WTW 285 (Discrete Structures)
Lecture 1 (Monday 15 July 2019)
Lecture Unit 1.1: Recursively defined sequences (Epp: §5.6) and
Lecture Unit 1.2: Solving recurrence relations by iteration (Epp: §5.7)

As an example of a recurrence relation, recall that combinations $\binom{n}{r}$ (where n and r are integers with $0 \leqslant r \leqslant n$) may be computed using either the explicit formula

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

or (from Pascal's formula) using the recurrence relation

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

with initial values $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all n.

A sequence that is generated by a recurrence relation is called a recursively defined sequence. For example, the recurrence relation

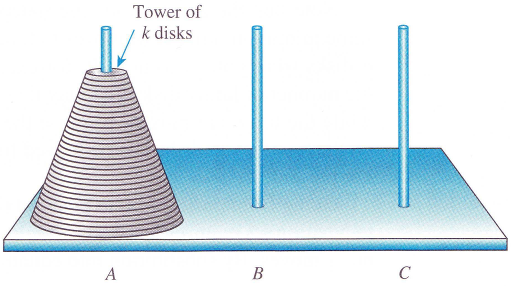
$$a_k := a_{k-1} + ka_{k-3}$$

with initial values $a_0 := 1$, $a_1 := 1$ and $a_2 := 2$, recursively defines the sequence

$$1, 1, 2, 5, 9, 19, 49, \ldots \ .$$

Tower of Hanoi

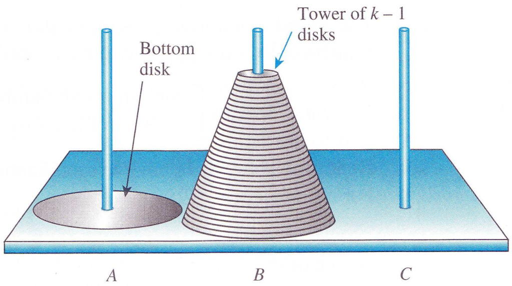
Suppose there are three poles and a tower of k discs of different sizes that are stacked in order of decreasing size on one of the poles.



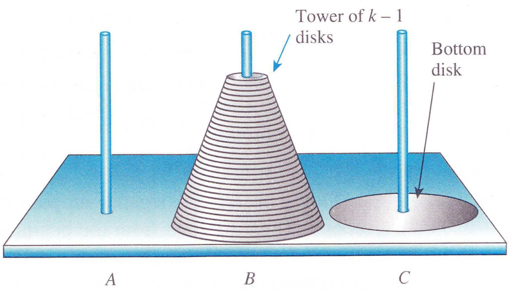
The entire tower of discs must be moved to any one of the other two poles B and C. The discs can only be moved one at a time from pole to pole and no disc may ever be placed on top of a smaller disc.

Let m_k denote the fewest number of moves in which a tower of k discs can be moved from one pole to any other pole in this manner. A recurrence relation for m_k will be derived.

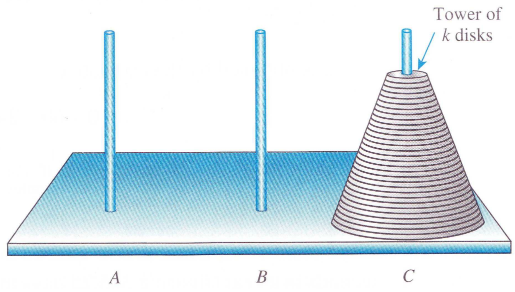
In order to move the bottommost disc, the top $k-1$ discs first have to be moved to a different pole, say from pole A to pole B; the fewest number of moves in which this can be done is m_{k-1}.



Then move the disc remaining on pole A to pole C; the fewest number of moves in which this can be done is 1.



Then move the $k-1$ discs on pole B, to pole C; the fewest number of moves in which this can be done is m_{k-1}.



Hence the fewest number of moves in which the tower of k discs can be moved from one pole to any other pole, is

$$m_k = m_{k-1} + 1 + m_{k-1} = 2m_{k-1} + 1.$$

The initial condition of the recurrence relation is

$$m_1 = 1$$

since the fewest number of moves in which one disc can be moved from one pole to another, is 1.

Example In the Tower of Hanoi problem, how many moves are necessary to move six discs from one pole to another?

Solution The value of m_6 must be computed using the recurrence relation

$$m_k = 2m_{k-1} + 1$$

with initial condition $m_1 = 1$.

$$\begin{aligned}
m_1 &= 1 \\
m_2 &= 2m_1 + 1 = 2 \times 1 + 1 = 3 \\
m_3 &= 2m_2 + 1 = 2 \times 3 + 1 = 7 \\
m_4 &= 2m_3 + 1 = 2 \times 7 + 1 = 15 \\
m_5 &= 2m_4 + 1 = 2 \times 15 + 1 = 31 \\
m_6 &= 2m_5 + 1 = 2 \times 31 + 1 = 63
\end{aligned}$$

Hence six discs can be moved using 63 moves, but cannot be moved using fewer than 63 moves.

Method of iteration

Given a recurrence relation that generates a sequence

$$m_k, m_{k+1}, m_{k+2}, \ldots,$$

an explicit function

$$f : \{k, k+1, k+2, \ldots\} \to \mathbb{R}$$

is called a solution of the recurrence relation when $f(n) = m_n$ for each n with $n \geqslant k$.

Recurrence relations can sometimes be solved using the method of iteration which involves writing down the first few terms of the recurrence relation and then guessing an explicit formula if there is clear pattern.

The following formulas for the summations of arithmetic and geometric series may be useful here:

$$\begin{aligned} 1+2+3+\cdots+n &= \frac{n(n+1)}{2} \quad \text{and} \\ 1+r+r^2+\cdots+r^n &= \frac{r^{n+1}-1}{r-1} \end{aligned}$$

when $r \neq 1$.

Once a solution has been guessed, its correctness must be proved using induction.

Example (Using the method of iteration to guess a solution for a recurrence relation)

Consider the Tower of Hanoi's recurrence relation

$$m_n = 2m_{n-1} + 1$$

with initial value $m_1 = 1$. Guess a solution for this recurrence relation.

Solution Observe that

$$\begin{aligned} m_1 &= 1 \\ m_2 &= 2m_1 + 1 = 2(1) + 1 = 2 + 1 \\ m_3 &= 2m_2 + 1 = 2(2+1) + 1 = 2^2 + 2 + 1 \\ m_4 &= 2m_3 + 1 = 2(2^2 + 2 + 1) + 1 = 2^3 + 2^2 + 2 + 1 \\ &\vdots \qquad\qquad\qquad \vdots \end{aligned}$$

Guess: $f(n) = \sum_{i=0}^{n-1} 2^i = \dfrac{2^{(n-1)+1} - 1}{2-1} = 2^n - 1$ with $n \geqslant 1$.

Example (Proving that an explicit formula solves a recurrence relation)

Consider the Tower of Hanoi's recurrence relation

$$m_n = 2m_{n-1} + 1$$

with initial value $m_1 = 1$. Prove that the explicit formula

$$f(n) := 2^n - 1 \text{ where } n \geqslant 1$$

solves this recurrence relation.

Solution It must be shown that $f(n) = m_n$ for each positive integer n; the proof is done using mathematical induction.

For the basis step, observe that $f(1) = 2^1 - 1$ and $m_1 = 1$ hence $f(1) = m_1$, as required.

For the inductive hypothesis, fix an arbitrary integer k with $k \geqslant 1$ for which $f(k) = m_k$. Since $f(k) = 2^k - 1$, this gives

$$2^k = f(k) + 1 = m_k + 1. \tag{$\star$}$$

For the inductive step, observe that

$$\begin{aligned} f(k+1) &= 2^{k+1} - 1 \\ &= 2 \cdot 2^k - 1 \\ &= 2(m_k + 1) - 1 \quad \text{(by } (\star)\text{)} \\ &= 2m_k + 1 \\ &= m_{k+1} \end{aligned}$$

as required.

By the principle of mathematical induction, $f(n) = m_n$ for each positive integer n.