Linear Algebra

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Abstract

The lecture note of 2025 Fall Linear Algebra by professor 李明穗（Amy Lee）.

Contents

Chapter 0

Introduction

Lecture 1

2 Sep. 13:20

0.1 Geometry

- linear
- To study geometry with linearity
- In a different dimension:
  - In 2D: **lines**
  - In 3D: **planes**
  - In nD: **hyperplanes**

0.2 Abstract Algebra

Definition 0.2.1 (Linear Algebra)**.** Here is the definition of Linear algebra.

- Algebra is the study of basic "mathematical structure."
  e.g. **Group**, **Ring**, **Field**, ...etc.
- Linear Algebra studies one of the structures called **vector space**.

Note. Followed by logical deduction from the basic definition, we can derive some theorems.

0.3 Applied Science

- **Mathematic**: ODE, PDE.
- **Linear Programming**: developing during World War II
- **Imange Processing**, **Computer Vision**, **Computer Graphic**, etc.

Chapter 1

Matrices and Gaussian Elimination

1.1 Introduction

The central problem of Linear Algebra is the solution of Linear Equations. The most important and simplest case is when the # of unknowns equals to the # of equations.

Note. There are two ways to solve linear equations:

- The method of elimination **(Gaussian Elimination)**
- Determinants **(Crammer's Rule)**

1.1.1 Four aspects to follow

(1) The geometry of linear equations.

Note. $n = 2, n = 3 \quad \rightarrow \quad$ higher dimensional space.

(2) The interpretation of elimination is a factorization of the coefficient matrix.

Definition. Some notation to define:

Definition 1.1.1 (Scalar, Matrix, Vector).

$$Ax = b \qquad \begin{cases} \alpha, \beta, \gamma : & \text{scalar} \\ A, B, C : & \text{matrix} \\ a, b, c : & \text{vector} \end{cases}$$

Definition 1.1.2 (Lower/Upper triangular matrix).

$$A = LU \qquad \begin{cases} L : & \text{lower triangular matrix} \\ U : & \text{upper triangular matrix} \end{cases}$$

Definition 1.1.3 (Transpose/Inverse).

$$A^T/A^{-1} : \qquad \begin{cases} A^T : & \text{Transpose of matrix A} \\ A^{-1} : & \text{Inverse of matrix A} \end{cases}$$

(3) Irregular case and Singular case **(no unique solution)**:

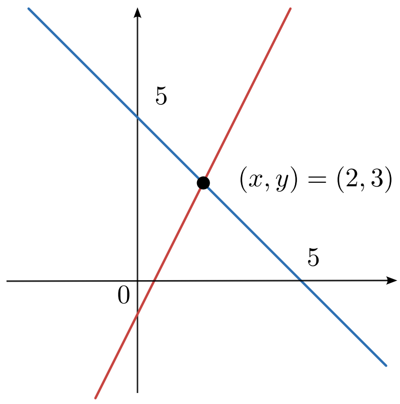
> **Note.** no solution or infinitely many solutions

(4) **The # of operations to solve the system by elimination**

1.2 Geometry of Linear Equation

> **Example.** Consider the linear equation below:
>
> $$\begin{cases} 2x - y &= 1 \\ x + y &= 5 \end{cases}$$

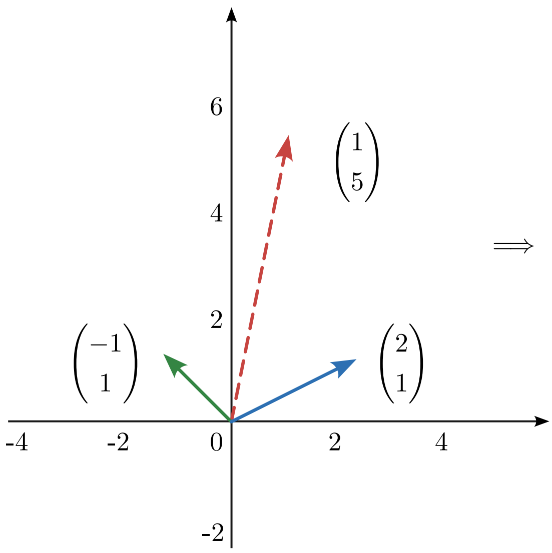
- **approach 1**: **row picture** $\rightarrow$ two lines in plane



$\Longrightarrow$ $(2, 3)$ is the intersection of two lines

Figure 1.1: Row Picture

- **approach 2**: **column picture**



$$\Longrightarrow \quad 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Figure 1.2: Column Picture

Lemma 1.2.1 (Linear Combination).

$$x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix}$$

To find the **Linear Combination** of $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ to reach $\begin{pmatrix}1\\5\end{pmatrix}$

Note. A vector is a $n\times 1$ array with n real numbers, c_n is

$$\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}$$

But in the text, we use

$$(c1,\cdots,c_n)$$

to represent.

Definition. Here are some operations on matrix:

Definition 1.2.1.

$$\alpha\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}_{n\times 1}=\begin{pmatrix}\alpha\cdot c_1\\\vdots\\\alpha\cdot c_n\end{pmatrix}_{n\times 1},\qquad \alpha\in\mathbb{R}$$

Definition 1.2.2.

$$\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}+\begin{pmatrix}d_1\\\vdots\\d_n\end{pmatrix}=\begin{pmatrix}c_1+d_1\\\vdots\\c_n+d_n\end{pmatrix}_{n\times 1}$$

Definition 1.2.3.

$$y\in\mathbb{R}$$

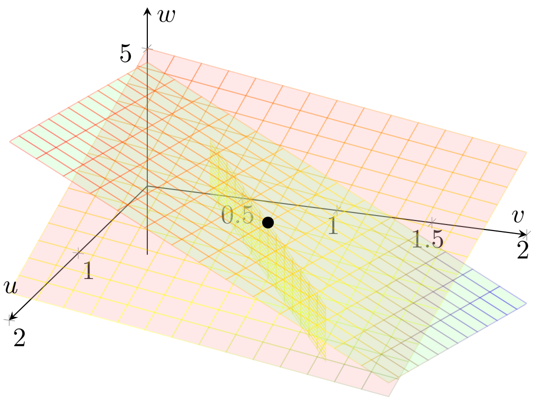
$$y\in\mathbb{R}^2\implies y=\begin{pmatrix}y_1\\y_2\end{pmatrix}_{2\times 1}\qquad y_1,y_2\in\mathbb{R}$$

$$y\in\mathbb{R}^3\implies y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}_{3\times 1}\qquad y_1,y_2,y_3\in\mathbb{R}$$

Example. Consider the linear equation below:

$$\begin{cases} 2u + v + w & = 5 \\ 4u - 6v & = -2 \\ -2u + 7u + 2w & = 9 \end{cases}$$

- Row picture



$$(u, v, w) = (1, 1, 2)$$

Lemma 1.2.2. in n-dimension, a line require $(n-1)$ equation.

Question. How to extend into n-dimensions?

Answer. Consider the following steps:

- Each equation represents a plane or hyperplane.
- The first equation produces a $(n-1)$-dimension plane in $\mathbb{R}^n$
- The second equation produces another $(n-1)$-dimension plane in $\mathbb{R}^n$
- Their intersection in smaller set of $(n-2)$-dimension
- $(n-3) \to (n-4) \to \cdots\cdots \to 3 \to 2 \to 1 \to$ point

Then we can find the final intersection. ⊛

- Column picture

$$u \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v \begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \qquad \Longleftarrow \qquad \begin{cases} 2u + v + w & = 5 \\ 4u - 6v & = -2 \\ -2u + 7u + 2w & = 9 \end{cases}$$

RHS is a linear combination of 3 column vectors.

Theorem 1.2.1. Solution to a linear equation:

$$\underbrace{(\textbf{intersection of to points})}_{\text{row pic.}} = \underbrace{(\textbf{coefficient of linear combination})}_{\text{column pic.}}$$

1.2.1 Singular Case

(1) Row Picture: In 3D case, they didn't intersect at a point.

- **Case 1**: two parallel

$$\begin{cases} 2u+v+w &= 5 \\ 4u+2v+2w &= 9 \end{cases}$$

- **Case 2**: three plane perpendicular ($\perp$)

$$\begin{cases} u+v+w &= 2 \ \cdots \ (1) \\ 2u+3w &= 5 \ \cdots \ (2) \\ 3u+v+4w &= 6 \ \cdots \ (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1)+(2)=(3) \quad ; \quad \text{LHS} \Rightarrow (1)+(2) \neq (3)$$

- **Case 2**: three plane have a whole line in common.

$$\begin{cases} u+v+w &= 2 \ \cdots \ (1) \\ 2u+3w &= 5 \ \cdots \ (2) \\ 3u+v+4w &= 7 \ \cdots \ (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1)+(2)=(3) \quad ; \quad \text{LHS} \Rightarrow (1)+(2) = (3)$$

- **Case 4**: three parallel

(2) Column Picture:

$$u\begin{pmatrix}1\\2\\3\end{pmatrix}+v\begin{pmatrix}1\\0\\1\end{pmatrix}+w\begin{pmatrix}1\\3\\4\end{pmatrix}=b$$

In the case above, three vectors are linear combination to each other, i.e. three vectors share the same plane.

Lemma 1.2.3 (Singular case)**.** If the three vectors are linear combination to each other (three vector share a common plane), it must be **singular case**.

- If $b=\begin{pmatrix}2\\5\\7\end{pmatrix}$, which is on the plane $\Rightarrow$ too many solution to produce b.
- If $b=\begin{pmatrix}2\\5\\6\end{pmatrix}$, which is not on the plane $\Rightarrow$ no solution.

1.2.2 Fundamental Linear Algebra Theorem (Geometry form)

Theorem 1.2.2 (Fundamental LA Theorem)**.** Consider a linear system

$$Ax = b, \quad A \in \mathbb{R}^{m\times n},\ x \in \mathbb{R}^n,\ b \in \mathbb{R}^m.$$

If the n hyperplanes have no only one intersection or infinitely many points, then the n columns lie in the same plane. (consistency of *row picture* and *column picture*)

> **Notation.** Logic notation:
>
> - If ..., then : $\Rightarrow$
> - If and only if : $\Leftrightarrow$

Lecture 2

9 Sep. 13:20

1.3 An Example of Gaussian Elimination

Example. Here is a linear equation.

$$\begin{cases} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{cases}$$

$$\begin{pmatrix} \boxed{\mathbf{2}} & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & \boxed{\mathbf{-8}} & -2 & -12 \\ 0 & 8 & 3 & 14 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & \boxed{\mathbf{1}} & 2 \end{pmatrix} \quad \text{"}\textbf{pivot}\text{"}$$

Then we get $w = 2$, we can plug in the equation i.e.

$$\begin{cases} 2u + v + 1w = 5 \\ -8v - 2w = -12 \\ w = 2 \end{cases} \quad \Longrightarrow \quad \text{Forward Elimination}$$

Then we substitute into 2nd, 1st equation to get $v = 1$ and $u = 1 \Longrightarrow$ Backend Elimination

Note. By definition, pivots cannot be zero!

Question. Under what circumstances could the elimination process break down?

Answer. Here are some situations.

- Something **must** go wrong in the singular case.
- Something **might** go wrong in the nonsingular case.

A zero appears in a pivot position!
If in the process, there are nonzero pivots, then there's only one solution. ⊛

Example.

$$\begin{pmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix}$$

(1) If $a_{11} = 0 \implies$ nonsingular

(2) If $a_{22} = 0 \implies$ nonsingular

(3) If $a_{33} = 1 \implies$ singular

Question. How many separate arithmetical operations does elimination require for n equations in n unknowns?

Answer. For a single operation.

$$\text{a single operation} = \text{each division \& each multiplication-subtraction}$$

⊛

- **FE**:

$$\underbrace{\begin{matrix} x & x & \cdots & x & = & x \\ \vdots & \vdots & & & & \vdots \\ x & x & \cdots & x & = & x \end{matrix}}_{n}$$

$$n(n-1) + (n-1)(n-2) + \cdots + (1^2 - 1) = \frac{n^3 - n}{3} \sim \frac{n^3}{3} \text{ steps}$$

- **RHS**:

$$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

- **BF**:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

1.4 Matrix Notation and Matrix Multiplication

$$\begin{cases} 2u + 4v + -2w &= 2 \\ 4u + 9v - 3w &= 8 \\ -2u - 3v + 7w &= 10 \end{cases} \implies u\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v\begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + w\begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

We can rewrite it in the below form.

$$A = \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}_{3\times 3}}_{\text{coefficient matrix}}, \quad x = \underbrace{\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{3\times 1}}_{\text{unknowns}}, \quad b = \underbrace{\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}_{3\times 1}}_{\text{RHS}} \implies x = \underbrace{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}_{3\times 1}}_{\text{solution}}$$

$$\boxed{Ax = b}$$

Definition 1.4.1. An $m \times n$ matrix, $A_{m\times n}$ over $\mathbb{R}$, is an array with m rows and n columns of real numbers, which can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \text{ where } a_{ij} \in \mathbb{R}, \begin{cases} i : \text{index of row} \\ j : \text{index of column} \end{cases}$$

- $\boxed{m \times n}$ is called the **dimensions (size)** of $A \implies$ dimension of a $()_{3\times 5}$ is 3×5
- $\boxed{a_{ij}}$ is called the **elements/entry/coefficient** of A
- **Addition**: $A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{m\times n}$

$$A + B = (a_{ij} + b_{ij})_{m\times n}$$

- **Multiplication**: $A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{n\times r}$

$$AB = (c_{ij})_{m\times r}, \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

- **Scalar Multiplication**:

$$\alpha A = (\alpha a_{ij})_{m\times n}$$

- $$A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$$

  In particular, if

$$A_{1\times n} B_{n\times 1} = \mathbf{v} \cdot \mathbf{w} = ()_{1\times 1}.$$

  Then it's the **inner product** of vector $\mathbf{v}$ and vector $\mathbf{w}$

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & 3 & -7 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\cdot(-1) & 4\cdot(2) & -2\cdot(2) \\ 4\cdot(-1) & 9\cdot(2) & -3\cdot(2) \\ -2\cdot(-1) & 3\cdot(2) & -7\cdot(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 22 \end{pmatrix}$$

$$(-1)\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 9 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$$

(1) by row: 3 inner product

(2) by column: a linear combination of 3 columns of A

Example (1A). Ax is a combination of columns of A

$$\begin{aligned} A_{m\times n}x_{n\times 1} &= \left(A_1|A_2|\cdots|A_n\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1(A_1) + x_2(A_2) + \cdots + x_n(A_n) = \left(\sum_{j=1}^{n} a_{ij}\ x_j\right)_{m\times 1} \end{aligned}$$

1.4.1 The Matrix Form of One Elimination Step

Definition (1B). Matrix form

Definition 1.4.2. zero matrix:

$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Definition 1.4.3. identity matrix:

$$I = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} = I_n = I_{n\times n}; \quad \begin{cases} A_{m\times n}I_n = A_{m\times n} \\ A_{m\times n} = A_{m\times n}I_n \end{cases}$$

Definition 1.4.4. elementary matrix (elimination matrix):

$$E_{ij} = \begin{array}{c}\begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & -\ell & \ddots & 0 \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \begin{array}{l} \\ \\ \\ \text{ith row} \\ \\ \end{array} \\ \text{jth column} \end{array} \qquad \ell : \text{multiplier}$$

$$E_{ij} \cdot A = \begin{pmatrix} \cdots & -\ell & \cdots & 1 \\ & & & \end{pmatrix} \begin{pmatrix} \quad \\ \quad \end{pmatrix} \begin{array}{l} \longleftarrow \text{i-th} \quad \Longrightarrow \quad \text{(i-th row)} + (-\ell)\text{(j-th column)} \\ \longleftarrow \text{j-th} \quad \Longrightarrow \quad \text{create zero at } (i, j) \text{ position!} \end{array}$$

Example.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_{21}} \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 0_{21} & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}_{EA}$$

Note. Here is two properties

1. $Ax = b \quad \Longrightarrow \quad E_{ij}Ax = E_{ij}b$
2. $E_{ij}A \neq AE_{ij}$

1.4.2 Matrix Multiplication

(1) The (i, j)-th entry of AB is the inner product of the **i-th** of A and the **j-th** of B.

(2) Each column of AB is the product of a matrix A and **a column of B**

$$\begin{aligned} \Longrightarrow \quad \text{column } j \text{ of } AB &= A \text{ times } \textbf{j-th} \text{ of } B \\ &= \text{linear combination of } \textbf{columns of } A \\ &= b_{1j} A_{\underbrace{\boxed{\bullet}}_{\text{any numbers}} 1} + b_{2j} A_{\bullet 2} + \cdots + b_{nj} A_{\bullet n} \end{aligned}$$

Example.

$$\underbrace{\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}}_{A_{2\times 3}} \underbrace{\begin{pmatrix} 5 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}}_{B_{3\times 3}} = \underbrace{\begin{pmatrix} 16 & 1 & 1 \\ 8 & 0 & -1 \end{pmatrix}}_{C_{2\times 3}}$$

$$\text{1st column of } AB = \begin{pmatrix} 16 \\ 8 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(3) Each row of AB is a product of a row of A and a matrix B.

$$\begin{aligned}\implies \quad \text{i-th row of } AB = & \text{ of } A \text{ times } B.\\ &= \text{linear combination of } \textbf{rows of } B\\ &= a_{i1}B_{1\bullet} + a_{i2}B_{2\bullet} + \cdots + a_{in}B_{n\bullet}\end{aligned}$$

Theorem 1.4.1. Let A, B and C be matrices (possibly rectangular). Assume that their dimension permit them to be added and multiplied in the following theorem

(1) The matrix multiplication is associative

$$(AB)C = A(BC)$$

(2) Matrix operations are distributive

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

(3) Matrix multiplication is noncommutative

$$AB \neq BA \quad \text{in general}$$

(4) Identity Matrix

$$A_{n\times n}I_n = I_nA_{n\times n} = A_{n\times n}$$

Example.

$$\underset{21}{E} = \begin{pmatrix} 1 & 0 & 0\\ \boxed{-2} & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \underset{31}{F} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ \boxed{1} & 0 & 1\end{pmatrix}, \quad \underset{32}{G} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & \boxed{-1} & 1\end{pmatrix}$$

(1)

$$\underset{21}{E}\,\underset{31}{F} = \begin{pmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} \boxed{=} \underset{31}{F}\,\underset{21}{E} = \begin{pmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$$

(2)

$$\underset{21}{E}\,\underset{32}{G} = \begin{pmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ 0 & -1 & 1\end{pmatrix} \boxed{\neq} \underset{32}{G}\,\underset{21}{E}$$

(3)

$$\underset{32}{G}\,\underset{31}{F}\,\underset{21}{E} = \underbrace{\begin{pmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ 3 & -1 & 1\end{pmatrix}}_{\text{"right order"}} \boxed{\neq} \underset{21}{E}\,\underset{31}{F}\,\underset{32}{G} = \begin{pmatrix} 1 & 0 & 0\\ -2 & 1 & 0\\ 1 & -1 & 1\end{pmatrix}$$

Note. The product of lower triangular matrices is a lower triangular matrix.

Lecture 3

1.5 Triangular Factors and Row Exchanges

16 Sep. 13:20

$$\boxed{Ax = b}$$

$$\implies LUx = b \implies \begin{cases} Lc = b \\ Ux = c \end{cases}$$

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = b$$

Remark. ℓ: multipliers

$$E_{ij}(\ell) : (\text{i-th row}) + (-\ell)(\text{j-th column})$$

$$\begin{pmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{pmatrix} \xrightarrow[R_3+(1)R_1]{R_2+(-2)R_1} \begin{pmatrix} 2 & 4 & -2 & 2 \\ 0_{21} & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{pmatrix} \xrightarrow{R_3+(-1)R_2} \begin{pmatrix} \boxed{2} & 4 & -2 & 2 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0_{32} & \boxed{4} & 8 \end{pmatrix} \text{ pivot}$$

$$E_{21}(2) = E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{31}(-1) = F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad E_{32}(1) = G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

i.e.

$$E_{21}E_{31}E_{32}Ax = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = Ux = c = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} = E_{21}E_{31}E_{32}b$$

Question. How can we undo the steps of Gaussian Elimination?

$$E^{-1}F^{-1}G^{-1}GFEA = A = \underbrace{E^{-1}F^{-1}G^{-1}}\boxed{U} = LU \quad \text{i.e.} \quad A = \underset{\text{factors of } A}{LU}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -(-2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(1) & 0 & 1 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(-1) & 1 \end{pmatrix}$$

$$E^{-1}F^{-1}G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{-1} & \boxed{1} & 1 \end{pmatrix} \implies \text{records everything that has been done so far}$$

1.5.1 Triangular Factorization

Theorem 1.5.1. If no exchanges are required, the original matrix A can be written as

$$A = LU$$

- The matrix L is lower triangular with 1's on the diagonal and the multipliers ℓ_{ij} (taken from elimiation) below the diagonal.
- The matrix U is the upper triangular matrix which appears after forward elimination and before back-substitution; its diagonal entries are the pivots.

Example.

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 6 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \text{提出2}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 6 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Question.

$$A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 2 & 6 & 5 \\ -1 & 4 & -2 \\ 1 & 2 & 3 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

"triangular matrix" 有三條對角線

Answer. ⊛

1.5.2 One Linear System = Two Triangular Systems

$$Ax = b \implies Ux = c \;\&\; Lc = b \implies A = LU$$

Remark. The LU form is unsysttematic in one aspect. U has pivots along its diagonal where L always has 1's.
We can rewrite U as

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12}/u_{11} & \cdots & u_{1n}/u_{11} \\ 0 & 1 & & u_{2n}/u_{22} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example.

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} \boxed{3} & 4 \\ 0 & \boxed{2/3} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2/3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 4/3 \\ 0 & 1 \end{pmatrix}}_{U}$$

Theorem 1.5.2. If

$$A = L_1 D_1 U_1 \text{ and } A = L_2 D_2 U_2$$

then

$$L_1 = L_2, D_1 = D_2, U_1 = U_2$$

i.e. if A has LDU decomposition, then it **is** unique.

1.5.3 Row Excahnge and Permutation Matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \langle \text{Permutation matrix } P_{ij} \rangle$$

Note. Permutation matrix is also an elementary matrix.

Example. Here are some of the example:

1°

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \qquad \boxed{R_2 \leftrightarrow R_3}$$

2°

$$PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{pmatrix} \qquad \boxed{R_2 \leftrightarrow R_3}$$

3°

$$AP = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 5 & 6 \end{pmatrix} \qquad \boxed{C_2 \leftrightarrow C_3}$$

Note. For the permutation matrix:

1° PA: Performing row exchange of A

2° AP: Performing column exchange of A

3° $PAx = Pb$; Should we permute the component of $x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ as well? **NONONONONO!!!**

Example.

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{pmatrix} \qquad Ax = b$$

(1) if $d = 0$, the problem is incurable. The matrix is singular.

(2) if $d \neq 0$, $P_{13}A = \begin{pmatrix} d & e & f \\ 0 & 0 & c \\ 0 & a & b \end{pmatrix}$; if $a \neq 0$, $P_{23}P_{13}A = \begin{pmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{pmatrix}$

$$\left|\begin{array}{cccc} & P_{23}P_{13} & \neq & P_{13}P_{23} \\ & & & \\ & 1 \quad\; 3 \quad\; 3 & & 1 \quad\; 1 \quad\; 2 \\ \text{row} & 2 \to 2 \to 1 & & 2 \to 3 \to 3 \\ & 3 \quad\; 1 \quad\; 2 & & 3 \quad\; 2 \quad\; 1 \end{array}\right|$$

Theorem 1.5.3. We seperate into two cases:

- In the non singular case, there's a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. In this case,
  (1) $Ax = b$ has a unique solution.
  (2) It is found by elimination with row exchange
  (3) With the rows reorders in advance, PA can be factored into LU $\langle PA = LU \rangle$
- In singular case, no reordering can produce a full set of pivots.

Example.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix} \xrightarrow[\cancel{\ell_{31}=2}\ \ell_{21}=2]{\cancel{\ell_{21}=1}\ \ell_{31}=1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{P_{23}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \langle \text{This is WORNG} \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

To summarize: A good code for Gaussian Elimination keeps a record of L, U and P. They allow the solution $(Ax = b)$ from two triangular systems. If the system $Ax = b$ has a unique solution, they we say:

1° The system is nonsingular or

2° The matrix is nonsingular

Otherwise, it is singular.

1.6 Inverse and Transpose

Definition 1.6.1. An $n \times n$ matrix A is invertible if $\exists$ an $n \times n$ matrix $B \ni BA = I = AB$

Theorem 1.6.1. If A is invertible, then the matrix B satisfying $AB = BA = I$ is unique!

Proof. Suppose $\exists c \neq B \ni AC = CA = I$

$$B = BI = B(AC) = (BA)C = IC = C \text{ i.e } B = C$$

we call this matrix B, the $\boxed{\text{inverse of } A}$, and denoted as $\boxed{A^{-1}}$ ■

Note. Not all $n \times n$ matrices have inverse.
e.g.

1°

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

2° if $Ax = \vec{0}$ has a $\overset{x \neq 0}{\underline{\text{nonzero solution}}}$, then A has no inverse!

$$x = A^{-1}(Ax) = A^{-1}\vec{0} = \vec{0} \quad (\rightarrow\leftarrow)$$

Note. The inverse of A^{-1} is A itself. i.e. $(A^{-1})^{-1} = A$.

Note. If $A = (a)_{1\times 1}$ and $a \neq 0$, then $A^{-1} = (\frac{1}{a})$. The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$ is

$$\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if } \det(A) \neq 0$$

Note.

$$A = \begin{pmatrix} d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & d_n \end{pmatrix} \quad d_i \neq 0,\ \forall i \quad \implies A^{-1} = \begin{pmatrix} 1/d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1/d_n \end{pmatrix}$$

Proposition 1.6.1. If A and B are invertible, then

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}\cdots A_2^{-1}A_1^{-1}$

1.6.1 The Calculation of A^{-1}: Gaussian-Jordan Method

$$A \cdot A^{-1} = I$$

$$A_{n\times n}B_{n\times n} = I_n$$

$$\begin{aligned}
&\implies A_{n\times n}(B_1|B_2|\cdots|B_n)_{n\times n} = (e_1|e_2|\cdots|e_n)_{n\times n} \\
&\implies (AB_1|AB_2|\cdots|AB_n)_{n\times n} = (e_1|e_2|\cdots|e_n)_{n\times n} \\
&\implies AB_1 = e_1;\ AB_2 = e_2;\ \cdots;\ AB_n = e_n \quad \longrightarrow \quad n \text{ linear systems:} Ax = b
\end{aligned}$$

Definition 1.6.2 (Gaussian-Jordan Method)**.** Instead of stopping at U and switching to back substitution, it continues by subtracting multipliers of a row from the rows above till it reaches a diagonal matrix. Then we divide each row by corresponding pivot.

$$(\underset{LU}{A}|I) \xrightarrow{\times L^{-1}} (U|L^{-1}) \xrightarrow{\times U^{-1}} (I|A^{-1})$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$$

1.6.2 Invertible = Nonsingular

Question. What kind of matrices are invertible?

Answer. Here are the example:

1° nonzero pivot Ch1 Ch4

2° nonzero determinants Ch4

3° independent columns (rows) Ch2

4° nonzero eigenvalues Ch5

which will in the whole course ⊛

Suppose a matrix A has full set of nonzero pivots. By definition, A is nonsingular and the n systems

$$Ax_1 = e_1,\ Ax_2 = e_2,\ \cdots\ , Ax_n = e_n$$

can be solved by elimination or Gaussian-Jordan Method.

Row exchanges maybe necessary, but the columns of A^{-1} are uniquely determined.

$$\begin{aligned} Ax = b \quad & PAx = Pb \\ & PAx_i = Pe_i \end{aligned}$$

$$\{Pe_1, Pe_2, \cdots, Pe_n\} = \{e_1, e_2, \cdots, e_n\}$$

Note. Compute A^{-1}:

1° $A(x_1|\cdots|x_n) = I = (e_1|\cdots|e_n) \iff Ax_i = e_i,\ i = i\cdots n$

2° Gauss-Jordan Method: $(\ A \mid I\ ) \longrightarrow (\ I \mid A^{-1}\ )$

Question. We have found a matrix $A^{-1} \ni AA^{-1} = I$. But is $A^{-1}A = I$

Answer. We can do this by recall.

> **As previously seen.** Recall that every Gauss-Jordan step is a multiplication of matrices on the left. There are three types of elementary matrices:

1° $\boxed{E_{ij}(\ell)}$: to subtract a multiple ℓ of j row from i row.

2° $\boxed{P_{ij}}$: to exchange row i and j

3° $\boxed{D_i(d)}$: to multiply row i by d i.e. $D_i(d) = \begin{pmatrix} 1 & & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & d & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & & 1 \end{pmatrix}$ $\rightarrow$ ith row

$$\begin{pmatrix} d_1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & d_n \end{pmatrix} = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & d_n \end{pmatrix}$$

$$\implies \quad DEEPEEA = I \quad \implies \quad A^{-1}A = I \therefore \text{ we have a left inverse!}$$

These are the operation of A^{-1} ⊛

Theorem 1.6.2. For nonsingular and invertible:

- Every nonsingular matrix is invertible.
- Every invertible matrix is nonsingular.

Theorem 1.6.3. A square matrix is invertible $\iff$ it is nonsingular

Lecture 4

23 Sep. 13:20

1.7 Transpose A^T

Proposition 1.7.1. Here are the proposition of transpose

- $(A+B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$

Proof. Here is the proof

1° $((A+B)^T)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A^T + B^T)_{ij}$

2° $((AB)^T)_{ij} = (AB)_{ji} = \boxed{\sum_{k=1}^{n} a_{jk}b_{ki}}$ $\quad (B^T A^T)_{ij} = \sum_{\ell=1}^{n} b^T_{i\ell}a^T_{\ell j} = \sum_{\ell=1}^{n} b_{\ell i}a_{j\ell} = \boxed{\sum_{\ell=1}^{n} a_{j\ell}b_{\ell i}}$

3°

■

Definition 1.7.1. A symmetric matrix is a matrix which equals its own transpose. i.e. $A = A^T$

Example.

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \text{ YES } \begin{pmatrix} 5 & 4 \\ 1 & 5 \end{pmatrix} \text{ NO } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ YES}$$

Note. A symmetric matrix is not necessarily invertible. If it is invertible, then its inverse is symmetric.

Theorem 1.7.1. If A is symmetric and if A can be factored as LDU, then $A = LDU^T$

Proof. Here is the proof.

1° $A = A^T$, $A = LDU \Rightarrow A^T = (LDU)^T = U^T D^T L^T = A = LDU$

2° By theorem 1.5.2, the theorem is correct.

LDU is unique if they exist. ■

Chapter 2

Vector Spaces and Linear Equation

2.1 Vector Spaces and Subspace

To answer the basic questions about the $\underset{1^\circ}{\text{existence}}$ and $\underset{2^\circ}{\text{uniquness}}$ of the solution of $Ax = b$, we need the concept of vector space.

$$\text{Field} \implies \text{Vector Space} \implies \text{Solution of } Ax = b$$

Definition 2.1.1 (Field). Let F be a set with two operations "+" and "$\bullet$" i.e.

$$+ : F \times F \longrightarrow F$$
$$\bullet : F \times F \longrightarrow F$$

and $+, \bullet$ are well-defined functions. If the system $\boxed{(F, +, \bullet)}$ satisfies the following conditions, the F is called a $\boxed{\text{Field}}$.
For $a, b, c \in F$

(1) $(a + b) + c = a + (b + c)$

(2) $a + b = b + a$

(3) $\exists\, 0 \in F \ni a + 0 = 0 + a = a$ 單位元素 (1st operation)

(4) $\forall a \in F,\ \exists (-a) \in F \ni a + (-a) = 0$ 反元素 (1st operation)

(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(6) $a \cdot b = b \cdot a$

(7) $\exists\, 1 \in F \ni a \cdot 1 = 1 \cdot a = a$ 單位元素 (2nd operation)

(8) $\forall a \neq 0 \in F,\ \exists\, a^{-1} \in F \ni a \cdot a^{-1} = a^{-1} \cdot a = 1$ 反元素 (2nd operation)

(9) $a \cdot (b + c) = ab + ac$ Distribution Law

Example.

$$\underset{\text{(real)}}{\mathbb{R}} \text{ (YES) } \underset{\text{(rational)}}{\mathbb{Q}} \text{ (YES) } \underset{\text{(integer)}}{\mathbb{Z}} \text{ (NO) } \underset{\text{(complex)}}{\mathbb{C}} \text{ (YES) } \mathbb{N} \text{ (NO)}$$

Definition 2.1.2 (vector space)**.** Let V be a set and F be a field. $\boxed{V \text{ is a vector space over } F}$ if $\underset{1^\circ}{\text{addition}}$ and $\underset{2^\circ}{\text{multiplication}}$ by scalar are defined on V and they satisfy.

$$+ : V \times V \longrightarrow V$$
$$\bullet : F \times V \longrightarrow V$$

(A1) addition is associated

(A2) addition is commutative

(A3) $\exists$ zero vector $\in V \ni 0 + v = v + 0, \ \forall v \in V$

(A4) $\forall v \in V, \ \exists (-v) \in V \ni (-v) + v = 0$

(M1) $1 \cdot v = v, \ v \in V, \ 1 \in F$

(M2) $(\lambda\mu) \cdot v = \lambda(\mu v) \ v \in V, \ \lambda, \mu \in F$

(M3) $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2 \ v_1, v_2 \in V, \ \lambda \in F$

(M4) $(\lambda + \mu)v = \lambda v + \mu v \ v \in V, \ \lambda, \mu \in F$

2.1.1 Algebraic Rules of Vector Algebra

Question. $n \in \mathbb{N}$, $\mathbb{R}^n/\mathbb{R}$ ($\mathbb{R}^n$ over $\mathbb{R}$) is a vector space?

Answer. **YES** ⊛

Example.

$$\mathbb{C}^n/\mathbb{C}, \ \mathbb{C}^n/\mathbb{R}, \ \mathbb{R}/\mathbb{R}$$

Question. $M_{2\times 2}(\mathbb{R})/\mathbb{R}$ is a vector space?

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}$$

Answer. **YES** ⊛

Question. V is a vector space?

$$V = \{\text{all } 3 \times 3 \text{ symmetric matrices over } \mathbb{R}\}$$

Answer. **YES** ⊛

Question. $\mathbb{R}^\infty/\mathbb{R}, \mathbb{R}^\infty = \{(a_1, a_2, \cdots\cdots) \mid a_i \in \mathbb{R}\}$

Answer. **YES** ⊛

Question. Let $V = \{f \mid f$ is a real-valued function defined on $[0,1]\}$ define $(rf)(x) = r \cdot f(x),\ r \in \mathbb{R}$

Answer. **YES**

$$\text{(zero vector)} = \text{(zero function)}$$

i.e. $f(x) = 0, \forall x \in [0,1]$ ⊛

Question. $V = \{\text{all positive } \mathbb{R}\}$

$$\begin{cases} x+y & = xy \\ c \cdot x & = x^c \end{cases}, \quad \text{is } V \text{ a v.s. over } \mathbb{R}$$

Answer. **YES**

1° (A1) $(x+y)+z = x+(y+z)$

2° (A2) $(x+y) = xy = yx = (y+x)$

3° (A3) zero vector: $x + 1 = x$

4° (A4) x + $\frac{1}{x}$ = zero vector = 1

5° (M3) $\lambda(x+y) = (x+y)^\lambda = (xy)^\lambda = x^\lambda y^\lambda = (\lambda x)(\lambda y) = \lambda x + \lambda y$

6° (M4) $(\lambda+\mu) \cdot x = x^{(\lambda+\mu)} = x^\lambda \cdot x^\mu = \lambda x \cdot \mu x = \lambda x + \mu x$

All conditions apply. ⊛

2.1.2 subspace

Definition 2.1.3 (subspace). A $\boxed{\text{subspace } W}$ of a vector space $(V, +, \cdot)$ over F is a nonempty subset of $V \ni (W, +, \cdot)$ itself is a vector space over F. W is a subspace of V over F if and only if W is closed under addition and scalar multiplication.

Question. Does the zero vector belong to subspace?

Answer. **YES**
$W = \{\text{zero vector}\}$ is the smallest possible vector space. ⊛

Remark. If W_1 and W_2 are subspaces of V over F. Then $W_1 \cap W_2 \neq \emptyset$

Note. If W is a subspace of V/F, then we use notation $\boxed{W \leq V}$.

Question. $V = \mathbb{R}^2/\mathbb{R}$ (xy-plane), What are the subspace of V?

Answer. Here are all subspace of V

(i) origin (one point)

(ii) $\mathbb{R}^2/\mathbb{R} \leq V$

(iii) all lines through origin

(iv) ~~2nd quadant~~ (no zero)

There are much more example. ⊛

Question. $V = M_{n\times n}(\mathbb{R})/\mathbb{R}$

$$S = \{n \times n \text{symmetric matrix}\}$$
$$U = \{n \times n \text{upper triangular matrix}\}$$
$$L = \{n \times n \text{lower triangular matrix}\}$$

Answer. **YES, YES, YES** ⊛

Theorem 2.1.1 (). Let V be a vector space over F. A nonempty subset W of V is a subspace of V, if and only if for each pair $x, y \in W$ and $\alpha \in F$:

1° The zero vector $\in W$.

2° $\alpha x + y \in W$

2.1.3 Column Space of A

Example.

$$A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$$

The first concern is to find all attainable r.h.s. vector b. For example:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = u \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + v \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

Theorem 2.1.2. The system is solvable if and only if the vector b can be expressed as a combination of columns of A

Note. The columns of $A_{m\times n}$ are vectors in $\mathbb{R}^m$, the rows of $A_{m\times n}$ are vectors in $\mathbb{R}^n$.

Example. Let $\mathcal{C}(A) = \{\text{all combinations of columns of } A\}$. Then, $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^m/\mathbb{R}$.

Proof. If b and $b' \in \mathcal{C}(A)$, $\exists\, x, x' \ni Ax = b$ & $Ax' = b'$

$$\forall \alpha \in \mathbb{R}, \quad A(\alpha x + x') = A(\alpha x) + A(x') = A\alpha x + Ax' = \alpha b + b' \in \mathcal{C}(A)$$

$\implies \mathcal{C}(A) \le \mathbb{R}^m/\mathbb{R}$ ∎

Definition 2.1.4. $\mathcal{C}(A)$ is called the $\boxed{\text{column space}}$ of A. Thus if $b \in \mathcal{C}(A)$, then $Ax = b$ is solvable.

- $A_{m\times n} = 0 \;\longrightarrow\; \mathcal{C}(A) = 0_{m\times 1}$
- $A_{m\times n} = I_m \;\longrightarrow\; \mathcal{C}(A) = \mathbb{R}^m$

Lecture 5

2.1.4 Nullspace of A

30 Sep. 13:20

Definition 2.1.5. Let $\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$, then $\mathcal{N}(A) \le \mathbb{R}^n/\mathbb{R}$. Then $\mathcal{N}(A)$ is called the $\boxed{\text{null space}}$ of A.

Proof. We proof it with the Theorem 2.1.1

- zero vector is in the $\mathcal{N}(A)$
- $x,\ x' \in \mathcal{N}(A) \;\Rightarrow\; Ax = 0,\ Ax' = 0$

$$A(x + x') = Ax + Ax' = 0 + 0 = 0 \;\Rightarrow\; x + x' \in \mathcal{N}(A)$$

$$A(\alpha x) = \alpha Ax = \alpha \cdot 0 = 0 \;\Rightarrow\; \alpha x \in \mathcal{N}(A), \forall \alpha \in \mathbb{R} \quad \therefore \mathcal{N}(A) \le \mathbb{R}^n/\mathbb{R}$$

∎

Note. The system $Ax = 0$ is called a homogeneous equation. (齊次)

Remark. The solution set of $Ax = b$ is NOT a subspace of $\mathbb{R}^n/\mathbb{R}$

$$x, x' \longrightarrow Ax = b,\ Ax' = b$$

$$A(x + x') = Ax + Ax' = 2b \neq b$$

Example.

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{N}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

Example.

$$\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{N}(A) = \left\{ \begin{pmatrix} t \\ t \\ -t \end{pmatrix}, t \in (-\infty, \infty) \right\}$$

$$\begin{aligned} \mathcal{C}(A) &= \{\text{all combinations of columns of } A\} \\ &= \text{column space of } A \leq \mathbb{R}^m/\mathbb{R} \end{aligned}$$

$$\begin{aligned} \mathcal{N}(A) &= \{x \in \mathbb{R}^n \mid Ax = 0\} \\ &= \text{null space of } A \leq \mathbb{R}^n/\mathbb{R} \end{aligned}$$

2.2 The Solution of m Equations in n Unknows

For $ax = b, \quad a, b, x \in \mathbb{R}$

(i) if $a \neq 0 \ \Rightarrow \ x = \dfrac{b}{a}$, unique

(ii) if $a = 0,\ b = 0 \ \Rightarrow$ infinitely many solutions.

(iii) if $a = 0,\ b = 0 \ \Rightarrow$ no x exists.

Now, consider $Ax = b$, if A is a square, then (i), (ii), (iii) may occur.

(i) A^{-1} exists $\longrightarrow \ x = A^{-1}b$, unique

(ii) A is singular (undetermined case)

(iii) inconsistent case.

With a rectangular matrix A, $x = A^{-1}b$ will never happen!

Definition. Here is the definition of two similar jargon.

Definition 2.2.1 (row echelon matrix). An $m \times n$ matrix R is called a $\boxed{\text{row echelon matrix}}$ if

(i) the nonzero rows come first and the pivots are the first nonzero etries in those rows.

(ii) below each pivot is a column of zeros

(iii) each pivot lies to the right of the pivot in the row above.

e.g.

$$\begin{pmatrix} \circledast & \circledast & \circledast & \circledast & \circledast \\ 0 & \circledast & \circledast & \circledast & 0 \\ 0 & 0 & 0 & \circledast & \circledast \end{pmatrix}$$

Definition 2.2.2 (row-reduced echelon matrix). An $m \times n$ matrix R is called a $\boxed{\text{row-reduced echelon matrix}}$ if

(i) the nonzero rows come first and the pivots are the first nonzero etries in those rows; pivots are normalized to be 1.

(ii) Above & Below each pivot is a column of zeros

(iii) each pivot lies to the right of the pivot in the row above.

e.g.

$$\begin{pmatrix} \boxed{1} & 0 & \circledast & 0 & \circledast \\ 0 & \boxed{1} & \circledast & 0 & \circledast \\ 0 & 0 & 0 & \boxed{1} & \circledast \end{pmatrix}$$

Theorem 2.2.1. To any $m \times n$ matrix A, there exists a permutation matrix P, a lower triangular matrix L with unit diagnal and an $m \times n$ echelon matrix $U \ni PA = LU$

OR

Every $m \times n$ matrix A is row equivalent to a row echelon matrix.

- Case 1. Homogeneous Case. $b_{m\times 1} = 0$

$$Ax = 0$$

We call the component of x , which correspond to columns with pivots the basic variables; and these correspond to columns with pivots the free variables.

$$\begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{cases} \text{basic variables: } u, w \\ \text{free variables: } v, y \end{cases}$$

The basic variables are then expressed in terms of free variables.

$$\begin{cases} 3w + y = 0 \\ u + 3v + 3w + 2y = 0 \end{cases} \implies \begin{cases} w = -\frac{1}{3}y \\ u = -3v - y \end{cases}$$

$$x = \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} -3v - y \\ v \\ -\frac{1}{3}y \\ y \end{pmatrix} = v \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

- $\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is obtain from x by setting $\begin{cases} v = 1 \\ y = 0 \end{cases}$
- $\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix}$ is obtain from x by setting $\begin{cases} v = 0 \\ y = 1 \end{cases}$

Theorem 2.2.2. If a homogeneous system $A_{m\times n}x = 0$ has more $\overset{n}{\text{unknows}}$ than $\overset{m}{\text{equations}}$ (m $<$ n), it has a nontrivial solution.

$$(A_{m\times n}) \longrightarrow (A_{m\times n})$$

at most m pivot, at most m basic variables, at least $(n - m)$ free variables.

Note. The nullspace is a subspace of the same $\boxed{\text{dimension}}$ as the number of $\boxed{\text{free}}$ variables.

- Case 2. Inhomogeneous Case: $b \neq 0$

$$Ax = b \ \rightarrow \ Ux = c \text{ where } c = L^{-1}b$$

$$\begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{pmatrix} \quad \longrightarrow \quad b_3 - 2b_2 + 5b_1 = 0$$

We know that $Ax = b$ is solvable $\Rightarrow \ b \in \mathcal{C}(A)$

  - 1 & 3: basic variables
  - $\mathcal{C}(A) =$ the set of combinations of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ & $\begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix}$

    , which is also $\left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \middle| \ b_3 - 2b_2 + 5b_1 = 0 \right\} \perp \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

Example.

$$b = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} w = 1 - \frac{1}{3}y \\ u = -2 - 3v - y \end{cases}$$

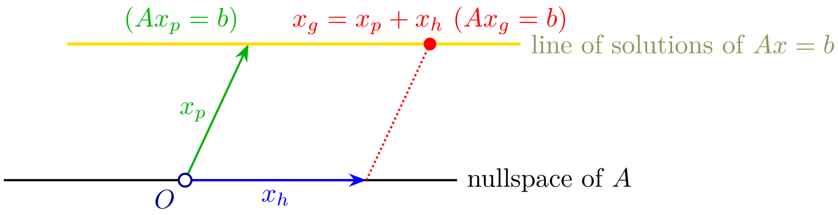
$$x = \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} -2 - 3v - y \\ v \\ 1 - \frac{1}{3}y \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{shift}} + \underbrace{v \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}}_{\text{solution to } Ax=0 \text{ (nullspace)}}$$

Shift: particular solution to $Ax = b$ (set all free variables to be zero)

$$\boxed{x_{\text{general}} = x_{\text{particular}} + x_{\text{homogeneous}}; \ \ x_g = x_p = x_h}$$

Generally, the general solution is fills a two-dimensiona; surface (but NOT a subspace since it doesn't contain the zero vector (origin))

It is paralled to the Nullspace of A



2.2.1 Steps to oobtain the solution to $Ax = b$

(i) Reduce $Ax = b$ to $Ux = c$ to determine basic/free variables.

(ii) Set all free variables to zero to find particular solution, x_p

(iii) set RHS = 0. Give each free variables 1 others 0, in terms, find the hoomogeneous sloution, x_h

$$\Longrightarrow \quad x_g = x_p + x_h$$

Definition 2.2.3 (rank). $\boxed{A_{m\times n}}$ if there are r pivots, there are r basic variables and $n-r$ free variables. The number of pivots, r, is called the $\boxed{\text{rank}}$ of the matrix.

Theorem 2.2.3. Suppose elimination reduce $A_{m\times}x = b$ to $Ux = c$ and there are r pivots and the last $(m-r)$ rows of U are zero. Then there is a solution only if last $(m-r)$ elements of c are zeros.

- If $r = m$, there's always a solution. The general solution is the sum of particular solution and a homogeneous solution.
- If $r = n$, there are No free variables and the null space contains $x = 0$ only. The number r is called the rank of A.

Two extreme case: $A_{m\times n}x = b$

(1) If $r = n \;\rightarrow$ No free variables $\rightarrow \mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} = \{0\}$

(2) If $r = m \;\rightarrow$ No zero rows in $U \;\rightarrow\; \mathcal{C}(A) = \mathbb{R}^m \;\Rightarrow\; \exists$ solution for all b

2.3 Linear Independence, Basis and Dimension

In the elimination process, we refer to the number, r, of pivots as the rank of A. This definition is purely computational rather than mathematical. We shall give a formal definition later.
Now we shall disscuss the following four ideas:

(i) linear independence or dependence

(ii) **spanning** a subspace

(iii) **basis** for a subspace

(iv) **dimension** of a subspace

Definition 2.3.1. Let V be a vector space over F. A nonempty subset S of V is said to ve linearly dependent if there exist distinct vectors $v_1, v_2, \cdots, v_n$ in S and scalar $\alpha_1, \alpha_2, \cdots, \alpha_n$ in F, not all of which are zero $\ni$

$$\alpha_1 v_1 + \alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

A set which is not linearly dependent is called linearly independent. If $S = \{v_1, v_2, \cdots, v_n\}$ then we say that $v_1, v_2, \cdots, v_n$ are linearly dependent/independent.