Digression. Suppose $(X(t))$ is a WSS RP.

$$\int_{-\infty}^{\infty} \underbrace{S_X(f)}_{PSD}\cdot df = \mathbb{E}(X(t)^2) \rightarrow \text{deterministic}$$

$$\int_{-\infty}^{\infty} S_X(f)\, e^{-j2\pi f \tau}\cdot df = R_X(\tau) \rightarrow \text{deterministic.}$$
$$= \mathbb{E}\, X(t)X(t+\tau).$$

how is $\mathbb{E}\,X(t)^2$ computed?

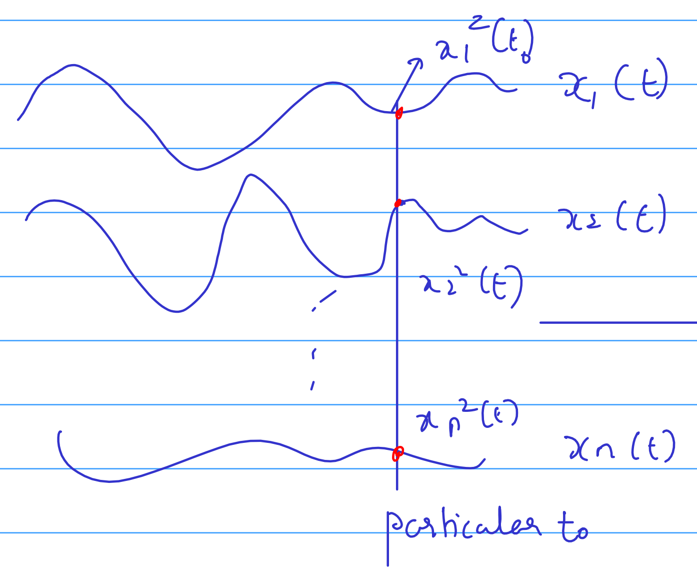
$\hookrightarrow \int f_{X(t)}(x)\cdot x^2\cdot dx$

$\downarrow$

interpret $\mathbb{E}$ as averages.

$X \sim f_X(x)$ $\quad x_1, x_2, x_3, \cdots, x_n$

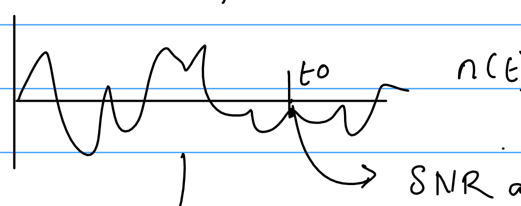
$$\bar{x}_n = \left(1/n \sum_1^n x_i\right) \longrightarrow \int f_X(x)\cdot x\cdot dx$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \; \mathbb{E}X.$$



$\longrightarrow \overline{x^2}(t) \longrightarrow \mathbb{E}\,X^2(t)$

ensemble averages.

Consider receiving a signal corrupted with noise
a receiver sees a particular sample path of noise



$\rightarrow$ SNR at t_0 using $\mathbb{E}\,N(t)^2$

$\swarrow$ power for this sample path (and for any sample path)

$$\lim_{T_0 \to \infty}\left(1/T_0 \int_{-T_0/2}^{T_0/2} (N(t))^2\cdot dt\right) = \text{Time averaged noise power}$$

(operational performance measure)

For a class of random processes which are called Ergodic

How are these operational quantities related to ensemble averages?

A WSS process for which the time average of the process = ensemble avg is called ergodic in the mean.

A WSS process $X(t)$ for which

$$\lim_{T_0\to\infty} \frac{1}{T_0}\int_{T_0} x(t)\,x(t+\tau)\cdot dt = \mathbb{E}\,X(t)X(t+\tau)$$ is called ergodic in autocorr. sense.

Examples for ergodicity

1) IID process : $(X(t))$ $\quad X(t) \sim F_X(x)$ and $X(t_1) \perp X(t_2)$

$$1/2T \int_{-T}^{T} X(t)\cdot dt = \mathbb{E}\,X(0) \quad \text{this holds?}$$ (ergodic in the mean sense)

(X_n). $$1/N \sum_{n=0}^{N-1} X_n \overset{?}{\doteq} \mathbb{E}\,X(0)$$ (Law of large numbers)

$\hookrightarrow$ var $\propto 1/N$

$$1/2T\int_{-T}^{T} X(t)\cdot X(t+\tau)\cdot dt = \mathbb{E}\,X(t)\cdot X(t+\tau)$$

(X_n) $$\frac{1}{2N+1}\sum_{n=-N}^{N} X_n\cdot X_{n+\tau} = \mathbb{E}\,X_n\cdot X_{n+\tau}$$

$$\mathbb{E}(\cdot) = \left(\frac{1}{2N+1}\sum_{n=-N}^{N}\mathbb{E}(X_n\cdot X_{n+\tau})\right) = \mathbb{E}(X_0\cdot X_{0+\tau}) = \mathbb{E}(X_n\cdot X_{n+\tau})$$

Recall Chebyshev inequality? X is a R.V

$$\Pr\{|X - \mathbb{E}X| > a\sigma\} \leq 1/a^2$$
$$\Pr\{|X - \mathbb{E}X| > \varepsilon\} \leq \sigma^2/\varepsilon^2$$

H.W: $$\frac{1}{(2N+1)^2}\,\text{var}\left(\sum_{n=-N}^{N} X_n\cdot X_{n+\tau}\right)$$

2) $O^{1/2}$ $\quad O^{3/4} \rightarrow \Pr\{\text{heads}\}$.

The time average is either $1/2$ or $3/4$.
what will be the ensemble average?

$\xrightarrow{1/2} O^{1/2}$

$\xrightarrow{1/2} O^{3/4}$

$$1/2(1/2) + 1/2(3/4) = 5/8$$

$\uparrow$ ensemble avg.

In going forward: operational performance measures (time avgs) = ensemble avgs.

Gaussian processes: $\rightarrow$ DT (X_n)
$\rightarrow$ CT $(X(t))$.

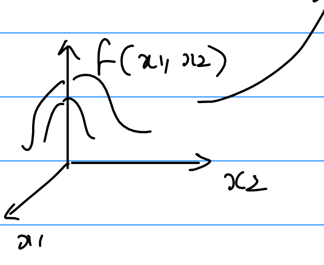
$\swarrow$ def. of Gaussian $(X(t))$

1) for any m, $t_1 < t_2 < t_3 \cdots < t_m$

$a_1 X(t_1) + a_2 X(t_2) + \cdots + a_m X(t_m)$ is a Gaussian RV

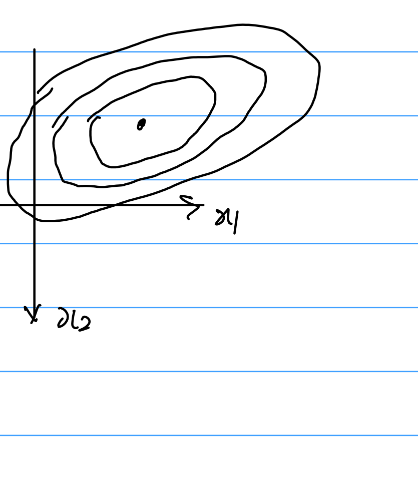
2) The joint distribution of $X(t_1) \cdots X(t_m)$ is multivariate Gaussian.

e.g $f_{X(t_1)\,X(t_2)}(x_1, x_2)$

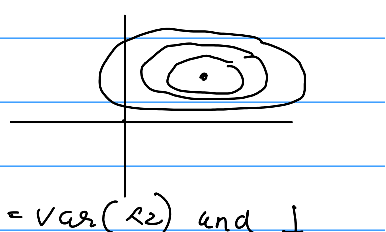


$$\frac{1}{(\sqrt{2\pi})^2}\,\frac{1}{\sqrt{\|\Sigma\|}} \times e^{-1/2(\bar{x}-\bar{\mu})^T \Sigma^{-1}(\bar{x}-\bar{\mu})}$$ where $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Sigma = \begin{pmatrix} \text{var}(x_1) & \text{cov}(x_1,x_2) \\ \text{cov}(x_1,x_2) & \text{var}(x_2) \end{pmatrix} \quad\quad \bar{\mu} = \begin{pmatrix} \mathbb{E}X_1 \\ \mathbb{E}X_2 \end{pmatrix}$$



if $X(t_1) \perp X(t_2)$
$\text{cov}(x_1, x_2) = 0$
then



if $\text{var}(x_1) = \text{var}(x_2)$ and $\perp$