WTW 285 (Discrete Structures)
Lecture 3 (Monday 22 July 2019)
Lecture Unit 1.1: Recursively defined sequences (Epp: §5.6) and
Lecture Unit 1.2: Solving recurrence relations by iteration (Epp: §5.7)

Fibonacci sequence

Assume the following about rabbits:

- **they are not fertile for the first month of life;**
- **when they are one month old, they become fertile and males and females pair for life;**
- **each fertile pair of rabbits gives birth to one pair (consisting of a male and female) of rabbits at the end of each month for the rest of their lives;**
- **no rabbits die.**

Note that it takes a pair of rabbits two months from birth to produce their first offspring.

Suppose that there are initially two rabbits (one male and one female). Let F_n denote the number of rabbit pairs that are alive after n months. The sequence

$$F_0, F_1, F_2, \ldots$$

is called the Fibonacci sequence. A recurrence relation for this sequence will be derived. Observe that, for sufficiently large n,

$$\begin{aligned}
F_n &= \text{number of rabbit pairs alive after } n \text{ months} \\
&= \big(\text{number of rabbit pairs alive after } n-1 \text{ months}\big) + \\
&\quad \big(\text{number of new rabbit pairs born after } n \text{ months}\big) \\
&= \big(\text{number of rabbit pairs alive after } n-1 \text{ months}\big) + \\
&\quad \big(\text{number of rabbit pairs conceived after } n-1 \text{ months}\big) \\
&= \big(\text{number of rabbit pairs alive after } n-1 \text{ months}\big) + \\
&\quad \big(\text{number of rabbit pairs fertile after } n-1 \text{ months}\big) \\
&= \big(\text{number of rabbit pairs alive after } n-1 \text{ months}\big) +
\end{aligned}$$

$$
\begin{aligned}
& \big(\text{number of rabbit pairs alive after } n-2 \text{ months}\big) \\
= \;& F_{n-1} + F_{n-2}\,,
\end{aligned}
$$

where it can now be seen that by "sufficiently large" is meant $n \geqslant 2$. The initial conditions that need to be defined are the values $F_0 = 1$ and $F_1 = 1$. Hence the Fibonacci sequence is recursively defined by the recurrence relation

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_n &= F_{n-1} + F_{n-2} \text{ when } n \geqslant 2.
\end{aligned}
$$

The first few terms of the sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots .$$

A method for finding an explicit formula that solves the recurrence relation of the Fibonaci sequence will be developed in Lecture Unit 1.3.

Compound interest

To say that interest on an account is paid/charged at an annual rate of $k\,\%$ compounded every m months means that, at the end of each period of m months, interest is added to the previous balance of the account at a rate of $\left(\frac{m}{12} \times k\right)\%$. The annual percentage rate (APR) of the account is the interest rate of the account over a period of one year.

Example Suppose that an amount of R $10\,000$ is invested in an account with an annual interest rate of $6\,\%$ compounded monthly.

1. Derive a recurrence relation for the balance of the account after k months.

2. Derive an explicit formula that solves the recurrence relation obtained in Question 1 and prove its correctness.

3. What will the value of the investment be in 5 years' time?

4. **What is the APR of this account?**

Solution

1. **Let P_k denote the balance of the account after k months. P_k is given by the following recurrence relation:**

$$P_k = P_{k-1} + 0.005P_{k-1} = 1.005P_{k-1}$$

with initial value $P_0 = 10\,000$.

2. **The method of iteration will be used to guess a solution to the above recurrence relation. Observe that**

$$\begin{array}{lcl} P_0 & = & 10\,000 \\ P_1 & = & 1.005P_0 = 1.005 \times 10\,000 \\ P_2 & = & 1.005P_1 = 1.005(1.005 \times 10\,000) = 1.005^2 \times 10\,000 \\ P_3 & = & 1.005P_2 = 1.005(1.005^2 \times 10\,000) = 1.005^3 \times 10\,000 \\ P_4 & = & 1.005P_3 = 1.005(1.005^3 \times 10\,000) = 1.005^4 \times 10\,000 \\ \vdots & & \qquad\qquad\qquad \vdots \end{array}$$

Guess: $P_k = 1.005^k \times 10\,000$ for $k \geqslant 0$.

It must now be proved that the formula

$$A_k = 1.005^k \times 10\,000$$

with $k \geqslant 0$ solves the recurrence relation derived in Question 1, i.e. that $A_k = P_k$ for each non-negative integer k. Mathematical induction is used for this.

For the basis step, observe that

$$A_0 = 1.005^0 \times 10\,000 = 10\,000 = P_0.$$

For the inductive hypothesis, let m be any non-negative integer such that $A_m = P_m$. Then $P_m = 1.005^m \times 10\,000$.

For the inductive step, observe that

$$\begin{array}{rcll} A_{m+1} & = & 1.005^{m+1} \times 10\,000 & \text{(definition of } A_k\text{)} \\ & = & 1.005\,(1.005^m \times 10\,000) & \end{array}$$

$$
\begin{aligned}
&= 1.005P_m \quad \text{(inductive hypothesis)} \\
&= P_{m+1} \quad \text{(definition of } P_k\text{)}.
\end{aligned}
$$

By the principle of mathematial induction, $A_k = P_k$ for each non-negative integer k. Hence the formula $A_k = 1.005^k \times 10\,000$ with $k \geqslant 0$ solves the recurrence relation derived in Question 1, i.e.

$$P_k = 1.005^k \times 10\,000$$

for $k \geqslant 0$.

3. $P_{60} = 1.005^{60} \times 10\,000 \approx \mathrm{R}\,13\,488.50$

4. **Since, for any non-negative integer k,**

$$\frac{P_{k+12}}{P_k} = \frac{1.005^{k+12} \times 10\,000}{1.005^k \times 10\,000} = 1.005^{12} \approx 1.0617,$$

the APR of this account is approximately $6.17\,\%$.

Other examples

Pay special attention to:

- **Epp: Example 5.7.2 (p. 307) that deals with an object falling through a vacuum, and**
- **Epp: Example 5.7.6 (p. 311) that deals with the number of edges in a complete graph. Note that an explicit formula for the number of edges can also be computed by pointing out that each set of two vertices determines an edge so there are**

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$$

edges in the graph K_n.