Aalto University

Distributed Algorithms 2020

9a Round elimination

Proving lower bounds

- **Claim:** solving problem X ***takes ≥ 5 rounds***

Proving lower bounds

- **Claim:** solving problem X ***takes ≥ 5 rounds***
- **Equivalent:** ***any 4-round algorithm A fails*** to solve problem X

Proving lower bounds

- **Claim:** solving problem X ***takes ≥ 5 rounds***
- **Equivalent:** ***any 4-round algorithm A fails*** to solve problem X
- How to show something like this?

Proving lower bounds

- **Claim:** solving problem *X* ***takes ≥ 5 rounds***
- **Equivalent:** ***any 4-round algorithm A fails*** to solve problem *X*
- How to show something like this?
  - huge number of possible 4-round algorithms

Proving lower bounds

- **Easy to do directly:**
  showing that 0-round algorithms fail

Proving lower bounds

- **Easy to do directly:**
  showing that 0-round algorithms fail
- **Hard to do directly:**
  showing that 4-round algorithms fail

Proving lower bounds

- **Easy to do directly:**
  showing that 0-round algorithms fail
- **Hard to do directly:**
  showing that 4-round algorithms fail
- Solution: ***round elimination technique***

Round elimination

problem X_0

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

$$X_1 = \mathrm{re}(X_0)$$

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

$\rightarrow$ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

$\rightarrow$ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

$X_2 = \mathrm{re}(X_1)$

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

$X_3 = \mathrm{re}(X_2)$

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

problem $X_4 = \mathrm{re}(X_3)$

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

→ A_4 solves problem $X_4 = \mathrm{re}(X_3)$ in 0 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

→ A_4 solves problem $X_4 = \mathrm{re}(X_3)$ in 0 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

→ A_4 solves problem $X_4 = \mathrm{re}(X_3)$ in 0 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

→ A_4 solves problem $X_4 = \mathrm{re}(X_3)$ in 0 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

→ A_4 solves problem $X_4 = \mathrm{re}(X_3)$ in 0 rounds

Round elimination

Assume: A_0 solves problem X_0 in 4 rounds

→ A_1 solves problem $X_1 = \mathrm{re}(X_0)$ in 3 rounds

→ A_2 solves problem $X_2 = \mathrm{re}(X_1)$ in 2 rounds

→ A_3 solves problem $X_3 = \mathrm{re}(X_2)$ in 1 round

→ A_4 solves problem $X_4 = \mathrm{re}(X_3)$ in 0 rounds

Round elimination

Assume: A_0 solves problem X_0 in 10 rounds

$\rightarrow A_1$ solves problem $X_1 = \mathrm{re}(X_0)$ in 9 rounds

$\ldots$

$\rightarrow A_{10}$ solves problem $X_{10} = \mathrm{re}(X_9)$ in 0 rounds

Round elimination turns problem X_0 into a new problem X_1 that **can be solved 1 round faster**

Round elimination turns problem X_0 into a new problem X_1 that **can be solved 1 round faster**

Bipartite locally verifiable problems

Locally verifiable

- Defined using “local constraints”

Locally verifiable

- Defined using "local constraints"
- **Example:** ***vertex coloring***
    - constraint on each edge: endpoints must have different colors

Locally verifiable

- Defined using “local constraints”
- **Example:** ***vertex coloring***
  - constraint on each edge: endpoints must have different colors
- **Example:** ***maximal independent set***
  - constraint on each edge: independence
  - constraint on each node: maximality

Locally verifiable

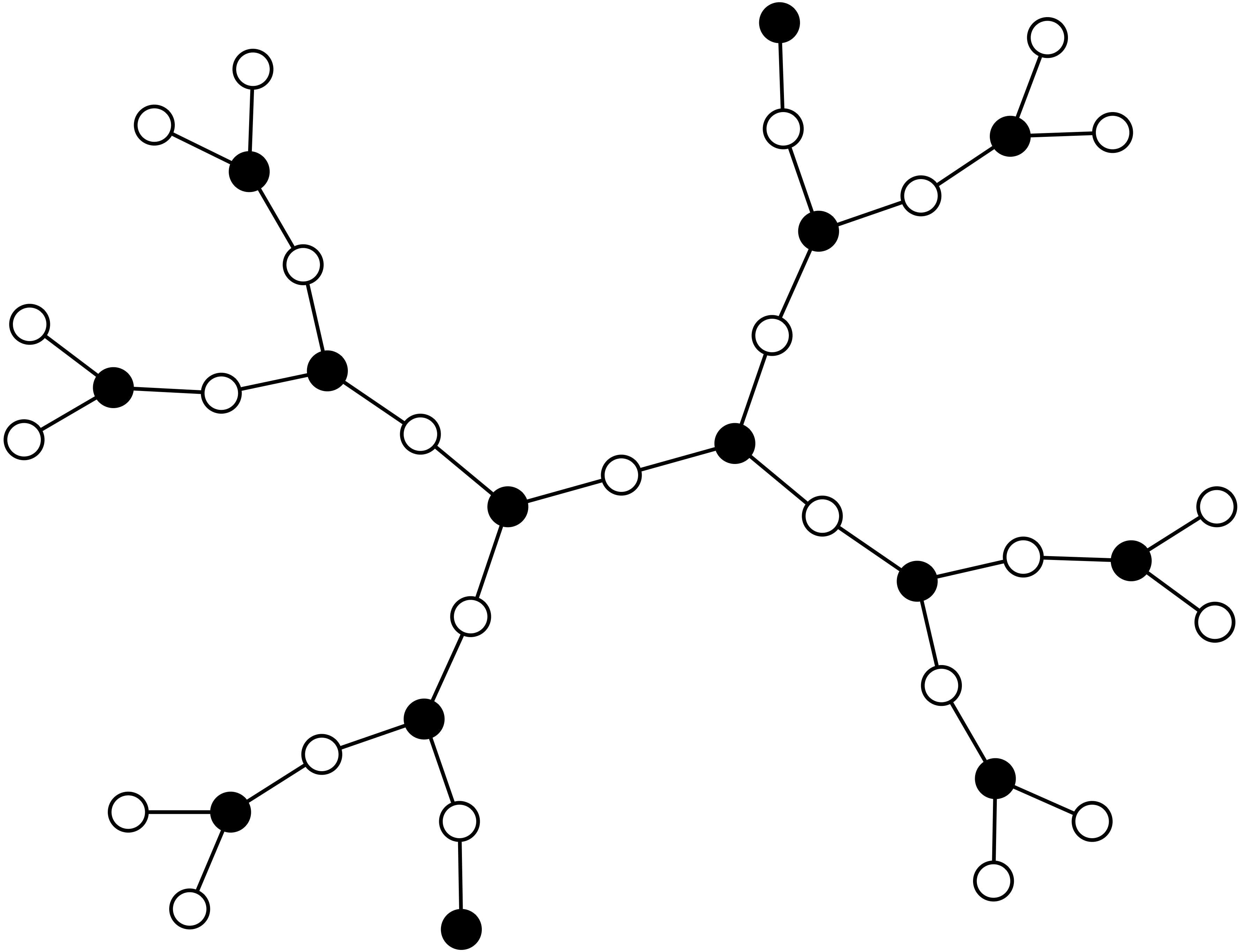
- **Not locally verifiable:** *spanning tree*
  - "connectivity" is a global constraint
  - "acyclicity" is a global constraint

Bipartite locally verifiable problem

Bipartite locally verifiable problem

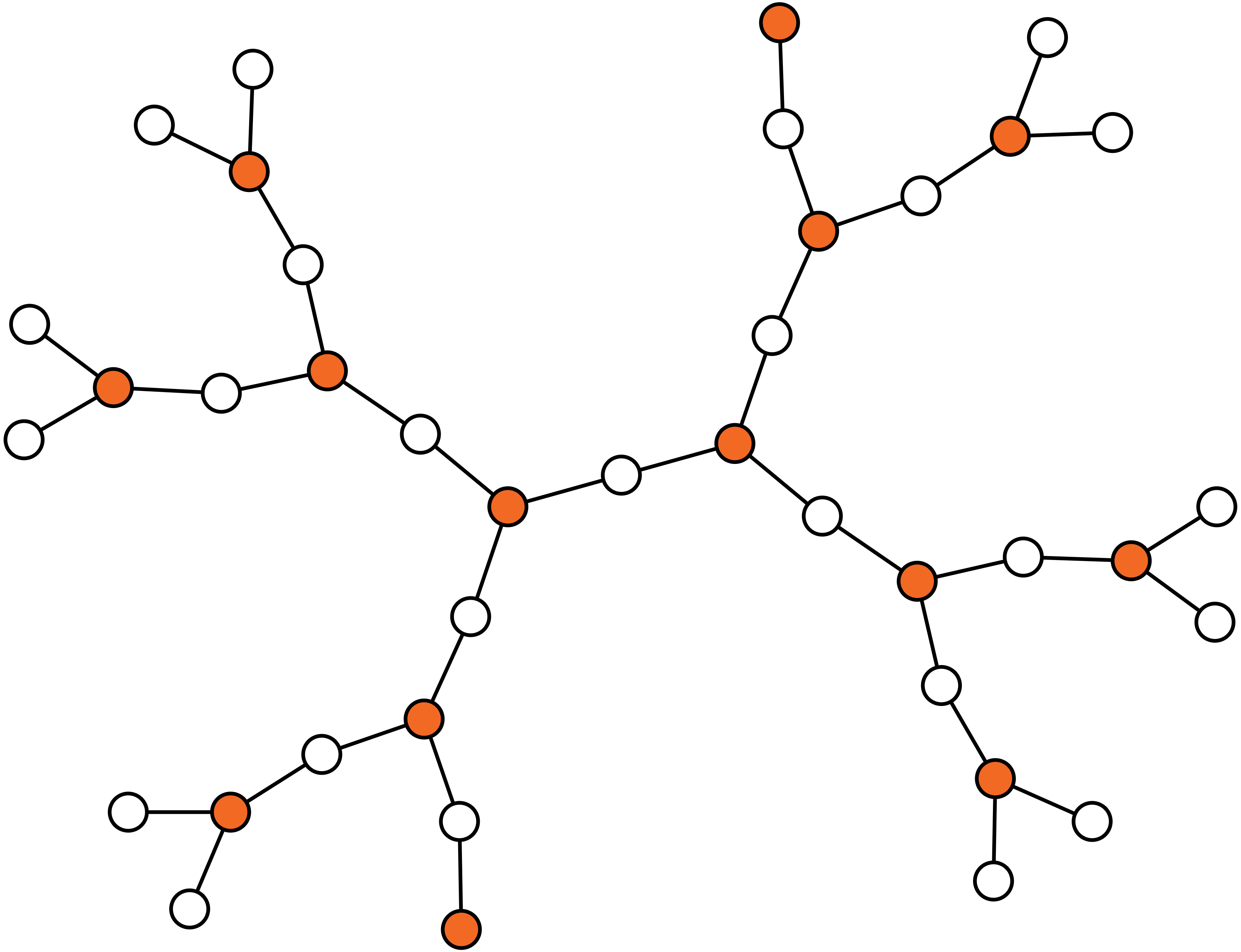
defined on regular trees

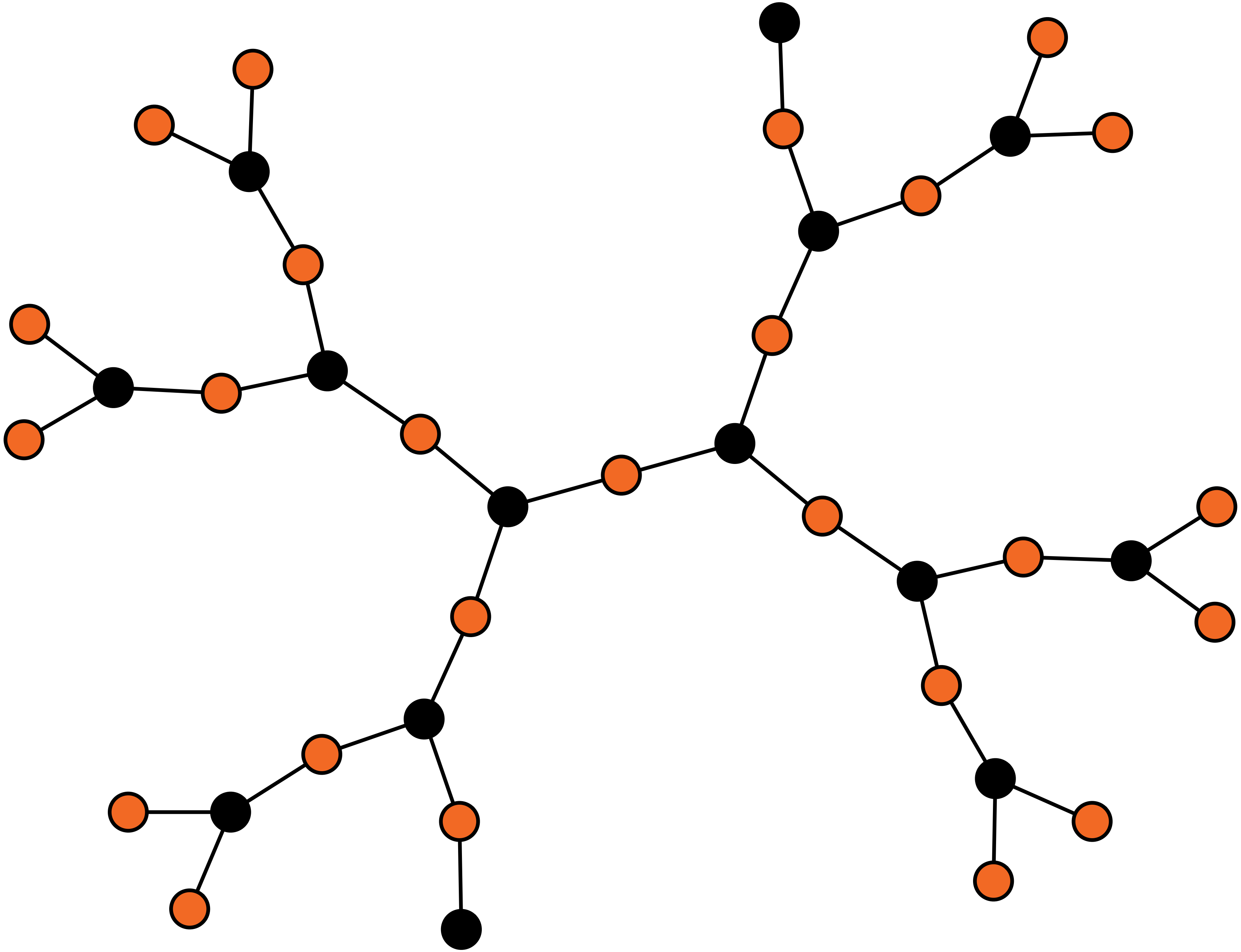
Weak 3-labeling



Weak 3-labeling

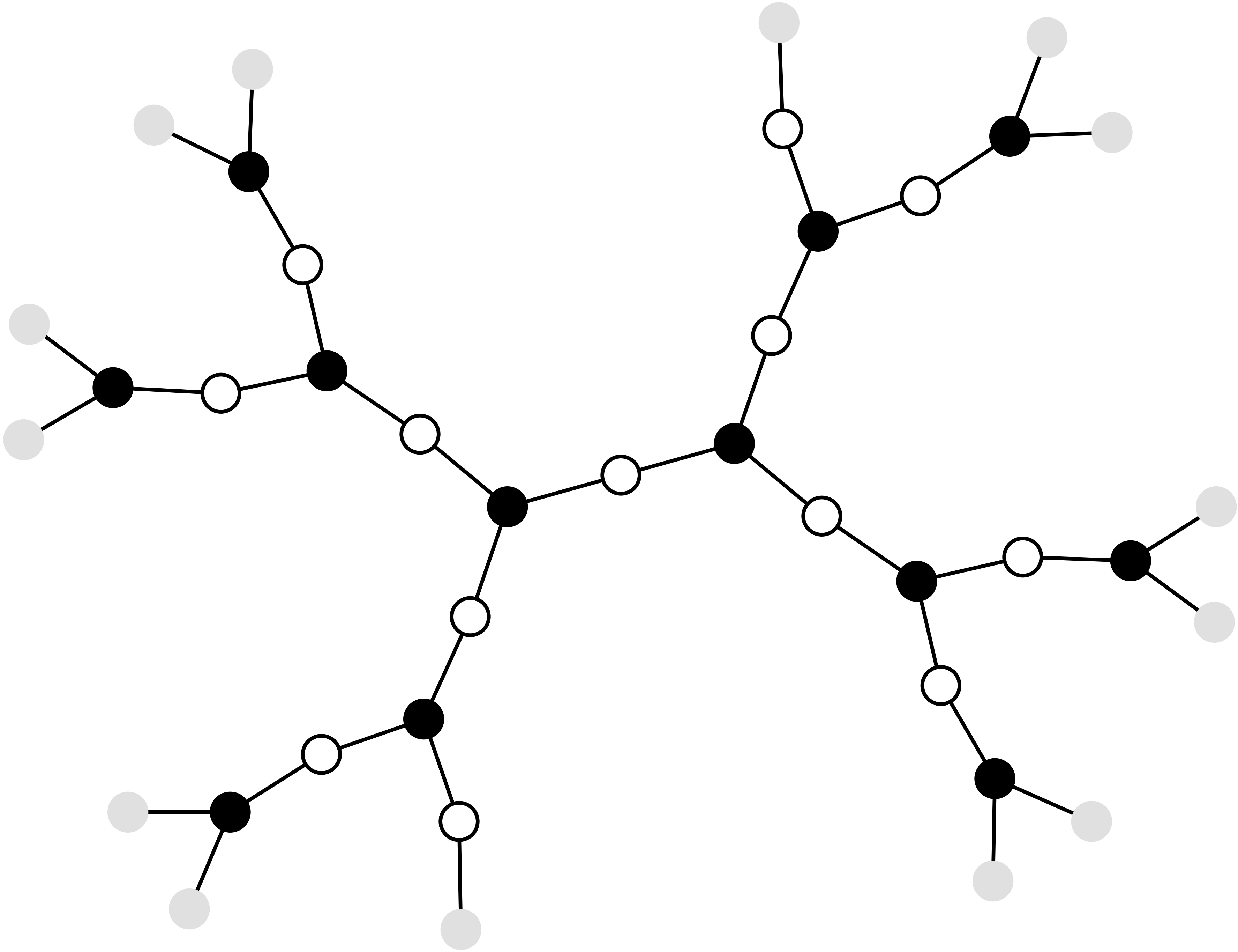
Weak 3-labeling



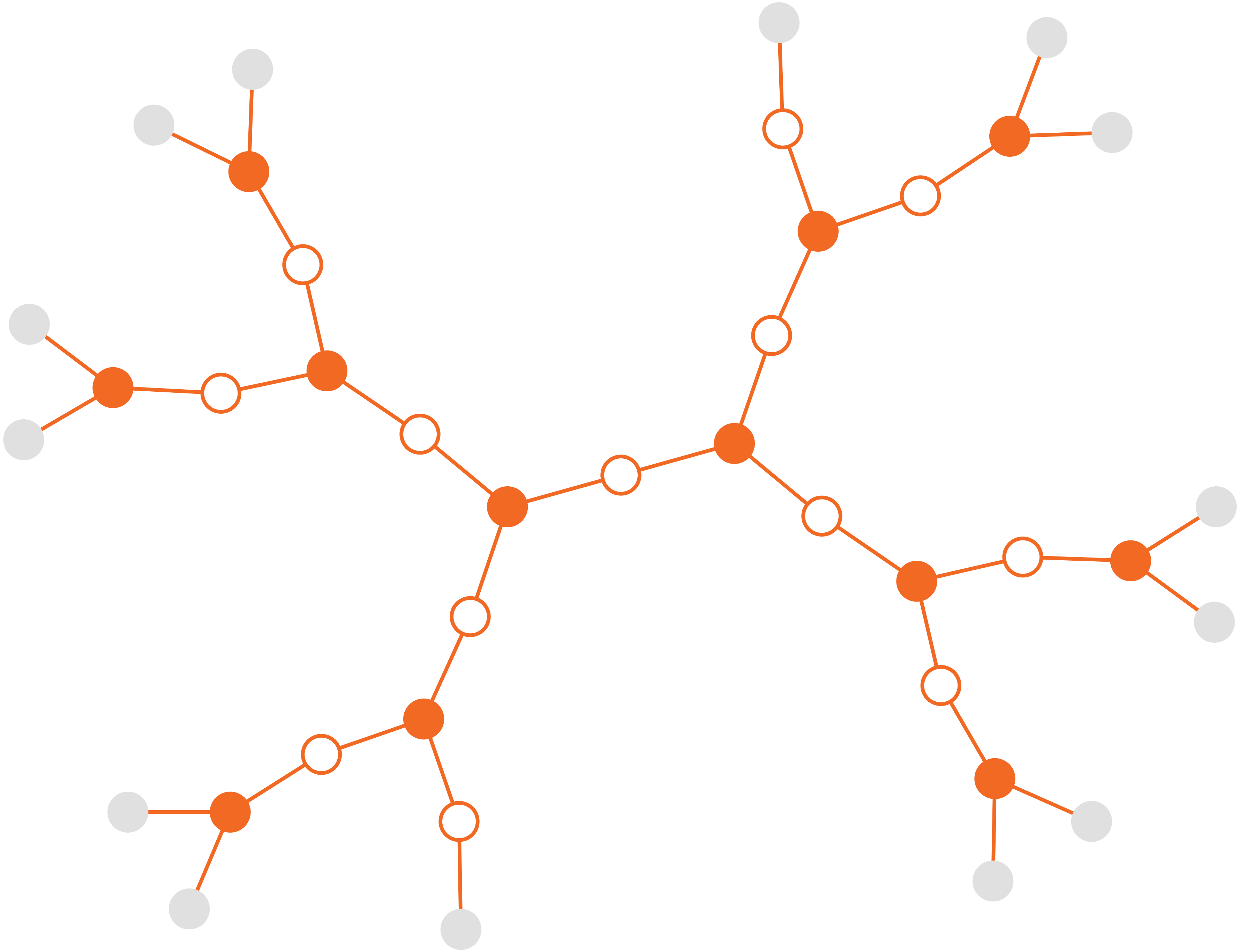


Weak
3-labeling

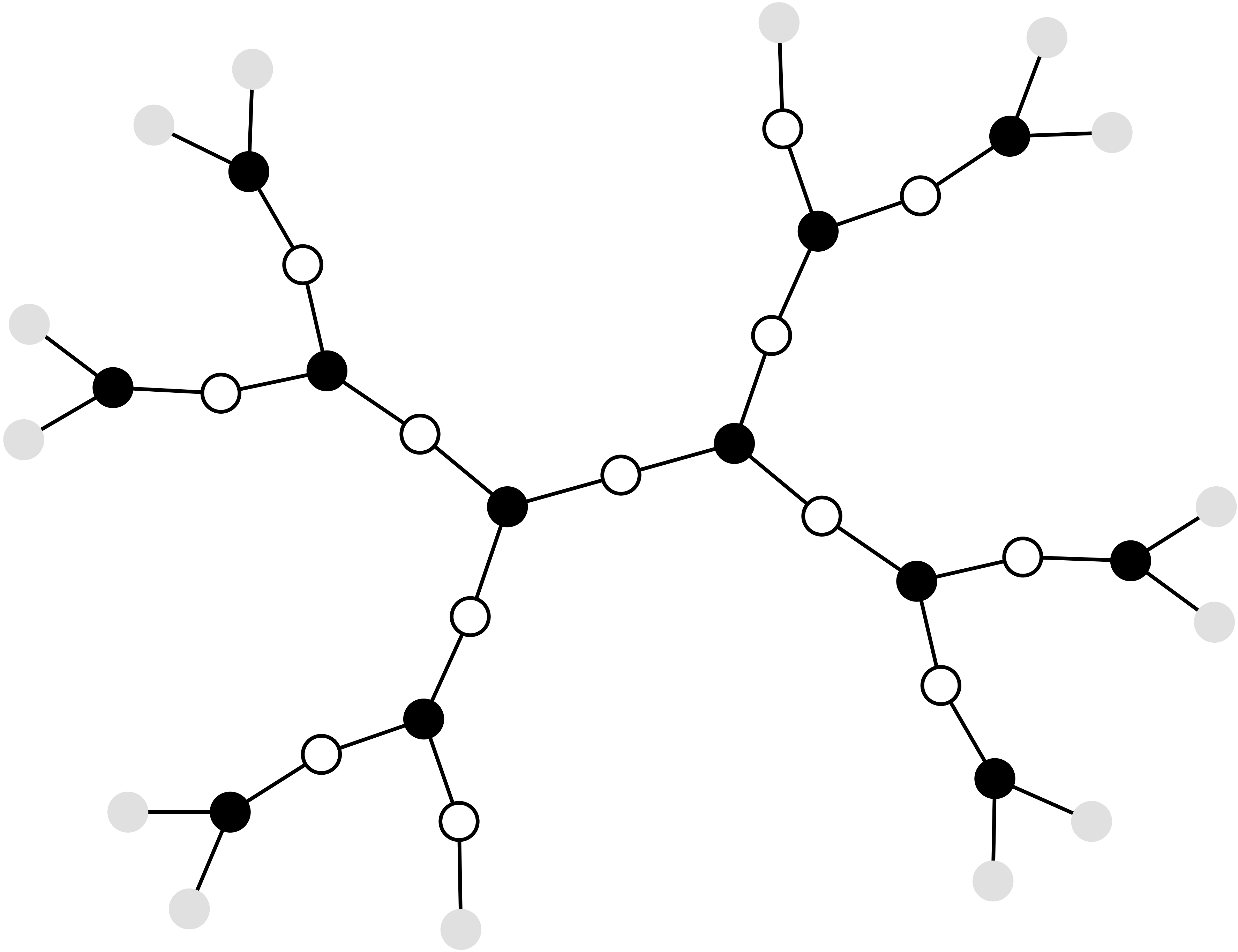
Weak 3-labeling



Weak 3-labeling

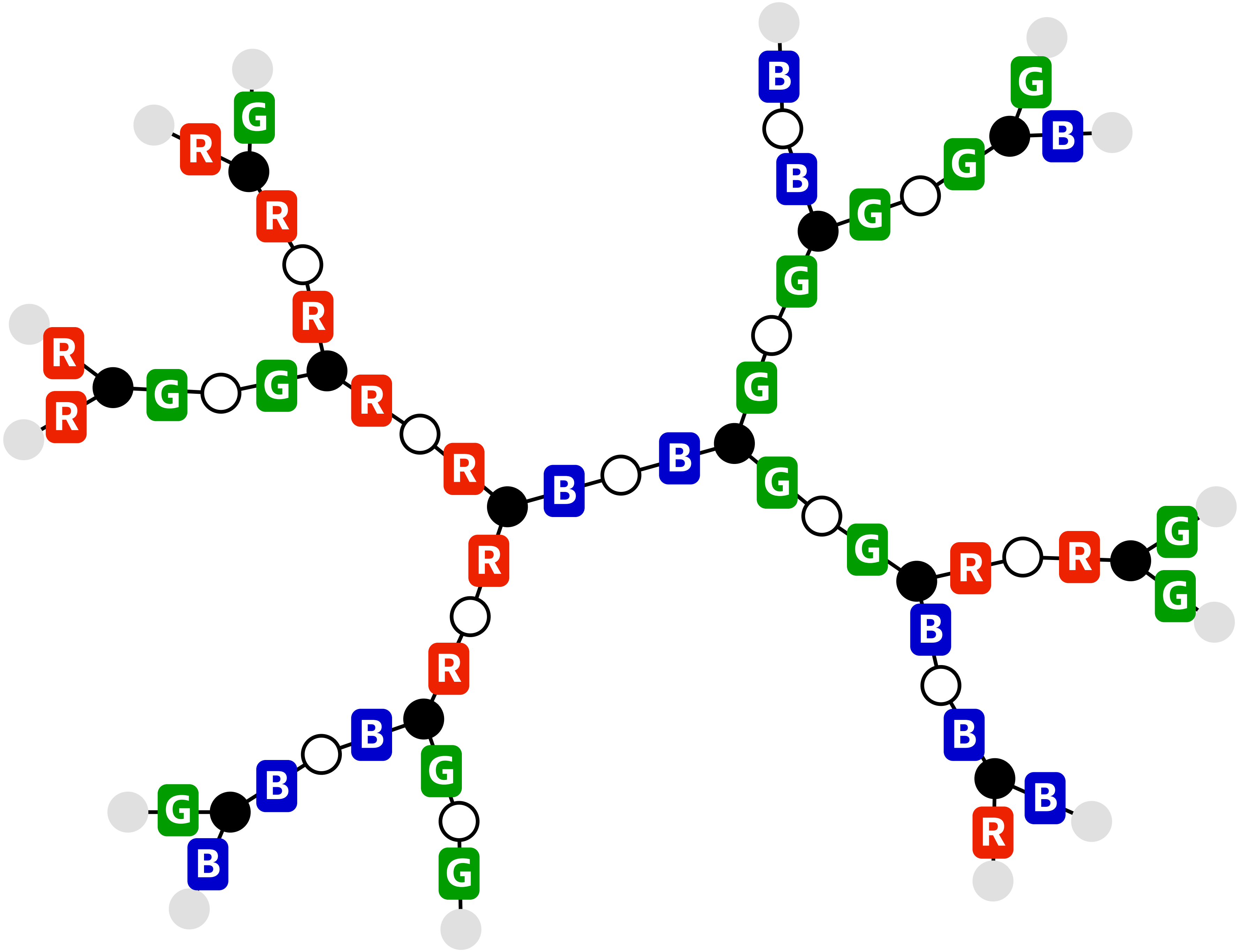


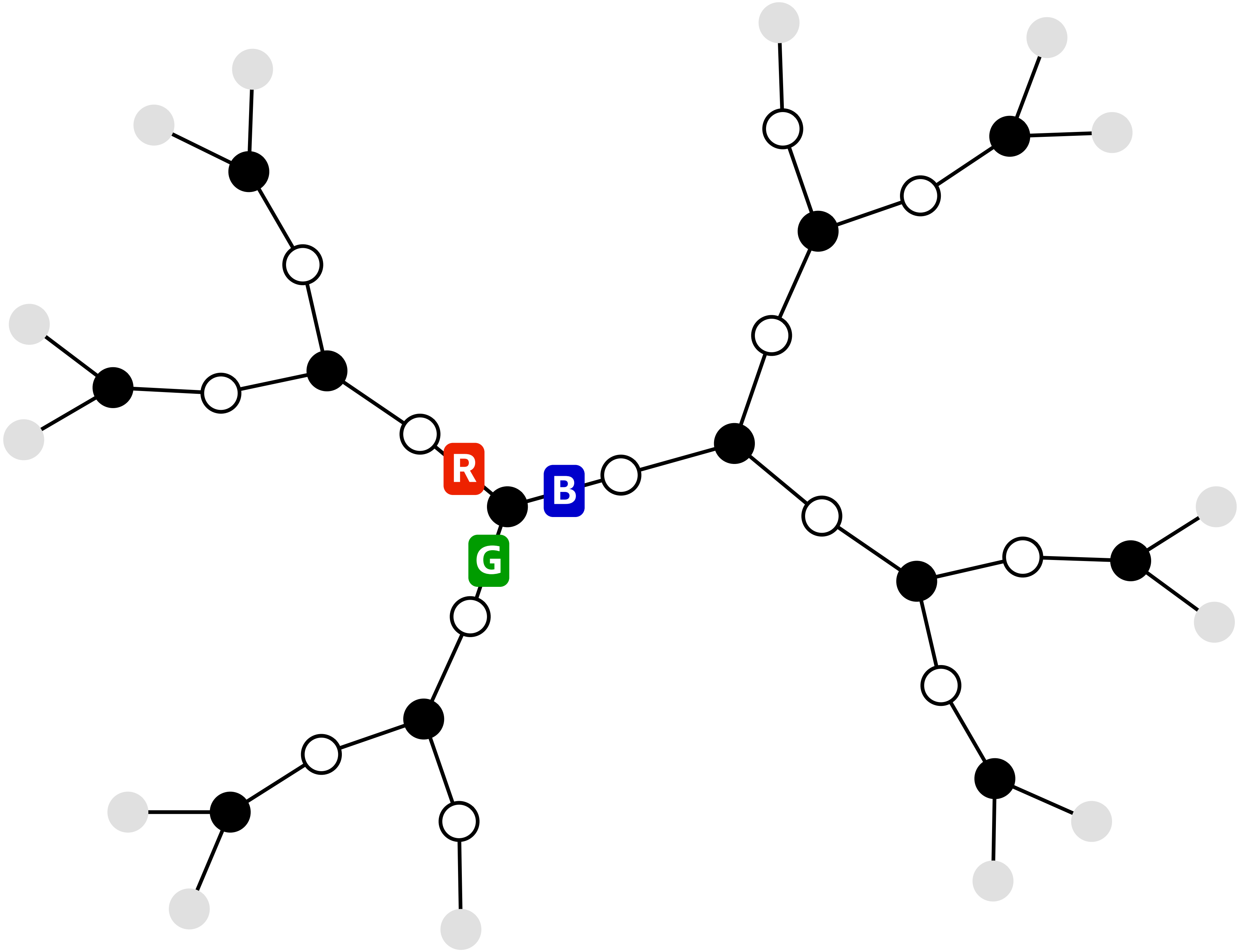
RGB



Weak 3-labeling

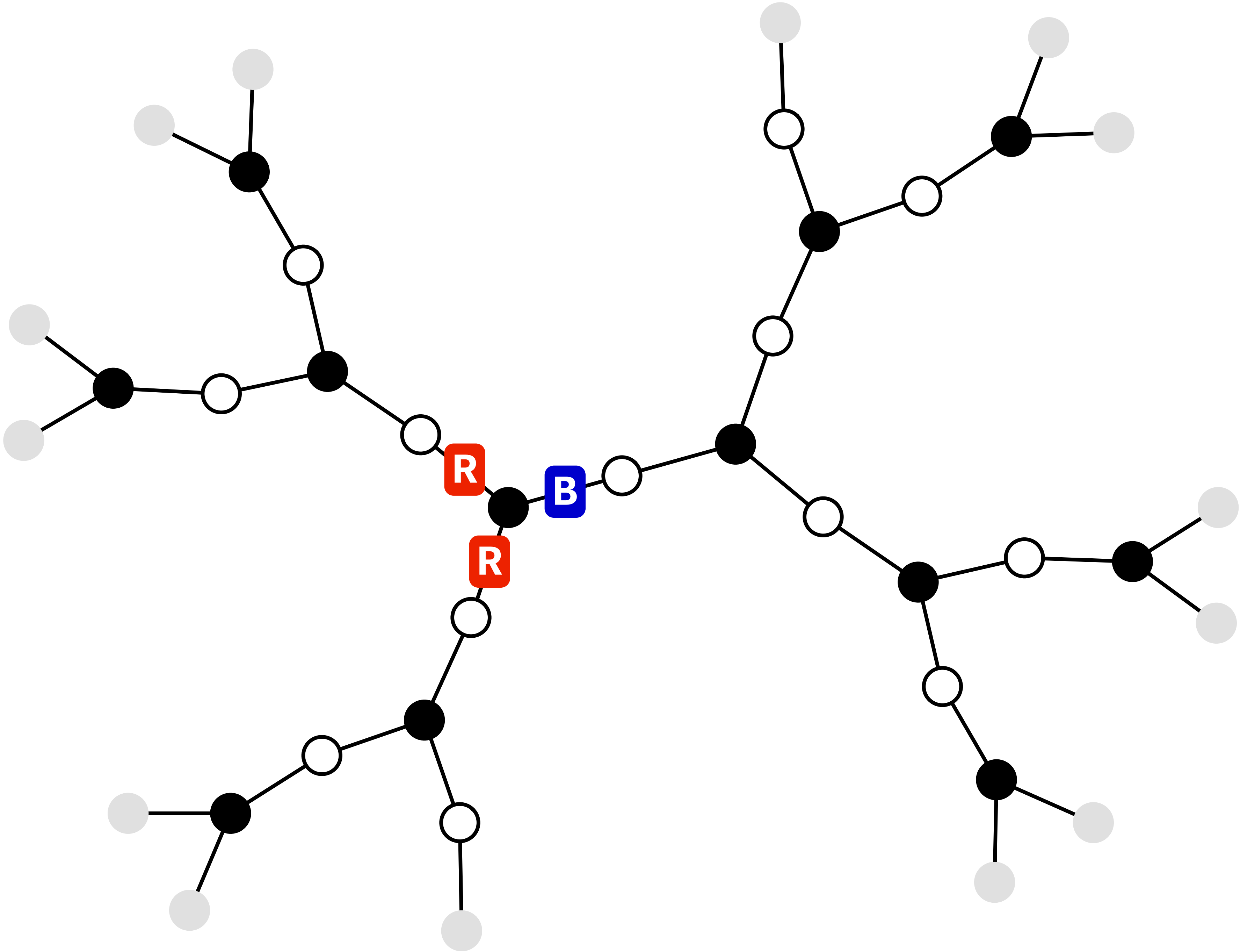
Weak 3-labeling



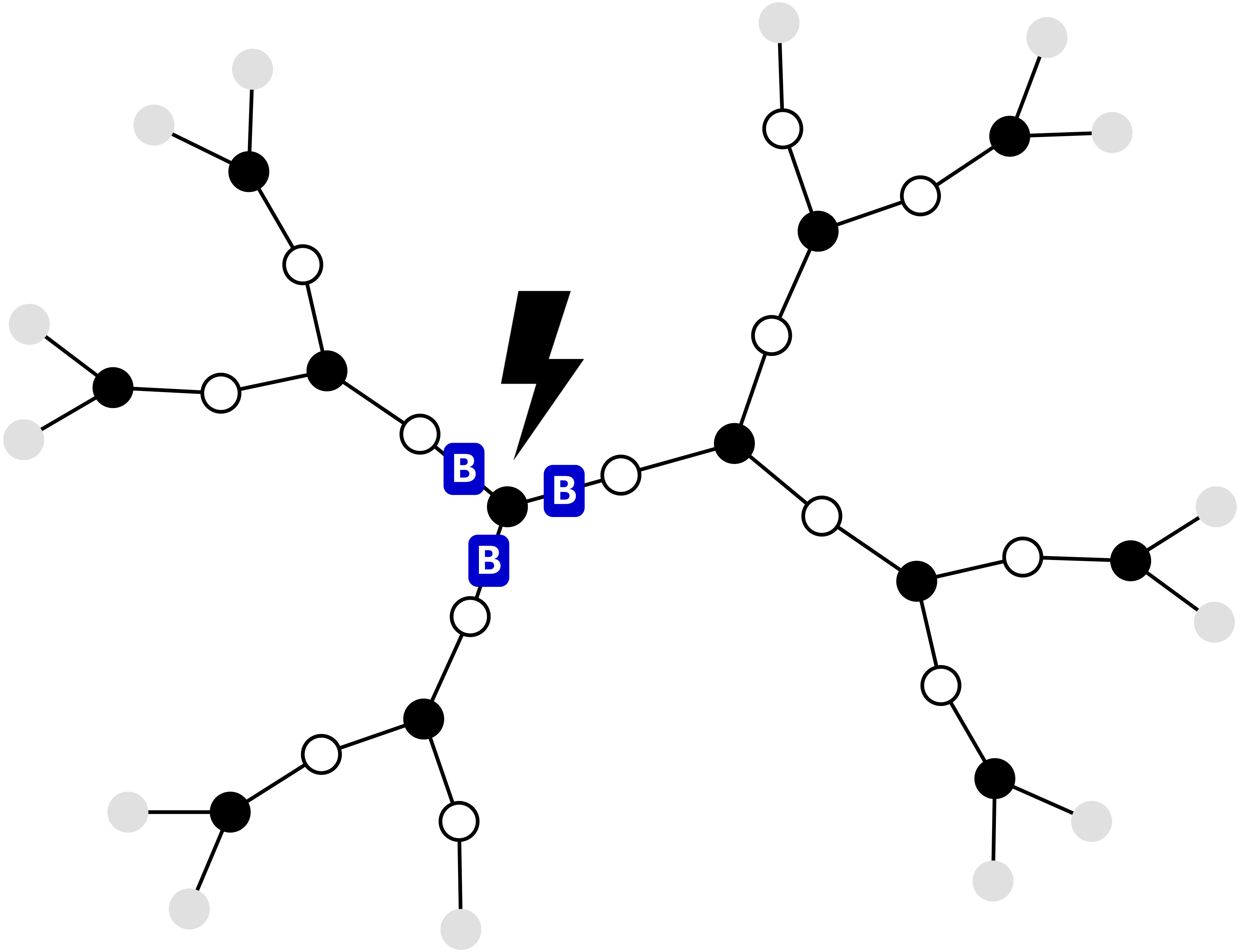


Weak 3-labeling

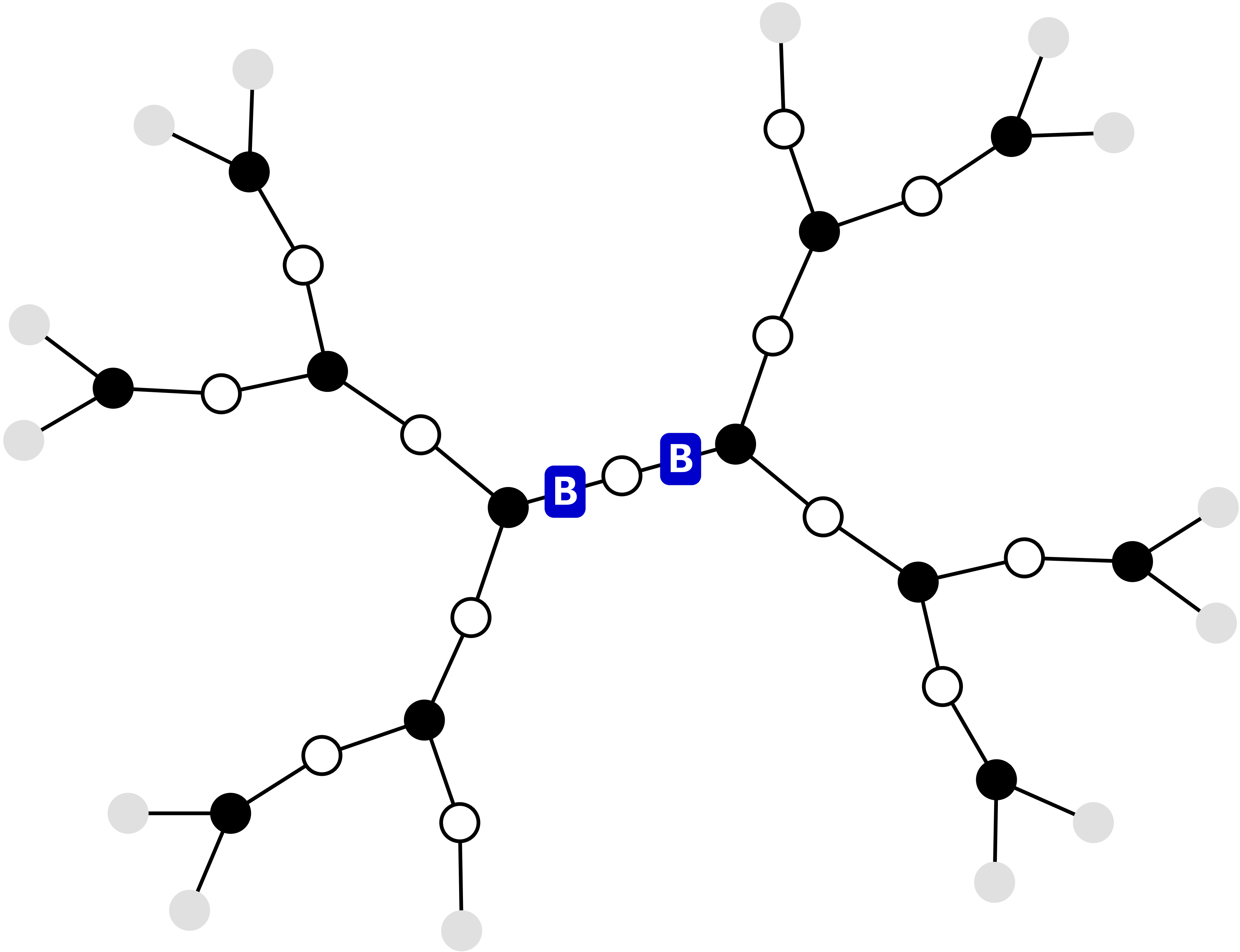
Weak 3-labeling



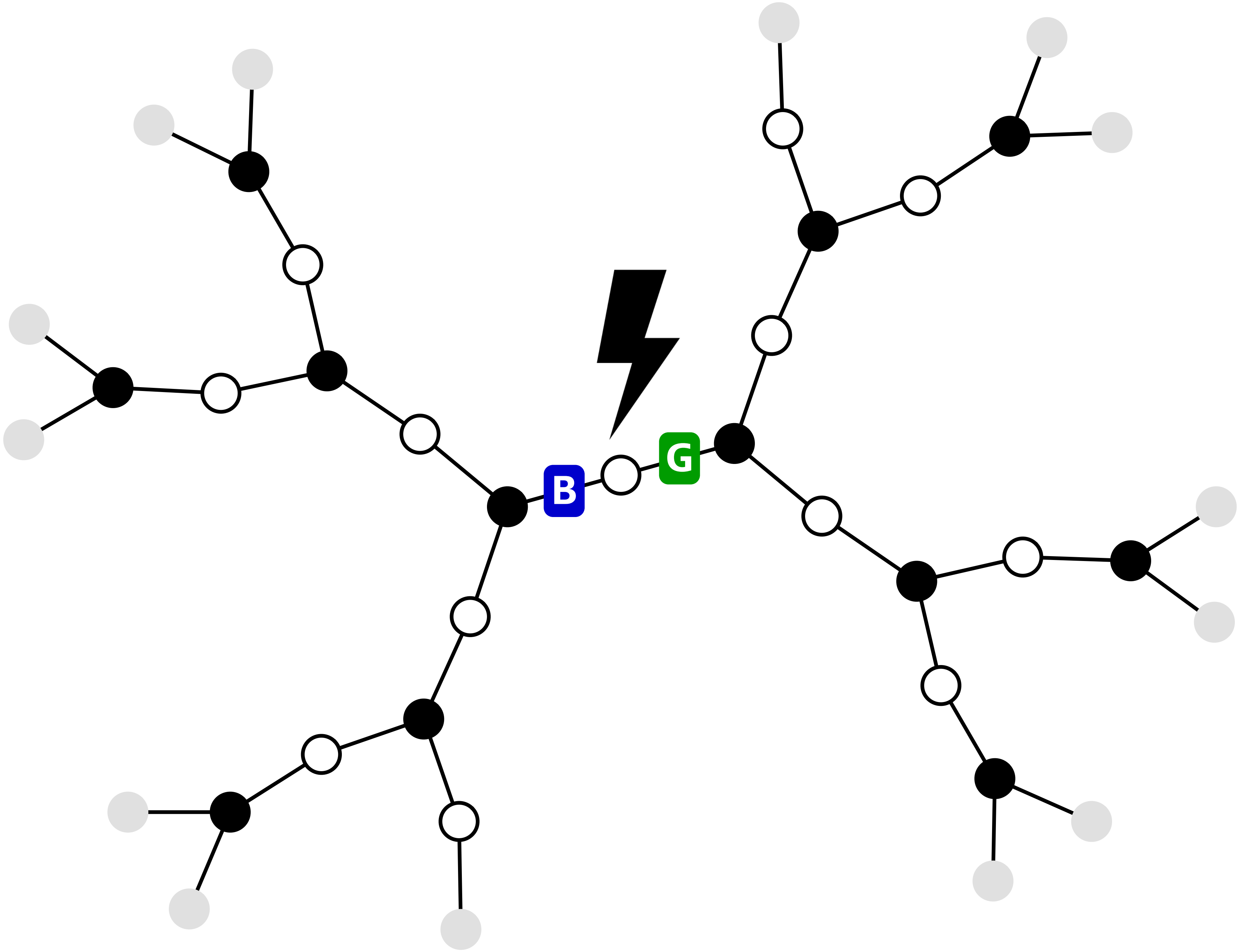
Weak 3-labeling



Weak 3-labeling



Weak 3-labeling



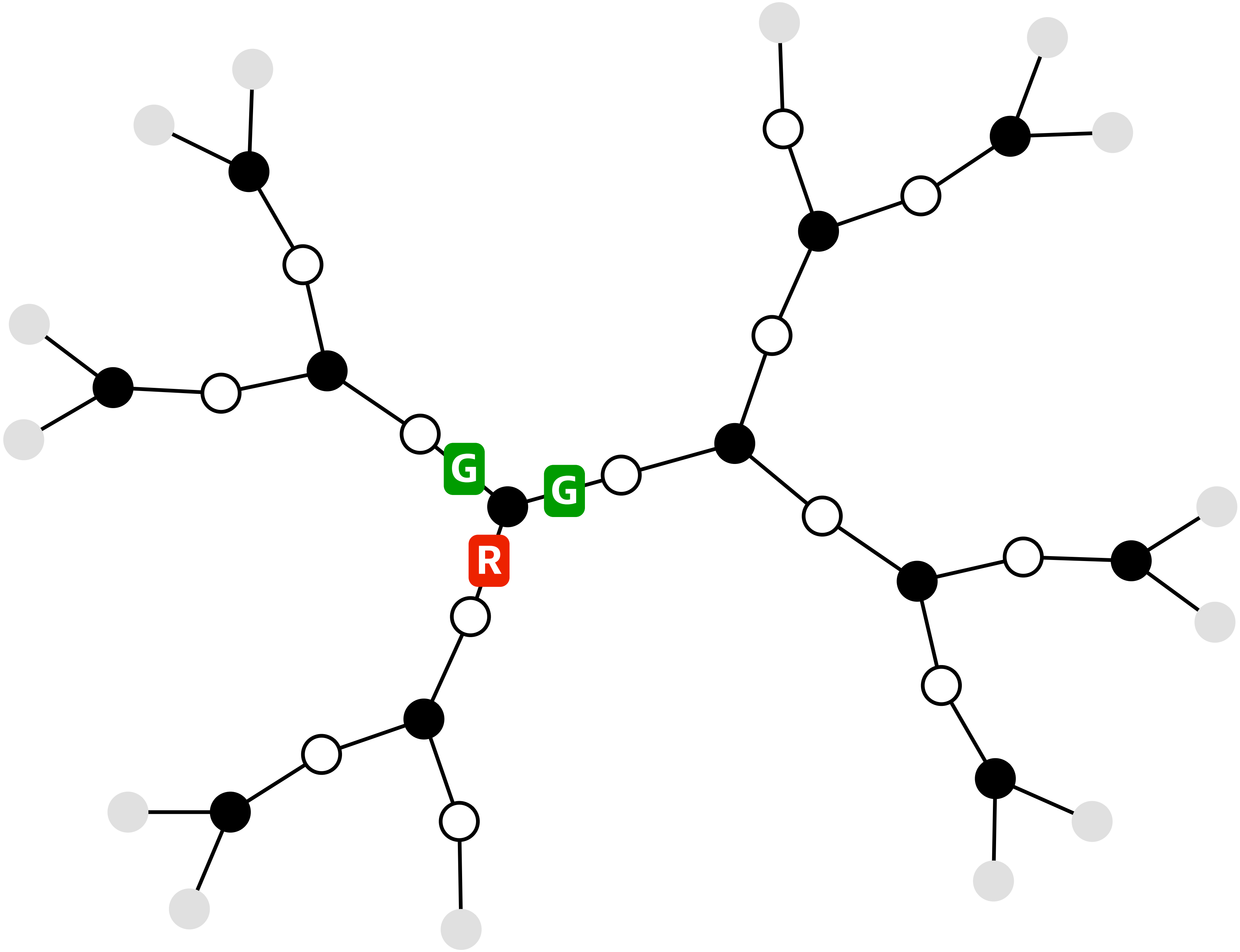
X_0: labels **R**, **G**, **B**

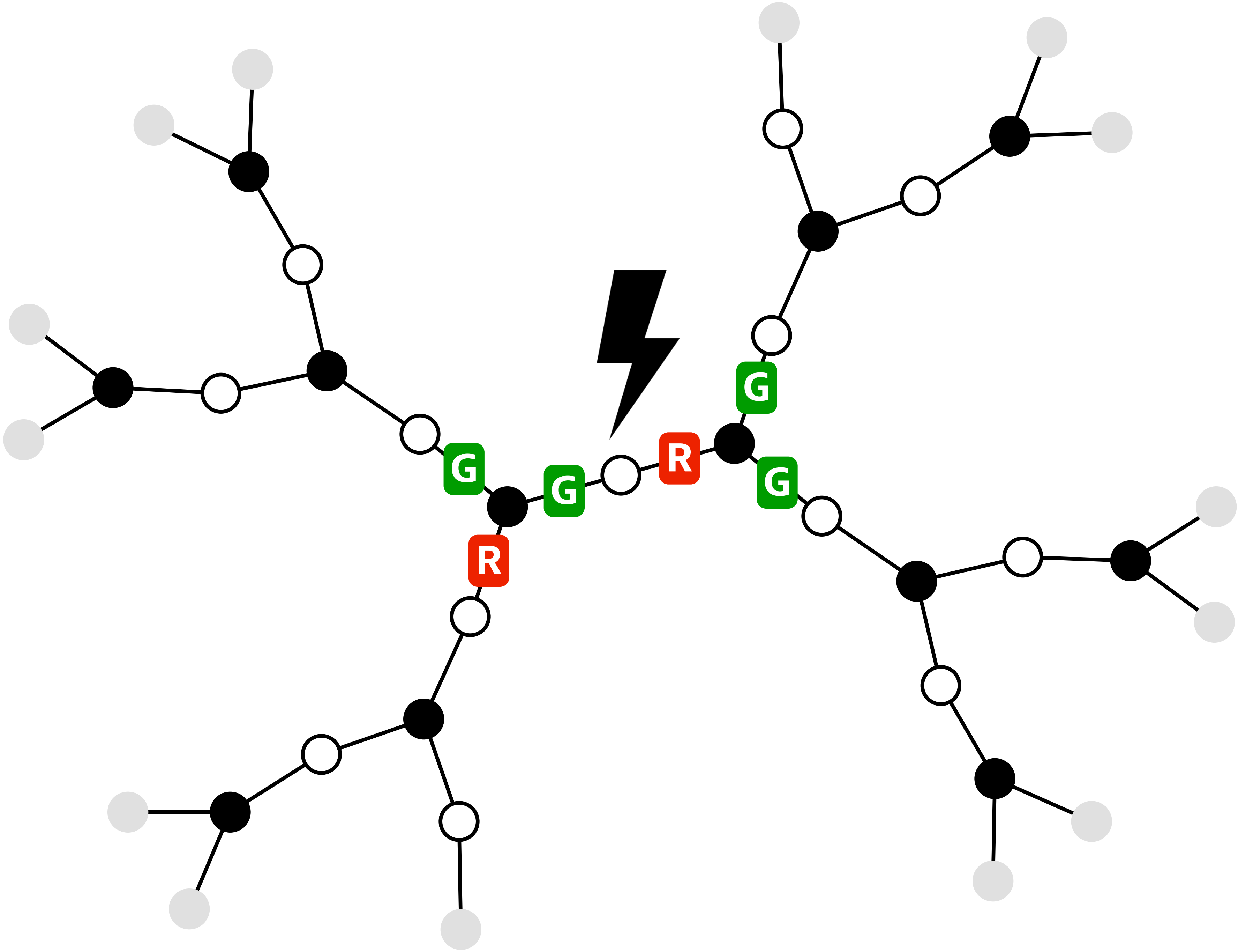
- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

Not solvable
in 0 rounds


G
G
R


G
G
G
G
R
R

X_0: labels R, G, B

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$X_1 = \mathrm{re}(X_0)$:

X_0: labels R, G, B

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$X_1 = \mathrm{re}(X_0)$:

X_0: labels R, G, B

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$X_1 = \mathrm{re}(X_0)$:

X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

X_1 = re(X_0): labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$X_1 = \mathrm{re}(X_0)$: labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

Not solvable
in 0 rounds

X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

Not solvable in 1 round

$X_1 = \mathrm{re}(X_0)$: labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

Not solvable in 0 rounds

$\boldsymbol{X_0}$**:** labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$\boldsymbol{X_1} = \mathbf{re}(\boldsymbol{X_0})$**:** labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

$\boldsymbol{X_2} = \mathbf{re}(\boldsymbol{X_1})$**:**

X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$X_1 = \mathrm{re}(X_0)$: labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

$X_2 = \mathrm{re}(X_1)$: labels **R**, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

$\boldsymbol{X_0}$**:** labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

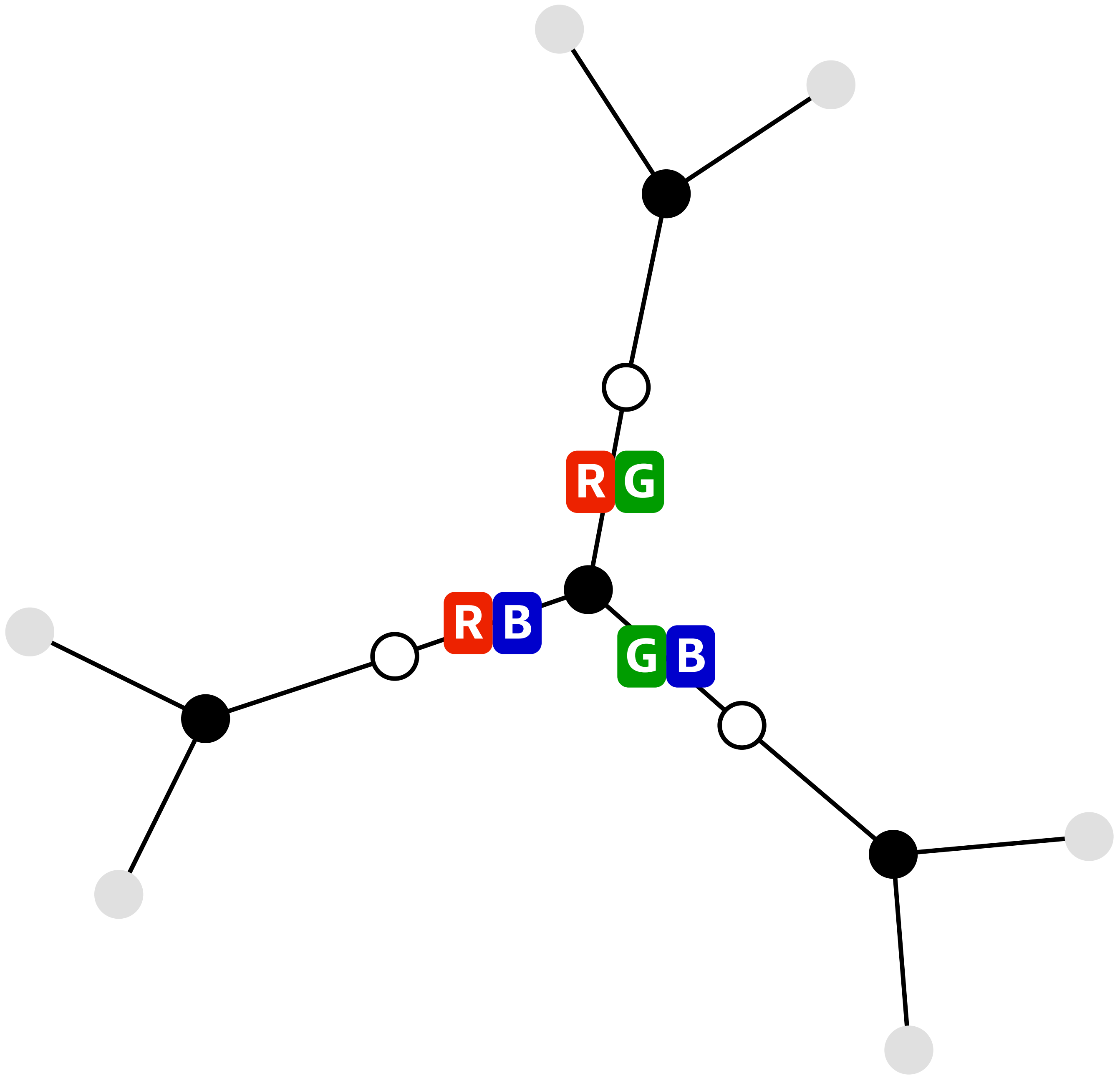
$\boldsymbol{X_1}$ **= re(**$\boldsymbol{X_0}$**):** labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

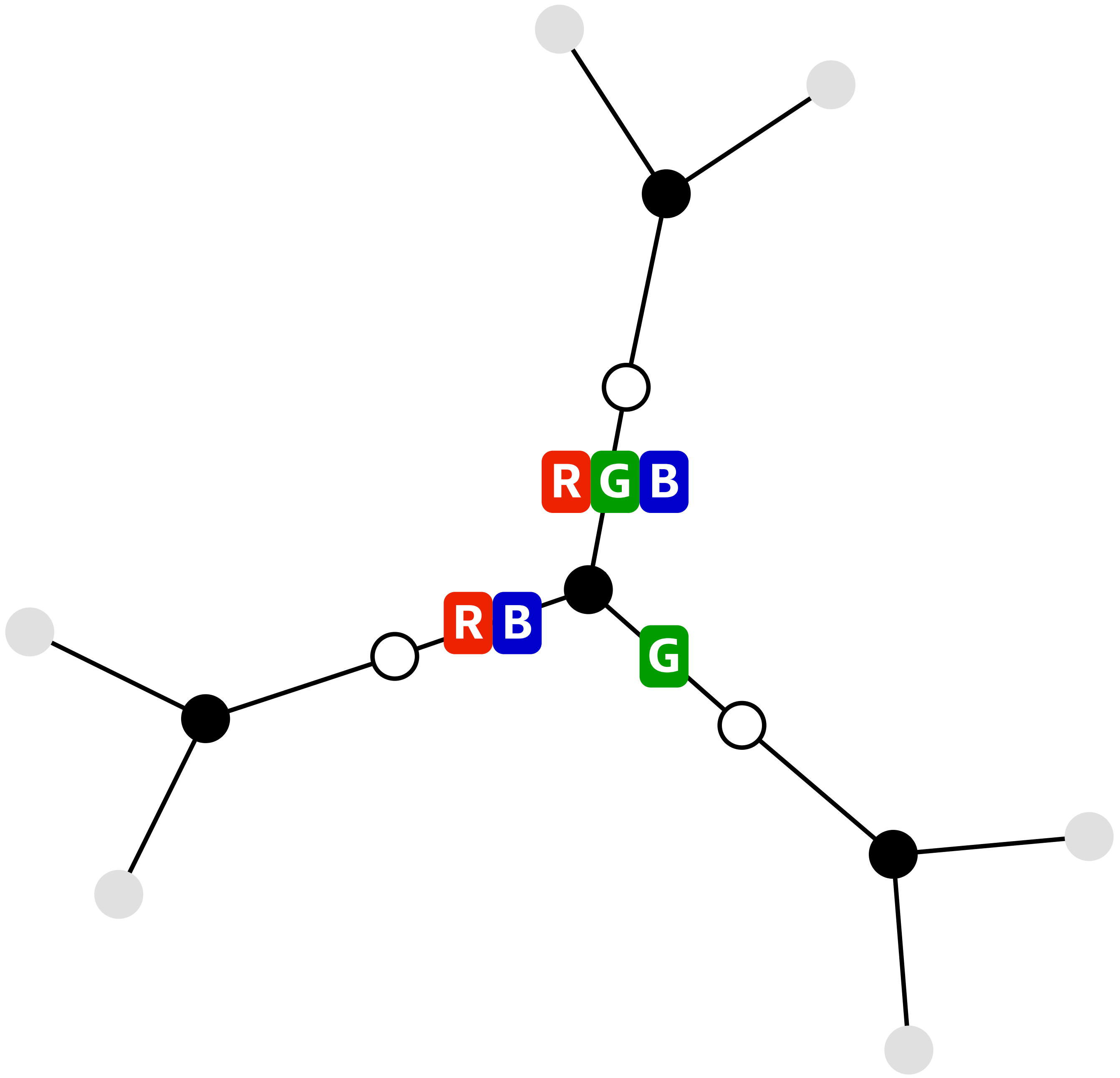
$\boldsymbol{X_2}$ **= re(**$\boldsymbol{X_1}$**):** labels **R**, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

- active (deg 3): not all with **R**,
  not all with **G**, not all with **B**

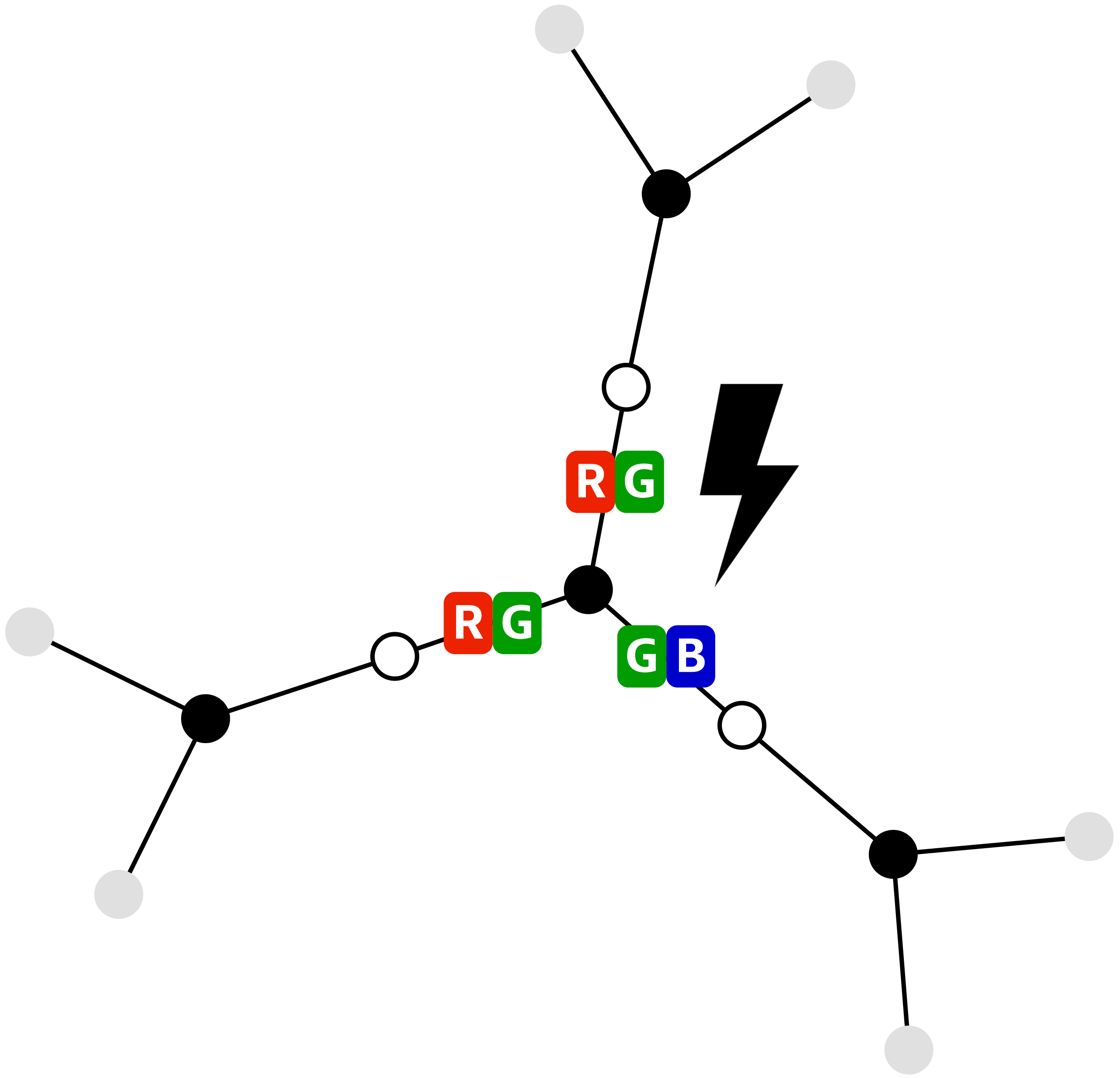
$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

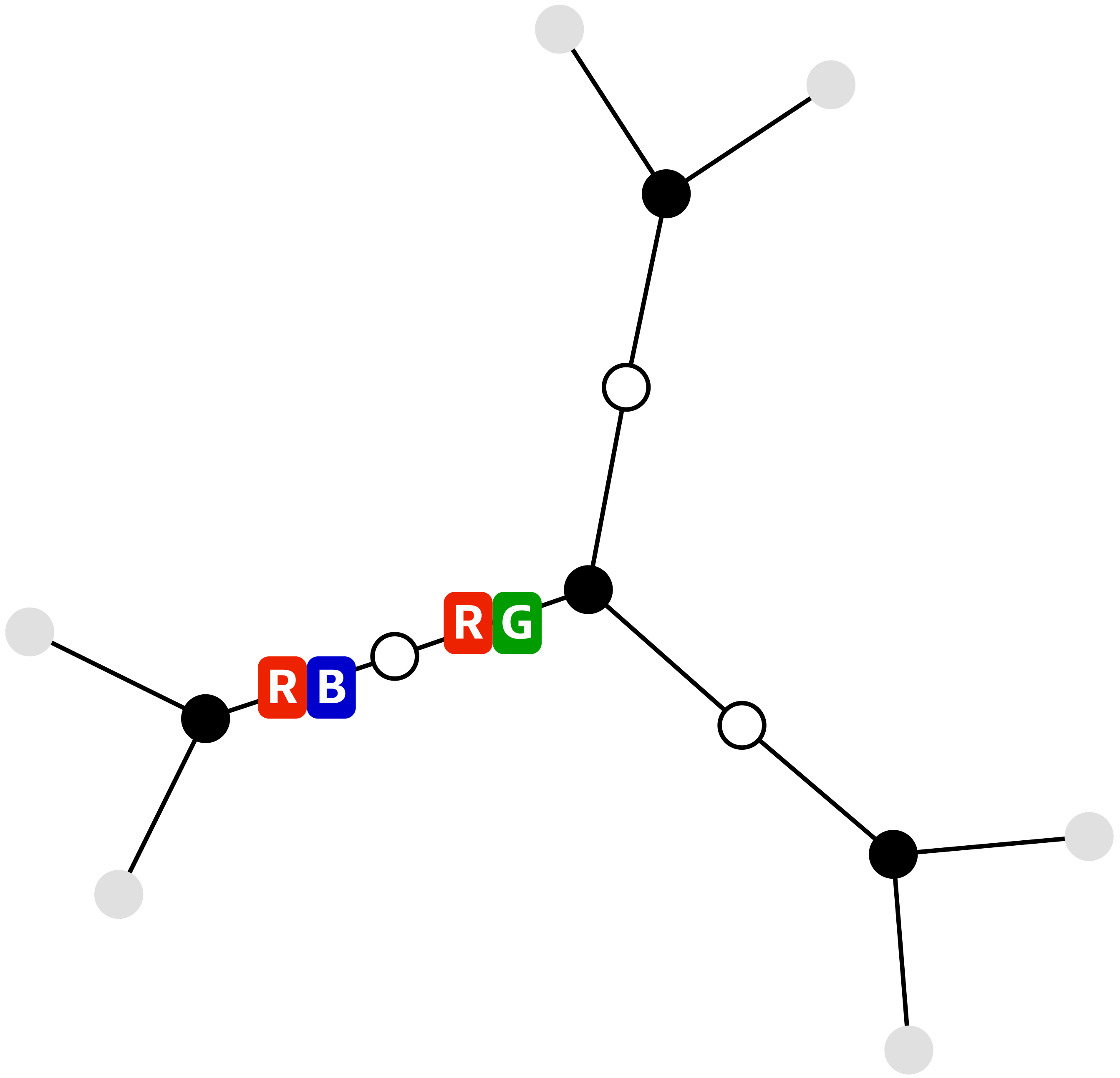
$X_1 = \mathrm{re}(X_0)$: labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

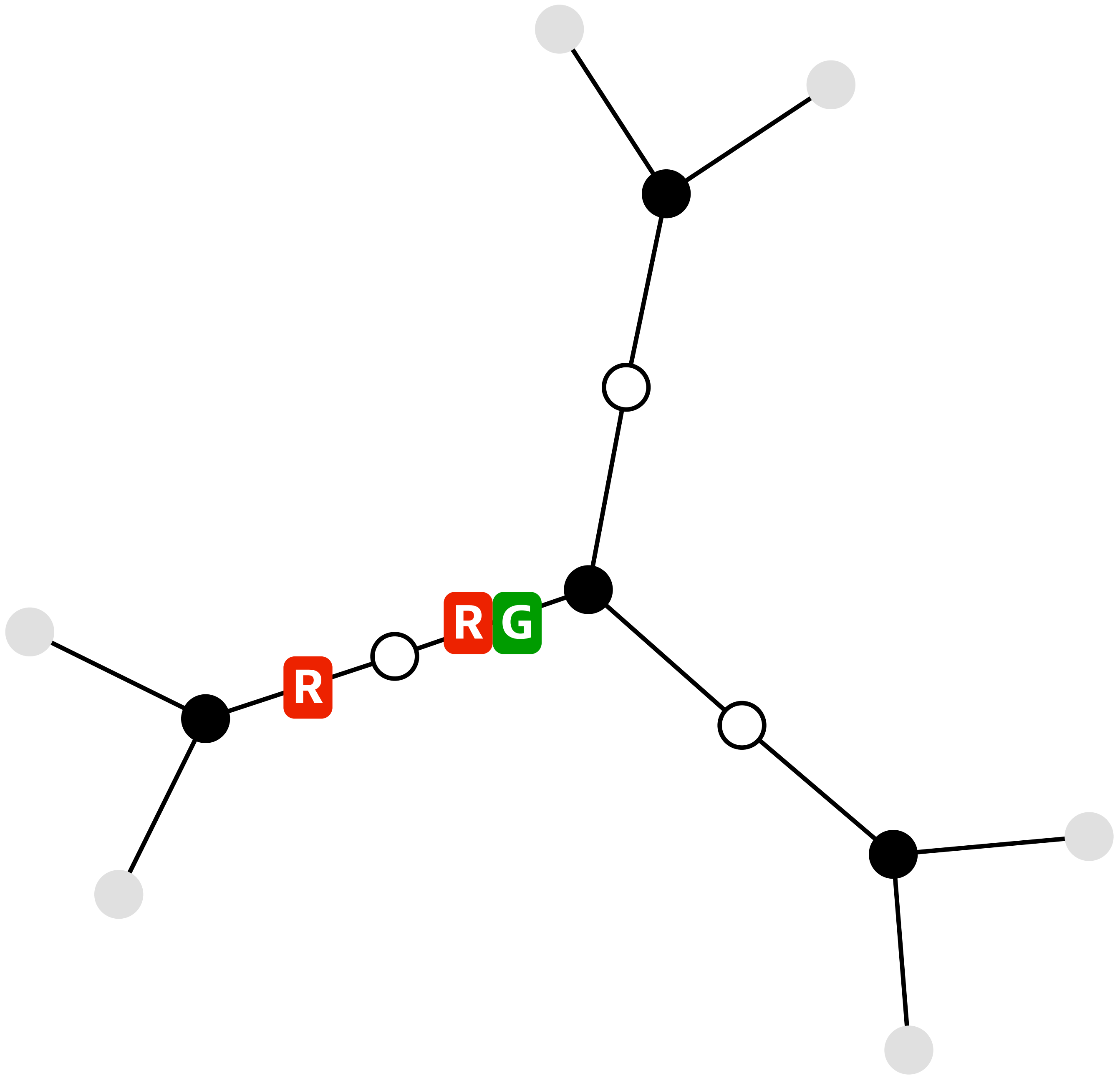
$X_2 = \mathrm{re}(X_1)$: labels **R**, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

- active (deg 3): not all with **R**, not all with **G**, not all with **B**
- passive (deg 2): non-empty intersection

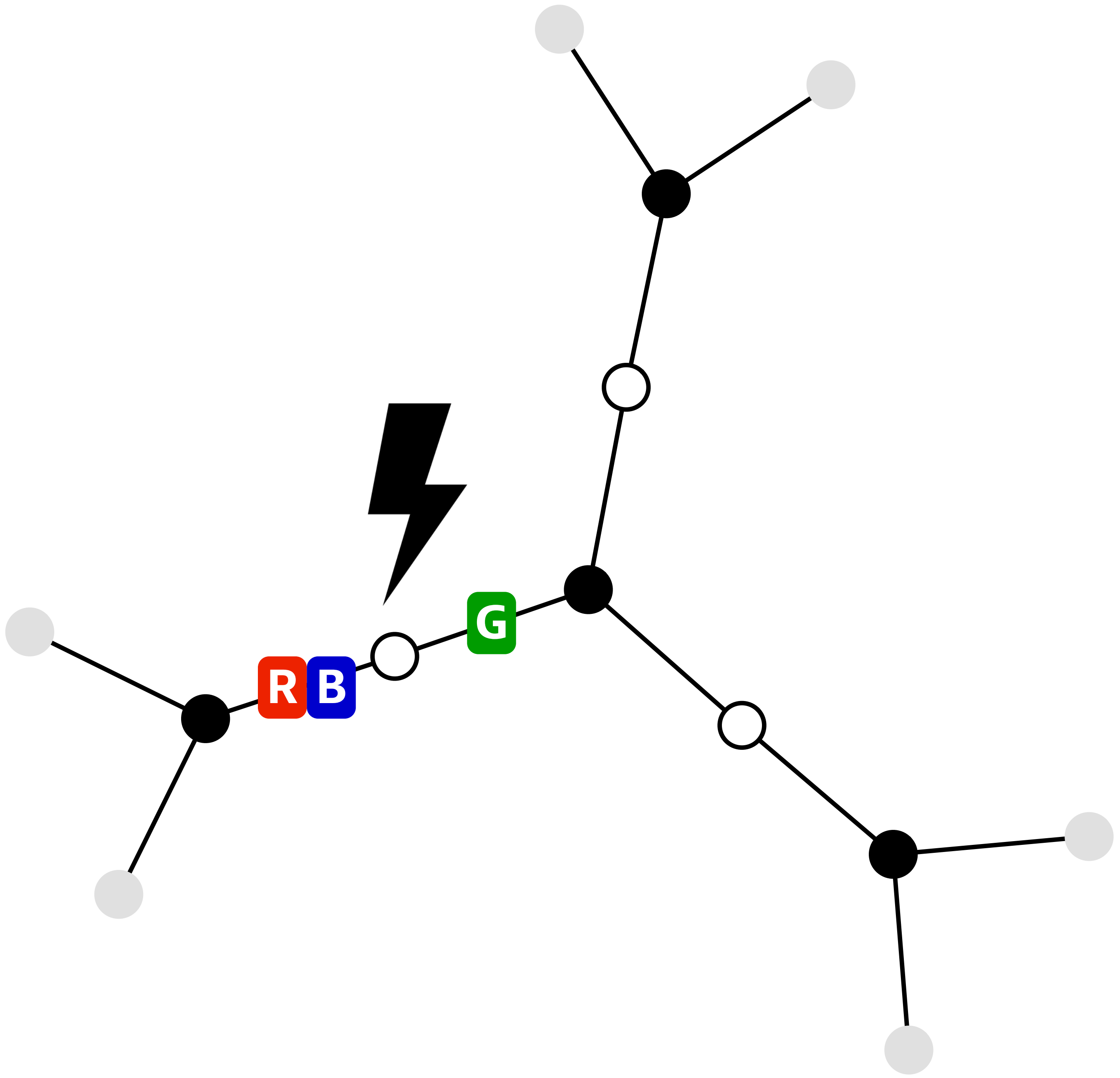
$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

X_1 = re(X_0): labels **R**, **G**, **B**

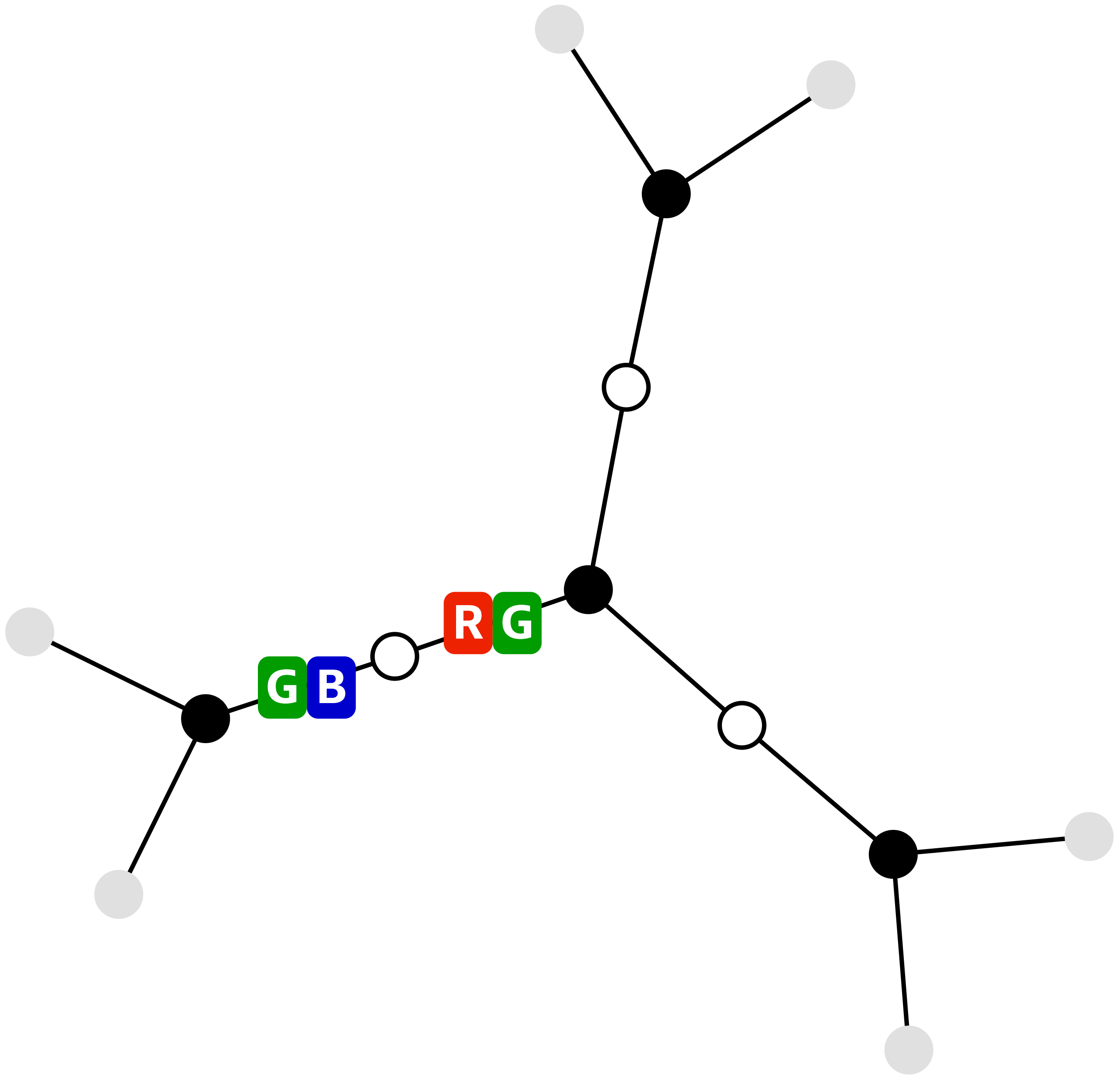
- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

X_2 = re(X_1): labels **R**, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

- active (deg 3): not all with **R**, not all with **G**, not all with **B**
- passive (deg 2): non-empty intersection

Solvable in 0 rounds

$$X_2 = \mathrm{re}(\mathrm{re}(X_0))$$



X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

X_1 = re(X_0): labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

X_2 = re(X_1): labels **R**, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

- active (deg 3): not all with **R**, not all with **G**, not all with **B**
- passive (deg 2): non-empty intersection

$T = 0$

X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

X_1 = re(X_0): labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

$T = 1$

X_2 = re(X_1): labels **R**, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

- active (deg 3): not all with **R**, not all with **G**, not all with **B**
- passive (deg 2): non-empty intersection

$T = 0$

X_0: labels **R**, **G**, **B**

- active (deg 3): not all **R**, not all **G**, not all **B**
- passive (deg 2): equality

$T = 2$

X_1 = re(X_0): labels **R**, **G**, **B**

- active (deg 2): equality
- passive (deg 3): not all **R**, not all **G**, not all **B**

$T = 1$

X_2 = re(X_1): labels **R**, **G**, **B**, **RG**, **RB**, **GB**, **RGB**

- active (deg 3): not all with **R**, not all with **G**, not all with **B**
- passive (deg 2): non-empty intersection

$T = 0$