Note that this lecture was given using the board in class. So the actual contents of this lecture note may differ from what was covered in class.

Review:

- CTRP $(X(t)),\ t \in \mathbb{R}$ - DTRP $(X_n),\ n \in \mathbb{Z}$

$$(X(t)) \longrightarrow \underbrace{\boxed{h(t)}}_{LTI} \longrightarrow (Y(t)) \qquad\qquad (X_n) \longrightarrow \underbrace{\boxed{h_n}}_{LSI} \longrightarrow (Y_n)$$

$$Y(t) = \int_{-\infty}^{\infty} X(\tau)\cdot h(t-\tau)\,d\tau \qquad\qquad Y_n = \sum_{k=-\infty}^{\infty} X_k h_{n-k}$$

R.V at time t R.V at time n.

Can we characterize FDD of $(Y(t))$ as a function of FDD of $(X(t))$ and the impulse response $h(t)$?

example:

$(X(t))$ is IID.

$h(t)$ is $\delta(t) + \delta(t-1)$

$\therefore\ Y(t) = X(t) + X(t-1)$

Is $(Y(t))$ IID? Check $Y(t)$ and $Y(t+1)$'s joint distribution.

No!

important to note that the structure of the FDD is not preserved here.

Q But are there FDD structures which are preserved when LTI filtering is done. We will answer this below.

If not IID can we find the FDD of $(Y(t))$?

$F_{Y(t_0)}$ is easy to find: it is the convolution of the pdfs of $X(t)$ and $X(t-1)$ (assuming pdfs can be defined).

important to note that getting the joint distribution is not that easy.

in applications (esp. like computation of SNR) summary statistics are enough.

Q So we now ask whether it is possible to find out summary statistics

we will answer this below.

A FDD structures which are preserved by LTI filtering.

- If $(X(t))$ is strictly stationary then $(Y(t))$ is strictly stationary. (also for DT)
- If $(X(t))$ is WSS then $(Y(t))$ is WSS.

example: (why should this property hold for SS $(X(t))$?)

let $(X(t))$ be SS and $h(t) = \delta(t) + \delta(t-1)$

then $Y(t) = X(t) + X(t-1)$

the distribution of $Y(t)$ is determined by the joint dist of $X(t), X(t-1)$

$Y(t+\tau)$ " $X(t+\tau), X(t+\tau-1)$

since $(X(t))$ is SS so $F_{X(t),X(t-1)} = F_{X(t+\tau)X(\tau+t-1)}$

we can think about $(Y(t))$'s FDDs in the same manner.

Going into the details of this invariance property for WSS processes will be useful for us.

<u>Invariance ppty for WSS processes:</u>

$$(X(t)) \longrightarrow \boxed{h(t)} \longrightarrow (Y(t))$$

WSS - - - - - - - ?

$(X(t))$ WSS $\Rightarrow$ a) $\mathbb{E}X(t) = \mu_X$ (a constant), b) $\mathbb{E}X(t_1)X(t_2) = R_X(t_1-t_2)$

$\mathbb{E}\,Y(t) = ?$

$$\mathbb{E}\,Y(t) = \mathbb{E}\int_{-\infty}^{\infty} X(\tau)\cdot h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \mu_X h(t-\tau)\,d\tau = \mu_X \int_{-\infty}^{\infty} h(u)\,du = \mu_Y.$$

$$\mathbb{E}\,Y(t_1)Y(t_2) = \mathbb{E}\left[\int_{-\infty}^{\infty} h(u)\cdot X(t_1-u)\,du \cdot \int_{-\infty}^{\infty} h(v)\cdot X(t_2-v)\,dv\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)\cdot h(v)\cdot \mathbb{E}\,X(t_1-u)\cdot X(t_2-v)\,du\,dv.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)\cdot h(v)\cdot R_X(t_1-t_2-u+v)\,du\,dv$$

{ from here it can be seen that this is a function of $t_1-t_2$

let $\omega = v - u$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(v-\omega)\cdot h(v)\cdot R_X(t_1-t_2+\omega)\,d\omega\,dv$$

$$= \int_{-\infty}^{\infty} R_X(t_1-t_2+\omega)\cdot \int_{-\infty}^{\infty} h(v)\cdot h(v-\omega)\,dv\,d\omega$$

$\omega = -\omega$

$$\text{or} = \int_{-\infty}^{\infty} R_X(t_1-t_2-\omega)\cdot \underbrace{\int_{-\infty}^{\infty} h(v)\cdot h(v+\omega)\,dv}_{g(\omega)}\,d\omega$$

$$= \int_{-\infty}^{\infty} R_X(t_1-t_2-\omega)\cdot g(\omega)\,d\omega = (R_X * g)_{t_1-t_2} = R_Y(t_1-t_2)$$