Linear Algebra

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Abstract

The lecture note of 2025 Fall Linear Algebra by professor 李明穗（Amy Lee）.

Contents

Chapter 0

Introduction

Lecture 1

0.1 Geometry

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- linear
- To study geometry with linearity
- In a different dimension:
  - In 2D: **lines**
  - In 3D: **planes**
  - In nD: **hyperplanes**

0.2 Abstract Algebra

Definition 0.2.1 (Linear Algebra)**.** Here is the definition of Linear algebra.

- Algebra is the study of basic "mathematical structure."
  e.g. **Group**, **Ring**, **Field**, ...etc.
- Linear Algebra studies one of the structures called **vector space**.

Note. Followed by logical deduction from the basic definition, we can derive some theorems.

0.3 Applied Science

- **Mathematic**: ODE, PDE.
- **Linear Programming**: developing during World War II
- **Imange Processing**, **Computer Vision**, **Computer Graphic**, etc.

Chapter 1

Matrices and Gaussian Elimination

1.1 Introduction

The central problem of Linear Algebra is the solution of Linear Equations. The most important and simplest case is when the # of unknowns equals to the # of equations.

> **Note.** There are two ways to solve linear equations:
>
> - The method of elimination **(Gaussian Elimination)**
> - Determinants **(Crammer's Rule)**

1.1.1 Four aspects to follow

(1) The geometry of linear equations.

> **Note.** $n = 2, n = 3 \quad \rightarrow \quad$ higher dimensional space.

(2) The interpretation of elimination is a factorization of the coefficient matrix.

> **Definition.** Some notation to define:
>
> **Definition 1.1.1** (Scalar, Matrix, Vector).
>
> $$Ax = b \qquad \begin{cases} \alpha, \beta, \gamma : & \text{scalar} \\ A, B, C : & \text{matrix} \\ a, b, c : & \text{vector} \end{cases}$$
>
> **Definition 1.1.2** (Lower/Upper triangular matrix).
>
> $$A = LU \qquad \begin{cases} L : & \text{lower triangular matrix} \\ U : & \text{upper triangular matrix} \end{cases}$$
>
> **Definition 1.1.3** (Transpose/Inverse).
>
> $$A^T / A^{-1} : \qquad \begin{cases} A^T : & \text{Transpose of matrix A} \\ A^{-1} : & \text{Inverse of matrix A} \end{cases}$$

(3) Irregular case and Singular case **(no unique solution)**:

> **Note.** no solution or infinitely many solutions

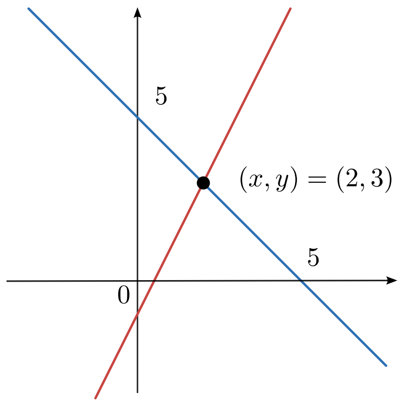
(4) **The # of operations to solve the system by elimination**

1.2 Geometry of Linear Equation

Example. Consider the linear equation below:

$$\begin{cases} 2x - y &= 1 \\ x + y &= 5 \end{cases}$$

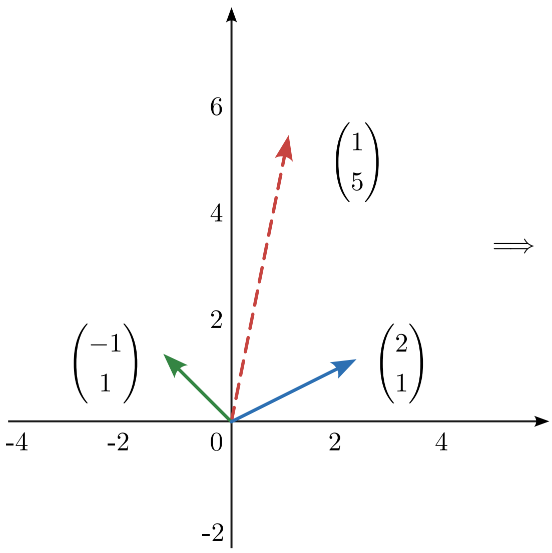
- **approach 1: row picture** $\to$ two lines in plane



$\Longrightarrow$ $(2,3)$ is the intersection of two lines

Figure 1.1: Row Picture

- **approach 2: column picture**



$$\Longrightarrow \quad 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Figure 1.2: Column Picture

Lemma 1.2.1 (Linear Combination).

$$x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix}$$

To find the **Linear Combination** of $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ to reach $\begin{pmatrix}1\\5\end{pmatrix}$

Note. A vector is a $n \times 1$ array with n real numbers, c_n is

$$\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}$$

But in the text, we use

$$(c1, \cdots, c_n)$$

to represent.

Definition. Here are some operations on matrix:

Definition 1.2.1.

$$\alpha\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}_{n\times 1}=\begin{pmatrix}\alpha\cdot c_1\\\vdots\\\alpha\cdot c_n\end{pmatrix}_{n\times 1}, \qquad \alpha\in\mathbb{R}$$

Definition 1.2.2.

$$\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}+\begin{pmatrix}d_1\\\vdots\\d_n\end{pmatrix}=\begin{pmatrix}c_1+d_1\\\vdots\\c_n+d_n\end{pmatrix}_{n\times 1}$$

Definition 1.2.3.

$$y\in\mathbb{R}$$

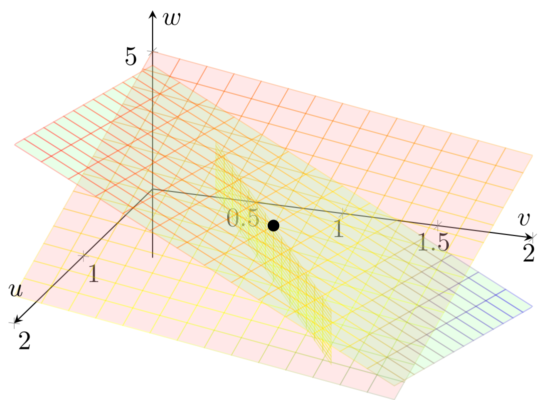
$$y\in\mathbb{R}^2 \implies y=\begin{pmatrix}y_1\\y_2\end{pmatrix}_{2\times 1} \qquad y_1,y_2\in\mathbb{R}$$

$$y\in\mathbb{R}^3 \implies y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}_{3\times 1} \qquad y_1,y_2,y_3\in\mathbb{R}$$

Example. Consider the linear equation below:

$$\begin{cases} 2u + v + w & = 5 \\ 4u - 6v & = -2 \\ -2u + 7u + 2w & = 9 \end{cases}$$

- Row picture



$$(u, v, w) = (1, 1, 2)$$

Lemma 1.2.2. in n-dimension, a line require $(n-1)$ equation.

Question. How to extend into n-dimensions?

Answer. Consider the following steps:

- Each equation represents a plane or hyperplane.
- The first equation produces a $(n-1)$-dimension plane in $\mathbb{R}^n$
- The second equation produces another $(n-1)$-dimension plane in $\mathbb{R}^n$
- Their intersection in smaller set of $(n-2)$-dimension
- $(n-3) \to (n-4) \to \cdots\cdots \to 3 \to 2 \to 1 \to$ point

Then we can find the final intersection. ⊛

- Column picture

$$u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \quad \Longleftarrow \quad \begin{cases} 2u + v + w & = 5 \\ 4u - 6v & = -2 \\ -2u + 7u + 2w & = 9 \end{cases}$$

RHS is a linear combination of 3 column vectors.

Theorem 1.2.1. Solution to a linear equation:

$$\underbrace{(\textbf{intersection of to points})}_{\text{row pic.}} = \underbrace{(\textbf{coefficient of linear combination})}_{\text{column pic.}}$$

1.2.1 Singular Case

(1) Row Picture: In 3D case, they didn't intersect at a point.

- **Case 1**: two parallel

$$\begin{cases} 2u+v+w &= 5 \\ 4u+2v+2w &= 9 \end{cases}$$

- **Case 2**: three plane perpendicular ($\perp$)

$$\begin{cases} u+v+w &= 2 \ \cdots \ (1) \\ 2u+3w &= 5 \ \cdots \ (2) \\ 3u+v+4w &= 6 \ \cdots \ (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1)+(2)=(3) \quad ; \quad \text{LHS} \Rightarrow (1)+(2) \neq (3)$$

- **Case 2**: three plane have a whole line in common.

$$\begin{cases} u+v+w &= 2 \ \cdots \ (1) \\ 2u+3w &= 5 \ \cdots \ (2) \\ 3u+v+4w &= 7 \ \cdots \ (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1)+(2)=(3) \quad ; \quad \text{LHS} \Rightarrow (1)+(2) = (3)$$

- **Case 4**: three parallel

(2) Column Picture:

$$u\begin{pmatrix}1\\2\\3\end{pmatrix}+v\begin{pmatrix}1\\0\\1\end{pmatrix}+w\begin{pmatrix}1\\3\\4\end{pmatrix}=b$$

In the case above, three vectors are linear combination to each other, i.e. three vectors share the same plane.

Lemma 1.2.3 (Singular case)**.** If the three vectors are linear combination to each other (three vector share a common plane), it must be **singular case**.

- If $b=\begin{pmatrix}2\\5\\7\end{pmatrix}$, which is on the plane $\Rightarrow$ too many solution to produce b.
- If $b=\begin{pmatrix}2\\5\\6\end{pmatrix}$, which is not on the plane $\Rightarrow$ no solution.

1.2.2 Fundamental Linear Algebra Theorem (Geometry form)

Theorem 1.2.2 (Fundamental LA Theorem)**.** Consider a linear system

$$Ax = b, \quad A \in \mathbb{R}^{m\times n},\ x \in \mathbb{R}^n,\ b \in \mathbb{R}^m.$$

If the n hyperplanes have no only one intersection or infinitely many points, then the n columns lie in the same plane. (consistency of *row picture* and *column picture*)

> **Notation.** Logic notation:
>
> - If ..., then : $\Rightarrow$
> - If and only if : $\Leftrightarrow$

Lecture 2

1.3 An Example of Gaussian Elimination

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Example. Here is a linear equation.

$$\begin{cases} 2u + v + w & = 5 \\ 4u - 6v & = -2 \\ -2u + 7v + 2w & = 9 \end{cases}$$

$$\begin{pmatrix} \boxed{\mathbf{2}} & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix} \implies \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & \boxed{\mathbf{-8}} & -2 & -12 \\ 0 & 8 & 3 & 14 \end{pmatrix} \implies \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & \boxed{\mathbf{1}} & 2 \end{pmatrix} \quad \text{"pivot"}$$

Then we get $w = 2$, we can plug in the equation i.e.

$$\begin{cases} 2u + v + 1w = 5 \\ -8v - 2w = -12 \\ w = 2 \end{cases} \implies \text{Forward Elimination}$$

Then we substitute into 2nd, 1st equation to get $v = 1$ and $u = 1 \implies$ Backend Elimination

Note. By definition, pivots cannot be zero!

Question. Under what circumstances could the elimination process break down?

Answer. Here are some situations.

- Something **must** go wrong in the singular case.
- Something **might** go wrong in the nonsingular case.

A zero appears in a pivot position!
If in the process, there are nonzero pivots, then there's only one solution. ⊛

Example.

$$\begin{pmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix}$$

(1) If $a_{11} = 0 \implies$ nonsingular

(2) If $a_{22} = 0 \implies$ nonsingular

(3) If $a_{33} = 1 \implies$ singular

Question. How many separate arithmetical operations does elimination require for n equations in n unknowns?

Answer. For a single operation.

$$\text{a single operation} = \text{each division \& each multiplication-subtraction}$$

⊛

- **FE**:

$$\underbrace{\begin{matrix} x & x & \cdots & x & = & x \\ \vdots & \vdots & & & & \vdots \\ x & x & \cdots & x & = & x \end{matrix}}_{n}$$

$$n(n-1) + (n-1)(n-2) + \cdots + (1^2 - 1) = \frac{n^3 - n}{3} \sim \frac{n^3}{3} \text{ steps}$$

- **RHS**:

$$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

- **BF**:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

1.4 Matrix Notation and Matrix Multiplication

$$\begin{cases} 2u + 4v + -2w &= 2 \\ 4u + 9v - 3w &= 8 \\ -2u - 3v + 7w &= 10 \end{cases} \Longrightarrow \quad u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + w \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

We can rewrite it in the below form.

$$A = \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}_{3\times 3}}_{\text{coefficient matrix}}, \quad x = \underbrace{\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{3\times 1}}_{\text{unknowns}}, \quad b = \underbrace{\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}_{3\times 1}}_{\text{RHS}} \quad \Longrightarrow \quad x = \underbrace{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}_{3\times 1}}_{\text{solution}}$$

$$\boxed{Ax = b}$$

Definition 1.4.1. An $m \times n$ matrix, $A_{m\times n}$ over $\mathbb{R}$, is an array with m rows and n columns of real numbers, which can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \text{ where } a_{ij} \in \mathbb{R}, \begin{cases} i : \text{index of row} \\ j : \text{index of column} \end{cases}$$

- $\boxed{m \times n}$ is called the **dimensions (size)** of $A \Longrightarrow$ dimension of a $()_{3\times 5}$ is 3×5
- $\boxed{a_{ij}}$ is called the **elements/entry/coefficient** of A
- **Addition**: $A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{m\times n}$

$$A + B = (a_{ij} + b_{ij})_{m\times n}$$

- **Multiplication**: $A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{n\times r}$

$$AB = (c_{ij})_{m\times r}, \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

- **Scalar Multiplication**:

$$\alpha A = (\alpha a_{ij})_{m\times n}$$

- $$A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$$

  In particular, if

$$A_{1\times n} B_{n\times 1} = \mathbf{v} \cdot \mathbf{w} = ()_{1\times 1}.$$

  Then it's the **inner product** of vector $\mathbf{v}$ and vector $\mathbf{w}$

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & 3 & -7 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\cdot(-1) & 4\cdot(2) & -2\cdot(2) \\ 4\cdot(-1) & 9\cdot(2) & -3\cdot(2) \\ -2\cdot(-1) & 3\cdot(2) & -7\cdot(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 22 \end{pmatrix}$$

$$(-1)\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 9 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$$

(1) by row: 3 inner product

(2) by column: a linear combination of 3 columns of A

Example (1A). Ax is a combination of columns of A

$$\begin{aligned} A_{m\times n}x_{n\times 1} &= \left(A_1|A_2|\cdots|A_n\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1(A_1) + x_2(A_2) + \cdots + x_n(A_n) = \left(\sum_{j=1}^{n} a_{ij}\ x_j\right)_{m\times 1} \end{aligned}$$

1.4.1 The Matrix Form of One Elimination Step

Definition (1B). Matrix form

Definition 1.4.2. zero matrix:

$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Definition 1.4.3. identity matrix:

$$I = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} = I_n = I_{n\times n}; \quad \begin{cases} A_{m\times n}I_n = A_{m\times n} \\ A_{m\times n} = A_{m\times n}I_n \end{cases}$$

Definition 1.4.4. elementary matrix (elimination matrix):

$$E_{ij} = \begin{array}{cc} \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & -\ell & \ddots & 0 \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} & \begin{array}{c} \\ \\ \\ \text{ith row} \\ \\ \end{array} \\ \begin{array}{ccccc} & & \text{jth column} & & \end{array} & \end{array} \qquad \ell : \text{multiplier}$$

$$E_{ij} \cdot A = \begin{pmatrix} \cdots & -\ell & \cdots & 1 \\ & & & \end{pmatrix} \begin{pmatrix} \ \\ \ \end{pmatrix} \begin{array}{l} \longleftarrow \text{ i-th} \implies \text{(i-th row)} + (-\ell)\text{(j-th column)} \\ \longleftarrow \text{ j-th} \implies \text{create zero at } (i, j) \text{ position!} \end{array}$$

Example.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_{21}} \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 0_{21} & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}_{EA}$$

Note. Here is two properties

1. $Ax = b \implies E_{ij}Ax = E_{ij}b$
2. $E_{ij}A \neq AE_{ij}$

1.4.2 Matrix Multiplication

(1) The (i, j)-th entry of AB is the inner product of the **i-th** of A and the **j-th** of B.

(2) Each column of AB is the product of a matrix A and **a column of B**

$$\begin{aligned} \implies \quad \text{column } j \text{ of } AB &= A \text{ times } \textbf{j-th} \text{ of } B \\ &= \text{linear combination of } \textbf{columns of } A \\ &= b_{1j}A\underbrace{\boxed{\bullet}}_{\text{any numbers}}{}_1 + b_{2j}A_{\bullet 2} + \cdots + b_{nj}A_{\bullet n} \end{aligned}$$

Example.

$$\underbrace{\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}}_{A_{2\times 3}} \underbrace{\begin{pmatrix} 5 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}}_{B_{3\times 3}} = \underbrace{\begin{pmatrix} 16 & 1 & 1 \\ 8 & 0 & -1 \end{pmatrix}}_{C_{2\times 3}}$$

$$\text{1st column of } AB = \begin{pmatrix} 16 \\ 8 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(3) Each row of AB is a product of a row of A and a matrix B.

$$\begin{aligned} \Longrightarrow \quad \text{i-th row of } AB &= \text{ of } A \text{ times } B. \\ &= \text{linear combination of } \textbf{rows of } B \\ &= a_{i1}B_{1\bullet} + a_{i2}B_{2\bullet} + \cdots + a_{in}B_{n\bullet} \end{aligned}$$

Theorem 1.4.1. Let A, B and C be matrices (possibly rectangular). Assume that their dimension permit them to be added and multiplied in the following theorem

(1) The matrix multiplication is associative

$$(AB)C = A(BC)$$

(2) Matrix operations are distributive

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

(3) Matrix multiplication is noncommutative

$$AB \neq BA \quad \text{in general}$$

(4) Identity Matrix

$$A_{n\times n}I_n = I_nA_{n\times n} = A_{n\times n}$$

Example.

$$\underset{21}{E} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \underset{31}{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{1} & 0 & 1 \end{pmatrix}, \quad \underset{32}{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-1} & 1 \end{pmatrix}$$

(1)

$$\underset{21}{E}\,\underset{31}{F} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \boxed{=} \underset{31}{F}\,\underset{21}{E} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(2)

$$\underset{21}{E}\,\underset{32}{G} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \boxed{\neq} \underset{32}{G}\,\underset{21}{E}$$

(3)

$$\underset{32}{G}\,\underset{31}{F}\,\underset{21}{E} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}}_{\text{"right order"}} \boxed{\neq} \underset{21}{E}\,\underset{31}{F}\,\underset{32}{G} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Note. The product of lower triangular matrices is a lower triangular matrix.

Lecture 3

1.5 Triangular Factors and Row Exchanges

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$$\boxed{Ax = b}$$

$$\implies \quad LUx = b \quad \implies \quad \begin{cases} Lc = b \\ Ux = c \end{cases}$$

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = b$$

Remark. ℓ: multipliers

$$E_{ij}(\ell) : \text{(i-th row)} + (-\ell)\text{(j-th column)}$$

$$\begin{pmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{pmatrix} \xrightarrow[R_3+(1)R_1]{R_2+(-2)R_1} \begin{pmatrix} 2 & 4 & -2 & 2 \\ 0_{21} & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{pmatrix} \xrightarrow{R_3+(-1)R_2} \begin{pmatrix} \boxed{2} & 4 & -2 & 2 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0_{32} & \boxed{4} & 8 \end{pmatrix} \quad \text{pivot}$$

$$E_{21}(2) = E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{31}(-1) = F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad E_{32}(1) = G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

i.e.

$$E_{21}E_{31}E_{32}Ax = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = Ux = c = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} = E_{21}E_{31}E_{32}b$$

Question. How can we undo the steps of Gaussian Elimination?

$$E^{-1}F^{-1}G^{-1}GFEA = A = \underbrace{E^{-1}F^{-1}G^{-1}}\boxed{U} = LU \quad \text{i.e.} \quad A = \underset{\text{factors of } A}{LU}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -(-2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(1) & 0 & 1 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(-1) & 1 \end{pmatrix}$$

$$E^{-1}F^{-1}G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{-1} & \boxed{1} & 1 \end{pmatrix} \implies \text{records everything that has been done so far}$$

1.5.1 Triangular Factorization

Theorem 1.5.1. If no exchanges are required, the original matrix A can be written as

$$A = LU$$

- The matrix L is lower triangular with 1's on the diagonal and the multipliers ℓ_{ij} (taken from elimiation) below the diagonal.
- The matrix U is the upper triangular matrix which appears after forward elimination and before back-substitution; its diagonal entries are the pivots.

Example.

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 6 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \text{提出2}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 6 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Question.

$$A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 2 & 6 & 5 \\ -1 & 4 & -2 \\ 1 & 2 & 3 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

"triangular matrix" 有三條對角線

Answer. ⊛

1.5.2 One Linear System = Two Triangular Systems

$$Ax = b \implies Ux = c \,\&\, Lc = b \implies A = LU$$

Remark. The LU form is unsysttematic in one aspect. U has pivots along its diagonal where L always has 1's.
We can rewrite U as

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12}/u_{11} & \cdots & u_{1n}/u_{11} \\ 0 & 1 & & u_{2n}/u_{22} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example.

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} \boxed{3} & 4 \\ 0 & \boxed{2/3} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2/3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 4/3 \\ 0 & 1 \end{pmatrix}}_{U}$$

Theorem 1.5.2. If

$$A = L_1 D_1 U_1 \text{ and } A = L_2 D_2 U_2$$

then

$$L_1 = L_2, D_1 = D_2, U_1 = U_2$$

i.e. if A has LDU decomposition, then it **is** unique.

1.5.3 Row Excahnge and Permutation Matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \langle \text{Permutation matrix } P_{ij} \rangle$$

Note. Permutation matrix is also an elementary matrix.

Example. Here are some of the example:

1°

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \qquad \boxed{R_2 \leftrightarrow R_3}$$

2°

$$PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{pmatrix} \qquad \boxed{R_2 \leftrightarrow R_3}$$

3°

$$AP = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 5 & 6 \end{pmatrix} \qquad \boxed{C_2 \leftrightarrow C_3}$$

Note. For the permutation matrix:

1° PA: Performing row exchange of A

2° AP: Performing column exchange of A

3° $PAx = Pb$; Should we permute the component of $x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ as well? **NONONONONO!!!**

Example.

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{pmatrix} \qquad Ax = b$$

(1) if $d = 0$, the problem is incurable. The matrix is singular.

(2) if $d \neq 0$, $P_{13}A = \begin{pmatrix} d & e & f \\ 0 & 0 & c \\ 0 & a & b \end{pmatrix}$; if $a \neq 0$, $P_{23}P_{13}A = \begin{pmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{pmatrix}$

$$\begin{array}{|c ccc c ccc|} & & P_{23}P_{13} & & \neq & & P_{13}P_{23} & \\ \\ & 1 & 3 & 3 & & 1 & 1 & 2 \\ \text{row} & 2 \rightarrow & 2 \rightarrow & 1 & & 2 \rightarrow & 3 \rightarrow & 3 \\ & 3 & 1 & 2 & & 3 & 2 & 1 \end{array}$$

Theorem 1.5.3. We seperate into two cases:

- In the non singular case, there's a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. In this case,
  (1) $Ax = b$ has a unique solution.
  (2) It is found by elimination with row exchange
  (3) With the rows reorders in advance, PA can be factored into LU $\langle PA = LU \rangle$
- In singular case, no reordering can produce a full set of pivots.

Example.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix} \xrightarrow[\cancel{\ell_{31}=2}\ \ell_{21}=2]{\cancel{\ell_{21}=1}\ \ell_{31}=1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{P_{23}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \langle \text{This is WORNG} \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

To summarize: A good code for Gaussian Elimination keeps a record of L, U and P. They allow the solution $(Ax = b)$ from two triangular systems. If the system $Ax = b$ has a unique solution, they we say:

1° The system is nonsingular or

2° The matrix is nonsingular

Otherwise, it is singular.

1.6 Inverse and Transpose

Definition 1.6.1. An $n \times n$ matrix A is invertible if $\exists$ an $n \times n$ matrix $B \ni BA = I = AB$

Theorem 1.6.1. If A is invertible, then the matrix B satisfying $AB = BA = I$ is unique!

Proof. Suppose $\exists c \neq B \ni AC = CA = I$

$$B = BI = B(AC) = (BA)C = IC = C \text{ i.e } B = C$$

we call this matrix B, the $\boxed{\text{inverse of } A}$, and denoted as $\boxed{A^{-1}}$ ■

Note. Not all $n \times n$ matrices have inverse.
e.g.

1°

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

2° if $Ax = \vec{0}$ has a $\overset{x \neq 0}{\underline{\text{nonzero solution}}}$, then A has no inverse!

$$x = A^{-1}(Ax) = A^{-1}\vec{0} = \vec{0} \quad (\rightarrow\leftarrow)$$

Note. The inverse of A^{-1} is A itself. i.e. $(A^{-1})^{-1} = A$.

Note. If $A = (a)_{1\times 1}$ and $a \neq 0$, then $A^{-1} = (\frac{1}{a})$. The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$ is

$$\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if } \det(A) \neq 0$$

Note.

$$A = \begin{pmatrix} d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & d_n \end{pmatrix} \quad d_i \neq 0,\ \forall i \quad \Longrightarrow A^{-1} = \begin{pmatrix} 1/d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1/d_n \end{pmatrix}$$

Proposition 1.6.1. If A and B are invertible, then

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}\cdots A_2^{-1}A_1^{-1}$

1.6.1 The Calculation of A^{-1}: Gaussian-Jordan Method

$$A \cdot A^{-1} = I$$

$$A_{n\times n}B_{n\times n} = I_n$$

$$\begin{aligned} &\Longrightarrow A_{n\times n}(B_1|B_2|\cdots|B_n)_{n\times n} = (e_1|e_2|\cdots|e_n)_{n\times n} \\ &\Longrightarrow (AB_1|AB_2|\cdots|AB_n)_{n\times n} = (e_1|e_2|\cdots|e_n)_{n\times n} \\ &\Longrightarrow AB_1 = e_1;\ AB_2 = e_2;\ \cdots;\ AB_n = e_n \quad \longrightarrow \quad n \text{ linear systems:} Ax = b \end{aligned}$$

Definition 1.6.2 (Gaussian-Jordan Method)**.** Instead of stopping at U and switching to back substitution, it continues by subtracting multipliers of a row from the rows above till it reaches a diagonal matrix. Then we divide each row by corresponding pivot.

$$(\underset{LU}{A}|I) \xrightarrow{\times L^{-1}} (U|L^{-1}) \xrightarrow{\times U^{-1}} (I|A^{-1})$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$$

1.6.2 Invertible = Nonsingular

Question. What kind of matrices are invertible?

Answer. Here are the example:

1° nonzero pivot Ch1 Ch4

2° nonzero determinants Ch4

3° independent columns (rows) Ch2

4° nonzero eigenvalues Ch5

which will in the whole course ⊛

Suppose a matrix A has full set of nonzero pivots. By definition, A is nonsingular and the n systems

$$Ax_1 = e_1, \ Ax_2 = e_2, \ \cdots \ , Ax_n = e_n$$

can be solved by elimination or Gaussian-Jordan Method.

Row exchanges maybe necessary, but the columns of A^{-1} are uniquely determined.

$$\begin{aligned} Ax = b \quad & PAx = Pb \\ & PAx_i = Pe_i \end{aligned}$$

$$\{Pe_1, Pe_2, \cdots, Pe_n\} = \{e_1, e_2, \cdots, e_n\}$$

Note. Compute A^{-1}:

1° $A(x_1|\cdots|x_n) = I = (e_1|\cdots|e_n) \iff Ax_i = e_i, \ i = i \cdots n$

2° Gauss-Jordan Method: $(\ A \mid I\ ) \longrightarrow (\ I \mid A^{-1}\ )$

Question. We have found a matrix $A^{-1} \ni AA^{-1} = I$. But is $A^{-1}A = I$

Answer. We can do this by recall.

> **As previously seen.** Recall that every Gauss-Jordan step is a multiplication of matrices on the left. There are three types of elementary matrices:

1° $\boxed{E_{ij}(\ell)}$: to subtract a multiple ℓ of j row from i row.

2° $\boxed{P_{ij}}$: to exchange row i and j

3° $\boxed{D_i(d)}$: to multiply row i by d i.e. $D_i(d) = \begin{pmatrix} 1 & & & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & d & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \end{pmatrix}$ $\to$ ith row

$$\begin{pmatrix} d_1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$$

$$\implies \quad DEEPEEA = I \quad \implies \quad A^{-1}A = I \therefore \text{ we have a left inverse!}$$

These are the operation of A^{-1} ⊛

Theorem 1.6.2. For nonsingular and invertible:

- Every nonsingular matrix is invertible.
- Every invertible matrix is nonsingular.

Theorem 1.6.3. A square matrix is invertible $\iff$ it is nonsingular