**STATS 210 – Probability, random variables and stochastic process**

**Final project ---- Auction Theory**

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##### Abstract/ Introduction

This topic of auction arises from Auction Theorey, which deals with how people act in auction markets and researches the properties of auction markets in different mechanisms. Auction theory is usually applied in making efficient decisions when allocating scarce and rival goods, especially in stock exchanges, auctions for the privatization of public-sector companies and the sale of licenses for use of the electromagnetic spectrum.

To explore the specific auction scenarios in this topic, we first set up a theoretical auction model, where there is one seller who try to sell items and n bidders with their own valuations and biddings. We discuss two types of auctions, the first-price auction and the second-price auction. In the first-price auction, the bidder who makes the highest bid wins the item and pays his/her bid. And in the second-price auction, the bidder who makes the highest bid wins the auction, and pays an amount equal to the second-highest bid.

By appling mathematics and probability, we mainly studied three aspects: (1) the optimal bidding strategies from the buyer side in both first price auction and second price auction (Task1, Task2 Q1 and Q2-a); (2) comparision of seller revenue in first price auction and second price auction(Q2-b); (3) strategies for the seller to maximize his revenue with a reserve price(Q3).

##### Task 1

For the First Price Auction **(**FPA), each buyer makes one bid to the seller and the seller sells the item to the highest bidder for that amount. The person 𝑖 value the item at , and the valuations are independent and identically distributed random variables. Each person draws their valuation uniformly from the interval (0,1). Suppose the other bidders bid their own valuations, meaning that their bidding function is 𝛽() = .

1. In an FPA where there is only one other bidder, and it is known that the other bidder uses the bidding function 𝛽() = . To find bidder A’s optimal bidding function which maximizes his expected utility in such a scenario, we make the following derivation.

Let random variable denote bidder A’s valuation and denote his bidding value. As a rational bidder, he will never bids more than , because bidding more than will make his utility smaller than 0 no matter he wins or loses. Therefore, a bidder’s bidding value satisfies . Let denote the other bidder’s valuation as well as his/her bidding value, and let denote his utility, which is given by

Since and follow the distribution of U(0,1), we can derive their probability density functions (PDF) satisfy , and their cumulative distribution functions (CDF) satisfy .

Our target is to find the optimal bidding function which maximizes a bidder’s expected utility , which is given by

Take the derivative of in terms of where we treat as an independent variable and as a constant, and let the derived function equals to zero:

Therefore, when my bidding function is , which means that when a bidder bids half of my valuation, his expected utility is maximized:

1. To extend our discussion, now we generalize the previous scenario by assuming that there are other bidders as bidder A’s competitors. We denote the valuations (as well as the bids) of the other bidders as , which are independent and identically distributed random variables. Let represent the highest bid among bidder A’s competitors’ bids:

Bidder A’s utility and its expected value are given by

( are i.i.d.)

Take the derivative of in terms of where we treat as an independent variable and as a constant, and let the derived function equals to zero:

Since

Therefore, when a bidder’s bidding function is , which means that when he bids of his valuation, his expected utility is maximized:

##### Task 2

##### 1. First Price Auction (FPA)

In the first price auction, suppose we know that all buyers draw their values either uniformly at random from the (0,1) interval, or from an exponential distribution with a certain parameter 𝜆. The person 𝑖 value the item at , and e**veryone else is extremely risk averse so that all of them have bidding function** 𝛽() =.

1. **FPA Uniform**

**Mathematical formulation**

As proved in Task1, bidder A’s expected utility is maximized when his bidding function is .

For n = 2, where there is only one other bidder, applying the optimal bidding function back to the function E[B], we can obtain bidder A’s optimal expected utility

.

A bidder’s optimal expected total profit over rounds of bidding can be obtained by multiplying k with . By Law of Large Number,

E[]

Let 1000, the expected cumulative profit over 1000 trials is

E[] ≈ 83.3

For the scenario that there are other bidders as bidder A’s competitors, applying this optimal bidding function back to E[B], we can obtain his optimal expected utility

.

Bidder A’s optimal expected total profit over rounds of bidding can be obtained by multiplying k with . By Law of Large Number,

E[]

When = 10, the expected cumulative profit over 1000 trials is

E[] 1000 ≈ 3.52

**Computer simulation**

We used a *for loop* to perform the auctions 1000 times. Each time, we used the code np.random.uniform to generate n-1 values, where n is the number of player. After that, we also used the same code to generate a value to represent the ith player’s value. Then, we used the beta function to detetmined the ith bidder’s bid price and decide whether his bid value is the highest or not. If it is the highest value, we calculated the profit and add it into the profit time series list.

According to our mathematical derivation, when there are other bidders bidding their valuations that obey U(0,1), a bidder’s optimal expected total profit over trials is

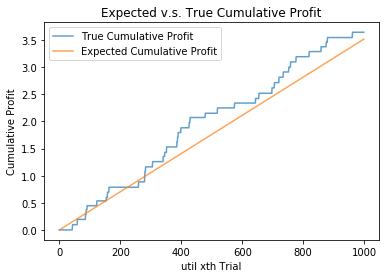
Letting = 1000, we plot this function when = 2 and = 10 respectively, label them by “expected cumulative profit” and compare their plot with the simulation results, which are shown in Figure 1.1 and Figure 1.2 :

Figure 1.1. FPA, uniform, n = 2 Figure 1.2. FPA, uniform, n = 10

Comparing the above two plots, we can find that the curve of n=2 is smoother than that of n = 10. This agrees with our intuition, because a more rugged curve essentially results from more occurrences of “0 utility”. When there are more competitors bidding for one item, the probability of my success will decrease, which will lead to more non-increasing portions in the curve. A mathematical proof is shown below:

When the valuations of bidder A’s competitors are , and as derived above,

P[]

E[P[]] E[]

Since ≥ 2 and is smaller than 1, E[P[]] decreases as increases. Therefore, as more people participate in the acution, bidder A’s cumulative profit does not increase as smoothly as when there is less competition between the bidders.

1. **FPA Exponential**

**Mathematical formulation**

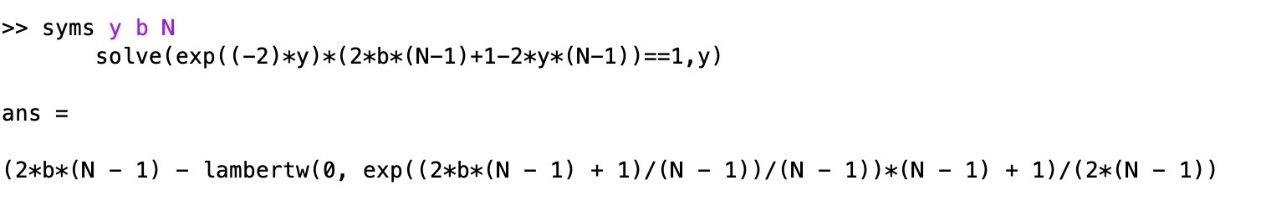
Now assume that everyone else and bidder A draw the valuations from the exponential distribution with parameter 𝜆 = 2. Knowing that everyone else continues to bid their valuation, bidder A’s aim is to empirically find a bidding function to compete against other bidders.

Similar in question(a), we denote the valuations (as well as the bids) of the other bidders as , which are independent and identically distributed random variables. Let represent the highest bid among bidder A’s competitors’ bids.

As derived in Task 1, his utility and its expected value are given by

Take the derivative of in terms of where we treat as an independent variable and as a constant, and let the derived function equals to zero:

Using the *solve* function in matlab to solve the above equation, we get



where LambertW is a Lambert W Function or product log function. It is the inverse function of  *f(w)=w.exp(w)* where *w* can be any complex number.

For n = 10, where there are 9 other bidders, applying the optimal bidding function back to the function E[B], we can obtain bidder A’s optimal expected utility

A bidder’s optimal expected total profit over rounds of bidding can be obtained by multiplying k with . By Law of Large Number,

Since this formula is structurally complicated and therefore hard to evaluate even with matlab, we use = 0.7 as a linear approximation of the optimal bidding function when we want to calculate the expected cumulative profit of bidder A. The feasibility of using this linear approximation is illustrated in the following figure, which shows that if we use = 0.7 as our bidding function, the cumulative profit over time is very close to that obtained by using the optimal bidding function. Therefore, we can define = 0.7 as our near-optimal bidding function.

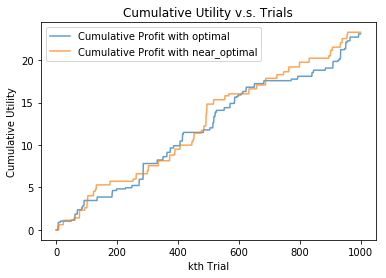


Figure 2. Comparison of optimal and near-optimal bidding function

Applying our near-optimal bidding function to E[B], we can get

0.3

0.3

E[] E[0.3] E[0.3]

Using the *int* function in matlab to integrate the above expression, we get

E[] 0.023794112796477487

Let 1000, the expected cumulative profit over 1000 trials is

E[] ≈ E[] ≈ 24

**Computer simulation**

We used the same *for loop* structure as the one in part a. Each time, we used the code np.random. exponential to generate n-1 values, where n is the number of player. After that, we also used the same code to generate a value to represent the ith player’s value. Then, we used the optimal bidding function derived above to detetmined the ith bidder’s bid price, with n=10. And then, we repeated the same process as part a to generate a list of cumulative profits.

We plot the cumulative utility until the th trial, which is shown as follows. As we can see, the total profit is above 24, and the result is stable.

Comparing the true cumulative utility with the expected cumulative untility, we find that the two graphs agree well with each other.



Figure 3. FPA, exponential, n = 10

##### 2. Second Price Auction

In a second-price sealed bid auction, the object is sold to the bidder with the highest bid. However, the winner does not pay his own bid, but the amount bid by the second-highest bidder instead. A bidder who does not bid the highest doesn’t get the object and doesn’t have to pay anything. Therefore, in this form of auction, a bidder’s own bid does not affect the amount that he/she would pay, and instead it only determines whether the bidder get the object or not.

Let’s consider such a scenario. [1] There are totally bidders and their corresponding bids are . Suppose that bidder A is the th bidder who bids , which is exactly his valuation of this item, and the highest bid among his competitors is = max{}. By bidding , bidder A will win if ≥ (gaining a profit of −) and lose if .

Suppose, however, that bidder A bids a value < . If < < , then he still wins and his proﬁt is still −. If < < , he still loses. However, if < < , then he loses. Thus, bidding less than cannot increase a bidder’s proﬁt, and it sometimes decreases a bidder’s probability to win and thus decreases the profit instead. Therefore, our optimal biding function for second price auction is the identity bidding function 𝛽() = .

**2.1 Mathematical formulation**

To calculate the expected profit of a buyer, who is called buyer A, we first define several functions and notations as below. Assuming there are totally buyers and therefore buyer A have competitors. We denote their valuations (as well as their bids) as , which are independent and identically distributed random variables. Let represent the highest bid among my competitors’ bids:

Therefore, buyer A’s utility is given by

Next, in order to calculate the expected profit of bidder A, we need to calculate the probability that A wins.

The expected profit E[B] is calculated by

* 1. **Computer simulation**

To simulate the process, we used a *for loop* to perform the auctions 1000 times. Each time, we used the code np.random.uniform to generate n-1 values, where n is the number of player. After that, we also used the code to generate a value to represent the ith player’s value. Then, we used the beta function to detetmined the ith bidder’s bid price, which is itself. We used the code max to find out the highest bidding value of the other bidders, and decide whether ith bidder’s bidding price is the highest. If it is the highest value, we used the ith bidder’s value to minus the highest bidding value of the other bidders to figure out the profit, and add the profit value into the profit time series list.

According to our mathematical derivation, when there are other bidders bidding their valuations that obey U(0,1), a bidder’s optimal expected total profit over trials is

Letting = 1000, we plot this function when = 2 and = 10 respectively, and compare them with the simulation results:

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描述已自动生成 Figure 4. SPA, uniform, n = 2 Figure 5. SPA, uniform, n = 10

As we can see that no matter when or , the simulated result and theoretical result are almost the same. By giving it a closer look, we find that the curve of is smoother than that of . This again, results from a decreased probability to win when there are more competitors, as it is analyzed above.

##### Seller Revenue

If a seller doesn’t know the valuation of the other bidders, he will not know in advance whether one type of auction will give him more revenue than the other, and his profit from either type of auction is a random variable. But given his beliefs about the probability distribution of bidders’ values, he can calculate his expected revenue from any type of auction.

* 1. **Mathematical formulation**

In a first price auction, we define the random variable as the highest valuation, as the distribution function of bidders’ values. So, the PDF of X can be written as

where n is the number of bidders.

Since bidders draw their values uniformly from the (0, 1) interval, we can have:

The expectation of the highest values is given by

Since everyone uses the bidding function , the expectation of the highest bid is ×

In a second price auction, we define the random variable as the second highest value, as the distribution function of bidders’ values. So, the PDF of X can be written as

where n is the number of bidders.

Since bidders draw their values uniformly from the (0, 1) interval, we can have:

The expectation of the second highest valuation is given by

Since everyone uses the bidding function , the expectation of the second highest bid is .

Since the expectation of a seller’s revenue under FPA and SPA are the same, his holding a first price auction or a second price auction is expected to make no difference, given that the bidders use the optimal bidding functions described above.

* 1. **Computer simulation**

To simulate the process, we used a *for loop* to perform the auctions 1000 times. Each time, we used the code np.random.uniform to generate n-1 values, where n is the number of player. After that, we used the beta function to get their biding prices. The beta function has been mentioned above, if the input parameter indicate it is a FPA, we used the second beta function with n=6. If the input parameter indicate it is a SPA, we used the forth beta function. Then, we sorted the biding prices and, depending on the input parameter, we chose the largest or second-largest value as the seller’s revenue. And then, we add the revenue value into the profit time series list.

We calculated the cumulative revenue of the seller in both first-price auction and second-price auction respectively and compared their trend. Here are the results:

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Figure 6. Comparation of Seller Revenue in FPA and SPA

From the figure of the simulation result, we can see they are almost the same.

##### First Price Auction With A Reserve

* 1. **Mathematical formulation**

According to the book *Auction theory*, the optimal reserve price has nothing to do with the numbers of the bidders [1]. And after searching for the optimal model, we find that the problem we faced can be simplified as a problem of choosing a time to sell the item, in order to maximize an expected reward. This is actually an optimal stopping problem [2][3].

Before solving the problem, we first define several parameters.

: the round of auctions

: the true rank of the round reserve price

: from the round, sell the item when come into the highest bid, which means the seller does not sell the item in the first rounds

: select the round to sell the item.

Because the probability of each round to be the best choice is equal, thus

Therefore

Now the problem is that we need to find an optimal r to maximize .

We take as , in order to find the largest value of , we can first know the change rate of the function:

When n is constant,  is monotonically decreasing with the maximum

When n is larger than 5,  is positive, so we can definitely say that we can find the optimal in the range of [5, n-1] which makes close to zero.

Therefore

which means

When we have the value of n, we can get the optimal r’.

Because and from the figures

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Figure 7. Comparing with 1 (left) and (right)

We can have that  and .

As a result:

When n gets larger, we can approximate as .

Thus, can be estimated as .

From function (1) we can find that when n is very large, we can view close to 1 [4] and thus  and .

In our case, .

Therefore, our strategy is that: in the 1st to the 37th auctions, we set the reserve price very high so that the iPhone will not be sold. From the 38th round, we set the reserve price as the highest bid price in the first 37 auctions. Then, in the following auctions, if the highest bid price in a auction is above the our reserve price, we will sell the iPhone to the winner.

##### What we learn from this project

By investigating in auction theory, in this project we are able to look into plentiful basic while significant probability concepts as well as their meritorious real-life application. Apart from calculating PDF/CDF/expectation of random variables, this project also encourages us to build our own model by defining functions of several random variables and to explore their attibutes. For the questions in two tasks, we revisit several special distributions such as uniform and expoenential distribution, the concept of multivariate random variables, sum of independent random variables, law of large numbers and central limit theorm.

Moreover, in simulation we get to practice with the data structure of array and some inbuilt modules in python, such as the random number generators from numpy. Using code *np.random.uniform(0,1)* and *np.random.exponential(0.5)* to realize the two distributions explored in this project largely facilitate our simulation. In addition, we learn to solve complicated equations using matlab, and in python the imported package of *sympy* enables us to deal with product log function.

##### Conclusion

In this work we explore some basic while significant topics regarding auctions. From both the perspective of a bidder and a seller, we investigate in the seller revenue (how much does the seller make), the buyer profit (the difference between the winning bid and the buyer's value of an item), the difference and the effects of different mechanisms, and the optimal usage of reserves to maximize a seller’s revenue in repetitive auctions.

First, in task 1 we explore the optimal bidding function 𝛽(·) that maximizes a bidder’s expected utility in situation “FPA, uniform, n=2”, which then leads up to the discussion of the situation “FPA, uniform, n>2”. To build these models, we define the utility of a bidder as a random variable that is determined by the value of two other random variables and as well as the bidder’s bidding function 𝛽(). Expressing the expected value of B in terms of x and 𝛽(), we take the derivative of E[B] in terms of 𝛽() and find the 𝛽() that maximizes the expectation of B, which gives us the optimal bidding function in this case: 𝛽() = x/2. Applying this function in simulation and cumulating the revenue of that buyer, we can plot the cumulative profit of that person until n rounds. Comparing the plot with the expected cumulative profit obtained through mathematical calculation, we find the two plots agree well with each other. Both of them increase monotonically, and the curve for n=2 is smoother than that for n=10 because of a bidder’s smaller probability to win when there are more competitors.

Second, in task 2 we firstly explore the optimal bidding function of “FPA, exponential, n>2”. Similarly, we express the expected value of B and take its derivative in terms of 𝛽(), which gives us a complicated equation with parameters x and n. By taking advantage of the product log function in matlab, we solve the equation and use it to simulate the situation of “FPA, exponential, n=10” for 1000 times. The effectiveness of the found bidding function is evaluated by an observation of whether bidder A’s cumulative utility can reach above 24. After running the program for several times, we agree on the good behavior of the derived bidding function and encourage larger number of trials in simulation for better stability.

Third, we talk about second price auction from a buyer’s view. We prove that in such a situation the optimal bidding function is identity bidding function for any n, and we calculate the expected utility of a bidder when n=2 and n=10. Comparing the theoretical result and the simulation result, we see that they agree with each other and the curve for n=10 is more rugged.

Fourth, we explore auction from the seller’s view, talking about whether a seller should hold a first price auction, or a second price one when the bidders use optimal bidding functions described above. In other words, we compare the revenue of a seller in situations “FPA, uniform, n>2” and “SPA, uniform, n>2”. Our calculation shows that the revenue of a seller is (n-1)/(n+1) in both of these mechanisms of auctions. Plotting the simulated cumulative revenue of the seller when n=6, we find that the simulation results of two mechanisms are almost the same, which well supports our calculation and conclusion.

Fifth, we develop a strategy for the seller to maximize his revenue from selling 𝑘 identical items over 𝑎 auctions with a reserve price for each auction. In the specific example that we investigate in, a seller sells 1 iPhone at 100 different auctions where at each auction 𝑛 bidders bid independently with a fixed bidding function and a fixed distribution. We start by applying the Optimal Stopping to auction, which inspires us to simplify this problem as a choice of a time to sell the item. Our strategy is to set a very high reserve in the 1st to the 37th auctions to avoid selling the item, after which we set the reserve price as the highest bid price in the first 37 auctions.

##### Our presentation link:

<https://duke.box.com/s/376sg2hlgdzk1c7950fpemyuusks8im0>

##### Reference

[1] Vijay Krishna, 2010, Auction theory, second edition, US.

[2] Chow, Y.S.; Robbins, H.; Siegmund, D. (1971). Great Expectations: The Theory of Optimal Stopping. Boston: Houghton Mifflin.

[3] Ferguson, Thomas S. (2007). Optima Stopping and Application. UCLA.

[4] Specoder, the optimal stopping problem, Accessed from <http://specoder.is-programmer.com/posts/29993>.

[5] “Aucnotes.Pdf.” n.d. Accessed March 22, 2020. <http://econ.ucsb.edu/~tedb/Courses/GameTheory/aucnotes.pdf>.