6.4 OPERATIONS ON FUZZY SETS

We now define three operations on fuzzy sets, two of them being binary in character and the third being unary in character. These operations extend and generalize the corresponding operations of CST.

Definitions: Let U be a domain and A, B be fuzzy sets on U. Then,

1. **Union** of A and B, denoted by $A \cup B$, is defined as that fuzzy set on U for which $(A \cup B)(x) = \max(A(x), B(x))$ for every x in U.
2. **Intersection** of A and B, denoted by $A \cap B$, is defined as that fuzzy set on U for which $(A \cap B)(x) = \min(A(x), B(x))$ for every x in U.
3. **Complement** of A, denoted by A', is defined as that fuzzy set on U, for which $(A')(x) = 1 - A(x)$ for every x in U.

We now show that the above defined operations on fuzzy sets are *nice* extensions of the corresponding operations on crisp sets.

Results: Let A and B be crisp sets on U. Then,

1. Crisp union of A and B = Fuzzy union of A and B
2. Crisp intersection of A and B = Fuzzy intersection of A and B
3. Crisp complement of A = Fuzzy complement of A

We establish (1) and leave the other two as exercises (see Problem 2).

Proof 1: Let A, B and C be the characteristic functions of the crisp sets A, B and $A \cup B$, respectively. Then, by the definition of crisp union, we have, for every x in U,

$$\begin{aligned} C(x) &= 0 \quad \text{if } A(x) = 0 = B(x) \\ &= 1 \quad \text{otherwise} \end{aligned}$$

It is easy to see that

$$\begin{aligned} \max[A(x), B(x)] &= 0 \text{ if } A(x) = 0 = B(x) \\ &= 1 \text{ otherwise} \end{aligned}$$

Hence

$$C(x) = \max[A(x), B(x)] \text{ for all } x$$

Thus, crisp union of A and B = $A \cup B$ = Fuzzy union of A and B, since the right side of Eq. (1) represents fuzzy union.

The following examples illustrate the above mentioned concepts and ideas.

Examples:

1. Let $U = \{a, b, c, d\}$ be the domain and A and B be fuzzy sets on U as given below:

	a	b	c	d
A	0.5	0.8	0.0	0.3
B	0.2	1.0	0.1	0.7

Then, for $A \cup B$, we have

$$(A \cup B)(a) = \max[A(a),\ B(a)]$$
$$= \max[0.5,\ 0.2]$$
$$= 0.5$$

Similarly, we obtain

$$(A \cup B)(b) = 1.0$$
$$(A \cup B)(c) = 0.1 \text{ and}$$
$$(A \cup B)(d) = 0.7$$

Thus, we finally obtain

	a	b	c	d
$A \cup B$	0.5	1.0	0.1	0.7

Note that this table could have been arrived at from the first table by taking the maximum of each column separately. In a similar way, performing the minimum of each column separately, we obtain $A \cap B$

Elements	a	b	c	d
$A \cap B$	0.2	0.8	0.0	0.3

And subtracting each entry of A from 1 in the first table, we obtain A'.

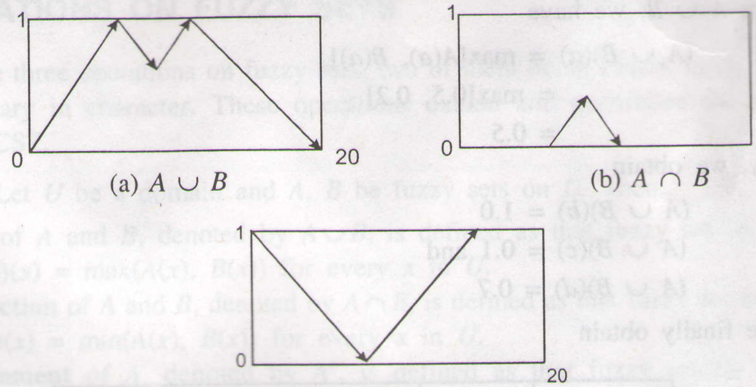
Elements	a	b	c	d
A'	0.5	0.2	1.0	0.7

2. Consider the fuzzy sets A and B given in (1). Then we see that $A(a) > B(a)$, $A(b) < B(b)$, $A(c) < B(c)$ and $A(d) < B(d)$; i.e. all the inequalities are not of the same type. Thus, neither A is contained in B nor B is contained in A. However, consider the following fuzzy sets C and D on U given by

Elements	a	b	c	d
C	0.4	0.8	0.1	0.7
D	0.2	0.7	0.1	0.4

It is clear that $C(x) \geq D(x)$ for all x in U. Thus, D is contained in C.

3. The profiles of A and B are given in Figures 6.1(a) and (b), respectively. Then, $A \cup B$, $A \cap B$ and A' have the profiles as shown in Figures 6.2(a), (b) and (c), respectively.



(a) $A \cup B$ (b) $A \cap B$

(c) A'

Figure 6.2

Remarks.

1. The three operations listed in Section 6.4 are called the *standard operations* on fuzzy sets. Non-standard operations on fuzzy sets are introduced and discussed in Annexure 6.4.
2. The three operations introduced in Section 6.4 together make the set $PF(U)$ into a complete, distributive and complemented lattice. However, this lattice is *not* ortho-complemented, i.e. $A \cup A'$ is not equal to U and $A \cap A'$ is not equal to □, in general.

6.5 PROPERTIES OF THE STANDARD OPERATIONS

Just as the operations of union, intersection and complementation on crisp sets satisfy certain properties, the corresponding three (standard) operations defined on $PF(U)$ satisfy the following list of properties:
For A, B, C in $PF(U)$, we have

1. $A \cup B = B \cup A$; $A \cap B = B \cap A$ (Commutativity)
2. $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$ (Associativity)
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributivity)
4. $A \cup A = A$; $A \cap A = A$ (Idempotence)
5. $A \cup$ □ $= A$; $A \cap$ □ $=$ □
   $A \cup U = U$; $A \cap U = A$ (Identity)
6. $A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$ (Absorption)
7. $(A')' = A$ (Involution)
8. $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$ (De Morgan)
9. $U' =$ □; □$' = U$ (Universal complementation)

Proof: Proofs are straightforward and are left to the reader as an exercise. However, we illustrate the method of proof by writing down the details for one of them.