Definition. Some notation to define:

Definition 1.1.1 (Scalar, Matrix, Vector).

$$Ax = b \qquad \begin{cases} \alpha, \beta, \gamma : & \text{scalar} \\ A, B, C : & \text{matrix} \\ a, b, c : & \text{vector} \end{cases}$$

Definition 1.1.2 (Lower/Upper triangular matrix).

$$A = LU \qquad \begin{cases} L : & \text{lower triangular matrix} \\ U : & \text{upper triangular matrix} \end{cases}$$

Definition 1.1.3 (Transpose/Inverse).

$$A^T / A^{-1} : \qquad \begin{cases} A^T : & \text{Transpose of matrix A} \\ A^{-1} : & \text{Inverse of matrix A} \end{cases}$$

(3) Irregular case and Singular case **(no unique solution)**:

> **Note.** no solution or infinitely many solutions

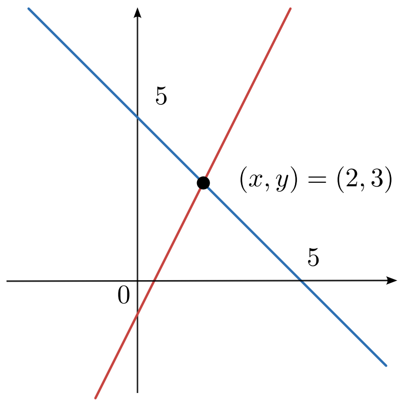
(4) **The # of operations to solve the system by elimination**

1.2 Geometry of Linear Equation

Example. Consider the linear equation below:

$$\begin{cases} 2x - y &= 1 \\ x + y &= 5 \end{cases}$$

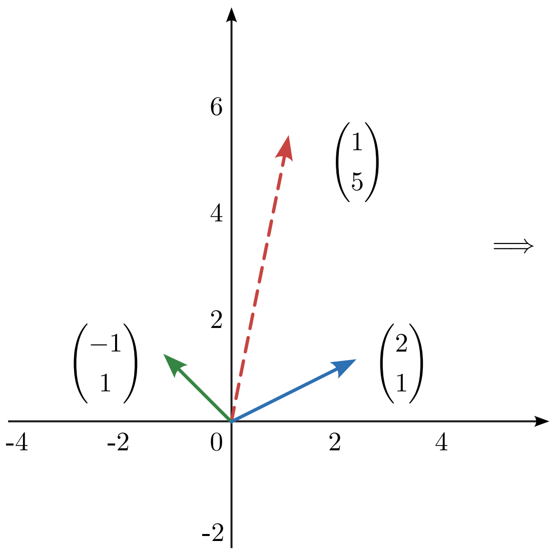
- **approach 1**: **row picture** $\to$ two lines in plane



$\Longrightarrow$ $(2, 3)$ is the intersection of two lines

Figure 1.1: Row Picture

- **approach 2**: **column picture**



$$\Longrightarrow \quad 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Figure 1.2: Column Picture

Lemma 1.2.1 (Linear Combination).

$$x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix}$$

To find the **Linear Combination** of $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ to reach $\begin{pmatrix}1\\5\end{pmatrix}$

Note. A vector is a $n\times 1$ array with n real numbers, c_n is

$$\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}$$

But in the text, we use

$$(c1,\cdots,c_n)$$

to represent.

Definition. Here are some operations on matrix:

Definition 1.2.1.

$$\alpha\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}_{n\times 1}=\begin{pmatrix}\alpha\cdot c_1\\\vdots\\\alpha\cdot c_n\end{pmatrix}_{n\times 1},\qquad \alpha\in\mathbb{R}$$

Definition 1.2.2.

$$\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}+\begin{pmatrix}d_1\\\vdots\\d_n\end{pmatrix}=\begin{pmatrix}c_1+d_1\\\vdots\\c_n+d_n\end{pmatrix}_{n\times 1}$$

Definition 1.2.3.

$$y\in\mathbb{R}$$

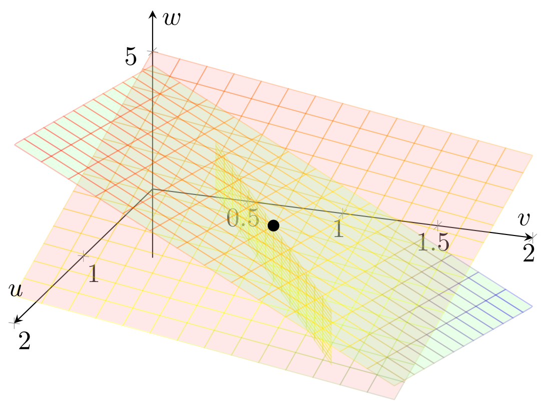
$$y\in\mathbb{R}^2\implies y=\begin{pmatrix}y_1\\y_2\end{pmatrix}_{2\times 1}\qquad y_1,y_2\in\mathbb{R}$$

$$y\in\mathbb{R}^3\implies y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}_{3\times 1}\qquad y_1,y_2,y_3\in\mathbb{R}$$

Example. Consider the linear equation below:

$$\begin{cases} 2u+v+w & =5 \\ 4u-6v & =-2 \\ -2u+7u+2w & =9 \end{cases}$$

- Row picture



$$(u,v,w)=(1,1,2)$$

Lemma 1.2.2. in n-dimension, a line require $(n-1)$ equation.

Question. How to extend into n-dimensions?

Answer. Consider the following steps:

- Each equation represents a plane or hyperplane.
- The first equation produces a $(n-1)$-dimension plane in $\mathbb{R}^n$
- The second equation produces another $(n-1)$-dimension plane in $\mathbb{R}^n$
- Their intersection in smaller set of $(n-2)$-dimension
- $(n-3)\to(n-4)\to\cdots\cdots\to 3\to 2\to 1\to$ point

Then we can find the final intersection. ⊛

- Column picture

$$u\begin{pmatrix}2\\4\\-2\end{pmatrix}+v\begin{pmatrix}1\\-6\\7\end{pmatrix}+w\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}5\\-2\\9\end{pmatrix} \quad \Longleftarrow \quad \begin{cases} 2u+v+w & =5 \\ 4u-6v & =-2 \\ -2u+7u+2w & =9 \end{cases}$$

RHS is a linear combination of 3 column vectors.

Theorem 1.2.1. Solution to a linear equation:

$$\underbrace{(\textbf{intersection of to points})}_{\text{row pic.}} = \underbrace{(\textbf{coefficient of linear combination})}_{\text{column pic.}}$$

1.2.1 Singular Case

(1) Row Picture: In 3D case, they didn't intersect at a point.

- **Case 1**: two parallel

$$\begin{cases} 2u + v + w &= 5 \\ 4u + 2v + 2w &= 9 \end{cases}$$

- **Case 2**: three plane perpendicular ($\perp$)

$$\begin{cases} u + v + w &= 2 \ \cdots \ (1) \\ 2u + 3w &= 5 \ \cdots \ (2) \\ 3u + v + 4w &= 6 \ \cdots \ (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1) + (2) = (3) \quad ; \quad \text{LHS} \Rightarrow (1) + (2) \neq (3)$$

- **Case 2**: three plane have a whole line in common.

$$\begin{cases} u + v + w &= 2 \ \cdots \ (1) \\ 2u + 3w &= 5 \ \cdots \ (2) \\ 3u + v + 4w &= 7 \ \cdots \ (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1) + (2) = (3) \quad ; \quad \text{LHS} \Rightarrow (1) + (2) = (3)$$

- **Case 4**: three parallel

(2) Column Picture:

$$u \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + v \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + w \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = b$$

In the case above, three vectors are linear combination to each other, i.e. three vectors share the same plane.

Lemma 1.2.3 (Singular case)**.** If the three vectors are linear combination to each other (three vector share a common plane), it must be **singular case**.

- If $b = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$, which is on the plane $\Rightarrow$ too many solution to produce b.
- If $b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$, which is not on the plane $\Rightarrow$ no solution.

1.2.2 Fundamental Linear Algebra Theorem (Geometry form)

Theorem 1.2.2 (Fundamental LA Theorem). Consider a linear system

$$Ax = b, \quad A \in \mathbb{R}^{m\times n},\ x \in \mathbb{R}^n,\ b \in \mathbb{R}^m.$$

If the n hyperplanes have no only one intersection or infinitely many points, then the n columns lie in the same plane. (consistency of *row picture* and *column picture*)

Notation. Logic notation:

- If ..., then : $\Rightarrow$
- If and only if : $\Leftrightarrow$

Lecture 2

1.3 An Example of Gaussian Elimination

9 Sep. 13:20

Example. Here is a linear equation.

$$\begin{cases} 2u + v + w & = 5 \\ 4u - 6v & = -2 \\ -2u + 7v + 2w & = 9 \end{cases}$$

$$\begin{pmatrix} \boxed{2} & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & \boxed{-8} & -2 & -12 \\ 0 & 8 & 3 & 14 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & \boxed{1} & 2 \end{pmatrix} \quad \text{"pivot"}$$

Then we get $w = 2$, we can plug in the equation i.e.

$$\begin{cases} 2u + v + 1w = 5 \\ -8v - 2w = -12 \\ w = 2 \end{cases} \quad \Longrightarrow \quad \text{Forward Elimination}$$

Then we substitute into 2nd, 1st equation to get $v = 1$ and $u = 1 \Longrightarrow$ Backend Elimination

Note. By definition, pivots cannot be zero!

Question. Under what circumstances could the elimination process break down?

Answer. Here are some situations.

- Something **must** go wrong in the singular case.
- Something **might** go wrong in the nonsingular case.

A zero appears in a pivot position!
If in the process, there are nonzero pivots, then there's only one solution. ⊛

Example.

$$\begin{pmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix}$$

(1) If $a_{11} = 0 \implies$ nonsingular

(2) If $a_{22} = 0 \implies$ nonsingular

(3) If $a_{33} = 1 \implies$ singular

Question. How many separate arithmetical operations does elimination require for n equations in n unknowns?

Answer. For a single operation.

$$\text{a single operation} = \text{each division \& each multiplication-subtraction}$$

⊛

- **FE**:

$$\underbrace{\begin{matrix} x & x & \cdots & x & = & x \\ \vdots & \vdots & & & & \vdots \\ x & x & \cdots & x & = & x \end{matrix}}_{n}$$

$$n(n-1) + (n-1)(n-2) + \cdots + (1^2 - 1) = \frac{n^3 - n}{3} \sim \frac{n^3}{3} \text{ steps}$$

- **RHS**:

$$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

- **BF**:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

1.4 Matrix Notation and Matrix Multiplication

$$\begin{cases} 2u + 4v + -2w &= 2 \\ 4u + 9v - 3w &= 8 \\ -2u - 3v + 7w &= 10 \end{cases} \implies u\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v\begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + w\begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

We can rewrite it in the below form.

$$A = \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}_{3\times 3}}_{\text{coefficient matrix}}, \quad x = \underbrace{\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{3\times 1}}_{\text{unknowns}}, \quad b = \underbrace{\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}_{3\times 1}}_{\text{RHS}} \quad \implies x = \underbrace{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}_{3\times 1}}_{\text{solution}}$$

$$\boxed{Ax = b}$$

Definition 1.4.1. An $m \times n$ matrix, $A_{m\times n}$ over $\mathbb{R}$, is an array with m rows and n columns of real numbers, which can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \text{ where } a_{ij} \in \mathbb{R}, \begin{cases} i : \text{index of row} \\ j : \text{index of column} \end{cases}$$

- $\boxed{m \times n}$ is called the **dimensions (size)** of $A \implies$ dimension of a $()_{3\times 5}$ is 3×5
- $\boxed{a_{ij}}$ is called the **elements/entry/coefficient** of A
- **Addition**: $A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{m\times n}$

$$A + B = (a_{ij} + b_{ij})_{m\times n}$$

- **Multiplication**: $A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{n\times r}$

$$AB = (c_{ij})_{m\times r}, \quad \text{where } \ c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

- **Scalar Multiplication**:

$$\alpha A = (\alpha a_{ij})_{m\times n}$$

-

$$A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$$

  In particular, if

$$A_{1\times n} B_{n\times 1} = \mathbf{v} \cdot \mathbf{w} = ()_{1\times 1}.$$

  Then it's the **inner product** of vector $\mathbf{v}$ and vector $\mathbf{w}$

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & 3 & -7 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\cdot(-1) & 4\cdot(2) & -2\cdot(2) \\ 4\cdot(-1) & 9\cdot(2) & -3\cdot(2) \\ -2\cdot(-1) & 3\cdot(2) & -7\cdot(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 22 \end{pmatrix}$$

$$(-1)\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 9 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$$

(1) by row: 3 inner product

(2) by column: a linear combination of 3 columns of A

Example (1A). Ax is a combination of columns of A

$$\begin{aligned} A_{m\times n}x_{n\times 1} &= \begin{pmatrix} A_1 | A_2 | \cdots | A_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1(A_1) + x_2(A_2) + \cdots + x_n(A_n) = \begin{pmatrix} \sum_{j=1}^{n} a_{ij}\ x_j \end{pmatrix}_{m\times 1} \end{aligned}$$

1.4.1 The Matrix Form of One Elimination Step

Definition (1B). Matrix form

Definition 1.4.2. zero matrix:

$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Definition 1.4.3. identity matrix:

$$I = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} = I_n = I_{n\times n}; \quad \begin{cases} A_{m\times n}I_n = A_{m\times n} \\ A_{m\times n} = A_{m\times n}I_n \end{cases}$$

Definition 1.4.4. elementary matrix (elimination matrix):

$$
E_{ij} = \begin{array}{cl}
\begin{pmatrix}
1 & \cdots & 0 & \cdots & 0 \\
0 & \ddots & & & \vdots \\
\vdots & & \ddots & & \vdots \\
\vdots & & -\ell & \ddots & 0 \\
0 & \cdots & 0 & \cdots & 1
\end{pmatrix} & \begin{array}{l} \\ \\ \\ \text{ith row} \\ \\ \end{array} \\
\begin{array}{ccccc} & & \text{jth column} & & \end{array} &
\end{array}
\qquad \ell : \text{multiplier}
$$

$$
E_{ij} \cdot A = \begin{pmatrix} \cdots & -\ell & \cdots & 1 \\ & & & \end{pmatrix} \begin{pmatrix} \quad \\ \quad \end{pmatrix} \begin{array}{l} \longleftarrow \text{ i-th} \implies (\text{i-th row}) + (-\ell)(\text{j-th column}) \\ \longleftarrow \text{ j-th} \implies \text{create zero at } (i,j) \text{ position!} \end{array}
$$

Example.

$$
\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_{21}} \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 0_{21} & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}_{EA}
$$

Note. Here is two properties

1. $Ax = b \implies E_{ij}Ax = E_{ij}b$
2. $E_{ij}A \neq AE_{ij}$

1.4.2 Matrix Multiplication

(1) The (i,j)-th entry of AB is the inner product of the **i-th** of A and the **j-th** of B.

(2) Each column of AB is the product of a matrix A and **a column of B**

$$
\begin{aligned}
\implies \quad \text{column } j \text{ of } AB &= A \text{ times } \textbf{j-th} \text{ of } B \\
&= \text{linear combination of } \textbf{columns of } A \\
&= b_{1j}A \underbrace{\boxed{\bullet}}_{\text{any numbers}}{}_{1} + b_{2j}A_{\bullet 2} + \cdots + b_{nj}A_{\bullet n}
\end{aligned}
$$

Example.

$$
\underbrace{\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}}_{A_{2\times 3}} \underbrace{\begin{pmatrix} 5 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}}_{B_{3\times 3}} = \underbrace{\begin{pmatrix} 16 & 1 & 1 \\ 8 & 0 & -1 \end{pmatrix}}_{C_{2\times 3}}
$$

$$
\text{1st column of } AB = \begin{pmatrix} 16 \\ 8 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}
$$

(3) Each row of AB is a product of a row of A and a matrix B.

$$
\begin{aligned}
\implies \quad \text{i-th row of } AB &= \text{ of } A \text{ times } B. \\
&= \text{linear combination of } \textbf{rows of } B \\
&= a_{i1}B_{1\bullet} + a_{i2}B_{2\bullet} + \cdots + a_{in}B_{n\bullet}
\end{aligned}
$$

Theorem 1.4.1. Let A, B and C be matrices (possibly rectangular). Assume that their dimension permit them to be added and multiplied in the following theorem

(1) The matrix multiplication is associative

$$(AB)C = A(BC)$$

(2) Matrix operations are distributive

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

(3) Matrix multiplication is noncommutative

$$AB \neq BA \quad \text{in general}$$

(4) Identity Matrix

$$A_{n\times n}I_n = I_nA_{n\times n} = A_{n\times n}$$

Example.

$$\underset{21}{E} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \underset{31}{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{1} & 0 & 1 \end{pmatrix}, \quad \underset{32}{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-1} & 1 \end{pmatrix}$$

(1)

$$\underset{21}{E}\,\underset{31}{F} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \boxed{=} \quad \underset{31}{F}\,\underset{21}{E} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(2)

$$\underset{21}{E}\,\underset{32}{G} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \boxed{\neq} \quad \underset{32}{G}\,\underset{21}{E}$$

(3)

$$\underset{32}{G}\,\underset{31}{F}\,\underset{21}{E} = \underset{\text{"right order"}}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}} \quad \boxed{\neq} \quad \underset{21}{E}\,\underset{31}{F}\,\underset{32}{G} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Note. The product of lower triangular matrices is a lower triangular matrix.

Lecture 3

1.5 Triangular Factors and Row Exchanges

16 Sep. 13:20

$$\boxed{Ax = b}$$

$$\implies LUx = b \implies \begin{cases} Lc = b \\ Ux = c \end{cases}$$

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = b$$

Remark. ℓ: multipliers

$$E_{ij}(\ell) : (\text{i-th row}) + (-\ell)(\text{j-th column})$$

$$\begin{pmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{pmatrix} \xrightarrow[R_3+(1)R_1]{R_2+(-2)R_1} \begin{pmatrix} 2 & 4 & -2 & 2 \\ 0_{21} & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{pmatrix} \xrightarrow{R_3+(-1)R_2} \begin{pmatrix} \boxed{2} & 4 & -2 & 2 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0_{32} & \boxed{4} & 8 \end{pmatrix} \text{ pivot}$$

$$E_{21}(2) = E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{31}(-1) = F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad E_{32}(1) = G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

i.e.

$$E_{21}E_{31}E_{32}Ax = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = Ux = c = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} = E_{21}E_{31}E_{32}b$$

Question. How can we undo the steps of Gaussian Elimination?

$$E^{-1}F^{-1}G^{-1}GFEA = A = \underbrace{E^{-1}F^{-1}G^{-1}}\boxed{U} = LU \quad \text{i.e.} \quad A = \underset{\text{factors of } A}{LU}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -(-2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(1) & 0 & 1 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(-1) & 1 \end{pmatrix}$$

$$E^{-1}F^{-1}G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{-1} & \boxed{1} & 1 \end{pmatrix} \implies \text{records everything that has been done so far}$$

1.5.1 Triangular Factorization

Theorem 1.5.1. If no exchanges are required, the original matrix A can be written as

$$A = LU$$

- The matrix L is lower triangular with 1's on the diagonal and the multipliers ℓ_{ij} (taken from elimiation) below the diagonal.
- The matrix U is the upper triangular matrix which appears after forward elimination and before back-substitution; its diagonal entries are the pivots.

Example.

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 6 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \text{提出2}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 6 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Question.

$$A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 2 & 6 & 5 \\ -1 & 4 & -2 \\ 1 & 2 & 3 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

"triangular matrix" 有三條對角線

Answer. ⊛

1.5.2 One Linear System = Two Triangular Systems

$$Ax = b \implies Ux = c \;\&\; Lc = b \implies A = LU$$

Remark. The LU form is unsysttematic in one aspect. U has pivots along its diagonal where L always has 1's.
We can rewrite U as

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12}/u_{11} & \cdots & u_{1n}/u_{11} \\ 0 & 1 & & u_{2n}/u_{22} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example.

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} \boxed{3} & 4 \\ 0 & \boxed{2/3} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2/3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 4/3 \\ 0 & 1 \end{pmatrix}}_{U}$$

Theorem 1.5.2. If

$$A = L_1 D_1 U_1 \text{ and } A = L_2 D_2 U_2$$

then

$$L_1 = L_2, D_1 = D_2, U_1 = U_2$$

i.e. if A has LDU decomposition, then it **is** unique.

1.5.3 Row Excahnge and Permutation Matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \langle \text{Permutation matrix } P_{ij} \rangle$$

Note. Permutation matrix is also an elementary matrix.

Example. Here are some of the example:

1°

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \qquad \boxed{R_2 \leftrightarrow R_3}$$

2°

$$PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{pmatrix} \qquad \boxed{R_2 \leftrightarrow R_3}$$

3°

$$AP = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 5 & 6 \end{pmatrix} \qquad \boxed{C_2 \leftrightarrow C_3}$$

Note. For the permutation matrix:

1° PA: Performing row exchange of A

2° AP: Performing column exchange of A

3° $PAx = Pb$; Should we permute the component of $x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ as well? **NONONONONO!!!**

Example.

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{pmatrix} \qquad Ax = b$$

(1) if $d = 0$, the problem is incurable. The matrix is singular.

(2) if $d \neq 0$, $P_{13}A = \begin{pmatrix} d & e & f \\ 0 & 0 & c \\ 0 & a & b \end{pmatrix}$; if $a \neq 0$, $P_{23}P_{13}A = \begin{pmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{pmatrix}$

$$\left| \begin{array}{c ccccc c ccccc} & & & P_{23}P_{13} & & & \neq & & & P_{13}P_{23} & & \\ & & & & & & & & & & & \\ & 1 & & 3 & & 3 & & 1 & & 1 & & 2 \\ \text{row} & 2 & \to & 2 & \to & 1 & & 2 & \to & 3 & \to & 3 \\ & 3 & & 1 & & 2 & & 3 & & 2 & & 1 \end{array} \right|$$

Theorem 1.5.3. We seperate into two cases:

- In the non singular case, there's a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. In this case,
  (1) $Ax = b$ has a unique solution.
  (2) It is found by elimination with row exchange
  (3) With the rows reorders in advance, PA can be factored into LU $\langle PA = LU \rangle$
- In singular case, no reordering can produce a full set of pivots.

Example.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix} \xrightarrow[\cancel{\ell_{31}=2}\ \ell_{21}=2]{\cancel{\ell_{21}=1}\ \ell_{31}=1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{P_{23}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \langle \text{This is WORNG} \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

To summarize: A good code for Gaussian Elimination keeps a record of L, U and P. They allow the solution ($Ax = b$) from two triangular systems. If the system $Ax = b$ has a unique solution, they we say:

1° The system is nonsingular or

2° The matrix is nonsingular

Otherwise, it is singular.

1.6 Inverse and Transpose

Definition 1.6.1. An $n \times n$ matrix A is invertible if $\exists$ an $n \times n$ matrix $B \ni BA = I = AB$

Theorem 1.6.1. If A is invertible, then the matrix B satisfying $AB = BA = I$ is unique!

Proof. Suppose $\exists c \neq B \ni AC = CA = I$

$$B = BI = B(AC) = (BA)C = IC = C \text{ i.e } B = C$$

we call this matrix B, the $\boxed{\text{inverse of } A}$, and denoted as $\boxed{A^{-1}}$ ∎

Note. Not all $n \times n$ matrices have inverse.
e.g.

1°

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

2° if $Ax = \vec{0}$ has a $\overset{x \neq 0}{\underline{\text{nonzero solution}}}$, then A has no inverse!

$$x = A^{-1}(Ax) = A^{-1}\vec{0} = \vec{0} \quad (\rightarrow\leftarrow)$$

Note. The inverse of A^{-1} is A itself. i.e. $(A^{-1})^{-1} = A$.

Note. If $A = (a)_{1\times 1}$ and $a \neq 0$, then $A^{-1} = (\frac{1}{a})$. The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$ is

$$\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if } \det(A) \neq 0$$

Note.

$$A = \begin{pmatrix} d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & d_n \end{pmatrix} \quad d_i \neq 0,\ \forall i \quad \Longrightarrow A^{-1} = \begin{pmatrix} 1/d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1/d_n \end{pmatrix}$$

Proposition 1.6.1. If A and B are invertible, then

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}\cdots A_2^{-1}A_1^{-1}$

1.6.1 The Calculation of A^{-1}: Gaussian-Jordan Method

$$A \cdot A^{-1} = I$$

$$A_{n\times n}B_{n\times n} = I_n$$

$$\begin{aligned}
&\Longrightarrow A_{n\times n}(B_1|B_2|\cdots|B_n)_{n\times n} = (e_1|e_2|\cdots|e_n)_{n\times n} \\
&\Longrightarrow (AB_1|AB_2|\cdots|AB_n)_{n\times n} = (e_1|e_2|\cdots|e_n)_{n\times n} \\
&\Longrightarrow AB_1 = e_1;\ AB_2 = e_2;\ \cdots;\ AB_n = e_n \quad \longrightarrow \quad n \text{ linear systems:} Ax = b
\end{aligned}$$

Definition 1.6.2 (Gaussian-Jordan Method)**.** Instead of stopping at U and switching to back substitution, it continues by subtracting multipliers of a row from the rows above till it reaches a diagonal matrix. Then we divide each row by corresponding pivot.

$$(\underset{LU}{A}|I) \xrightarrow{\times L^{-1}} (U|L^{-1}) \xrightarrow{\times U^{-1}} (I|A^{-1})$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$$

1.6.2 Invertible = Nonsingular

Question. What kind of matrices are invertible?

Answer. Here are the example:

1° nonzero pivot Ch1 Ch4

2° nonzero determinants Ch4

3° independent columns (rows) Ch2

4° nonzero eigenvalues Ch5

which will in the whole course ⊛

Suppose a matrix A has full set of nonzero pivots. By definition, A is nonsingular and the n systems

$$Ax_1 = e_1, \ Ax_2 = e_2, \ \cdots \ , Ax_n = e_n$$

can be solved by elimination or Gaussian-Jordan Method.

Row exchanges maybe necessary, but the columns of A^{-1} are uniquely determined.

$$\begin{aligned} Ax = b \quad & PAx = Pb \\ & PAx_i = Pe_i \end{aligned}$$

$$\{Pe_1, Pe_2, \cdots, Pe_n\} = \{e_1, e_2, \cdots, e_n\}$$

Note. Compute A^{-1}:

1° $A(x_1|\cdots|x_n) = I = (e_1|\cdots|e_n) \iff Ax_i = e_i, \ i = i\cdots n$

2° Gauss-Jordan Method: $(\ A \mid I\ ) \longrightarrow (\ I \mid A^{-1}\ )$

Question. We have found a matrix $A^{-1} \ni AA^{-1} = I$. But is $A^{-1}A = I$

Answer. We can do this by recall.

> **As previously seen.** Recall that every Gauss-Jordan step is a multiplication of matrices on the left. There are three types of elementary matrices:

1° $\boxed{E_{ij}(\ell)}$: to subtract a multiple ℓ of j row from i row.

2° $\boxed{P_{ij}}$: to exchange row i and j

3° $\boxed{D_i(d)}$: to multiply row i by d i.e. $D_i(d) = \begin{pmatrix} 1 & & & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & d & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \end{pmatrix} \rightarrow$ ith row

$$\begin{pmatrix} d_1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$$

$$\implies \quad DEEPEEA = I \quad \implies \quad A^{-1}A = I \therefore \text{ we have a left inverse!}$$

These are the operation of A^{-1} ⊛

Theorem 1.6.2. For nonsingular and invertible:

- Every nonsingular matrix is invertible.
- Every invertible matrix is nonsingular.

Theorem 1.6.3. A square matrix is invertible $\iff$ it is nonsingular

Lecture 4

23 Sep. 13:20

1.7 Transpose A^T

Proposition 1.7.1. Here are the proposition of transpose

- $(A+B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$

Proof. Here is the proof

1° $((A+B)^T)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A^T + B^T)_{ij}$

2° $((AB)^T)_{ij} = (AB)_{ji} = \boxed{\sum_{k=1}^{n} a_{jk} b_{ki}}$ $\quad (B^T A^T)_{ij} = \sum_{\ell=1}^{n} b^T_{i\ell} a^T_{\ell j} = \sum_{\ell=1}^{n} b_{\ell i} a_{j\ell} = \boxed{\sum_{\ell=1}^{n} a_{j\ell} b_{\ell i}}$

3°

■

Definition 1.7.1. A symmetric matrix is a matrix which equals its own transpose. i.e. $A = A^T$

Example.

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \text{ YES } \begin{pmatrix} 5 & 4 \\ 1 & 5 \end{pmatrix} \text{ NO } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ YES}$$

Note. A symmetric matrix is not necessarily invertible. If it is invertible, then its inverse is symmetric.

Theorem 1.7.1. If A is symmetric and if A can be factored as LDU, then $A = LDU^T$

Proof. Here is the proof.

1° $A = A^T$, $A = LDU \Rightarrow A^T = (LDU)^T = U^T D^T L^T = A = LDU$

2° By theorem 1.5.2, the theorem is correct.

LDU is unique if they exist. ∎

Chapter 2

Vector Spaces and Linear Equation

2.1 Vector Spaces and Subspace

To answer the basic questions about the $\underset{1^\circ}{\text{existence}}$ and $\underset{2^\circ}{\text{uniquness}}$ of the solution of $Ax = b$, we need the concept of vector space.

$$\text{Field} \implies \text{Vector Space} \implies \text{Solution of } Ax = b$$

Definition 2.1.1 (Field). Let F be a set with two operations "$+$" and "$\bullet$" i.e.

$$+ : F \times F \longrightarrow F$$
$$\bullet : F \times F \longrightarrow F$$

and $+, \bullet$ are well-defined functions. If the system $\boxed{(F, +, \bullet)}$ satisfies the following conditions, the F is called a $\boxed{\text{Field}}$.
For $a, b, c \in F$

(1) $(a + b) + c = a + (b + c)$

(2) $a + b = b + a$

(3) $\exists\, 0 \in F \ni a + 0 = 0 + a = a$ 單位元素 (1st operation)

(4) $\forall a \in F,\ \exists(-a) \in F \ni a + (-a) = 0$ 反元素 (1st operation)

(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(6) $a \cdot b = b \cdot a$

(7) $\exists\, 1 \in F \ni a \cdot 1 = 1 \cdot a = a$ 單位元素 (2nd operation)

(8) $\forall a \neq 0 \in F,\ \exists\, a^{-1} \in F \ni a \cdot a^{-1} = a^{-1} \cdot a = 1$ 反元素 (2nd operation)

(9) $a \cdot (b + c) = ab + ac$ Distribution Law

Example.

$$\underset{\text{(real)}}{\mathbb{R}}\ \text{(YES)} \quad \underset{\text{(rational)}}{\mathbb{Q}}\ \text{(YES)} \quad \underset{\text{(integer)}}{\mathbb{Z}}\ \text{(NO)} \quad \underset{\text{(complex)}}{\mathbb{C}}\ \text{(YES)} \quad \mathbb{N}\ \text{(NO)}$$

Definition 2.1.2 (vector space)**.** Let V be a set and F be a field. $\boxed{V \text{ is a vector space over } F}$ if $\underset{1^\circ}{\text{addition}}$ and $\underset{2^\circ}{\text{multiplication}}$ by scalar are defined on V and they satisfy.

$$+ : V \times V \longrightarrow V$$
$$\bullet : F \times V \longrightarrow V$$

(A1) addition is associated

(A2) addition is commutative

(A3) $\exists$ zero vector $\in V \ni 0 + v = v + 0, \ \forall v \in V$

(A4) $\forall v \in V, \ \exists (-v) \in V \ni (-v) + v = 0$

(M1) $1 \cdot v = v, \ v \in V, \ 1 \in F$

(M2) $(\lambda\mu) \cdot v = \lambda(\mu v) \ v \in V, \ \lambda, \mu \in F$

(M3) $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2 \ v_1, v_2 \in V, \ \lambda \in F$

(M4) $(\lambda + \mu)v = \lambda v + \mu v \ v \in V, \ \lambda, \mu \in F$

2.1.1 Algebraic Rules of Vector Algebra

Question. $n \in \mathbb{N}$, $\mathbb{R}^n/\mathbb{R}$ ($\mathbb{R}^n$ over $\mathbb{R}$) is a vector space?

Answer. **YES** ⊛

Example.

$$\mathbb{C}^n/\mathbb{C}, \ \mathbb{C}^n/\mathbb{R}, \ \mathbb{R}/\mathbb{R}$$

Question. $M_{2\times 2}(\mathbb{R})/\mathbb{R}$ is a vector space?

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}$$

Answer. **YES** ⊛

Question. V is a vector space?

$$V = \{\text{all } 3 \times 3 \text{ symmetric matrices over } \mathbb{R}\}$$

Answer. **YES** ⊛

Question. $\mathbb{R}^\infty/\mathbb{R}, \mathbb{R}^\infty = \{(a_1, a_2, \cdots\cdots) \mid a_i \in \mathbb{R}\}$

Answer. **YES** ⊛

Question. Let $V = \{f \mid f$ is a real-valued function defined on $[0,1]\}$ define $(rf)(x) = r \cdot f(x),\ r \in \mathbb{R}$

Answer. **YES**

$$(\text{zero vector}) = (\text{zero function})$$

i.e. $f(x) = 0, \forall x \in [0,1]$ ⊛

Question. $V = \{\text{all positive } \mathbb{R}\}$

$$\begin{cases} x+y & = xy \\ c \bullet x & = x^c \end{cases}, \quad \text{is } V \text{ a v.s. over } \mathbb{R}$$

Answer. **YES**

1° (A1) $(x+y)+z = x+(y+z)$

2° (A2) $(x+y) = xy = yx = (y+x)$

3° (A3) zero vector: $x + 1 = x$

4° (A4) x + $\frac{1}{x}$ = zero vector = 1

5° (M3) $\lambda(x+y) = (x+y)^\lambda = (xy)^\lambda = x^\lambda y^\lambda = (\lambda x)(\lambda y) = \lambda x + \lambda y$

6° (M4) $(\lambda+\mu)\cdot x = x^{(\lambda+\mu)} = x^\lambda \cdot x^\mu = \lambda x \cdot \mu x = \lambda x + \mu x$

All conditions apply. ⊛

2.1.2 subspace

Definition 2.1.3 (subspace). A $\boxed{\text{subspace } W}$ of a vector space $(V, +, \bullet)$ over F is a nonempty subset of $V \ni (W, +, \bullet)$ itself is a vector space over F. W is a subspace of V over F if and only if W is closed under addition and scalar multiplication.

Question. Does the zero vector belong to subspace?

Answer. **YES**
$W = \{\text{zero vector}\}$ is the smallest possible vector space. ⊛

Remark. If W_1 and W_2 are subspaces of V over F. Then $W_1 \cap W_2 \neq \emptyset$

Note. If W is a subspace of V/F, then we use notation $\boxed{W \leq V}$.

Question. $V = \mathbb{R}^2/\mathbb{R}$ (xy-plane), What are the subspace of V?

Answer. Here are all subspace of V

(i) origin (one point)

(ii) $\mathbb{R}^2/\mathbb{R} \leq V$

(iii) all lines through origin

(iv) ~~2nd quadant~~ (no zero)

There are much more example. ⊛

Question. $V = M_{n\times n}(\mathbb{R})/\mathbb{R}$

$$S = \{n \times n\text{symmetric matrix}\}$$
$$U = \{n \times n\text{upper triangular matrix}\}$$
$$L = \{n \times n\text{lower triangular matrix}\}$$

Answer. **YES, YES, YES** ⊛

Theorem 2.1.1 (). Let V be a vector space over F. A nonempty subset W of V is a subspace of V, if and only if for each pair $x, y \in W$ and $\alpha \in F$:

1° The zero vector $\in W$.

2° $\alpha x + y \in W$

2.1.3 Column Space of A

Example.

$$A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$$

The first concern is to find all attainable r.h.s. vector b. For example:

$$\begin{pmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix} = u\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} + v\begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$$

Theorem 2.1.2. The system is solvable if and only if the vector b can be expressed as a combination of columns of A

Note. The columns of $A_{m\times n}$ are vectors in $\mathbb{R}^m$, the rows of $A_{m\times n}$ are vectors in $\mathbb{R}^n$.

Example. Let $\mathcal{C}(A) = \{\text{all combinations of columns of } A\}$. Then, $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^m/\mathbb{R}$.

Proof. If b and $b' \in \mathcal{C}(A)$, $\exists\, x, x' \ni Ax = b$ & $Ax' = b'$

$$\forall \alpha \in \mathbb{R}, \quad A(\alpha x + x') = A(\alpha x) + A(x') = A\alpha x + Ax' = \alpha b + b' \in \mathcal{C}(A)$$

$\Longrightarrow \mathcal{C}(A) \leq \mathbb{R}^m/\mathbb{R}$ ∎

Definition 2.1.4. $\mathcal{C}(A)$ is called the $\boxed{\text{column space}}$ of A. Thus if $b \in \mathcal{C}(A)$, then $Ax = b$ is solvable.

- $A_{m\times n} = 0 \longrightarrow \mathcal{C}(A) = 0_{m\times 1}$
- $A_{m\times n} = I_m \longrightarrow \mathcal{C}(A) = \mathbb{R}^m$

Lecture 5

2.1.4 Nullspace of A

30 Sep. 13:20

Definition 2.1.5. Let $\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$, then $\mathcal{N}(A) \leq \mathbb{R}^n/\mathbb{R}$. Then $\mathcal{N}(A)$ is called the $\boxed{\text{null space}}$ of A.

Proof. We proof it with the Theorem 2.1.1

- zero vector is in the $\mathcal{N}(A)$
- $x,\ x' \in \mathcal{N}(A) \ \Rightarrow\ Ax = 0,\ Ax' = 0$

$$A(x + x') = Ax + Ax' = 0 + 0 = 0 \ \Rightarrow\ x + x' \in \mathcal{N}(A)$$

$$A(\alpha x) = \alpha Ax = \alpha \cdot 0 = 0 \ \Rightarrow\ \alpha x \in \mathcal{N}(A), \forall \alpha \in \mathbb{R} \quad \therefore \mathcal{N}(A) \leq \mathbb{R}^n/\mathbb{R}$$

∎

Note. The system $Ax = 0$ is called a homogeneous equation. (齊次)

Remark. The solution set of $Ax = b$ is NOT a subspace of $\mathbb{R}^n/\mathbb{R}$

$$x, x' \longrightarrow Ax = b,\ Ax' = b$$

$$A(x + x') = Ax + Ax' = 2b \neq b$$

Example.

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{N}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

Example.

$$\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{N}(A) = \left\{ \begin{pmatrix} t \\ t \\ -t \end{pmatrix}, t \in (-\infty, \infty) \right\}$$

$$\begin{aligned} \mathcal{C}(A) &= \{\text{all combinations of columns of } A\} \\ &= \text{ column space of } A \leq \mathbb{R}^m/\mathbb{R} \end{aligned}$$

$$\begin{aligned} \mathcal{N}(A) &= \{x \in \mathbb{R}^n \mid Ax = 0\} \\ &= \text{ null space of } A \leq \mathbb{R}^n/\mathbb{R} \end{aligned}$$

2.2 The Solution of m Equations in n Unknows

For $ax = b, \quad a, b, x \in \mathbb{R}$

(i) if $a \neq 0 \ \Rightarrow \ x = \dfrac{b}{a}$, unique

(ii) if $a = 0,\ b = 0 \ \Rightarrow$ infinitely many solutions.

(iii) if $a = 0,\ b = 0 \ \Rightarrow$ no x exists.

Now, consider $Ax = b$, if A is a square, then (i), (ii), (iii) may occur.

(i) A^{-1} exists $\longrightarrow \ x = A^{-1}b$, unique

(ii) A is singular (undetermined case)

(iii) inconsistent case.

With a rectangular matrix A, $x = A^{-1}b$ will never happen!

Definition. Here is the definition of two similar jargon.

Definition 2.2.1 (row echelon matrix)**.** An $m \times n$ matrix R is called a $\boxed{\text{row echelon matrix}}$ if

(i) the nonzero rows come first and the pivots are the first nonzero etries in those rows.

(ii) below each pivot is a column of zeros

(iii) each pivot lies to the right of the pivot in the row above.

e.g.

$$\begin{pmatrix} \circledast & \circledast & \circledast & \circledast & \circledast \\ 0 & \circledast & \circledast & \circledast & 0 \\ 0 & 0 & 0 & \circledast & \circledast \end{pmatrix}$$

Definition 2.2.2 (row-reduced echelon matrix)**.** An $m \times n$ matrix R is called a $\boxed{\text{row-reduced}}$ $\boxed{\text{echelon matrix}}$ if

(i) the nonzero rows come first and the pivots are the first nonzero etries in those rows; pivots are normalized to be 1.

(ii) Above & Below each pivot is a column of zeros

(iii) each pivot lies to the right of the pivot in the row above.

e.g.

$$\begin{pmatrix} \boxed{1} & 0 & \circledast & 0 & \circledast \\ 0 & \boxed{1} & \circledast & 0 & \circledast \\ 0 & 0 & 0 & \boxed{1} & \circledast \end{pmatrix}$$

Theorem 2.2.1. To any $m \times n$ matrix A, there exists a permutation matrix P, a lower triangular matrix L with unit diagnal and an $m \times n$ echelon matrix $U \ni PA = LU$

OR

Every $m \times n$ matrix A is row equivalent to a row echelon matrix.

- Case 1. Homogeneous Case. $b_{m\times 1} = 0$

$$Ax = 0$$

We call the component of x , which correspond to columns with pivots the basic variables; and these correspond to columns with pivots the free variables.

$$\begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{cases} \text{basic variables: } u, w \\ \text{free variables: } v, y \end{cases}$$

The basic variables are then expressed in terms of free variables.

$$\begin{cases} 3w + y = 0 \\ u + 3v + 3w + 2y = 0 \end{cases} \implies \begin{cases} w = -\frac{1}{3}y \\ u = -3v - y \end{cases}$$

$$x = \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} -3v - y \\ v \\ -\frac{1}{3}y \\ y \end{pmatrix} = v\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

– $\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is obtain from x by setting $\begin{cases} v = 1 \\ y = 0 \end{cases}$

– $\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix}$ is obtain from x by setting $\begin{cases} v = 0 \\ y = 1 \end{cases}$

Theorem 2.2.2. If a homogeneous system $A_{m\times n}x = 0$ has more $\overset{n}{\text{unknowns}}$ than $\overset{m}{\text{equations}}$ (m $<$ n), it has a nontrivial solution.

$$(A_{m\times n}) \longrightarrow (A_{m\times n})$$

at most m pivot, at most m basic variables, at least $(n - m)$ free variables.

Note. The nullspace is a subspace of the same $\boxed{\text{dimension}}$ as the number of $\boxed{\text{free}}$ variables.

- Case 2. Inhomogeneous Case: $b \neq 0$

$$Ax = b \ \rightarrow \ Ux = c \text{ where } c = L^{-1}b$$

$$\begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\implies \begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{pmatrix} \quad \longrightarrow \quad b_3 - 2b_2 + 5b_1 = 0$$

We know that $Ax = b$ is solvable $\Rightarrow$ $b \in \mathcal{C}(A)$

- 1 & 3: basic variables
- $\mathcal{C}(A) =$ the set of combinations of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ & $\begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix}$

  , which is also $\left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \middle| \ b_3 - 2b_2 + 5b_1 = 0 \right\} \perp \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

Example.

$$b = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad \implies \quad \begin{cases} w = 1 - \frac{1}{3}y \\ u = -2 - 3v - y \end{cases}$$

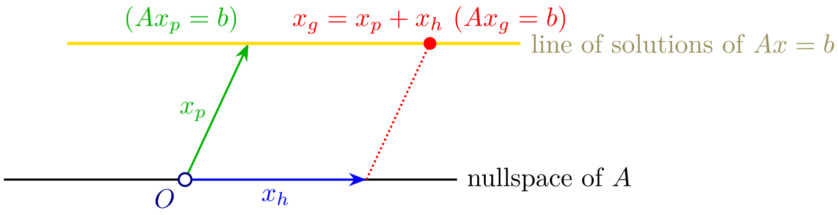
$$x = \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} -2 - 3v - y \\ v \\ 1 - \frac{1}{3}y \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{shift}} + \underbrace{v \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}}_{\text{solution to } Ax=0 \text{ (nullspace)}}$$

Shift: particular solution to $Ax = b$ (set all free variables to be zero)

$$\boxed{x_{\text{general}} = x_{\text{particular}} + x_{\text{homogeneous}}; \ \ x_g = x_p = x_h}$$

Generally, the general solution is fills a two-dimensiona; surface (but NOT a subspace since it doesn't contain the zero vector (origin))

It is paralled to the Nullspace of A



2.2.1 Steps to oobtain the solution to $Ax = b$

(i) Reduce $Ax = b$ to $Ux = c$ to determine basic/free variables.

(ii) Set all free variables to zero to find particular solution, x_p

(iii) set RHS $= 0$. Give each free variables 1 others 0, in terms, find the hoomogeneous sloution, x_h

$$\Longrightarrow \quad x_g = x_p + x_h$$

Definition 2.2.3 (rank). $\boxed{A_{m\times n}}$ if there are r pivots, there are r basic variables and $n-r$ free variables. The number of pivots, r, is called the $\boxed{\text{rank}}$ of the matrix.

Theorem 2.2.3. Suppose elimination reduce $A_{m\times}x = b$ to $Ux = c$ and there are r pivots and the last $(m-r)$ rows of U are zero. Then there is a solution only if last $(m-r)$ elements of c are zeros.

- If $r = m$, there's always a solution. The general solution is the sum of particular solution and a homogeneous solution.
- If $r = n$, there are No free variables and the null space contains $x = 0$ only. The number r is called the rank of A.

Two extreme case: $A_{m\times n}x = b$

(1) If $r = n$ $\rightarrow$ No free variables $\rightarrow$ $\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} = \{0\}$

(2) If $r = m$ $\rightarrow$ No zero rows in U $\rightarrow$ $\mathcal{C}(A) = \mathbb{R}^m$ $\Rightarrow$ $\exists$ solution for all b

2.3 Linear Independence, Basis and Dimension

In the elimination process, we refer to the number, r, of pivots as the rank of A. This definition is purely computational rather than mathematical. We shall give a formal definition later.
Now we shall disscuss the following four ideas:

(i) linear independence or dependence

(ii) $\boxed{\textbf{spanning}}$ a subspace

(iii) $\boxed{\textbf{basis}}$ for a subspace

(iv) $\boxed{\textbf{dimension}}$ of a subspace

Definition 2.3.1. Let V be a vector space over F. A nonempty subset S of V is said to ve $\boxed{\text{linearly dependent}}$ if there exist distinct vectors $v_1, v_2, \cdots, v_n$ in S and scalar $\alpha_1, \alpha_2, \cdots, \alpha_n$ in F, not all of which are zero $\ni$

$$\alpha_1 v_1 + \alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

A set which is not linearly dependent is called linearly independent. If $S = \{v_1, v_2, \cdots, v_n\}$ then we say that $v_1, v_2, \cdots, v_n$ are linearly dependent/independent.

Lecture 6

Remark (1). To show that $v_1, \cdots, v_n$ are linearly independent. We vaerify if 14 Oct. 13:20

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \text{ for some } c_i \in F$$

then c_i must be zero for all i.

Example. In $\mathbb{R}^2$, if v_1, v_2 are not colinear(共線) then they are linearly independent.

$$v_1(\neq 0) \text{ and } v_2(\neq 0) \text{ are linearly dependent } \iff v_1, v_2 \text{ are on the same } \textit{line}$$

Any 3 vectors in $\mathbb{R}^2$ are linearly dependent.

Remark (2). If $v_1 = v_2$, then the set $\{v_1, \cdots v_n\}$ is linearly dependent.

$$\alpha v_1 + (-\alpha) v_2 = 0$$

Remark (3). Any set which contain a linear dependent subset is linearly dependent.

Remark (4). Any subset of a linearly independent set is linearly independent.

Remark (5). Any set which contain 0 vector is linearly dependent.

Example.

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ \underset{v_1}{-2} & \underset{v_2}{0} & \underset{v_3}{-4} & \underset{v_4}{0} \end{pmatrix}$$

The columns of A are linearly dependent.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

$$(v_1 \ v_2 \ v_3 \ v_4) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \quad \Longrightarrow 4v_1 + (-3)v_2 + 2v_3 + 0v_4 = 0$$

Example.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

The columns of A are linearly independent

Note. We showed that the nullspace of A is $\{0\}$ only. That is exactly the same as saying the columns of A are linearly independent.

Example.

$$U = \begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition 2.3.1 (2F). The r nonzero rows of echelon matrix U are linearly independent, and so are r columns that contain pivots.

Example. In $\mathbb{R}^n$, $e_1, e_2, \cdots, e_n$ are linearly independent.

To summarize: To check any set of vectors $v_1, v_2, \cdots, v_n (\in \mathbb{R}^n)$ are linearly independent.
Let $A = (v_1|v_2|\cdots|v_n)_{m\times n}$, then solve $Ax_{n\times 1} = 0$.

1° if $\exists$ solution $x \neq 0$, then v_i's are linearly dependent.

2° if there are no free variables (i.e. $\text{rank}(A) = n$), nullspace $= \{0\}$ then v_i's are linearly independent.

3° if $\text{rank}(A) < n$, then v_i's are linearly dependent.

4° special case: if $v_i \in \mathbb{R}^{>}$ and $n > m$, then v_i's are linearly dependent.

Proposition 2.3.2. A set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $\boxed{n > m}$.

2.3.1 Spanning a Subspace

Definition 2.3.2 (2H). Let S be a subset vectors in V/F.
$\boxed{\text{The subspace spanned by } S}$ is defined to be the intersection W of all subspaces of V which contain S.
When S is finite, $S = \{v_1, \cdots, v_n\}$, we call W the subspace spanned by $v_1, \cdots, v_n$ and denoted as $W = \langle v_1, \cdots, v_n \rangle$ or $W = \text{span}(S) = \langle S \rangle$.

Theorem 2.3.1. [The subspace spanned by a nonempty subset S] of a vector space V is [the set T of all linear combinations of vectors in S].

Proof. We need to show $W = T$.

Claim. $W = T$ if and only if $W \subseteq T$ and $T \subseteq W$.

- Let W be the subspace spanned by S, $S \subseteq W$ (S 不一定有包含 0 vector 所以不能用 $\leq$).
  So $\overset{T}{\text{every linear combination of vectors in } S}$ is in W. $\implies T \subseteq W$.
  ($\because W$ is a subspace which is a vector space)

- on the other hand, T is a subspace containing S.
  ($\because x, y \in T,\ \alpha \in F \ \Rightarrow\ \alpha x + y \in T$)

So, $W \subseteq T$ by definition $\Rightarrow W = T$.
(Intersection of all subspaces containin S) ∎

Example. $\mathcal{C}(A)$ = space spanned by columns of A.

Example. $w_1 = (1,0,0),\ w_2 = (0,1,0),\ w_3 = (0,0,1)$, span a space $\mathbb{R}^3$.
$w_1 = (1,0,0),\ w_2 = (0,1,0),\ w_3 = (-3,0,0)$, span a plane $\mathbb{R}^2$.

Note. Spanning involves the columns space, independence involves the null space.

2.3.2 Basis

Definition 2.3.3 (2I). A $\boxed{\text{basis}}$ for a vector space is a set of vectors that satisfies

(i) it is linearly independent AND

(ii) it span the vector space

If the basis of V is finite, then V is finite-dimensional (f-dim).

Remark (1). There's one and only one way to write every $v \in V$ as a linear combination of the basis elements.

Remark (2). In $\mathbb{R}^n$,

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1} \quad \begin{matrix} \uparrow \\ i^{th} \\ \downarrow \end{matrix}$$

then $\{e_1, \vdots, e_n\}$ is a basis for $\mathbb{R}^n$. The basis is called the standard basis.

$$\forall x = (x_1, \cdots, x_n) \in \mathbb{R}^n,\ x = \sum_{i=1}^{n} x_i e_i$$

The standard basis is not the only basis for $\mathbb{R}^n$. In fact, there are infinitely many bases for $\mathbb{R}^n$. For any nonsingular matrix $A_{n\times n}$, the columns of A are the basis for $\mathbb{R}^n$.

Example.

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix}_{3\times 4} \longrightarrow U = \begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{3\times 4}$$

The columns of U that contain pivots are a basis for $\mathcal{C}(U)$.

Note that $\mathcal{C}(U)$ is generate by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, which is a xy-plane within $\mathbb{R}^3$.

Remark. $\mathcal{C}(U)$ is NOT same as $\mathcal{C}(A)$.

Theorem 2.3.2 (2J). Any two bases for V contain the same number of vectors. This number is called the *dimension* of V.

Proof. Suppose $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are bases for V, and suppose $m < n$.
For $j = 1, \ldots, n$,

$$w_j = a_{1j}v_1 + \cdots + a_{mj}v_m \quad \text{for some } a_{ij} \in F.$$

Let

$$w = [w_1, \ldots, w_n] = VA = [v_1, \ldots, v_m] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

The matrix A is $m \times n$ with $m < n$. By Theorem 2C, $\exists$ nontrivial C such that $AC = 0$.

$$VAC = WC = 0.$$

Hence the columns of W are linearly dependent. But the columns of W are basis elements, contradiction $\Rightarrow$ $m \not< n$.
Similarly, we can show that $n \not< m$, so we conclude $m = n$. ■

Theorem 2.3.3 (2L). Any linearly independent set in a finite-dimensional vector space V can be extended to a basis. Any spanning set of V can be reduced to a basis.

Proof. Let $v_1, \dots, v_k$ be linearly independent over F. Then $\langle v_1, \dots, v_k \rangle \leq V$.
If $\langle v_1, \dots, v_k \rangle = V$, then $\langle v_1, \dots, v_k \rangle$ is a basis of V. Otherwise, $\exists\, x \in V$ such that $x \notin \langle v_1, \dots, v_k \rangle$.
Then $x, v_1, \dots, v_k$ are linearly independent. If not, $\exists\, c \neq 0$, and $\exists\, \alpha_1, \dots, \alpha_k$, not all zero, such that

$$cx + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0.$$

$$\Rightarrow\ x = c^{-1}\alpha_1 v_1 + c^{-1}\alpha_2 v_2 + \cdots + c^{-1}\alpha_k v_k.$$

$$\Rightarrow\ x \in \langle v_1, \dots, v_k \rangle, \text{ contradiction.}$$

Then repeat the process, i.e. is $\langle x, v_1, \dots, v_k \rangle = V$? Since V is finite-dimensional, the process will terminate after finite steps.
The 2nd half of the theorem can be proved similarly (exercise). ■

2.4 The Four Fundamental Subspaces

Usually there are two ways to describe a subspace

(i) a set of vectors that span the space.
(e.g. the column space of $A_{m\times n}$, $\mathcal{C}(A)$)

(ii) a list of constraints that imposed on a subspace.
(e.g. the null space of $A_{m\times n}$, $\mathcal{N}(A) = \{x \mid Ax = 0\}$)

Here we will discuss four fundamental subspaces associated to $A_{m\times n}$

(1) the column space of A denoted by $\mathcal{C}(A)$

(2) the null space of A denoted by $\mathcal{N}(A)$

(3) the row space of A the columns spaces of A^T, denoted by $\mathcal{C}(A^T)$

(4) the left null space of A denoted by $\mathcal{N}(A^T)$, i.e. $\{y \mid A^T y = 0\}$

- If $A_{m\times n}$, then $\mathcal{C}(A), \mathcal{N}(A^T) \leq \mathbb{R}^m$ and $\mathcal{N}(A), \mathcal{C}(A^T) \leq \mathbb{R}^n$.

2.4.1 Row space $\mathcal{C}(A^T)$

The row space of A (the subspace spanned by the rows of A), $\mathcal{C}(A^T)$. For an echelon matrix, its r nonzero rows are independent and its row space is r-dimensional.

Proposition 2.4.1 (2M). The row space of A has the same dimension r as the row space of echelon form U of A, and they have the same basis.

$$\mathcal{C}(A^T) = \mathcal{C}(U^T)$$

But in general, $\mathcal{C}(A) \neq \mathcal{C}(U)$.

Lecture 7

2.4.2 Nullspace $\mathcal{N}(A)$

21 Oct. 13:20

The nullspace of $A_{m\times n}$, $\{x \mid Ax = 0\} = \{x \mid Ux = 0\}$

$$\therefore \text{The nullspace of } A \text{ is the same as the nullspace of } U$$

Proposition 2.4.2 (2N)**.** The nullspace $\mathcal{N}(A)$ is of dimension $n - r$
A basis of $\mathcal{N}(A)$ can be constructed by reducing to $Ux = 0$ which has $n - r$ free variables corresponding to the columns if U that do not contain pivots. Let each free variable 1, in turn, and others 0, and solve $Ux = 0$. The $n - r$ vectors produced in this manner will be a basis of $\mathcal{N}(A)$.

$$\dim(\mathcal{N}(A)) = n - r$$

The $\mathcal{N}(A)$ is also called the $\boxed{\text{kernel of } A}$, $\ker(A)$, and its dimension is called the $\boxed{\text{nullity of } A}$.

$$\ker(A) = \mathcal{N}(A)$$

2.4.3 Column space $\mathcal{C}(A)$

The $\mathcal{R}$ in $\mathcal{R}(A)$ stands for "**range**" which is consistent with the usual idea if range of f

Let $f(x) = A_{m\times n}x_{n\times 1}$, the

- the domain of f is $\mathbb{R}^n$
- the range of f is $\{b \in \mathbb{R}^m \mid Ax = b\} = \mathcal{C}(A) = \mathcal{R}(A)$
- the kernel of f is $\{x \in \mathbb{R}^n \mid Ax = f(x) = 0\} = \mathcal{N}(A) = \ker(A)$

If U is the echelon form of A, $\mathcal{C}(A) \neq \mathcal{C}(U)$, but they have the same dimension. For U, the columns with pivots form a basis of $\mathcal{C}(U)$. Then, the corresponding columns in A form a basis of $\mathcal{C}(A)$. Since the two systems $Ax = 0$, $Ux = 0$ are equivalent and have the same solutions. A nontrivial solution x means a linear combination of columns of U, hence the same linear combination of columns of A.

So, if the set of columns of U is independent, then so are the corresponding columns of A and vice versa.

To find a basis of $\mathcal{C}(A)$, we pick those columns of A, which correspond to the columns of U with pivots.

Proposition 2.4.3 (2O)**.** The dimension of the column space = rank r, which also equals the dimension of the row space.

$$\therefore \# \text{ of independent columns} = \# \text{ of independent rows} = r$$

or more formally,

$$\text{rank}(A) = r = \text{row rank} = \text{column rank}$$

2.4.4 Left nullspace $\mathcal{N}(A^T)$

$$\underset{n\times m}{A^T}\underset{m\times 1}{y} = \underset{n\times 1}{0} = (\underset{1\times m}{y^T}\underset{m\times n}{A})^T$$

$$(\#\text{ of basic variables}) + (\#\text{ of free variables}) = (\#\text{ of variables}) = n$$

$$\boxed{\dim(\mathcal{C}(A)) + \dim(\mathcal{N}(A)) = \#\text{ of columns of } A}$$

For A^T, which has m columns, the column space of A^T is the row space of A which has dimension $\text{rank}(A)$. So,

$$\dim(\mathcal{N}(A^T)) = m - \text{rank}(A)$$

i.e.

$$\boxed{\dim(\mathcal{C}(A^T)) + \dim(\mathcal{N}(A^T)) = \#\text{ of columns of } A^T}$$

Proposition 2.4.4 (2P). The left nullspace $\mathcal{N}(A^T)$ is of dimension $m - r$
The left nullspace contain the coefficients that make the rows of A combined to a zero vector (linear dependent).

$$\text{To find } y \ni y^T A = 0$$

$$\text{Suppose that } PA = LU \longrightarrow \underset{m\times m}{\boxed{L^{-1}P}}\underset{m\times n}{A} = \underset{m\times n}{U}$$

The last $m-r$ rows of $L^{-1}P$ must be a basis for the left nullspace. ($\therefore$ the last $m-r$ rows of $L^{-1}P$ are independent and $\dim(\mathcal{N}(A^T))$ is $m-r$ $\rightarrow$ it is a basis of $\mathcal{N}(A^T)$)

Theorem 2.4.1 (Fundamental Theorem of Linear Algebra). Let A be n arbitrary $m \times n$ matrix, then

$$\dim(\mathcal{C}(A)) = \dim(\mathcal{C}(A^T)) = \text{rank}(A)$$

$$\dim(\mathcal{N}(A)) = n - \text{rank}(A); \quad \dim(\mathcal{N}(A^T)) = m - \text{rank}(A)$$

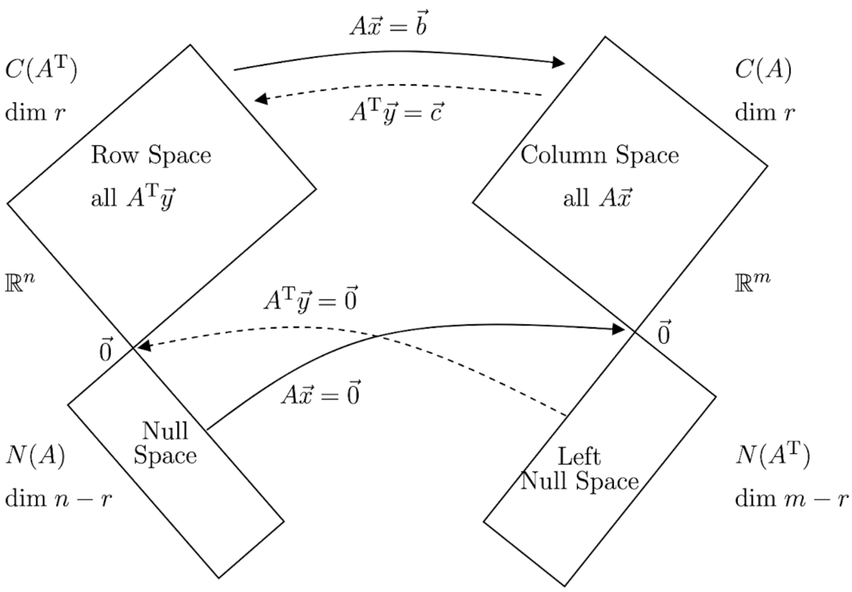


Figure 2.1: Fundamental Theorem of Linear Algebra

Example. Find out the basis for the four fundamental subspaces of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \\ 4 & 3 & 6 & 1 \end{pmatrix} \longrightarrow U = \begin{pmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad r = 2$$

1° $\mathcal{C}(A)$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right\} \quad \dim(\mathcal{C}(A)) = r = 2$$

2° $\mathcal{N}(A)$

$$Ax = 0 \longrightarrow Ux = 0 \longrightarrow U \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + \frac{2}{3}x_3 + \frac{1}{3}x_4 = 0 \end{cases}$$

(a) $x_3 = 1,\ x_4 = 0 \longrightarrow \begin{pmatrix} -1 \\ -2/3 \\ 1 \\ 0 \end{pmatrix} = v_2$

(b) $x_3 = 0,\ x_4 = 1 \longrightarrow \begin{pmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{pmatrix} = v_2$

Hence, $\mathcal{B} = \mathcal{N}(A)$ is $\{v_1, v_2\}$ and

$$\dim(\mathcal{N}(A)) = n - r = 4 - 2 = 2$$

3° $\mathcal{C}(A^T)$

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ 0 \end{pmatrix} \longrightarrow \mathcal{B} = \{S_1^T, S_2^T\}, \quad \dim(\mathcal{C}(A^T)) = r = 2$$

4° $\mathcal{N}(A^T) \longrightarrow \mathcal{N}(B)$

$$B = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 3 \\ 1 & 4 & 6 \\ 0 & 1 & 1 \end{pmatrix} = A^T \longrightarrow \begin{pmatrix} \boxed{1} & 2 & 4 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{cases} y_1 + 2y_3 = 0 \\ y_2 + y_3 = 0 \end{cases}$$

$$z = 1 \longrightarrow \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \quad \therefore \mathcal{B} = \left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}, \quad \dim(\mathcal{N}(A^T)) = m - r = 3 - 2 = 1$$

Check orthogonality

Proposition 2.4.5 (2Q). We can find the existence and uniqueness of solution of $Ax = b$.

- **Existence** of inverse:
  The system $Ax = b$ has at least one solution x for each b iff the columns span $\mathbb{R}^m$ ($r = m$). In this case,
  $$\exists\, n \times m \text{ "right" inverse } C \ni AC = I$$
  This is possible only if $m \leq n$.

- **Uniqueness** of inverse:
  The system $Ax = b$ has at most one solution x for each b iff the columns are independent ($r = n$). In this case,
  $$\exists\, n \times m \text{ "left" inverse } B \ni BA = I$$
  This is possible only if $m \geq n$.

Proof. We separetely prove the two parts.

- **Existence**:
  $$Ax = b \text{ has a solution for each } b \;\Leftrightarrow\; b \in \mathcal{C}(A), \forall b \in \mathbb{R}^m \;\Rightarrow\; \mathcal{C}(A) = \mathbb{R}^m$$
  Let $e_1, e_2, \cdots, e_m$ be the standard basis of $\mathbb{R}^m$.

  Then $\exists\, x_1, x_2, \cdots, x_m \ni Ax_i = e_i, \forall i = 1, 2, \cdots, m$

  Let $C = (x_1 \mid x_2 \mid \cdots \mid x_m)$, then $AC = A(x_1 \mid x_2 \mid \cdots \mid x_m) = (e_1 \mid e_2 \mid \cdots \mid e_m) = I_m$.

- **Uniqueness**:
  $$Ax = b \text{ has at most one solution for each } b \in \mathbb{R}^m$$
  $\Leftrightarrow\; \forall b \in \mathbb{R}^m$, if b can be represented as linear combination of columns of A, then it is unique

Hence, proof is complete. ∎

Example.

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}_{2\times 3} \qquad m = 2, n = 3, r = 2 \quad \longrightarrow \quad \exists \text{ right inverse } C \ni AC = I$$

1°
$$AC = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ 0 & 1/5 \\ c_{31} & c_{32} \end{pmatrix} = I_2 \quad \Rightarrow \quad C \text{ is not unique}$$

2°
$$\begin{pmatrix} 1/4 & 0 \\ 0 & 1/5 \\ c_{31} & c_{32} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{impossible since LHS is } 3 \times 2$$

3°
$$A_2 = \begin{pmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}_{3\times 2} \qquad m = 3, n = 2, r = 2 \quad \longrightarrow Ax = b \qquad \begin{pmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Note. The following statements about a square matrix $A_{n\times n}$ are equivalent:

(1) A is nonsingular (invertible)

(2) The columns of A span $\mathbb{R}^n$, so $Ax = b$ has only one solution $\forall b \in \mathbb{R}^n$

(3) The columns of A are independent, so $Ax = 0$ has only one trivial solution $x = 0$

(4) The rows of A span $\mathbb{R}^n$

(5) The rows of A are independent

(6) Elimination can be completed: $PA = LDU$ with all $d_i \neq 0$

(7) $\exists\ A^{-1} \ni AA^{-1} = A^{-1}A = I_n$

(8) Determinant of A $\det(A) \neq 0$

(9) Zero is NOT an eigenvalue of A

(10) A^TA is $\boxed{\text{positive definite}}$ (正定)

2.5 Graph and Network

skip

2.6 Linear Transformation

We have seen that a matrix move subspaces around. For example, A maps $\mathcal{N}(A)$ to the zero vector and move all vectors into its column space $\mathcal{C}(A)$. Let A be an $n \times n$ matrix and $x \in \mathbb{R}^n$, so A transforms x into $Ax \in \mathcal{C}(A)$.

2.6.1 Notation of Linear Transformation

Example. Here are some examples of linear transformations:

1°

$$A = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} = c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{(scaling by } c\text{)}$$

2°

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad \text{(rotation by } 90^\circ\text{)}$$

3°

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \quad \text{(reflection about } x_1 = x_2\text{)}$$

4°

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad \text{(projection onto } x_1\text{-axis)}$$

Lecture 8

Definition 2.6.1 (2T)**.** Let V, W be vector spaces over a field $\mathbb{F}$. A linear transform from V to W is a function $T : V \to W$ such that preserves the operations on V and W, i.e. 28 Oct. 13:20

$$\begin{cases} T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), & \forall\ \mathbf{u}, \mathbf{v} \in V; \\ T(c\mathbf{u}) = cT(\mathbf{u}), & \forall\ \mathbf{u} \in V, c \in \mathbb{F}. \end{cases}$$

Example.

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$$

T is a linear transform.

Example.

$$A = \frac{d}{dt} : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_{n-1}(\mathbb{R})$$

$$p(t) \in \mathbb{P}_n(\mathbb{R}),\ p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

See the attributes below:

$$AP = \frac{d}{dt}(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}$$

The nullspace of A is all constant polynomials.

$$\mathcal{C}(AP) = \mathbb{P}_{n-1}(\mathbb{R})$$

the basis is $\{1, t, t^2, \ldots, t^{n-1}\}$ and $\operatorname{rank}(\mathcal{C}(A)) = n$.

$$\operatorname{nullity}(A) + \operatorname{rank}(A) = 1 + n = \dim(\mathbb{P}_n(\mathbb{R})).$$

Example.

$$A = \int_0^t : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_{n+1}(\mathbb{R})$$

See the attributes below:

$$AP = \int_0^t (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)\ dt = a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \cdots + \frac{a_n t^{n+1}}{n+1} + C$$

The nullspace of A is all constant polynomials.

$$\mathcal{N}(AP) = \{0\}$$

The range of A

$$\mathcal{C}(AP) = \mathbb{P}_{n+1}(\mathbb{R}) - \{\text{constant}\}/\{0\}$$

Example.

$$T : \mathbb{R}^3 \to \mathbb{R}$$

$$T : (x_1, x_2, x_3) \mapsto 2x_1 + 3x_2 - x_3,\ \ x_i \in \mathbb{R}$$

T is a linear transform.

Example.

$$T : \mathbb{R}^3 \to \mathbb{R}$$

$$T : (x_1, x_2, x_3) \mapsto 2x_1^2 + 3x_2 - x_3,\ x_i \in \mathbb{R}$$

T is NOT a linear transform.

$$\because\ T(x+y) \neq T(x) + T(y)$$

Theorem 2.6.1. Let $T : V \to W$ be a linear transform, where V, W are vector spaces over a field $\mathbb{F}$.

(i) If M is a subspace of V, then

$$T(M) = \{x \in W \mid \exists\ \mathbf{m} \in M,\ \text{such that } T(\mathbf{m}) = x\}$$

is a subspace of W.

(ii) If N is a subspace of W, then

$$T^{-1}(N) = \{\mathbf{v} \in V \mid T(\mathbf{v}) \in N\}$$

is a subspace of V.

Proof. Here is the proof:

(i) Let $M \leq V$, $y_1, y_2 \in T(M) \subseteq W$, and $\alpha \in \mathbb{F}$.

$$y_1, y_2 \in T(M) \Rightarrow \exists x_1, x_2 \in M \text{ s.t. } T(x_1) = y_1,\ T(x_2) = y_2$$

Then

$$T(\alpha x_1 + x_2) = \alpha T(x_1) + T(x_2)$$

since T is a linear transformation.
Also

$$\alpha x_1 + x_2 \in M$$

since M is a subspace of V.
Therefore

$$\alpha y_1 + y_2 = \alpha T(x_1) + T(x_2) = T(\alpha x_1 + x_2) \in T(M)$$

so $T(M)$ is a subspace of W.

(ii) Let $x_1, x_2 \in T^{-1}(N)$ and $\alpha \in \mathbb{F}$.

$$T(\alpha x_1 + x_2) = \alpha T(x_1) + T(x_2) \in N$$

since $N \leq W$ and $T(x_1), T(x_2) \in N$.
Therefore

$$\alpha x_1 + x_2 \in T^{-1}(N)$$

and $T^{-1}(N)$ is a subspace of V.

■

Definition 2.6.2. $T : V \to W$ over a field $\mathbb{F}$ is a linear transform. Then $T^{-1}(\mathbf{O}_W)$ is called the $\boxed{\text{nullspace (kernel) of } T}$, where $\mathbf{O}_W$ is the zero vector in W. $T(V)$ is called the $\boxed{\text{range (image) of } T}$.

$$\dim(T^{-1}(\mathbf{O}_W)) = \text{nullity}(T)$$

$$\dim(T(V)) = \text{rank}(T)$$

2.6.2 Matrix Representation of Linear Transformations

Question. What is the transformation taken $A : \mathbb{R}^2 \to \mathbb{R}^3$

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\in \mathbb{R}^2} \to \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\in \mathbb{R}^3}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\in \mathbb{R}^2} \to \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}_{\in \mathbb{R}^3}$$

Answer.

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}_{3\times 2} = (T)^{\{e_1, e_2, e_3\}}_{\{e_1, e_2\}}$$

⊛

Example.

$$T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R}), \quad \text{i.e. } T(f) = \frac{d}{dt}(f)$$

The ordered basis of two vector spaces are

$$\begin{cases} \mathcal{B}_1 = \mathcal{B}(\mathbb{P}_3(\mathbb{R})) : & \{1, t, t^2, t^3\} \\ \mathcal{B}_2 = \mathcal{B}(\mathbb{P}_2(\mathbb{R})) : & \{1, t, t^2\} \end{cases}$$

Then we have

$$\left(T\right)^{\mathcal{B}_1}_{\mathcal{B}_2} = \begin{array}{c} \\ 1 \\ t \\ t^2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & t & t^2 & t^3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}_{3\times 4} \end{array} \qquad \text{e.g. } \left(T\right) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3\times 1}$$

Example.

$$T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R}), \quad \text{i.e. } T(f) = \frac{d}{dt}(f)$$

We have to handle the t^3 term, which means

$$\left(T\right)^{\mathcal{B}_1}_{\mathcal{B}_1} = \left(T\right)_{\mathcal{B}_1} = \begin{array}{c} \\ 1 \\ t \\ t^2 \\ t^3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & t & t^2 & t^3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 4} \end{array}, \qquad \text{this is called a } \langle\text{differentiation matrix}\rangle$$

Example.

$$\int_0^t : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_4(\mathbb{R})$$

The ordered basis of two vector spaces are

$$\begin{cases} \mathcal{B}_1 = \mathcal{B}(\mathbb{P}_3(\mathbb{R})): & \{1, t, t^2, t^3\}, \quad \dim(\mathbb{P}_3(\mathbb{R})) = 4 \\ \mathcal{B}_2 = \mathcal{B}(\mathbb{P}_4(\mathbb{R})): & \{1, t, t^2, t^3, t^4\}, \quad \dim(\mathbb{P}_4(\mathbb{R})) = 5 \end{cases}$$

hence we have

$$\left(T\right)_{\mathcal{B}_2}^{\mathcal{B}_1} = \begin{array}{c} \\ 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & t & t^2 & t^3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}_{5\times 4} \end{array} \qquad \text{which is called an } \langle\text{integration matrix}\rangle$$

and we also have

$$\mathcal{C}(T) = \text{span}\{t, t^2, t^3, t^4\}, \quad \text{rank}(T) = 4$$

$$\mathcal{N}(T) = \{0\}, \quad \text{nullity}(T) = 0$$

Example.

$$\mathbb{P}_2(\mathbb{R}) \xrightarrow{\int_t} \mathbb{P}_3(\mathbb{R}) \xrightarrow{\frac{d}{dt}} \mathbb{P}_2(\mathbb{R})$$

$$\left(\frac{d}{dt} \int_0^t\right) = \left(\frac{d}{dt}\right)_{3\times 4} \left(\int_0^t\right)_{4\times 3} = \left(I\right)_{3\times 3} \qquad \text{Diff is the left inverse of Int}$$

2.6.3 Rotation Q, Projection P, Reflection R

We introduce three important linear transformations in $\mathbb{R}^2$:

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1° Rotation: Q rotates vectors by an angle θ.

$$\left(Q\right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}_{2\times 2}$$

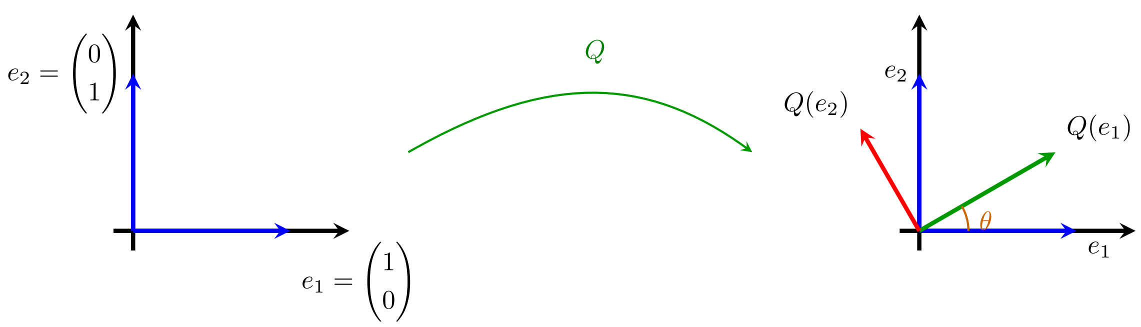
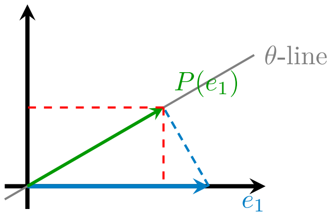


Figure 2.2: Rotation in $\mathbb{R}^2$

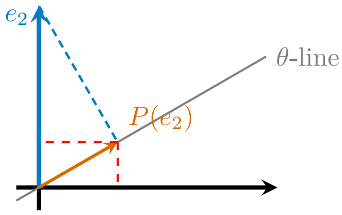
$$Q\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix},\quad Q\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$$

- $Q_{-\theta}\cdot Q_\theta=\mathbf{1}_{\mathbb{R}^2}$
- $Q_\theta\cdot Q_\theta=Q_{2\theta}$
- $Q_\theta\cdot Q_\phi=Q_{\theta+\phi}$

2° Projection: P projects vectors onto the θ-line.



$$P(e_1)=\begin{pmatrix}\cos^2\theta\\\cos\theta\sin\theta\end{pmatrix}$$



$$P(e_2)=\begin{pmatrix}\sin\theta\cos\theta\\\sin^2\theta\end{pmatrix}$$

Figure 2.3: Projection onto a line at angle θ

$$P=\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}$$

Here are some properties of projection:

- $P^2=P$
- Symmetric: $P^T=P$
- P^{-1} does not exist.

1°

$$\begin{aligned}P\left(\alpha\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\right)&=\alpha P\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\\&=\alpha\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}=\alpha\begin{pmatrix}\cos^3\theta+\cos\theta\sin^2\theta\\\sin\theta\cos^2\theta+\sin^3\theta\end{pmatrix}=\alpha\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\end{aligned}$$

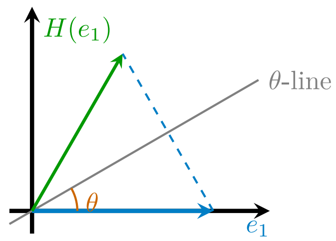
2°

$$\begin{aligned}P\left(\alpha\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}\right)&=\alpha P\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}\\&=\alpha\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}=\alpha\begin{pmatrix}-\sin\theta\cos^2\theta+\cos\theta\sin\theta\\-\sin^2\theta\cos\theta+\sin^3\theta\end{pmatrix}=\alpha\begin{pmatrix}0\\0\end{pmatrix}\end{aligned}$$

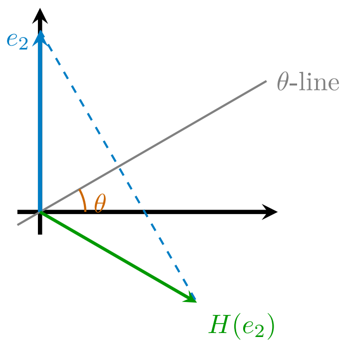
Thus,

$$\boxed{\alpha\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}\quad\text{is in the nullspace of } P.}$$

3° Reflection: R reflects vectors across the θ-line.



$$H(e_1) = \cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cos^2\theta - 1 \\ 2\sin\theta\cos\theta \end{pmatrix}$$



$$H(e_2) = \sin\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \sin\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\sin\theta\cos\theta \\ 2\sin^2\theta - 1 \end{pmatrix}$$

Figure 2.4: Reflection across a line at angle θ

$$H = \begin{pmatrix} 2\cos^2\theta - 1 & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 2\sin^2\theta - 1 \end{pmatrix}$$

Here are some properties of reflection:

- $H^2 = I$
- $H^{-1} = H$
- $H = 2P - I$ $(Hx + x = 2Px)$

Note. If first basis vector is on the θ-line, and the second basis vector is perpendicular to the θ-line, then

$$P^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2P^* - I, \quad Q^* = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Chapter 3

Orthogonality

3.1 Perpendicular Vectors and Orthogonal Subspaces

There are three important concepts in this section:

(i) The length of vector

(ii) The test for perpendicularity

(iii) How to create perpendicular vectors form linearly independent vectors

Now we start to discuss:

(i) **The length of vector**:
The length (or norm) of a vector, in $\mathbb{R}^n$, that satisfies the Pythagorean theorem is defined as:

> **Definition 3.1.1.** Let $\mathbf{x} \in \mathbb{R}^n$ be
>
> $$\mathbf{x} = (x_1, x_2, \dots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in \mathbb{R}^n$$
>
> then
>
> $$\|\mathbf{x}\|^2 = \sum_{i=1}^{n} x_i^2 = \mathbf{x}^T \mathbf{x}$$

(ii) **The test for perpendicularity**:

> **Definition 3.1.2.** Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then if $\mathbf{x} \perp \mathbf{y}$, then by Pythagorean theorem, we have
>
> $$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} - \mathbf{y}\|^2$$
>
> Then we can deduce that
>
> $$x_1^2 + x_2^2 + \cdots + x_n^2 + y_1^2 + y_2^2 + \cdots + y_n^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2$$
>
> then we have
>
> $$\mathbf{x}^T \mathbf{y} = 0$$

Definition 3.1.3 (Inner Product). Let V be a vector space over a field $\mathbb{F}$ $(\mathbb{R}, \mathbb{C})$. An inner product on V is a function that assigns to every ordered pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in V, a scalar in $\mathbb{F}$, denoted as

$$\langle \mathbf{x}, \mathbf{y} \rangle$$

$\forall\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in V,\ \ c \in \mathbb{F}$, we have

(a) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$

(b) $\langle c\ \mathbf{x}, \mathbf{y} \rangle = c\ \langle \mathbf{x}, \mathbf{y} \rangle$

(c) $\langle \mathbf{y}, \mathbf{x} \rangle = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$ (where $\overline{a + bi} = a - bi$ complex conjugate)

(d) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, if $\mathbf{x} \neq \mathbf{0}$

Note (1). If $\mathbb{F} = \mathbb{R}$, (c) will reduce to $\langle \mathbf{y}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.

Note (2). Inner product is linear in the first component.

Definition 3.1.4 (Standard Inner Product). Let $V = \mathbb{R}^n/\mathbb{R}$, defined

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

This is called the $\boxed{\text{standard inner product}}$ on $\mathbb{R}^n$.

Proposition 3.1.1. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

- Let $\langle \mathbf{x}, \mathbf{y} \rangle = x^T y$ be standard inner product.
- Let $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ if and only if $\mathbf{x} \perp \mathbf{y}$.

Example. If $\langle , \rangle$ is any inner product on V, and $r > 0$, we define

$$\langle \mathbf{x}, \mathbf{y} \rangle' = r \langle \mathbf{x}, \mathbf{y} \rangle$$

1° $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle' = r\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = r\langle \mathbf{x}, \mathbf{z} \rangle + r\langle \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle' + \langle \mathbf{y}, \mathbf{z} \rangle'$

2° $\langle c\mathbf{x}, \mathbf{y} \rangle' = r\langle c\mathbf{x}, \mathbf{y} \rangle = c \cdot r\langle \mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle'$

Example. Let $V = \{f \mid f :$ real-valued continuous functions on $[0, 1]\} = \mathcal{C}([0, 1])$. For $f, g \in V$, define

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

Example. Let $V = \mathbb{C}^n$, $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. For $\mathbf{x}, \mathbf{y} \in V$, define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \overline{\mathbf{y}} = \sum_{i=1}^{n} x_i \overline{y_i}$$

$$\langle \mathbf{y}, \mathbf{x} \rangle = \mathbf{y}^{\mathsf{T}} \overline{\mathbf{x}} = \overline{\mathbf{x}^{\mathsf{T}} \overline{\mathbf{y}}} = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}.$$

Example. Let $V = \mathbb{C}$, $\mathbb{C}$ is a vector space over $\mathbb{C}$. If $\mathbf{x}, \mathbf{y} \in \mathbb{C}$, $x = a + bi$, $y = c + di$, define

$$\langle \mathbf{x}, \mathbf{y} \rangle = (a + bi)(c - di)$$

1° $\langle \mathbf{y}, \mathbf{x} \rangle = (c + di)(a - bi) = \overline{(a + bi)(c - di)} = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$

2° $\langle \mathbf{x}, \mathbf{x} \rangle = (a + bi)(a - bi) = a^2 + b^2 > 0$ if $\mathbf{x} \neq 0$