# MATH-4007 Calculus 2 class 14

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**Description.** Find the following the indefinite integral

$$\int x(\ln x)^2 \, \mathrm{d}x$$

**Correct Ans.**

$$\frac{x^2(\ln x)^2}{2} - x^2 \ln x + \frac{x^2}{4} + C$$

**Reason.** See the following steps:

1° let $u = \ln x$ ($\mathrm{d}x = \frac{1}{x}\mathrm{d}u$), then do the substitution to get

$$\int x(\ln x)^2 \, \mathrm{d}x = \int e^u u^2 (x \, \mathrm{d}u) = \int u^2 e^{2u} \, \mathrm{d}u$$

2° Use integration by parts twice:

| | D | I |
|---|---|---|
| + | $u^2$ | $e^{2u}$ |
| - | $2u$ | $\frac{1}{2}e^{2u}$ |
| + | $2$ | $\frac{1}{4}e^{2u}$ |

we have

$$\begin{aligned}\int u^2 e^{2u} \, \mathrm{d}u &= \frac{1}{2}u^2 e^{2u} - \frac{1}{2}ue^{2u} + \frac{1}{2}\int e^{2u} \, \mathrm{d}u \\ &= \frac{1}{2}u^2 e^{2u} - \frac{1}{2}ue^{2u} + \frac{1}{4}e^{2u} + C \\ &= \frac{x^2(\ln x)^2}{2} - x^2 \ln x + \frac{x^2}{4} + C\end{aligned}$$

**Description.** Find the following the indefinite integral

$$\int \frac{5}{[(ax)^2 - b^2]^{3/2}} \mathrm{d}x$$

**Correct Ans.**

$$-\frac{5}{ab\sqrt{b^2 - a^2x^2}} + C$$

**Reason.** We have to convert it into a standard form

$$\sec^2\theta = 1 + \tan^2\theta$$

So we let $x = \dfrac{b}{a}\sec\theta$ and we get

$$\begin{aligned}
\int \frac{5}{[(ax)^2 + b^2]^{3/2}} \mathrm{d}x &= \int \frac{5}{[a^2(\frac{b}{a}\sec\theta)^2 + b^2]^{3/2}} \cdot \frac{b}{a}\sec\theta\tan\theta \ \mathrm{d}\theta \\
&= \int \frac{5}{[b^2\sec^2\theta + b^2]^{3/2}} \cdot \frac{b}{a}\sec\theta\tan\theta \ \mathrm{d}\theta \\
&= \int \frac{5}{b^3(\sec^2\theta)^{3/2}} \cdot \frac{b}{a}\sec\theta\tan\theta \ \mathrm{d}\theta \\
&= \frac{5}{ab^2}\int \frac{\cos\theta}{\sin^2\theta} \ \mathrm{d}\theta
\end{aligned}$$

then, let $u = \sin\theta$ ($\mathrm{d}\theta = \frac{1}{\cos\theta}\mathrm{d}u$), we have

$$\begin{aligned}
\frac{5}{ab^2}\int \frac{\cos\theta}{\sin^2\theta} \ \mathrm{d}\theta &= \frac{5}{ab^2}\int \frac{\cos\theta}{u^2} \cdot \frac{1}{\cos\theta}\mathrm{d}u \\
&= \frac{5}{ab^2}\int u^{-2}\mathrm{d}u \\
&= \frac{5}{ab^2} \cdot (-u^{-1}) + C \\
&= -\frac{5}{ab^2\sin\theta} + C
\end{aligned}$$

Finally, we have to convert back to $x$:

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{ax}{b}\right)^2} = \frac{\sqrt{b^2 - a^2x^2}}{b}$$

Thus, the final answer is

$$-\frac{5}{ab^2} \cdot \frac{b}{\sqrt{b^2 - a^2x^2}} + C = -\frac{5}{ab\sqrt{b^2 - a^2x^2}} + C$$

**Description.** Find the following the indefinite integral

$$\int -6\ln(x^2-1)\ \mathrm{d}x$$

**Correct Ans.**

$$?$$

**Reason.** Do the factorization first:

$$\int -6\ln(x^2-1)\ \mathrm{d}x = \int -6\ln(x-1)\ \mathrm{d}x + \int -6\ln(x+1)\ \mathrm{d}x$$

Then, we can do the integration by parts for each part:

1° for $\int -6\ln(x-1)\ \mathrm{d}x$:

| | D | I |
|---|---|---|
| + | $-6\ln(x-1)$ | $1$ |
| - | $-\frac{6}{x-1}$ | $x$ |

we have

$$\begin{aligned}\int -6\ln(x-1)\ \mathrm{d}x &= -6x\ln(x-1) + 6\int \frac{x}{x-1}\ \mathrm{d}x \\ &= -6x\ln(x-1) + 6\int \left(1+\frac{1}{x-1}\right)\mathrm{d}x \\ &= -6x\ln(x-1) + 6x + 6\ln|x-1| + C\end{aligned}$$

2° for $\int -6\ln(x+1)\ \mathrm{d}x$:

| | D | I |
|---|---|---|
| + | $-6\ln(x+1)$ | $1$ |
| - | $-\frac{6}{x+1}$ | $x$ |

we have

$$\begin{aligned}\int -6\ln(x+1)\ \mathrm{d}x &= -6x\ln(x+1) + 6\int \frac{x}{x+1}\ \mathrm{d}x \\ &= -6x\ln(x+1) + 6\int \left(1-\frac{1}{x+1}\right)\mathrm{d}x \\ &= -6x\ln(x+1) + 6x - 6\ln|x+1| + C\end{aligned}$$

**Description.** Find the following the indefinite integral

$$\int x^2 \arctan(5x)\ \mathrm{d}x$$

**Correct Ans.**

$$\frac{x^3}{3}\arctan(5x) - \frac{5}{3}\cdot\frac{1}{1250}\left(1 + 25x^2 - \ln|1 + 25x^2|\right) + C$$

**Reason.** Doing the integration by parts follow the LIATE rule:

| | D | I |
|---|---|---|
| + | $\arctan(5x)$ | $x^2$ |
| - | $\frac{5}{1+25x^2}$ | $\frac{x^3}{3}$ |

We get

$$\int x^2 \arctan(5x)\ \mathrm{d}x = \frac{x^3}{3}\arctan(5x) - \frac{5}{3}\int \frac{x^3}{1+25x^2}\ \mathrm{d}x$$

Use substitution $u = 1 + 25x^2$ to solve the remaining integral.

$$\begin{aligned}\int \frac{x^3}{1+25x^2}\ \mathrm{d}x = \int \frac{x^3}{u}\cdot\frac{\mathrm{d}u}{50x} &= \frac{1}{50}\int \frac{(u-1)/25}{u}\mathrm{d}u \\ &= \frac{1}{1250}\left(u - \ln|u|\right) + C\end{aligned}$$

Thus, the final answer is

$$\frac{x^3}{3}\arctan(5x) - \frac{5}{3}\cdot\frac{1}{1250}\left(1 + 25x^2 - \ln|1 + 25x^2|\right) + C$$

**Description.** Let $I_n = \displaystyle\int sec^n(x)\ \mathrm{d}x$. Prove that

$$I_n = \frac{1}{n-1}\tan x \sec^{n-2}(x) + \frac{n-2}{n-1} I_{n-2}$$

**Reason.** First imply the integration by parts:

| | D | I |
|---|---|---|
| + | $\sec^{n-2}(x)$ | $\sec x \tan x$ |
| - | $(n-2)\sec^{n-3}(x)\sec x \tan x$ | $\sec x$ |

Then we get

$$\begin{aligned}
I_n &= \int \sec^{n-2}(x)\ \sec^2(x)\, dx \\
&= \tan(x)\sec^{n-2}(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\, dx \\
&= \tan(x)\sec^{n-2}(x) - (n-2)\int \sec^{n-2}(x)(\sec^2 x - 1)\, dx \\
&= \tan(x)\sec^{n-2}(x) - (n-2)\int \underbrace{\sec^{n}(x)}_{I_n} dx + (n-2)\int \underbrace{\sec^{n-2}(x)}_{I_{n-2}} dx \\
&= \tan(x)\sec^{n-2}(x) - (n-2)I_n + (n-2)I_{n-2}
\end{aligned}$$

Rearranging the equation, we have

$$I_n = \frac{1}{n-1}\tan x \sec^{n-2}(x) + \frac{n-2}{n-1} I_{n-2}$$