POLITECNICO MILANO 1863

DIPARTIMENTO DI FISICA

Diffuse Optical Tomography Project - Phase 3

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Scenario

We are considering an infinite medium made of an unperturbed homogeneous background with absorption coefficient $\mu_a = 0.1\,cm^{-1}$ and scattering coefficient $\mu_s = 10\,cm^{-1}$, which is then in **diffusion regime**, and a point-like absorbing defect characrerized by the absorption coefficient $\Delta\mu_a = 0.1\,cm^{-1}$.
A cubical volume space of $8\,cm^3$ is discretized in 512 voxels of $1\,cm^3$. 64 source-detector (SD) pairs, assumed at null distance from one and other, are placed onto each pixel of the bottom surface of the space, each of them perform a time resolved measure from 0 to $7\,ns$ in 8 temporal gates of duration $t_{gate} = 1\,ns$ .This measure is the contrast $M(\vec{r}_{SD}, \vec{r}_P, t_{gate})$, defined as the ratio of the unperturbed fluence of the background medium $\Phi_0(\vec{r}_{SD}, t_{gate})$ and the perturbation $\delta\Phi(\vec{r}_{SD}, \vec{r}_P, t_{gate})$ due to the absorbing defect, i.e.:$M \stackrel{\Delta}{=} \frac{\delta\Phi}{\Phi_0}$

The solution of this equation is found by applying the **Born approximation** to the **Diffusion Equation** and is:

$$M(\vec{r}_{SD}, \vec{r}_P, t_{gate}) = \frac{V\cdot\delta\mu_a}{2\pi\cdot D\cdot|r_p - r_{SD}|} exp(\frac{|r_p - r_{SD}|^2}{D\cdot c\cdot t_{gate}})$$

where V is the volume of a voxel, $D = \frac{1}{3\cdot\mu_s}$ is the diffusion coefficient and $|\vec{r}_p - \vec{r}_{SD}\ | = \sqrt{(x_{SD} - x_p)^2 + (y_{SD} - y_p)^2 + (z_{SD} - z_p)^2}$ is the distance between particular SD pair and perturbation.

Goal

The goal of this project is to retrieve a vector representing the perturbation position A from the vector of measurements M i.e. solving for A the vectorial problem: $M = W \cdot A \implies A = W^{-1} \cdot M$

The measurements vector is affected by noise, so the inversion of W is not trivial.

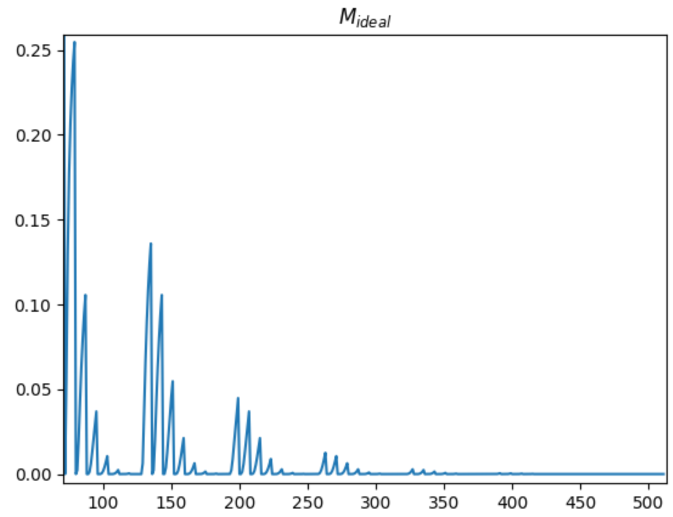
Noise:

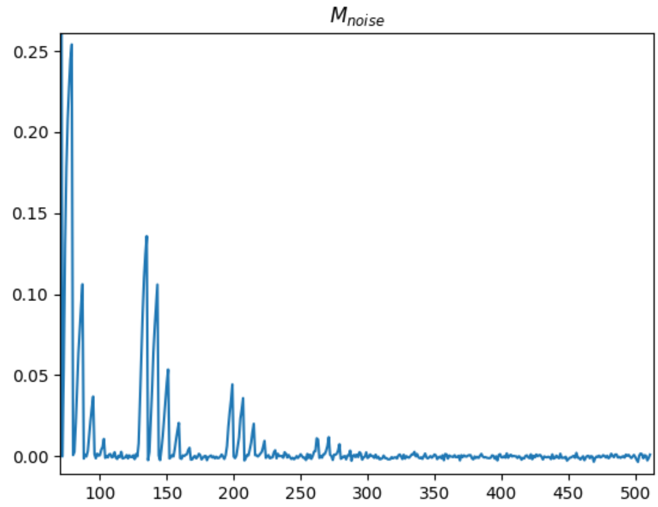
We start from the measurement vector found in the previous phase of the project (forward problem) and, to make it similar to a real measurement, we add noise with Poisson distribution to simulate **quantum shot noise**.

We assume that for a single temporal gate, the detectors count 10^6 photons in case of homogeneous medium. So we generate the noisy measurements vector using P as the number of detected photons for Φ_0 and the method $random.poisson()$ of the Phython's library Numpy:

$$M_{noise} = \frac{np.random.poisson(P*(1+M)) - np.random.poisson(P)}{np.random.poisson(P)}$$

This is how a portion of the compressed measurement vector looks like after noise is applied:



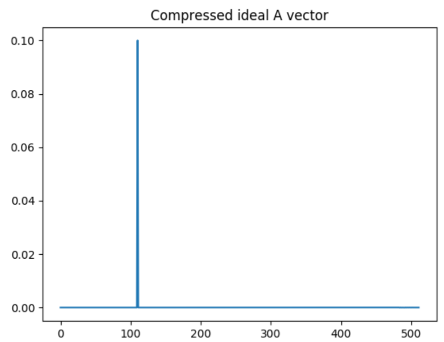


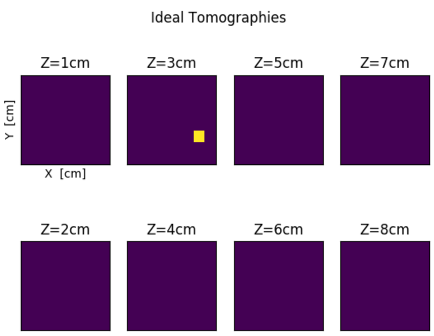
Regularization:

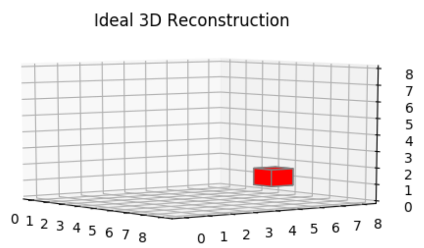
In case of no noise, retrieving the perturbation A is easy and can be done by applying **singular values decomposition (SVD)** to the sensitivity matrix W and solving the system:

$$W = U \cdot S \cdot V^T \implies A = V \cdot S^{-1} \cdot U^T \cdot M$$

Where U and V are the matrices containing respectively left and right singular vectors and S is a diagonal matrix containing singular values.
In this ideal case is possible to retrieve the anisotropy at every depth with maximum spatial resolution.



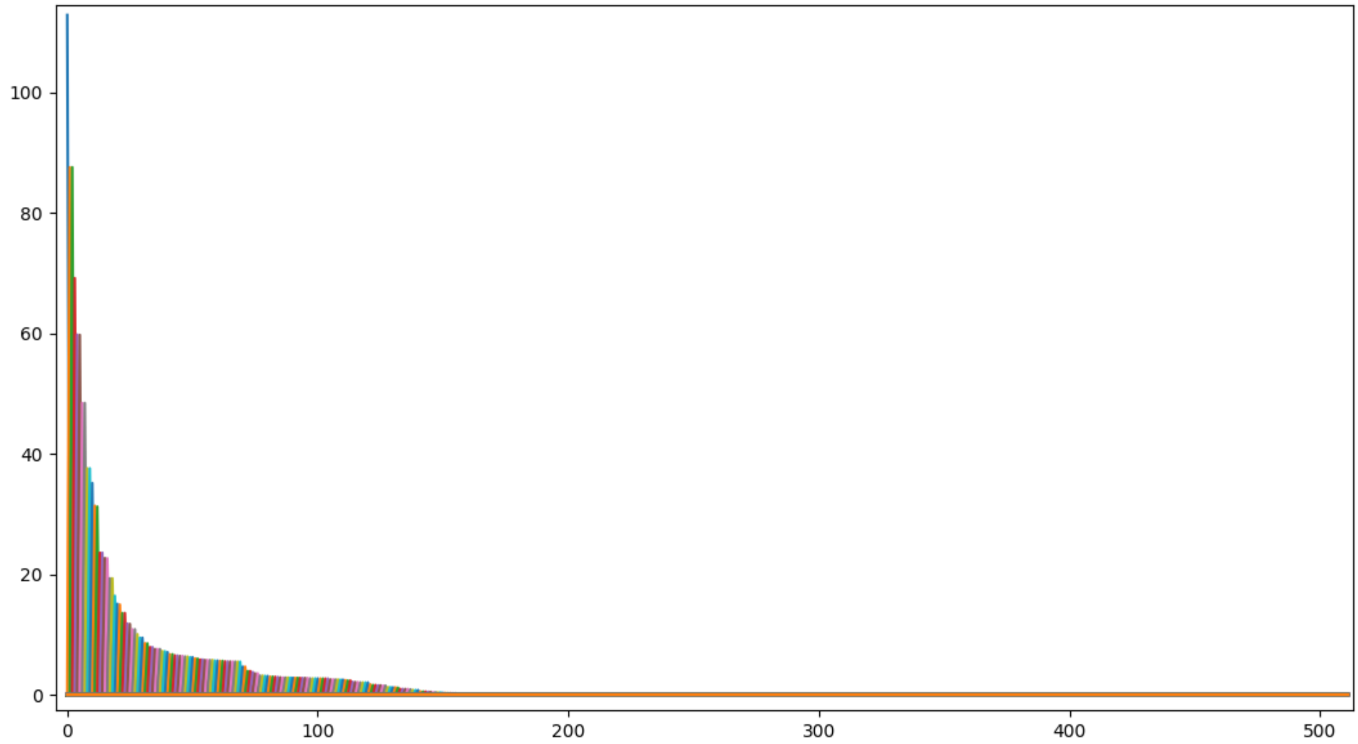




TSVD:

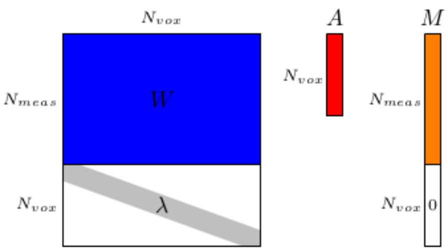
In the case of noisy measurements using the previous approach does not give meaningful information. In fact the deeper singular values, i.e. the capability of detect something in deep regions, are very small so their information is dominated by the noise making the reconstruction problem very ill-posed.

One way to solve the problem is by regularizing it, and the simplest method is the one of **Truncated Singular Values Decomposition (TSVD)** in which the smallest singular values weight is not considered so that during inversion they do not lead to infinite amplification of the noise.
The curves of singular values are visualized and a truncation at the 160^{th} singular value seems reasonable:



Arridge's method:

To optimize the results it can be employed the Arridge's method, which is similar to the Tichonov regularization but with the advantage of lower memory consumption.
This method consists in the expansion of the sensitivity matrix W with a diagonal matrix of dimensions $[N_{vox}, N_{vox}]$ with a parameter λ on the principal diagonal and the measurement vector M with a vector of lenght N_{vox} filled with zeroes.



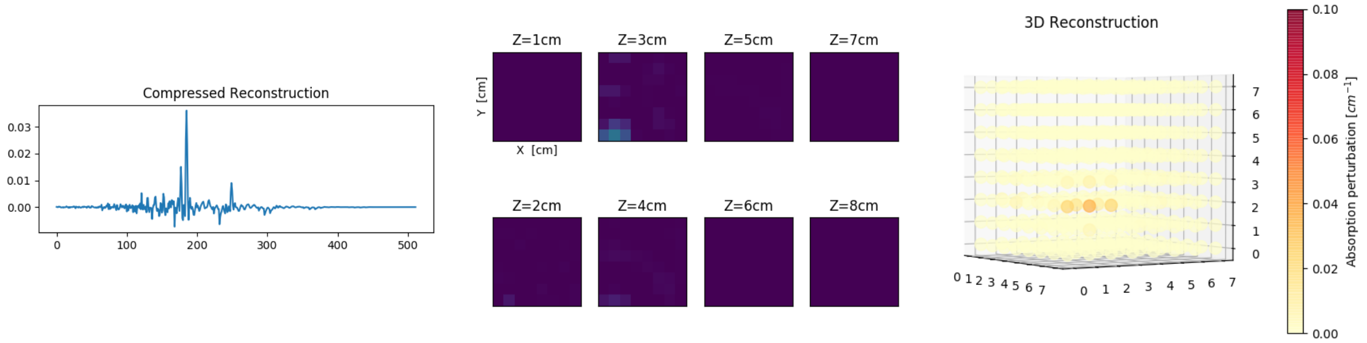
In this apporach a parameter **λ** appears, it is the so called **regularization parameter**. It is possible thus investigate the effects, on the solution, of the various values of λ. It turns out that increasing the value: $\lambda \rightarrow \infty$ the results will not be dependent on the measurements: the resulting image will be completely artificial and give a flat reconstruction. On the contrary, if we do not apply any regularization: $\lambda \rightarrow 0$
then the non-regularized problem will give only noise. In the intermediate case, we are truncating the amount of information coming from the sensitivity matrix W .

Reconstruction:

In the case of **TSVD** the perturbation is retrieved very similarly to the ideal case, the formula is:

$$A = V \cdot S_{trunc}^{-1} \cdot U^T \cdot M_{noise}$$

The computed results are the following:

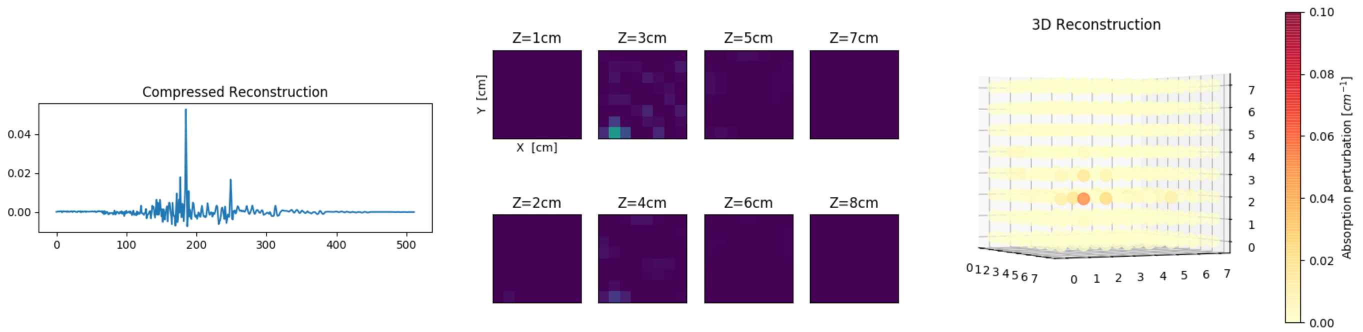


We can see that at 3 cm of depth the information spreads over space and the retrieved value of absorption coefficient perturbation at the actual position is not accurate. Using the Arridge's method and tuning the regularization parameter is possible to slightly increase the performance in terms of accuracy of measurement and spatial resolution.

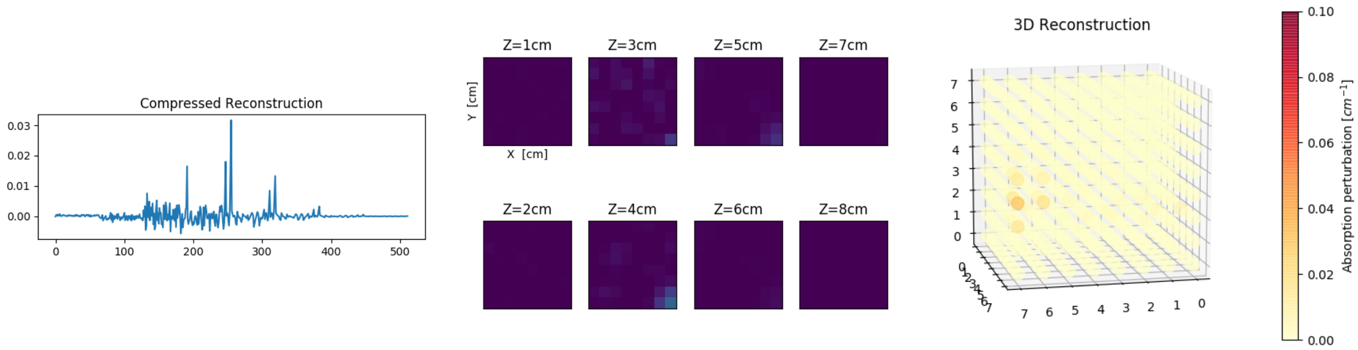
In the case of **Arridge's method** the perturbation is retrieved by solving the problem:

$$A = W_{Arridge}^{-1} \cdot M_{Arridge}$$

The solution is computed for a regularization parameter $\lambda = 0.01 \cdot max(S)$ with the Numpy method $np.linalg.lstsq()$, which returns the least-squares solution to a linear matrix equation. The results are the following:



We see that the spatial resolution increased, so we can try to localize also inhomogeneities that are located 1cm deeper:



Singular Vectors:

It is interesting and instructive to show the plot of singular vectors of the matrix W.
Considering singular vectors, we create a new basis with independent orthogonal components for the spaces considered. Each singular vector will have a different symmetry and it will thus be associated to a singular value, that will describe its weight in the projection. In the regularization procedure, therefore, we are increasing the weight of one of these vectors and decreasing the one of the others, also in terms of depth.
The peculiar shapes that these vectors form are very common in physics and many phenomena are characterized by these figures, some examples are the Zernike polynomials that describe the abberations of optical systems and the Spherical Harmonics that give the shape to the various configurations of atomic orbitals. Increasing the order we create higher spatial frequencies and all their orthogonal representations.
Below the plot of some orders of left singular vectors: