WTW 285 (Discrete Structures)
Lecture 2 (Wednesday 17 July 2019)
Lecture Unit 1.1: Recursively defined sequences (Epp: §5.6) and
Lecture Unit 1.2: Solving recurrence relations by iteration (Epp: §5.7)

Stirling numbers of the second kind (Epp: p. 578)

Definition Let $S_{n,r}$ (called a **Stirling number of the second kind**) denote the number of ways to partition a set $\{x_1, x_2, \ldots, x_n\}$ with n elements, into r (non-empty) subsets.

Example Compute the Stirling numbers $S_{3,1}$, $S_{3,2}$ and $S_{3,3}$.

Solution $S_{3,1} = 1$, $S_{3,2} = 3$ and $S_{3,3} = 1$.

Example Compute $S_{5,3}$.

Solution One must count the number of different ways to partition $X = \{x_1, x_2, x_3, x_4, x_5\}$ into three subsets.

- Case 1: The partition has the form $\{\square, \square, \square\}\{\square\}\{\square\}$.

  There are $\binom{5}{3} = 10$ ways to construct the subset with three elements and $\binom{2}{1} \times \binom{1}{1} \div 2! = 1$ ways to construct the other two subsets. Hence there are $10 \times 1 = 10$ such partitions. They are:

  $$\begin{array}{ll} \{x_1, x_2, x_3\}\{x_4\}\{x_5\} & \{x_1, x_4, x_5\}\{x_2\}\{x_3\} \\ \{x_1, x_2, x_4\}\{x_3\}\{x_5\} & \{x_2, x_3, x_4\}\{x_1\}\{x_5\} \\ \{x_1, x_2, x_5\}\{x_3\}\{x_4\} & \{x_2, x_3, x_5\}\{x_1\}\{x_4\} \\ \{x_1, x_3, x_4\}\{x_2\}\{x_5\} & \{x_2, x_4, x_5\}\{x_1\}\{x_3\} \\ \{x_1, x_3, x_5\}\{x_2\}\{x_4\} & \{x_3, x_4, x_5\}\{x_1\}\{x_2\} \end{array}$$

- Case 2: The partition has the form $\{\square\}\{\square, \square\}\{\square, \square\}$.

  There are $\binom{5}{1} = 5$ ways to construct the subset with one element and $\binom{4}{2} \times \binom{2}{2} \div 2! = 3$ ways to construct the other two subsets with two elements each.

Hence there are $5 \times 3 = 15$ **such partitions. They are:**

$\{x_1\}\{x_2, x_3\}\{x_4, x_5\}$	$\{x_4\}\{x_1, x_2\}\{x_3, x_5\}$
$\{x_1\}\{x_2, x_4\}\{x_3, x_5\}$	$\{x_4\}\{x_1, x_3\}\{x_2, x_5\}$
$\{x_1\}\{x_2, x_5\}\{x_3, x_4\}$	$\{x_4\}\{x_1, x_5\}\{x_2, x_3\}$
$\{x_2\}\{x_1, x_3\}\{x_4, x_5\}$	$\{x_5\}\{x_1, x_2\}\{x_3, x_4\}$
$\{x_2\}\{x_1, x_4\}\{x_3, x_5\}$	$\{x_5\}\{x_1, x_3\}\{x_2, x_4\}$
$\{x_2\}\{x_1, x_5\}\{x_3, x_4\}$	$\{x_5\}\{x_1, x_4\}\{x_2, x_3\}$
$\{x_3\}\{x_1, x_2\}\{x_4, x_5\}$	
$\{x_3\}\{x_1, x_4\}\{x_2, x_5\}$	
$\{x_3\}\{x_1, x_5\}\{x_2, x_4\}$	

In total, there are $10 + 15 = 25$ **partitions of** X **into three subsets. Hence** $S_{5,3} = 25$**.**

Proposition **Stirling numbers of the second kind are given by the recurrence relation**

$$S_{n,r} = S_{n-1,r-1} + rS_{n-1,r}$$

for all integers n **and** r **with** $1 < r < n$ **and with initial conditions** $S_{n,1} = 1$ **and** $S_{n,n} = 1$ **for all integers** $n \geqslant 1$**.**

Proof $S_{n,1} = 1$ **because the only partition of the set** $\{x_1, x_2, \ldots, x_n\}$ **into one subset is**

$$\{x_1, x_2, \ldots, x_n\},$$

and $S_{n,n} = 1$ **because the only partition of** $\{x_1, x_2, \ldots, x_n\}$ **into** n **subsets is**

$$\{x_1\}\{x_2\} \cdots \{x_n\}.$$

To show that $S_{n,r} = S_{n-1,r-1} + rS_{n-1,r}$ **when** $1 < r < n$, **count the number of partitions of** $\{x_1, x_2, \ldots, x_n\}$ **into** r **subsets:**

- **Case 1: One of the** r **subsets is** $\{x_n\}$**.**

  The other $r-1$ **subsets are obtained by partitioning** $\{x_1, x_2, \ldots, x_{n-1}\}$ **into** $r-1$ **subsets and this can be done in** $S_{n-1,r-1}$ **ways.**

- **Case 2: None of the** r **subsets are** $\{x_n\}$**.**

  Start by partitioning $\{x_1, x_2, \ldots, x_{n-1}\}$ **into** r **subsets; this can be done in** $S_{n-1,r}$ **ways. Then add** x_n **as an element to one of these** r **subsets; this can be done in** r **ways. Hence there are** $S_{n-1,r} \times r$ **such partitions.**

In total there are therefore $S_{n-1,r-1} + rS_{n-1,r}$ partitions of the set $\{x_1, x_2, \dots, x_n\}$ into r subsets so $S_{n,r} = S_{n-1,r-1} + rS_{n-1,r}$. $\square$

Comment **Stirling numbers of the first kind** **$s_{n,r}$ (with $0 \leqslant r \leqslant n$) count the number of ways of seating n people around r circular tables so that there is at least one person at each table.**

Example **By using the recurrence relation for Stirling numbers of the second kind, draw up a table that contains all the values $S_{n,r}$ for $1 \leqslant n \leqslant 7$.**

Solution **To be done in class.**