STATS 210

Lab Report 2

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1. Problem 1
   Math proof:

	j=0	j=1
i=0	1/8	1/8
i=1	1/4	1/2

Figure 1.1 The joint PMF

According to the figure above, the joint PMF of X and Y is

$$p_{XY}(x,y) = \begin{cases} 0.125, & x=0, y=0 \\ 0.125, & x=0, y=1 \\ 0.25, & x=1, y=0 \\ 0.5, & x=1, y=1 \end{cases}$$

And according to the definition, the marginal PMF of X and Y can be expressed as below.

$p_X(X=0) = \sum_0^1 p_{XY}(0,y) = 0.125 + 0.125 = 0.25.$

$p_X(X=1) = \sum_0^1 p_{XY}(1,y) = 0.25 + 0.5 = 0.75.$

$p_Y(Y=0) = \sum_0^1 p_{XY}(x,0) = 0.125 + 0.25 = 0.375$

$p_Y(Y=1) = \sum_0^1 p_{XY}(x,1) = 0.125 + 0.5 = 0.625$

Therefore, the marginal PMF are:

$$p_X(x) = \begin{cases} 0.25, & x=0 \\ 0.75, & x=1 \end{cases} \qquad p_Y(y) = \begin{cases} 0.375, & y=0 \\ 0.625, & y=1 \end{cases}$$

Computer simulation:

When doing the simulation, I use the numbers to represent the points. When the number is in the range of (0,0.125), it represents for (0,0); when the number is in the range of [0.125,0.25), it represents for (0,1); when the number is in the range of [0.25,0.5), it represents for (1,0); when the number is in the range of (0.5,1), it represents for (1,1). And then we can compute the joint and marginal PMF of X and Y. The result of my simulation is shown in Figure 1.2.

```
joint pmf:  {(1, 1): 0.5037, (0, 1): 0.1238, (0, 0): 0.1244, (1, 0): 0.2481}
P(x):  {1: 0.7518, 0: 0.2482}
P(Y):  {1: 0.6275, 0: 0.3725}
```

Figure 1.2 The result of computer simulation

Comparison between the two results:

We can draw a conclusion that the result of the computer simulation is close to the result of mathematical calculation.

2. Problem 2

Mathematical solution:

$$\mu_x = \frac{1}{4} \times 0 + \frac{3}{4} \times 1 = \frac{3}{4} \qquad \mu_y = \frac{3}{8} \times 0 + \frac{5}{8} \times 1 = \frac{5}{8}$$

$$\sigma_X{}^2 = (0 - \tfrac{3}{4})^2 \times \tfrac{1}{4} + (1 - \tfrac{3}{4})^2 \times \tfrac{3}{4} = \frac{3}{16}$$

$$\sigma_Y{}^2 = (0 - \tfrac{5}{8})^2 \times \tfrac{3}{8} + (1 - \tfrac{5}{8})^2 \times \tfrac{5}{8} = \frac{15}{64}$$

$$\sigma_{XY} = \mathrm{E[XY]} - \mathrm{E[X]E[Y]} = \frac{1}{2} - \frac{15}{32} = \frac{1}{32}$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\,\sigma_Y} = \frac{\frac{1}{32}}{\frac{\sqrt{3}}{4} \times \frac{\sqrt{15}}{8}} = \frac{1}{3\sqrt{5}} \approx 0.149$$

Computer simulation:

Based on Problem 1, I calculated the correlation coefficient according to the formula.

$$\hat{\rho}_{X,Y} = \frac{\frac{1}{M}\sum_{m=1}^{M} x_m y_m - \bar{x}\bar{y}}{\sqrt{\left(\frac{1}{M}\sum_{m=1}^{M} x_m^2 - \bar{x}^2\right)\left(\frac{1}{M}\sum_{m=1}^{M} y_m^2 - \bar{y}^2\right)}}$$

Here is my result.

```
corelation coefficient is 0.1520026600240423
error: 0.019664841723645216
```

Figure 2.1 The result of computer simulation

Comparison between the two results:

As we can see the error is very small, therefore the simulation result is very close to the mathematical result.

3. Problem 3

Math calculation:

Because X and Y are two independent geometric random variables, and $\mathrm{Z} = \mathrm{X} + \mathrm{Y}$, thus $P(Z)$ can be expressed as below.

$$f_x(x) = p(1-p)^{x-1},\ \ f_y(y) = p(1-p)^{y-1}$$

$$\begin{aligned} f(Z) = \int_{x=1}^{z} f_x(x) f_y(y) dx \\ &= \int_{x=1}^{z} f_x(x) f_y(z-x) dx \\ &= \int_{x=1}^{z} p(1-p)^{x-1} p(1-p)^{z-x-1} dx \\ &= p^2 \int_{x=1}^{z} (1-p)^{z-2} dx \\ &= p^2 (z-1)(1-p)^{z-2} \end{aligned}$$

$$(z = 2,3,4 \ldots \ldots)$$

Computer simulation:

After simulating two geometric random variables, I sum each pair together to get the new random variables and then calculate its PMF. The results are as below.

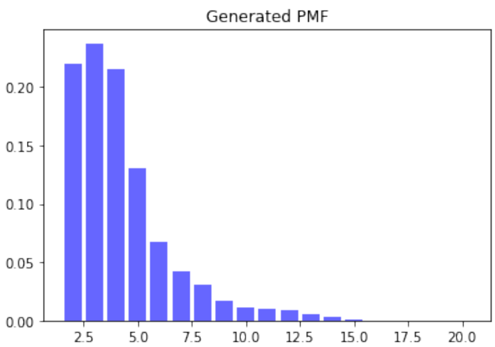

Figure 3.1 The result of computer simulation.

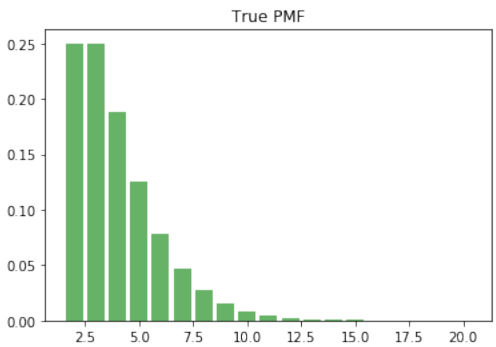

Figure 3.2 The result of mathematical calculation

Comparison between the two results:

When combine the figures together, we can say that the two results are very close.

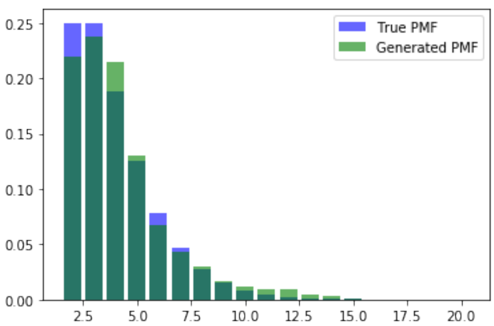

Figure 3.3 The comparison of the two results

4. Problem 4

Math calculation:

$\left|\det\left(\frac{\partial(w,z)}{\partial(x,y)}\right)\right| = \begin{vmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial x} \end{vmatrix} = \begin{vmatrix} 2x & 10y \\ -10x & 2y \end{vmatrix}$, when x=1, y=2, it equals to 208.

The Jacobian matrix actually represents the transformation and the determinant of the Jacobian matrix is the ratio of the areas.

Computer simulation:

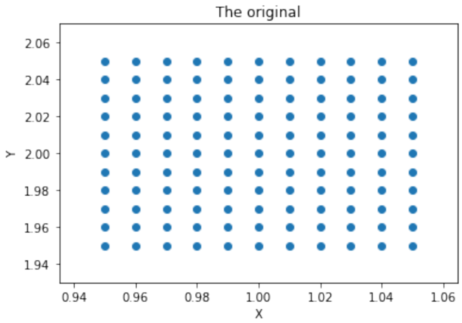

Figure 4.1 The original points

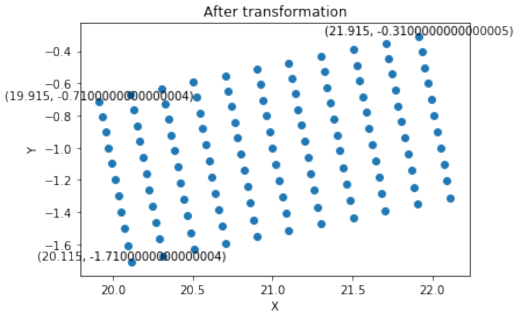

Figure 4.2 The transformed points

Because the figure composed by the points can be roughly viewed as rectangles, we can calculate their area through the coordinates of three vertexes. The original area is about `0.009999999999999995` , the updated area is `2.0799999999999996` ,then the ratio is `208.00000000000006`

Comparison between the two results:

The error is `error: 2.7328566760003855e-16` , which is so small that we can ignore it. Thus, we can say that the results of the simulation and calculation are very close.

5. Problem 5

Math calculation:

The sample mean converges to $\frac{1}{N}\sum_{i=1}^{N} X \approx \int x f_x(x)dx = \mu_x = 1$.

Computer simulation:

We can view it as draw N numbers from a normal distribution random variables and then calculate their mean. In my code, I use np.random.normal(1,1,n) to generate a list of random variables and then calculate its mean according to different n. As we can see, as n increases, the sample mean converges to 1.

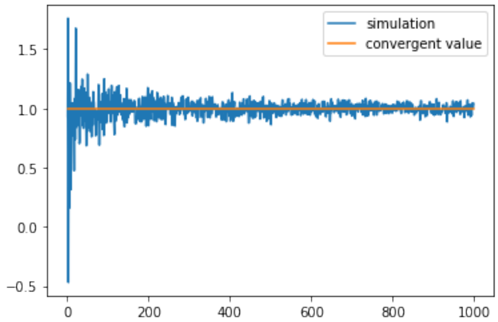

Figure 5.1 The comparison of simulation results and the convergent value

Comparison between the two results:

From the figure, we can say that the sample mean is really convergent to 1.

6. Problem 6

Math calculation:

According to law of large numbers, we know that the two sample mean must converge.

Variance method:

We can compare the speed of converge via the derivative of the variance of the sample mean. Since the variance reflects how the sample means fluctuate, and its derivative reflects the decrease of the fluctuation which means the how fast the sample mean converges.

The variance of sample mean should be the variance of a random distribution divided by n.

$\sigma^2 = \sigma_X^2 \div n$

$\left|\frac{d\sigma^2}{dn}\right| = \left|\frac{\sigma_X^2}{n^2}\right|$.

From the formula we can see that as the σ_X^2 gets larger, the sample mean converges faster. Thus the normal distribution converges faster.

The Chebyshev law of large numbers method:

$P[|\overline{S_n} - \mu_x| < \varepsilon] \leq 1\text{-}\frac{{\sigma_x}^2}{n\varepsilon^2}$ (1)

When ε goes to extremely small, the sample mean converges to the mean of random variables. And the derivative of formula (1) can express the speed of convergence. The process is similar with above. Through this method, we can also get the conclusion that the convergence speed of normal distribution is faster.

Computer simulation:

Similar with Problem 5, I use np.random.uniform (0,2,n) and np.random.normal (1,4,n) separately to generate a list of samples and then calculate their sample mean according to the different n. And below are their figures.

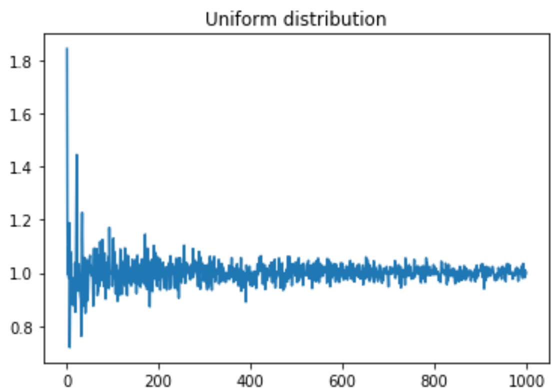

Figure 6.1 The sample mean of uniform distribution random variables

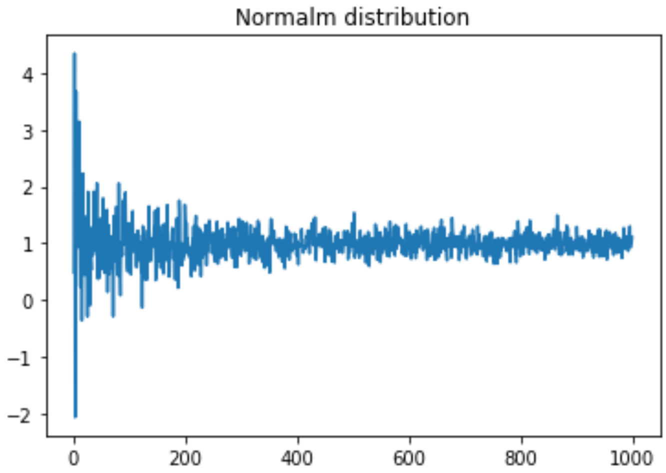

Figure 62 The sample mean of normal distribution random variables

Comparison between the two results:

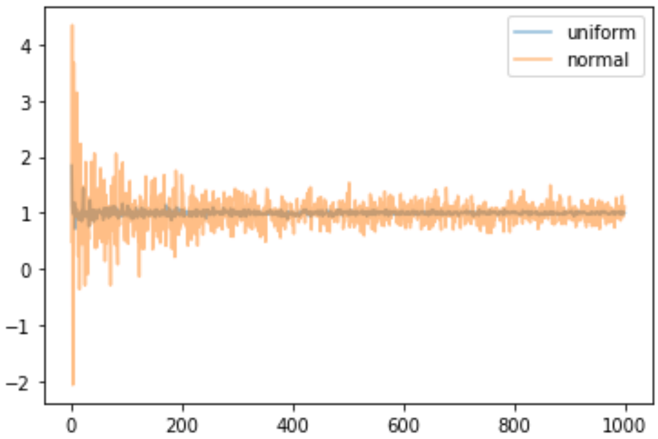

Figure 6.3 The comparison of uniform distribution and normal distribution

The figure shows that the sample mean of normal distribution converges faster (has more room to converge), corresponding with the proof.