Algorithm Design and Analysis

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Abstract

The lecture note of 2025 Fall Algorithm Design and Analysis by professor 呂學一. 希望我可以活著度過這學期～～～～

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Chapter 0

Introduction

Lecture 1

0.1 Design and Analysis

0.1.1 Design

Remark. Find the point to cut into the problem.

Question (Coffee and Milk). 把 500 毫升的咖啡倒入 10 毫升，再從 510 毫升牛奶咖啡取 10 毫升倒入 490 毫升牛奶中，試問兩邊比例？

Answer. 兩邊都固定 500 毫升，一邊少的必定出現在另一邊，**切入點對了根本不用計算** ⊛

0.1.2 Analysis

Question (Card). 把牌洗亂（平均）需要幾次？

Note. **定義何為亂？**
排列出現機率皆為

$$\frac{1}{52!}$$

七次是充分必要條件（嚴謹分析 on paper）n card should shuffle $\frac{3}{2}\log_2 n + \theta$ times. ⊛

Definition 0.1.1 (亂). With n-cards, we have to let the probbability of every combination become

$$\frac{1}{n!}$$

Question (Top-in shuffle). Consider Top-in shuffle with the cards. How to get it "randomly" ?

Answer. Define the k-th section to be 初始底牌從底下數上來是 k-th card.

1. bottom $k-1$ cards must be 亂
2. 每次都可以用 n/k 次將他洗亂，因為出現機率皆為 k/n

We can shuffle $n \cdot H_n$ times. ⊛

Theorem 0.1.1. 底下 $k-1$ 張卡片永遠是亂的

Proof. 考慮 top-in shuffle，利用數學歸納法

- 第一輪要插入底牌下方，只有 1 個空隙，因此必須插入，因此插入的機率是
$$\frac{1}{1!}$$
- 底下如果有 k 張牌，假設下面 k 張是亂的，表示他的排列 $k!$ 種，每種順序機率都是
$$\frac{1}{k!}$$
- 再插入一張，共有 $k+1$ 個空隙，排起來每種順序出現的機率為
$$\frac{1}{(k+1)} \cdot \frac{1}{k!} = \frac{1}{(k+1)!}$$
符合亂的定義

■

第 k 階段插入到下面都是從 n 個空隙裡面找到 k 個空隙插入，因此出現機率必定為 $\frac{k}{n}$，因此需要 shuffle 次數為

$$\frac{n}{k}$$

接著考慮第 n 階段，底牌不是亂的，因此要再洗一次，因此最終的和為

$$\sum_{i=1}^{n} \frac{n}{i} = n \cdot \sum_{i=1}^{n} \frac{1}{i} = n \cdot H_n$$

Note. choose another card to be "bottom", 可以減少第一次的 $1/n$ 就可以少 $n/1$ 次 shuffle. 因此可以把次數減少為：

$$n \cdot H_n - n$$

Remark. **簡單的分析點交換就可以造成巨大的影響**

0.2 Jargons

Definition 0.2.1 (Problems). 「問題」(Problem) 是一個對應關係，就是一個函數

- 演算法核心是在探討問題的解決難易度
- 有些問題確定很難，就不用妄想想出簡單演算法

Definition 0.2.2 (Instance). 「個例」(instance)，也就是問題的合法輸入

Definition 0.2.3 (Computation Model). 「計算模型」(Computation Model)，也就是遊戲規則，同個問題在不同的規則下可能難易度不同

- Comparision base & Computation base

Definition 0.2.4 (Algorithm). 「演算法」Algorithm is a detail step-by-step instruction

- 符合規則
- 詳細步驟

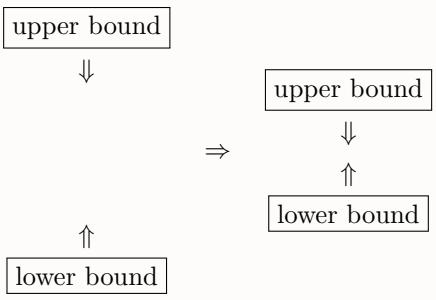
Definition 0.2.5 (Hardness). 「難度」(Hardness)，想知道一個「**問題**」有多難解，用最厲害的一個「**解法**」，對於每個「**個例**」，都至少要用多少「**工夫**」才能解完

- 魔方問題：對於所有解法，存在至少一個初始 instance 讓解法需要 20 次才能轉完，切入點是找到一個固定的初始狀態，這是一個已經最佳化的問題

Theorem 0.2.1 (Confirm Hardness). 用 **upper bound** 和 **lower bound** 去夾起來決定難度

- 當 **upper bound** = **lower bound** 的時候，我們才知道問題的確切難度
- 有些情況，就算夾起來也不一定可以確定難度

Proof.



■

Note. 我們在這門課都討論 worst case

Chapter 1

Complexity for a Problem

Lecture 2

1.1 函數成長率 (Rate of Growth)

Question (棋癡國王與文武大臣). 國王愛下棋，文武大臣要獎賞

- 武大臣每下一個棋子，獎賞多一袋米，起始為一袋米
- 文大臣每下一個棋子，獎賞雙倍，起始為一粒米

Answer. 棋盤 64 格

- 武大臣：n 袋米
- 文大臣：2^n 粒米

2^n 的成長率遠遠高於 n，單位的影響不及成長率 ⊛

1.2 成長率的比較

Note. 雖然 200 年前就有 Asymptotic Notation 的概念，但直到 1970 年代才被演算法分析之父 Donald Ervin Knuth 正式定義到 CS 領域內。

Question (Why Asymptotic Notation). 為什麼要用 Asymptotic Notation ?

Answer. 問題難度通常單位不一致

- $n = 3$ 魔方問題要 20 轉
- n 個信封的老大問題要 $n - 1$ 次比較

兩者難度無法比較 ⊛

Definition 1.2.1 (Rate of Growth). 沒有人有明確定義，但是成長率很好比較，有很多東西也是無法定義但可以比較，e.g. 無限集合可以比大小。

1.3 Big Oh Notation

Definition 1.3.1 (Big Oh Notation). For functions $f, g : \mathbb{N} \to \mathbb{R}$, we write

$$f(n) = O(g(n))$$

to satisfy the extistence of positive constants c and n_0 such that the inequality

$$0 \leq f(n) \leq c \cdot g(n)$$

holds for all integer $n \geq n_0$.

Note. $f(n), g(n)$ should be non-negative for sufficiently large n.

The definition of

$$f(n) = O(g(n))$$

says that there exist a positive constant c such that the value of $f(n)$ is upper-bounded by $c \cdot g(n)$ for all sufficiently large positive n.

Remark. 因此 $O(g(n))$ 可以理解成一個成長率不高過 g 的函數所成的集合

1.3.1 等號左邊也有 Big-Oh

Definition 1.3.2. The equality $O(g(n)) = O(h(n))$ signifies that

$$f(n) = O(h(n))$$

holds for all functions $f(n)$ with

$$f(n) = O(g(n))$$

i.e. $O(g(n)) = O(h(n))$ signifies that $f(n) = O(g(n))$ implies $f(n) = O(h(n))$.

The equality $=$ in $O(g(n)) = O(h(n))$ is more like $\subseteq$, i.e., $O(g(n)) \subseteq O(h(n))$.

Theorem 1.3.1. $O(g(n)) = O(h(n))$ if and only if $g(n) = O(h(n))$.

Proof. Consider the two directions separately.

- For the ($\Rightarrow$) case: We can easily proof that

  $$g(n) = O(g(n))$$

  then we can deduce that

  $$g(n) = O(g(n)) = O(h(n))$$

- For the ($\Leftarrow$) case:

> **As previously seen** (Definition 1.3.1).
>
> $$g(n) = O(h(n)) \quad \Rightarrow \quad \exists\, c_1, n_1 > 0,\ \forall n \geq n_1,\ 0 \leq g(n) \leq c_1 \cdot h(n)$$

Let f be the function such that $f(n) = O(g(n))$. Then, by definition, we can deduce that

$$\exists\, c_2, n_2 > 0,\ \forall n \geq n_2,\ 0 \leq f(n) \leq c_2 \cdot g(n).$$

Assume $n \geq \max\{n_1, n_2\}$. Then, we have

$$0 \leq f(n) \leq c_2 \cdot g(n) \leq c_2 \cdot (c_1 \cdot h(n)) = (c_1 c_2) \cdot h(n).$$

Thus, we can conclude that

$$f(n) = O(g(n)) = O(h(n))$$

Hence,

$$O(g(n)) = O(h(n)) \Leftrightarrow g(n) = O(h(n)).$$

■

1.4 Big-Oh 的運算

Question. 所以，Big-Oh 相加的意思是什麼？

Definition 1.4.1 (Big-Oh Addition). The equality

$$O(g_1(n)) + O(g_2(n)) = O(h(n))$$

signifies that the equality

$$f_1(n) + f_2(n) = O(h(n))$$

holds for any functions $f_1(n)$ and $f_1(n)$ with

$$f_1(n) = O(g_1(n))$$

$$f_2(n) = O(g_2(n)).$$

That is, $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$ together imply $f_1(n) + f_2(n) = O(h(n))$.

Remark. 雖然 $O(g_1(n)) + O(g_2(n))$ 看起來像是兩個集合的聯集，但相同集合想法**無法帶到減乘除**。

Definition 1.4.2 (Big-Oh $\circ$). The equality

$$O(g_1(n)) \circ O(g_2(n)) = O(h(n))$$

$$g_1(n) \circ O(g_2(n)) = O(h(n))$$

集合的復合操作

Notation.

$$\{f_1(n) \circ f_2(n) \mid f_1(n) \in S_1 \text{ and } f_2(n) \in S_2\}$$

可以被理解成

- 把 $=$ 解成 $\subseteq$
- 把 $g_1(n)$ 理解成 $\{g_1\}$
- $O(g1(n))$ 解為成長率不超過 g_1 的成長率的所有函數所組成的集合

Remark. 減乘除應被理解成與剛剛加法類似的模式，而無法被理解為集合的運算

Definition 1.4.3 (Big-Oh $-$, $\cdot$, $/$)**.** (Take $-$ as the example) The equality

$$O(g_1(n)) - O(g_2(n)) = O(h(n))$$

signifies the equality

$$f_1(n) - f_2(n) = O(h(n))$$

holds for any functions $f_1(n)$ and $f_2(n)$ with

$$f_1(n) = O(g_1(n))$$

$$f_2(n) = O(g_2(n))$$

Question. Proof or disproof:

$$O(n)^{O(\log_2 n)} = O(2^n)$$

Answer. First, we take log on both sides:

$$\text{LHS} = O(\log n) \cdot O(\log n) = (O(\log n))^2$$

$$\text{RHS} = O(n)$$

LHS grows slower than RHS, therefore the original statement is true. ⊛

Remark. log 的底數不影響成長率，因此可忽略。

Definition 1.4.4 (Big-Oh 套 Big-Oh)**.** The equality

$$O(O(g(n))) = O(h(n))$$

signifies that the equality

$$O(f(n)) = O(h(n))$$

holds for any function f with

$$f(n) = O(g(n))$$

i.e. $f(n) = O(g(n))$ implies $O(f(n)) = O(h(n))$.

Theorem 1.4.1. $g(n) = O(h(n))$ if and only if $O(O(g(n))) = O(h(n))$

Proof. Consider the two directions separately.

- For the ($\Rightarrow$) case:

  > **As previously seen** (Definition 1.3.1).
  >
  > $$g(n) = O(h(n)) \implies \exists\, c_0, n_0 > 0,\ \forall n \geq n_0,\ 0 \leq g(n) \leq c_0 \cdot h(n)$$

  $f(n) = O(O(g(n)))$ signifies that for $c_1, c_2, n_1, n_2 > 0$

  $$\forall n \geq n_1,\ 0 \leq f(n) \leq c_2 \cdot u(n); \quad \forall n \geq n_2,\ 0 \leq u(n) \leq c_1 \cdot g(n)$$

  Get all together, we have

  $$0 \leq f(n) \leq c_2 \cdot (c_1 \cdot g(n)) \leq c_2 c_1 c_0 \cdot h(n) \quad \implies \quad f(n) = O(h(n))$$

  Thus, we can conclude that

  $$O(O(g(n))) = O(h(n))$$

- We can easily proof that

  $$g(n) \subseteq O(g(n)) \subseteq O(O(g(n)))$$

  Then we can get

  $$g(n) = O(O(g(n))) = O(h(n))$$

Hence, $g(n) = O(h(n)) \Leftrightarrow O(O(g(n))) = O(h(n))$ ■

Theorem 1.4.2 (Rules of Computation in Big-Oh)**.** The following statements hold for functions $f, g : \mathbb{N} \to \mathbb{R}$ such that there is a constant n_0 such that $f(n)$ and $g(n)$ for any integer $n \geq n_0$:

- **Rule 1**: $f(n) = O(f(n))$.
- **Rule 2**: If c is a positive constant, then $c \cdot f(n) = O(f(n))$.
- **Rule 3**: $f(n) = O(g(n))$ if and only if $O(f(n)) = O(g(n))$.
- **Rule 4**: $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$.
- **Rule 5**: $O(f(n) \cdot g(n)) = f(n) \cdot O(g(n))$

Proof. For **Rule 5**: By the Definition 1.3.1, $u(n) = O(f(n) \cdot g(n))$ signifies that there exist positive constants c_1 and n_1 such that the inequality

$$\exists\, c_0, n_0 > 0,\ \forall n \geq n_0,\ 0 \leq u(n) \leq c_0 \cdot f(n) \cdot g(n)$$

the definition of $u(n) = f(n) \cdot O(g(n))$ is

$$\exists\, c_1, n_1 > 0,\ \forall n \geq n_1,\ 0 \leq u(n) \leq f(n) \cdot c_1 \cdot g(n)$$

which are equivilence to each other. ■

1.5 More Asymptotic Notation

Definition 1.5.1 (Little-oh). For any function $f, g : \mathbb{N} \to \mathbb{R}$, we write

$$f(n) = o(g(n))$$

to signify that for any constant $c > 0$, there is a positive constant $n_0(c)$ such that

$$0 \leq f(n) < c \cdot g(n)$$

holds for each integer $n \geq n_0(c)$

> **Note.** $n_0(c)$ is a function of c. When we $n_0(c)$ is a constant, we means that it does not depend on n.

白話來說 $f(n) = o(g(n))$ 的定義是說，不管是多小的常數 c，要 n 夠大 (i.e., $n \geq n_0(c)$)，

$$0 \leq f(n) < c \cdot g(n)$$

都還是成立。

Example.

$$n = o(n^2)$$

Observe that for any positive constant c, as long as $n > \frac{1}{c}$, we have

$$0 \leq n < c \cdot n^2$$

Therefore, we may let $n_0(c) = \frac{1}{c} + 1$ and have $n = o(n^2)$ proved.

Definition 1.5.2 (Other notation). The other notation can be defined via O and o notation:

- We write $f(n) = \Omega(g(n))$ if
$$g(n) = O(f(n)).$$
- We write $f(n) = \Theta(g(n))$ if
$$f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$$
- We write $f(n) = \omega(g(n))$ if
$$g(n) = o(n)$$

Limit notation 可以幫我們判斷各種 Asymptotic Notation:

- If
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$
, the we can guess $f(n) = o(g(n))$.
- If
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$$
, the we can guess $f(n) = \Theta(g(n))$.

- If
$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=\infty$$
, the we can guess $f(n)=\omega(g(n))$.

然而，極限不一定應可以推至 Asymptotic Notation:

- Let $f(n)=g(n)=(-1)^n$. We have
$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=1,$$
but $f(n)\neq O(g(n))$, $f(n)\neq\Omega(g(n))$, and $f(n)\neq\Theta(g(n))$.
- Let $f(n)=(-1)^n$ and $g(n)=n\cdot(-1)^n$. We have
$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=0,$$
but $f(n)\neq o(g(n))$.
- Let $f(n)=2+(-1)^n$ and $g(n)=2-(-1)^n$. We have
$$f(n)=\Theta(g(n)),$$
but $\lim_{n\to\infty}\frac{f(n)}{g(n)}$ does not exist.

Question. Can we just use $\leq$ instead of $<$ in the definition of o ?

Answer. In most part of it will be right. However there will be a special situation:

$$o(0)=0$$

which is definetly wrong. ⊛

Question. 為何不都用 $\exists\, c_0, n_0$ 或都用 $\forall c, n_0(c)$?

Answer. 如果都用 $\exists\, c_0, n_0$，那 o 就會退化，變成 O 而已，並且 $<, \leq$ 是沒有太大差別的

Proof. Suppose that $\hat{n}_0(c)$ is the constant ensured by the $\leq$-version. We simply let

$$n_0(c)=\max(m_0,\hat{n}_0(c/2)).$$

As a result, for any positive constant c, if $n\geq n_0(c)$, we have $g(n)>0$ and thus

$$\begin{aligned}0<f(n)&\leq\frac{c}{2}\cdot g(n)\\&<c\cdot g(n).\end{aligned}$$

■

證畢，由此可知符號並無太大影響，不可讓 o 退化 ⊛

Lecture 3

1.6 問題的難度

如果，

- P 不比 Q 難且
- Q 不比 P 簡單

那兩個問題的難度相同（兩者等價）

Definition 1.6.1. We say that the (worst-case) time complexity of Problem P is $\Theta(f(n))$ if

- the time complexity of Problem P is $O(f(n))$, i.e.

  there exists an $O(f(n))$-time algorithm that solves Problem P

- the time complexity of Problem P is $\Omega(f(n))$, i.e.

  any algorithm that solves Problem P requires $\Omega(f(n))$ time (in the worst case).

  對於任何演算法，只要存在一組 instance 可以達成，一組 $\Omega(f(n))$ 即可推出

Note. 若沒有特別說，n 代表的是 input(instance) size，儲存資料所需的容量

Note. 「正確的演算法」就是對於所有合法輸入都可以對應出正確的輸出，的解決問題方法

1.7 演算法複雜度比較

$$f(n)=O(g(n)):\begin{cases}O(f(n))=O(g(n))\\ o(f(n))=O(g(n))\\ \Theta(f(n))=O(g(n))\end{cases}$$

$$f(n)=\Omega(g(n)):\begin{cases}\Omega(f(n))=\Omega(g(n))\\ \omega(f(n))=\Omega(g(n))\\ \Theta(f(n))=\Omega(g(n))\end{cases}$$

$$f(n)=\Theta(g(n)):\begin{cases}\Theta(f(n))=\Theta(g(n))\end{cases}$$

$$f(n)=o(g(n)):\begin{cases}O(f(n))=o(g(n))\\ o(f(n))=o(g(n))\\ \Theta(f(n))=o(g(n))\end{cases}$$

$$f(n)=\omega(g(n)):\begin{cases}\Omega(f(n))=\omega(g(n))\\ \omega(f(n))=\omega(g(n))\\ \Theta(f(n))=\omega(g(n))\end{cases}$$

Comparing Algorithm A and B, We say that Algorithm A is no worse than Algorithm B in terms of worst-case time complexity if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that

- Algorithm A runs in time $O(f(n))$
- Algorithm B runs in time $\Omega(f(n))$ (in the worst case)

> **Remark.** 第一句 Big-Oh 並沒有出現「in the worst case」是因為我們在此處分析的是「**worst case complexity**」，所以其實在 lower bound 分析的時和通常也不說。

Comparing Algorithm A and B, We say that Algorithm A is strictly better than Algorithm B in terms of worst-case time complexity if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that

- Algorithm A runs in time $O(f(n))$
- Algorithm B runs in time $\omega(f(n))$ (in the worst case)

or

- Algorithm A runs in time $o(f(n))$
- Algorithm B runs in time $\Omega(f(n))$ (in the worst case)

1.8 分析演算法複雜度下界

儘管有些 case 可以，但 Big-Omega 不可以跟 Big-Oh 一樣分析（多增加）

> **Remark.** **Ω-time 必須要一組一組 instance 分析**

1.9 問題上下界 vs 演算法上下界

- 一個問題 P 的任何正確演算法 A 的複雜度上界都是問題 $O(f(n))$ 都是問題 P 的複雜度上界
- 一個問題 P 的複雜度下界 $\Omega(f(n))$ 都是 P 的任何正確演算法 A 的複雜度下界

Chapter 2

演算法的設計與分析

2.1 Half Sorted

Definition 2.1.1 (Half Sorting Problem). An n-element array A is half-sorted if

$$A[i] \leq A\left[\left\lfloor \frac{i}{2} \right\rfloor\right]$$

holds for each index i with $2 \leq i \leq n$.

Half-sorting Problem:

- Input:

  An array A of n distinct numbers.

- Output:

  A half-sorted array that is reordered from A.

Note. 正確的輸出未必唯一，因此輸入輸出就不是一個函數，而是一個「relation」

2.1.1 排序法 Sorting method

Theorem 2.1.1. 歸約 Reduction（問題重整），把問題的難度如果問題 P 可以「多項式時間歸約」成問題 Q，就寫作

$$P \leq_p Q$$

意思是：只要能解決問題 Q，就能透過快速轉換來解決問題 P，所以：

- 如果 Q 是容易的（有快速演算法），那麼 P 也會是容易的。
- 如果 P 已知很難，那麼 Q 至少也不會比較容易。

Note. 把問題的性質變強，便可以順便證明性質較弱的問題

因此，我們知道用排序法一定可以解決半排法，我們可以把半排問題「歸約」到「排序」問題，因此我們首先分析一下快速排序法：

Listing 2.1: Quicksort in Python

```python
def qsort(A, l, r):
    if l >= r:
        return
    (i, j, k) = (l, r, A[l])

    while i != j:
        while A[j] > k and i < j:
            j -= 1
        while A[i] <= k and i < j:
            i += 1
        if i < j:
            (A[i], A[j]) = (A[j], A[i])

    (A[l], A[i]) = (A[i], k)

    qsort(A, l, i-1)
    qsort(A, i+1, r)
```

我們必須分析他的正確性及複雜度

Theorem 2.1.2. The function `qsort()` is correct.

Proof. First, we know that every round of `qsort()` will let the array become:

$$A[l \,..\, p-1] < A[p] < A[p+1 \,..\, r] \quad A[p] = \text{pivot}$$

(How to proof)

Let m be the number of elements in the array. By the induction, we can start with

- Case $m = 1$: The array is well sorted.
- Case $\forall t \leq m \to (m+1)$: Every round of iteration we can get a p such that

  $$\forall x \in A[l \,..\, p-1],\ x \leq A[p], \quad \forall y \in A[p+1 \,..\, r],\ y \geq A[p]$$

  We assume that array with length equal to t, $\forall t \leq m$, has been sorted. Then we can know that that `qsort(A, l, p-1)`, `qsort(A, p+1, r)` is well sorted. Thus, we can combined $A[l \,..\, p-1]$, $A[p]$, $A[p+1 \,..\, r]$ to get a well-sorted array $A[l \,..\, r]$ with length m.

Hence, by induction, `qsort()` is correct. ■

Then, we can stat to analyze the time complexity (worst case):

2.1.2 順調法

Definition 2.1.2 (順調法).

為了方便觀察我們可以將這個陣列化成樹的形式（不是真的改變資料結構）

- Each $A[i]$-to-root path is increasing

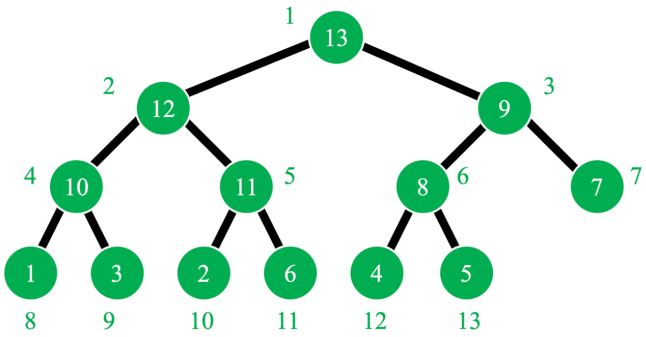


Figure 2.1: Display with Tree structure

2.1.3 逆調法

Lecture 4

2.2 Sorting Problem

Note. quick sort

Note. half sort sort

2.2.1 排序問題下界

解決了排序問題的下界就可以一次解決

- The (worst-case) time complexity of the comparison-based sorting problem is $\Omega(n \log n)$.
- The $O(n \log n)$-time analysis for the Half-Sort-Sort algorithm is tight.
- Learning Reduction

Definition 2.2.1 (Permutation Problem). For the instance

- Input: An array A of n distinct integers.
- Output: Reorder the n-index array $B = [1, 2, \cdots, n]$ such that

$$A[B[1]] < A[B[2]] < \cdots < A[B[n]].$$

排列難度 $\leq$ 排序難度。If the comparison-based sorting problem can be solved in $O(f(n))$ time, then so can the comparison-based permutation problem.

2.3 Amortized Analysis

Chapter 3

Advanced Analysis Techniques

Lecture 5

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3.1 Greedy Algorithm

3.2 Devide and Conquer

3.3 Dynamic Programming

Chapter 4

Graph Theory: Path and Shortest Path Problems

Lecture 8

Definition 4.0.1 (path). Let G be an n-vertex m-weighted directed graph with weight w (which can be positive, negative or zero). The weight of a path P of G is defined as

$$w(P) = \sum_{xy \in E(P)} w(xy)$$

For vertices u and v of G, we call a path of G from u to v a **uv-path** of G.

Definition 4.0.2 (distance). For vertices u and v of G, the **distance** from u to v in G, denoted by $d_G(u, v)$, is defined as

$$d_G(u,v) = \begin{cases} \infty & \text{if there is no uv-path in } G \\ w(P) & \forall Q \in \text{uv-paths},\ w(P) \le w(Q) \\ -\infty & \text{otherwise} \end{cases}$$

Comment (1). 這個 P 就叫做 **shortest** uv**-path**

Comment (2). 真正的 path 是不允許重複經過點的。這裡定義的 path 其實在真正的 graph theory 裡面叫做 **walk**。$V(P)$, $E(P)$ 都是 multiset。

4.1 Single-Source Shortest Path Problem

Problem 4.1.1 (Single-Source Distance Problem). Given

- Input: a directed graph G with edge weights $w : E(G) \to \mathbb{R}$ and a **source** vertex $r \in V(G)$.
- Output: $d_G(r, v)$ for all vertices $v \in V(G)$.

Note. 我們可以用下面這個問題可以規約 (reduce) 到上面的問題

Problem 4.1.2 (Single-Source Shortest Path Problem)**.** Given

- Input: a directed graph G with edge weights $w : E(G) \to \mathbb{R}$ and a **source** vertex $r \in V(G)$.
- Output: a (shortest-path) tree T of G rooted at r such that if G contains a shortest rv-path of G, then rv-path of T is a shortest rv-path of G.

所以我們應該要解決 Single-Source Distance Problem，先做兩個假設 $m = \Omega(n)$

Comment (1)**.** 我們可以用 DFS 先處理掉 r 無法到達的點，所以可以假設

$$d_G(r, v) < \infty, \ \forall v \in V(G)$$

Comment (2)**.** r 固定，簡寫 $d(v) := d_G(r, v)$

4.1.1 Bellman-Ford Algorithm

Algorithm. For each vertex $v \in V(G)$, we use $d[v]$ to estimate $d(v)$.

- Initialization
$$d[i] = \begin{cases} 0 & i = r \\ \infty & \text{otherwise} \end{cases}$$
- Repeat $n - 1$ times relaxation step: for each edge $uv \in E(G)$, 更新
$$d[v] = \min\{d[v], \ d[u] + w(uv)\}$$
- Relaxation 結束後，For each edge $uv \in E(G)$, if $d[v] > d[u] + w(uv)$, then
$$d[v] = -\infty$$
- For each vertex $v \in V(G)$, 如果他可以被任何 u which $d[u] = -\infty$ reach（DFS $O(m + n)$），則
$$d[v] = -\infty$$

Note. The running time is $O(mn)$.

Proof. 我們先做一些觀察

Observation (1)**.** 在 $n - 1$ 次 relaxation 後，$\forall v \in V(G)$, $d[v] \geq d(v)$，永遠不會小於真正的 $d(v)$.

Observation (2). If P is a shortest rs-path of G for some $s \in V(G)$,

- 在這條 rs-path 上的每一個 v of P，這條 rv-path 也會是 shortest rv-path of G.
- 對於每一個 edge uv of P, if 在先前的 relaxation step 後，會有

$$d[u] = d(u)$$

在這次 relaxation step 後，會有

$$d[v] = d(v)$$

現在我們來證明 Bellman-Ford Algorithm 的正確性。我們分三種情況討論，Case 1 已經在之前就證明可以用 DFS 處理掉了。

- Case 2: $d(v) \neq -\infty$. Let P be a shortest rv-path of G. for each vertex u_j of P, where $j = 0, 1, \ldots, |V(P)| - 1$, $u_0 = r$ and $u_{|V(P)|-1} = v$. 根據我們的 Obs.2，我們知道在第 i 次 relaxation step 後

$$d[u_j] = d(u_j) \quad \forall j \in \{0, \cdots, \min(i, |V(P)| - 1)\}$$

- Case 3: $d(v) = -\infty$: 因為到達不了的點一經被處理掉了，因此必定存在 rv-path P of G, which contain a cycle C such that $w(C) < 0$. 所以我們可以

  Claim. At the end of n-th round,

  $$d[u] = -\infty \quad \forall u \in V(C)$$

  By $u \in V(P)$, we have $d[v] = -\infty$ at the end.

  To prove this claim, we assume for contradiction. The n-th of round 並沒有成功更新 $d[u]$ for all $u \in V(C)$. 我們嘗試對每一個邊做 relaxation 都應該失敗。Thus,

  $$d[x] + w(xy) \geq d[y] \quad \forall xy \in E(C)$$

  把 C 上所有這種 inequality 全部加起來，我們會得到

  $$\sum_{xy \in E(C)} w(xy) \geq 0$$

  contradiction to $w(C) < 0$.

Hence, Bellman-Ford Algorithm is correct. ■

4.1.2 Lawler's Algorithm

Remark. 針對 Acyclic Graph 的 Algorithm，Since the input graph has no cycle, it has no negative cycle.

Algorithm. 只需要 One Relaxation Step 就可以了

- 用 $O(m+n)$ 做一次 Topological Sort on the input directed acyclic graph G to get a topological order u_i, $\forall i \in \{1, \ldots, n\}$

- Initialization

$$d[u_i] = \begin{cases} 0 & i = 0 (d[r] = 0) \\ \infty & \text{otherwise} \end{cases}$$

- For i from 1 to n, we do relaxation step for each $u_i v$

$$d[v] = \min\{d[v],\ d[u_i] + w(u_i v)\}$$

Note. The running time is $O(m + n)$.

Proof. 因為這是一個 DAG，所以做完一次 Topological Sort 後，我們就可以知道每個點的 outgoing edge 順序，也就是知道他們在 shortest path 裡面的順序，因此即便我們不知道這條 $u_i v$-path 在哪裡，但我們可以保證在我們處理到 u_i 的時候，$d[u_i]$ 已經是正確的 $d(u_i)$ 了。因此我們只需要做一次 relaxation step 就可以了。 ■

4.1.3 Dijkstra's algorithm

Remark. 本質上是針對 Non-Negative Weighted Graph 的 Greedy Algorithm，可以做更近一步的簡化，Since the input graph has no negative edge, it has no negative cycle.

Algorithm. One round of estimate improvement suffices, although we cannot rely on topological sort (since G may contain cycles).

- Initialization

$$d[v] = \begin{cases} 0 & v = r \\ \infty & \text{otherwise} \end{cases}$$

- 有 n 次 iteration，每次 iteration 從還沒被處理過的點中選出 $d[u]$ 最小的點 u，並對 u 的每一個 outgoing edge uv 做 relaxation

Note. The running time is $O(m + n \log n)$.

Proof. Let's prove the correctness by contradiction.

1° Let v be the first vertex selected in S with $d[v] \neq d(v)$. We have

$$d[v] > d(v)$$

接下來我們考慮 v 被加入 S 的情況

2° Let P be the shortest rv-path of G.

3° Let xy be an arbitrary edge of P such that $x \in S$ and $y \notin S$. (必定存在因為 $r \in S$ 而 $v \notin S$)

4° 因為我們正要處理 v，所以 xy is already processed, we have

$$d[y] = d(y),\ y \neq v$$

5° Since G are nonnegative and y precedes v in P, we have

$$d[y] \leq d(v)$$

6° By 1°, 5°, and 4° we have

$$d[v] > d(v) \geq d[y] = d(y)$$

which contradicts the selection of v. 我們就不該選到 v 因為他並不是最小的那個，還有一個 y 更小

Hence, Dijkstra's algorithm is correct. ■

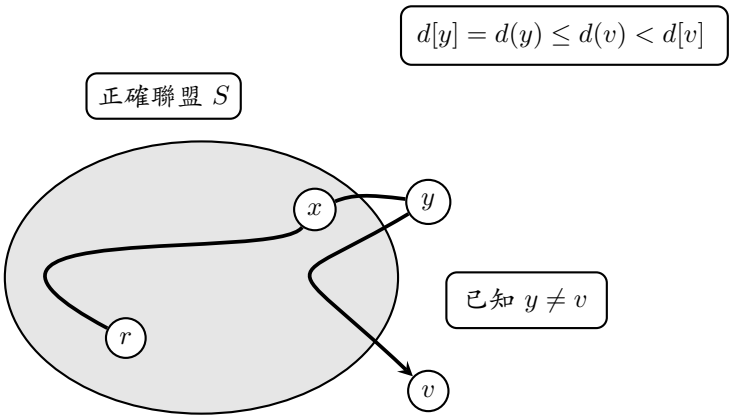


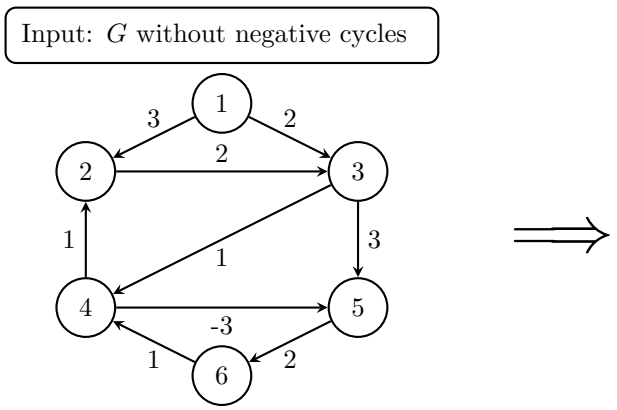
Figure 4.1: Dijkstra's Algorithm Correctness

Lecture 9

4.2 All-Pairs Shortest Path Problem

Problem 4.2.1 (All-Pairs Distance Problem)**.** Given

- Input: an edge-weighted directed graph G with $V(G) = \{1, 2, \cdots, n\}$ edge weights $w : E(G) \to \mathbb{R}^+$, without negative cycles.
- Output: $d_G(i, j)$ for all $i, j \in V(G)$.



Output: $d_G(i, j)$ for all $i, j \in V(G)$

d	1	2	3	4	5	6
1	0	3	2	3	0	2
2	∞	0	2	1	-2	0
3	∞	∞	0	1	-2	0
4	∞	∞	∞	0	-3	-1
5	∞	∞	∞	∞	0	2
6	∞	∞	0	∞	∞	0

Figure 4.2: All-Pairs Distance Problem Example

Algorithm (Naive Solution). Sloving the single-source shortest path problem for each vertex using Dijkstra, Lawler, or Bellman-Ford algorithm.

4.2.1 A Naive DP Solution

Definition 4.2.1. Let $w_k(i,j)$ be the length of the shortest ij-path in G having at most k edges. It will be ∞ if no such path exists.

$$\begin{cases} w_1(i,j) & = w(ij) \\ w_{n-1}(i,j) & = d_G(i,j) \end{cases}$$

Algorithm. Use the recurrence relation for $w_k(i,j)$ is

$$\begin{cases} w_1(i,j) & = w(ij) \\ w_{2k}(i,j) & = \min\limits_{1\le t\le n} (w_k(i,t) + w_k(t,j)) \end{cases}$$

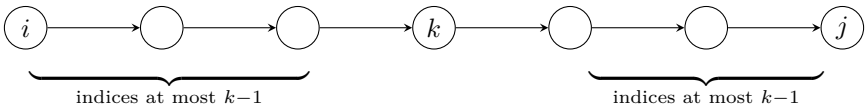
- For each (i,j,k), take $O(n)$ time to compute $w_{2k}(i,j)$ from $w_k(i,j)$.
- For each k, there are n^2 pairs of (i,j), using $O(n^3)$ time to compute all w_{2k} from all w_k.
- It take $O(\log n)$ iterations to compute $d_G = w_{n-1}$ from w_1.

Note. The running time is $O(n^3 \log n)$.

4.2.2 Floyd and Warshall's DP algorithm

Definition 4.2.2. Let $d_k(i,j)$ be the length of the shortest ij-path in G whose intermediate vertices are at most k. It will be ∞ if no such path exists.

$$\begin{cases} d_0(i,j) & = w(ij) \\ d_n(i,j) & = d_G(i,j) \end{cases}$$



Algorithm (Floyd and Warshall's DP Algorithm). Using the recurrence relation for $d_k(i,j)$ is

$$\begin{cases} d_0(i,j) & = w(ij) \\ d_k(i,j) & = \min\{d_{k-1}(i,j),\ d_{k-1}(i,k) + d_{k-1}(k,j)\} \end{cases}$$

- For each (i,j,k), take $O(1)$ time to compute $d_k(i,j)$ from $d_{k-1}(i,j)$.
- For each k, there are n^2 pairs of (i,j), using $O(n^2)$ time to compute all d_k from all d_{k-1}.
- It take n iterations to compute $d_G = d_n$ from d_0.

Note. The running time is $O(n^3)$.

4.2.3 Johnson's Reweighting Technique

Algorithm (Naive Solution with Dijkstra). 如果我們可以拿到一個 nonnegative edge-weight 的 graph，我們就可以簡單地用 Dijkstra's algorithm 來解 All-Pairs Shortest Path Problem

- For each vertex i of G, run Dijkstra's algorithm + Quake heap in $O(m + n\log n)$ time to compute $d_G(i, j)$ for all $j \in V(G)$.

Note. The running time is $O(nm + n^2 \log n)$.

所以我們需要一個方法把有負邊權的 graph 轉換成 nonnegative edge-weight 的 graph，Reweighting w into $\hat{w}$ such that

- $\hat{w}$ is nonnegative
- If $\hat{w}$ is the reweighted shortest ij-path, then the original shortest ij-path is w.

Algorithm (Johnson's Reweighting Technique). Following these steps:

- Assign a weight $h(i)$ to each vertex i of G.
- Let
$$\hat{w}(ij) = w(ij) + h(i) - h(j)$$
- Then for any ij-path P, we have
$$\hat{w}(P) = w(P) + h(i) - h(j)$$

Remark. P is a shortest ij-path in G with respect to $\hat{w}$ if and only if it is a shortest ij-path in G with respect to w.

$(4+5+3-1)+3-2 = (8+2+1+1)$ $\qquad$ $(7+2+3)+3-2 = (0+2+1)$

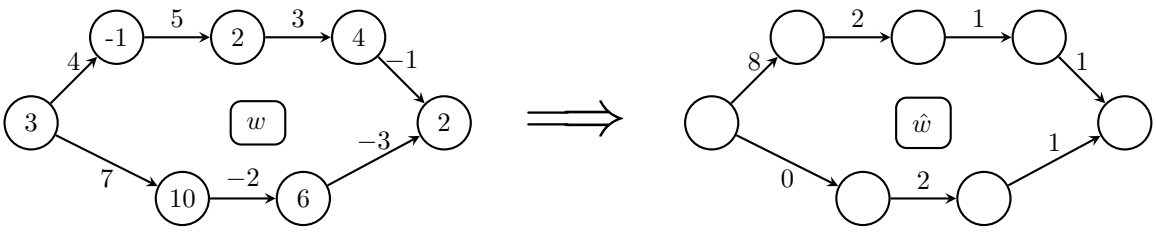


Figure 4.3: Reweighting Example

挑戰是在哪裡找 $h(i)$ 使得 $\hat{w}$ nonnegative？如果有了，我們就可以用 Dijkstra's algorithm 來解 All-Pairs Shortest Path Problem。

Algorithm (Johnson's Technique: Finding $h(i)$)**.** Following these steps:

- Let graph H be obtained by adding a new vertex s to G and adding an edge s_i of weight 0 for each vertex i of G.

  > **Note.** H has no negative cycle iff G has no negative cycle.

- Let $h(i)$ be the distance from s to i in H, i.e.

$$h(i) = d_H(s, i)$$

- The $d_H(s, i)$ can be computed using Bellman-Ford algorithm in $O(m + n)$ time.

Proof. To see that $\hat{w}$ is nonnegative, observe the Figure 4.4.

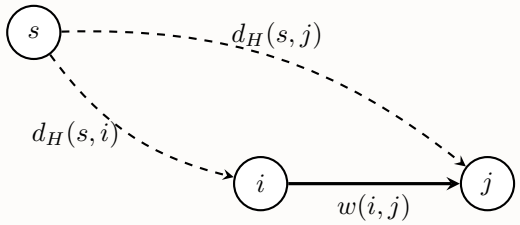


Figure 4.4: Proof of correctness of Johnson's Reweighting Technique

By observation, we have

$$\begin{aligned}
\hat{w}(ij) &= w(ij) + h(i) - h(j) \\
&\geq \underbrace{(w(ij) + d_H(s,i))}_{\text{shortest } sj\text{-path which contain } i} - \underbrace{d_H(s,j)}_{\text{shortest } sj\text{-path}} && \text{(Triangle Inequality)} \\
&= 0
\end{aligned}$$

■

Recall. Using Naive Solution with

- General edge weights: Bellman-Ford algorithm in $O(mn^2)$ time, which can be $\Theta(n^4)$ when $m = \Theta(n^2)$.
- Acyclic edge weights: Lawler's algorithm in $O(mn + n^2)$ time.
- Nonnegative edge weights: Dijkstra's algorithm in $O(mn + n^2 \log n)$ time.

Using Floyd–Marshall ' s DP algorithm in for general edge weights in $O(n^3)$ time.

Algorithm (Johnson's algorithm). Using Johnson's Reweighting Technique + Dijkstra's algorithm

- Obtain $h(i)$ for all vertex i using Bellman-Ford algorithm in $O(mn)$ time, and get $\hat{w}$ from w in $O(m)$ time.
- For each vertex i of G, run Dijkstra's algorithm + Quake heap in

$$O(m + n \log n)$$

time on G with edge weights $\hat{w}$ to compute $d_{\hat{G}}(i, j)$ for all $j \in V(G)$. Then obtain a shortest-paths tree of $G(\hat{G})$ rooted at i.

- Compute $d_G(i, j)$ for all $j \in V(G)$ using

$$d_G(i, j) = d_{\hat{G}}(i, j) + h(j) - h(i)$$

in $O(n^2)$ time.

Note. The running time is $O(mn + n^2 \log n)$.

Chapter 5

Graph Theory: Maximum Flow Problem

Problem 5.0.1 (Maximum Flow Problem)**.** Give

- Input: A directed graph G with edge capacities

$$c : E(G) \to \mathbb{R}^+$$

, and two distinct vertices $s, t \in V(G)$ called **source** and **sink** respectively.

- Output: A "st-flow" with maximum "(flow) value".

Comment. 在這個問題下我們允許 multiple/parallel edges 不需要合併成 simple network

Definition. Here are some definitions related to flows:

Definition 5.0.1 (st-flow)**.** A st-flow is a function

$$f : E(G) \to \mathbb{R}^+ \cup \{0\}$$

that satisfies the following two conditions:

- Capacity constraint:

$$f(e) \leq c(e) \quad \forall e \in E(G)$$

- Conservation law:

$$\sum_{uv \in E(G)} f(uv) = \sum_{vu \in E(G)} f(vu) \quad \forall v \in V(G) \setminus \{s, t\}$$

Definition 5.0.2 (Flow value)**.** The flow value of a flow f is defined as

$$|f| = \sum_{sv \in E(G)} f(sv) - \sum_{us \in E(G)} f(us)$$

5.1 Ford–Fulkerson's Algorithm

Intuition. We can reduce the Maximum Flow Problem into reachability problem for a sequence of **residual graphs** R.

Definition 5.1.1 (Residual Graph). The residual graph $R(f)$ with respect to a flow f of G with $V(G) = V(R(f))$ is defined as follows for each $uv \in E(G)$:

- If $f(uv) < c(uv)$, then $R(f)$ contains an edge uv with capacity

$$c_{R(f)}(uv) = c(uv) - f(uv)$$

- If $f(uv) > 0$, then $R(f)$ contains a reverse edge vu with capacity

$$c_{R(f)}(vu) = f(uv)$$

Comment (1). $R(f)$ 跟 G 一樣，所有 $c(uv)$ 都會是正的，不會是 0 或負的。

Comment (2). G 最多讓 $flow$ 增加 2 倍，因為每條邊 uv 在 $R(f)$ 裡面最多會有兩條邊：uv 和 vu，只要兩個條件都達成。

Lemma 5.1.1. For any st-flow f in G, we have the following properties:

- If $d_{R(f)} = \infty$, then f is a maximum st-flow in G.
- If $d_{R(f)} < \infty$, and g is an st-flow in $R(f)$, then $f + g$ remains an st-flow in G, where

$$(f + g)(uv) = f(uv) + g(uv) - g(vu), \quad \forall uv \in E(G)$$

Note. 這裡的 $g(uv), g(vu)$ 都是由原始的 uv-edge 產生的，因此原始圖若有 vu edge 必須分開處理，不能混在上面兩個式子裡面。

Proof. Let f' be the maximum st-flow in G, but not f. We defined h as follows:

$$h(uv) = f(uv) - f'(uv), \quad \forall uv \in E(G)$$

Since f and f' are both st-flows in G, we have conservation law for f and f', so h satisfies conservation law as well.

$$\sum_{uv \in E(G)} h(uv) = \sum_{vu \in E(G)} h(vu) \quad \forall v \in V(G) \setminus \{s, t\}$$

Now consider some vertex $x, y, z \in V(G) \setminus \{s, t\}$. If $h(xy) > 0$, because h satisfies conservation law, there must exist some $h(yz) > 0$. Continuing this process, we can find a path P,

$$P = s \to v_1 \cdots \to v_k \to t \quad \text{such that } h(v_i v_{i+1}) > 0 \quad \forall i = 0, 1, \cdots, k$$

If $h(uv) > 0$, we have

$$f(uv) > f'(uv) \geq 0 \tag{1}$$

we know $f'(uv)$ can not exceed $c(uv)$, so

$$f'(uv) \leq c(uv) \tag{2}$$

by (1) and (2), we have

$$f(uv) < c(uv)$$

which means

$$c_{R(f)}(uv) = c(uv) - f(uv) > 0$$

Therefore, all edges in $R(f)$ along path P have positive capacities. Which is a st-path in $R(f)$, contradicting the assumption that $d_{R(f)} = \infty$.
Now we start to prove the second property. We need to show that $f+g$ satisfies capacity constraint and conservation law.

- Capacity constraint: For any $uv \in E(G)$, we havee some constraint:
  - $g(uv) < c_{R(f)}(uv) = c(uv) - f(uv) \leq c(uv)$
  - $g(vu) \leq c_{R(f)}(vu) = f(uv)$

  to maximize $(f+g)(uv)$, we set $g(uv)$ to its maximum and $g(vu)$ to its minimum, so we have

$$(f+g)(uv) = f(uv) + g(uv) - g(vu) \leq f(uv) + (c(uv) - f(uv)) - 0 = c(uv)$$

- Conservation law: For any $v \in V(G) \setminus \{s,t\}$, we have

$$\begin{aligned}
\sum_{uv \in E(G)} (f+g)(uv) &= \sum_{uv \in E(G)} (f(uv) + g(uv) - g(vu)) \\
&= \sum_{uv \in E(G)} f(uv) + \sum_{uv \in E(G)} g(uv) - \sum_{uv \in E(G)} g(vu) \\
&= \sum_{uv \in E(G)} f(uv) + 0 && \text{(by conservation law of } g\text{)} \\
&= \sum_{vu \in E(G)} f(vu) && \text{(by conservation law of } f\text{)} \\
&= \sum_{vu \in E(G)} (f(vu) + g(vu) - g(uv)) && \text{(by conservation law of } g\text{)} \\
&= \sum_{vu \in E(G)} (f+g)(vu)
\end{aligned}$$

■

Algorithm 5.1: Ford-Fulkerson Algorithm

Input: A flow network $G = (V, E)$ with capacity $c(u, v)$; source s; sink t.

Output: A maximum flow f.

Initialize $f(u, v) \leftarrow 0$ for all $(u, v) \in E$

Compute residual capacity

$$c_{R(f)}(uv) = \begin{cases} c(uv) - f(uv) & \text{if } f(uv) < c(uv) \\ f(uv) & \text{if } f(uv) > 0 \end{cases}$$

while $\exists$ *st − augmenting path* P *in* $R(f)$ **do**

  Obtain an st-path P of $R(f)$, let $q = \min_{uv \in P} c_{R(f)}(u, v)$

  Obtain a st-flow g of $R(f)$ by setting

$$g(uv) = \begin{cases} q & \text{if } uv \in P \\ 0 & \text{otherwise} \end{cases}$$

  Update flow $f \leftarrow f + g$

end

return f

correctness. We separately prove three things:

- Initialization: f is a valid flow in G with value 0.
- According to Lemma 5.1.1（關鍵觀察）: In every round g is a valid flow in $R(f)$, so $f + g$ is a valid st-flow in G.
- Termination: When the algorithm terminates, $d_{R(f)}(s, t) = \infty$, so by Lemma 5.1.1, f is a maximum st-flow in G.

Proof complete. ■

Definition 5.1.2 (augmenting path). In Ford-Fulkerson algorithm, obtain a st-path P of $R(f)$, let $q = \min\limits_{uv \in P} c_{R(f)}(u, v)$. The path P is called an **augmenting path** with respect to flow f.

Definition 5.1.3 (saturating flow). In Ford-Fulkerson algorithm, obtain a st-flow g of $R(f)$ by setting

$$g(uv) = \begin{cases} q & \text{if } uv \in P \\ 0 & \text{otherwise} \end{cases}$$

. The flow g is called a **saturating flow** with corresponding to P.

Lecture 10

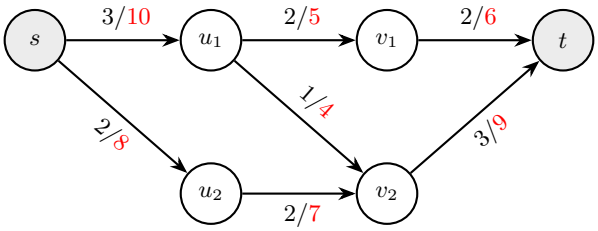


Figure 5.1: Example of Maxflow Problem

考慮 Ford-Fulkerson algorithm 的複雜度分析，如果所有容量都是整數，則每次增廣至少增加 1 單位的流量，而 m 條管線的容量總和是 $C = \sum_{e \in E} c(e)$，因此最多增廣 C 次，每次找增廣路徑花費 $O(m)$ 的時間，總複雜度是

$$T(m, C) = O(C) \cdot O(m) = O(mC)$$

但，這個算是「多項式時間」(polynomial time) 演算法嗎？還是是「指數時間」(exponential time) 演算法？

Remark. Complexity is according to the input size of an instance.

Example. For an $n \times n$ matrix multiplication problem, the input size is $N = \Theta(n^2)$ (set all the number is of size $O(1)$).

The complexity is

- Linear-time if $T(N) = O(N) = O(n^2)$.
- Polynomial-time if $T(N) = O(N)^{O(1)} = O(n)^{O(1)}$.
- Exponential-time if $T(N) = O(1)^N = O(1)^{n^2}$ or more.

Definition 5.1.4 (Complexity of Linear/Quadratic/Polynomial-time Algorithms)**.** For any instance I, define its input size as a non-negative integer function

$$N = \text{size}(I),$$

其中 N 表示描述輸入實例所需的位元數或其他適當的度量方式。令 $T(N)$ 為某演算法在輸入大小為 N 時的最壞情況執行時間。我們對時間複雜度作如下分類：

- Linear-time algorithm：若 $T(N) = O(N)$。
- Quadratic-time algorithm：若 $T(N) = O(N^2)$。
- Polynomial-time algorithm：若存在常數 k 使得 $T(N) = O(N)^{O(1)}$。
- Exponential-time algorithm：若存在常數 $c > 1$ 使得 $T(N) = O(1)^N$ or more。

Example. For a prime testing problem, given an integer N as input.

We have to consider the input size.

- If input size is $N = O(1)$, then the method of checking all integers from 2 to $\sqrt{N}$ is

  $$O(\sqrt{N}) = O(1)$$

  which is a linear-time algorithm.

- If the size of N is not constrainted, then the input size is $\Theta(\log N)$ (bits to represent N). The time of checking all integers from 2 to $\lfloor\sqrt{N}\rfloor$ is $\Omega(\sqrt{N})$

  – According to

  $$(\log N)^{O(1)} = o(\sqrt{N}) = o(N^{1/2})$$

  this algorithm is not polynomial-time.

  – According to

  $$O(N^{1/2}) = O(1)^{O(\log N)}$$

  this algorithm is singly exponential-time.

Note. 所以根據 maximum flow problem 的 input size 是

- 若 $C = O(1)$，input size 是 $O(m) \cdot O(1) = O(m)$，所需要花的時間是 $O(mC) = O(m)$，演算法是 linear-time。
- 若無大小限制，input size 是 $O(m) \cdot O(\log C) = O(m \log C)$，所需要花的時間是

  $$O(mC) \neq O(m \log C)^{O(1)}$$

  因此演算法不是 polynomial-time。

Remark. Ford-Fulkerson algorithm 還會出現無限迴圈的問題，例如下圖的圖，設計出非整數的容量，會導致無限迴圈。Ford-Fulkerson 的論文就有提出其他反例。

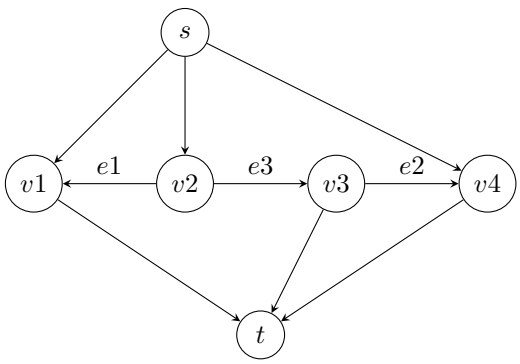


Figure 5.2: A graph that may cause infinite loop in Ford-Fulkerson algorithm

5.2 Edmonds-Karp Algorithm

這是史上第一個被證明是 polynomial-time 的 maximum flow algorithm。

Theorem 5.2.1. If one make sure that the augmenting st-path in $R(f)$ is always the shortest path from s to t (in terms of number of edges), then the Ford-Fulkerson algorithm runs in $O(m^2n)$ time.

Comment (1). 在保證每次 augmenting path 都是 shortest path 的前提下，Edmonds-Karp algorithm 保證在 mn round 內結束。

Comment. 無需假設 G 所有容量都是整數。

We need two lemmas to prove the above theorem.

Notation. $d_f^*(s,u)$ is the shortest distance from s to u in unweighted version of $R(f)$.

Lemma 5.2.1 (現邊觀察). If in $R(f+g)$ exists uv edge which is not in $R(f)$, then

$$d^*_{R(f+g)}(s,u) = d^*_{R(f)}(s,v) + 1$$

Proof. Let P be the shortest augmenting st-path of $R(f)$. If $R(f)$ don't have the uv edge, P can't go through uv edge. If it does not go through vu edge, too. Then $g(uv) = g(vu) = 0$. We get

$$(f+g)(uv) = f(uv) + g(uv) - g(vu) = f(uv)$$
$$(f+g)(vu) = f(vu) + g(vu) - g(uv) = f(vu)$$

Then we get $R(f+g) = R(f)$ which can not have uv edge, which is contradiction. So P must go through vu edge. Then we have

$$d^*_{f+g}(s,u) = d^*_f(s,v) + 1$$

Proof complete. ■

Lemma 5.2.2 (遞增觀察). Let P be the shortest augmenting st-path of $R(f)$. Let g be the saturating flow for $R(f)$ correspond to P. Then for any $v \in V(G)$ we have

$$d^*_{R(f+g)}(s,v) \geq d^*_{R(f)}(s,v)$$

Proof. Assume for contradiction that there exists some $v \in V(G)$ such that

$$d^*_{R(f+g)}(s,v) < d^*_{R(f)}(s,v) \tag{1}$$

Thus, $d^*_{f+g}(s,v) \neq \infty$. Let v be such vertex with the smallest $d^*_{f+g}(s,v)$. We know $v \neq s$ since $d^*_{f+g}(s,s) = d^*_f(s,s) = 0$. Let Q be the unweighted shortest sv-path in $R(f+g)$ (u could be s). Let uv be the last edge of Q. We have

$$d^*_{R(f)}(s,u) \leq d^*_{R(f+g)}(s,u) \tag{2}$$

If $uv \subseteq R(f)$, then equation (2), and $uv \in Q$ imply

$$d^*_{R(f)}(s,v) \leq d^*_{R(f)}(s,u) + 1 \leq d^*_{R(f+g)}(s,u) + 1 = d^*_{R(f+g)}(s,v)$$

which is contradiction to equation (1).
If $uv \not\subseteq R(f)$, by 現邊觀察 ($uv \not\subseteq R(f)$ and $uv \subseteq Q \subseteq R(f+g)$), then equation (2) and $uv \subseteq Q$ imply

$$d^*_{R(f)}(s,v) = d^*_{R(f)}(s,u) - 1 \leq d^*_{R(f+g)}(s,u) - 1 = d^*_{R(f+g)}(s,v) - 2$$

which is contradiction to equation (1), too. ■

Now let compute the time complexity of Edmonds-Karp algorithm.

Time Complexity. Since each augmenting path can be found by BFS in $O(m)$ time, we only need to proof that algorithm halt in $O(mn)$ rounds.

> **Claim.** Every round "saturates" at least one edge in the shortest st-path P found in that round, which is $O(m)$ edges in $G \cup G^r$, causing them to be removed from the residual graph of the next round. Thus, we can just show that each $uv \subseteq G \cup G^r$ being removed $O(n)$ times in total.

Suppose that an edge uv of $G \cup G^r$ is not in $R(f)$

- appears in $R(f+g)$ and
- removed in $R(f+g+\cdots+g'+h)$ for the first time after $R(f+g)$

where h is the saturating flow of $R(f+g+\cdots+g'+h)$ corresponding to the shortest augmenting st-path in $R(f+g+\cdots+g')$ saturates uv. Thus, $uv \in E(P)$. We have

$$\begin{aligned} d^*_{R(f)}(s,u) &= d^*_{R(f)}(s,v) + 1 && \text{by 現邊觀察} \\ &\leq d^*_{R(f+g+\cdots+g')}(s,v) + 1 && \text{by 遞增觀察} \\ &= d^*_{R(f+g+\cdots+g'+h)}(s,u) + 2 && uv \in E(P) \end{aligned}$$

Since $d^*_H(s,u) \in \{0,1,\cdots,n-1,\infty\}$ for any residual graph H, uv can at most appear and disappear $O(n)$ times in the residual graphs throughout the algorithm. Thus, the algorithm halts in $O(mn)$ rounds. and thus run in $O(m^2n)$ time. ■

Edmonds-Karp 的分析是用邊來看：

- 每條邊 uv 只要「出現之後消失」一次，就會讓 $d^*_R(s,u)$ 增加，
- 每個節點 u 的 $d^*_R(s,u)$ 只有 $O(n)$ 個可能的值.

所以每個邊「出現之後消失」$O(n)$ 次，每回合至少消失一條邊，一共 $O(mn)$ 回合。如果用節點 u 來看，能不能根據一樣的證明，得知

- 節點 u 的任一 outgoing edge「出現之後消失」一次，則 $d^*_R(s,u)$ 的值會增加，
- 節點 u 的 $d^*_R(s,u)$ 只有 $O(n)$ 個可能的值，

所以 u 的所有 outgoing edges 全部只能「出現之後消失」$O(n)$ 次，而每回至少讓一條邊消失，所以總共只有 $O(n^2)$ 回合？

答案是錯誤的，

因為節點 u 的任一 outgoing edge『出現之後消失』一次，則 $d^*_{R(f)}(s,u)$ 的值會增加至少 2

是錯的，邊消失一次，$d^*_{R(f)}(s,u)$ 不一定會增加，因為可能有其他替代路徑，不一定是 shortest path

5.3 Bipartite Matching via Maximum Flow

Problem 5.3.1 (Maximum matching of bipartite graph). Given

- Input: An undirected "bipartite" graph G
- Output: A matching $M \subseteq E(G)$ with maximum $|M|$.

Definition 5.3.1 (Bipartite Graph). A graph G is **bipartite** if there are disjoint vertex subsets U, V of G with $U \cup V = V(G)$ such that every edge of G has one endpoint in U and the other in V.

Definition 5.3.2 (matching). A edge subset $M \subseteq E(G)$ is a **matching** of G if $M = \emptyset$ or the minimal subgraph H of G with $E(H) = M$ which has $\max_{e \in E(H)} \deg(H) = 1$.

We can reduce the maximum matching problem of bipartite graph to the maximum flow problem by the below construction.
Let $G(s,t)$ be the unit-capacity graph obtained from G by adding

- new source vertex s with edges su for all $u \in U$
- new sink vertex t with edges vt for all $v \in V$

Observation. G has a maximum matching with k edges if and only if $G(s,t)$ has a maximum flow with value k.

We separately prove the two directions.

- ($\Rightarrow$) Let M be a maximum matching of G we have to make sure it follow capacity constraints and Conservation law.
  - Capacity constraints: 因為是一對一 matching，所以每個 $u \in U$ 和 $v \in V$ 最多只有一條 flow 1 的邊經過。
  - Conservation law: For each vertex $u \in U$, at most one edge su has flow 1, and for each vertex $v \in V$, at most one edge vt has flow 1, and for each $uv \in M$, at most exists one flow from $U \to V$.
- ($\Leftarrow$) Define $M = \{uv \in E(G) : f(uv) = 1\}$. We have to make sure M is a matching. And by the proposition below, $|M| = k$.

Proposition 5.3.1. If $G(s,t)$ has a maximum flow with value k, then $G(s,t)$ has an "integral" flow with value k. Given that each edge of $G(s,t)$ has unit capacity, the set of edges in the middle part of $G(s,t)$ with non-zero flow forms a matching of G with k edges. Due to f is closed under $\{+,-\}$.

Chapter 6

Computational Geometry

6.1 Nearest Point Pair

Problem 6.1.1 (Nearest Point Pair Problem)**.** Given

- Input: A set P of n points in the plane.
- Output: A pair of points $p, q \in P$ such that the Euclidean distance $d(p, q)$ is minimized.

Comment. Not losing generality, we can assume that

$$|P| = 2^k, \quad k \in \mathbb{N}$$

A naive algorithm is to compute the distance of each pair of points, then solve a 老大問題，which takes $O(n^2)$ time.

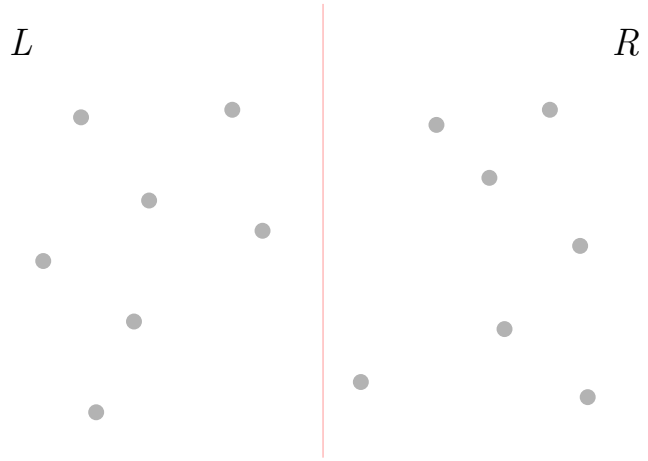


Figure 6.1: Divide-and-Conquer Strategy for Nearest Point Pair Problem

We can use divide-and-conquer strategy to solve this problem follow the above graph.

Algorithm. Pre sort P by y-coordinate as P_y, which would take

$$O(n \log n)$$

Then, use divide-and-conquer strategy to solve the problem

1° Spend $O(n)$ time to split P into two halves P_L and P_R,

$$P_L = \{p \mid p \in P : x(p) \leq x_{\text{mid}}\}, \quad P_R = \{p \mid p \in P : x(p) > x_{\text{mid}}\}$$

with x_{mid} being the median x-coordinate of points in P. We can use the minimum-selection algorithm to find the median in $O(n)$ time.

2° Output a closest pair among the following three pairs:

- The closest pair in P_L (recursively solved).
- The closest pair in P_R (recursively solved).
- The closest pair (p, q) with $p \in P_L$ and $q \in P_R$, which can be solved in $O(n)$ time as below.

> **Note.** Follow the graph below, the node will exist in the vertical strip with width $2d$ centered at the dividing line. For each point p in the strip, we only need to check at most 8 points in the box. So the complexity is
>
> $$O(n) \times O(1) = O(n)$$

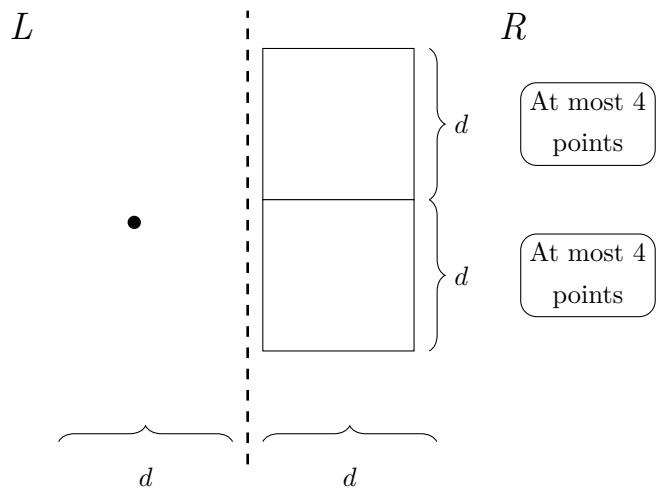


Figure 6.2: Finding Closest Pair Across the Dividing Line

把所有距離分割線不超過 d 的點挑出來，根據演算法一開始的準備動作，我們只要花 $O(n)$ 時間就可以把這些點根據它們的 y 座標排好。我們稱這個 sorted list 為 M^*。

- M^* 裡面在 L 中的點的形成 L^*, 在 R 中的點形成 R^*.

再花 $O(n)$ 時間就能夠整理出資料結構使得每個點 $p \in M^*$ 都能在 $O(1)$ 時間查得：

- 在 L^* 中 y 座標與 p 的 y 座標最近的上下（按照 M^* 的順序）各一個節點。

- 在 R^* 中 y 座標與 p 的 y 座標最近的上下（按照 M^* 的順序）各一個節點。

For each point p in L^*,

- check four points of R^* above p in M^*.
- check four points of R^* below p in M^*.

只需要在 M^* 中由上而下走過每個點，每個點在 M^* 的順序中上下各取八個點一定會找到在對面那一側的 $2d \times d$ 的方格內跟自己最近的點（如果格子內有點的話）。

Note. By master theorem, the time complexity of this divide-and-conquer algorithm is

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

which gives

$$T(n) = O(n \log n)$$

Therefore, the overall time complexity including the initial sorting step is

$$O(n \log n) + O(n \log n) = O(n \log n)$$

6.2 Convex Hull

Problem 6.2.1 (Convex Hull)**.** Given

- Input: A set P of n points in the plane.
- Output: The convex polygon with a minimum perimeter enclosing all points in P.

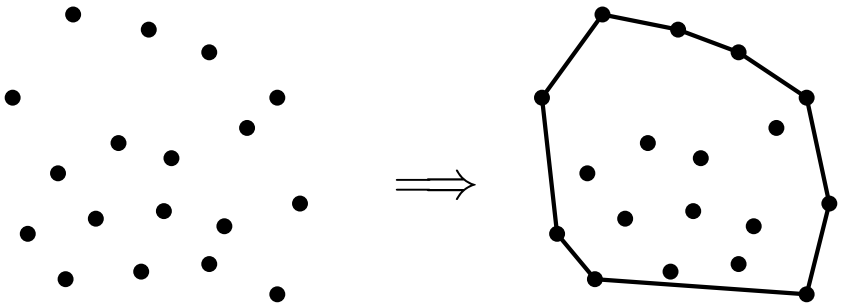


Figure 6.3: Convex Hull Example

A naive algorithm is to check the $p \in P$ with minimum x-coordinate, which must be a vertex on the convex hull. Then, chose the next vertex with maximum slope to the current vertex. Then, rotate the whole graph until we return to the starting vertex. This algorithm takes $O(n^2)$ time.

Algorithm. Now we introduce a new method with $O(n \log n)$ time complexity.

1° Find the point p^* with the lowest x-coordinate in $O(n)$ time (老大問題)

2° Sort all the points in P by the slope of the line segment p^*p for each $p \in P \setminus \{p^*\}$ in $O(n \log n)$ time.

3° Chose the initial convex hull H to be the triangle formed by p^* and the first two points in the sorted list.

4° For each remaining point p in the sorted list, do:

- Beacuse the one we chose the points based on the slope from p^*, the point p must be outside the current convex hull H.
- Move along the boundary of H in clockwise direction from the initial point, and remove all vertices q of H such that the line segment qp makes a left turn with respect to the edge of H incident to q.
- Do this until we reach a vertex r of H such that the line segment rp makes a right turn with respect to the edge of H incident to r.
- Add the edge rp^* and p^*p to H to form the new convex hull.

This step takes $O(n)$ time in total because each point is added and removed at most once.

Therefore, the overall time complexity is

$$O(n) + O(n \log n) + O(n) = O(n \log n)$$

■

6.2.1 Application of Convex Hull: Farthest Point Pair

Problem 6.2.2 (Farthest Point Pair Problem). Given

- Input: A set P of n points in the plane.
- Output: A pair of points $p, q \in P$ such that the Euclidean distance $d(p, q)$ is maximized.

We can use the convex hull to solve this problem by doing the bitonic boss problem.

Problem 6.2.3 (Bitonic Boss Problem). Given

- Input: A bitonic sequence $A[1],\ A[2],\ \cdots,\ A[n]$ of distinct positive integers.
- Output: the index i with $1 \leq i \leq n$ such that

$$A[i] = \max_{1 \leq j \leq n} A[j]$$

在 convex hull 上的點依照極角排序後，距離最大的兩個點一定會是 bitonic boss problem 的解。因此，我們可以先用

$$O(n \log n)$$

時間求出 convex hull，然後在 convex hull 上的點用 bitonic boss problem 找出距離最大的兩個點，which takes

$$O(h)$$

time, where h is the number of points on the convex hull. Therefore, the overall time complexity is

$$O(n \log n) + O(h) = O(n \log n)$$

Chapter 7

B-tree, 23-tree, 234-tree, RB-tree

Chapter 8

Hashing, Finger Pringting Function