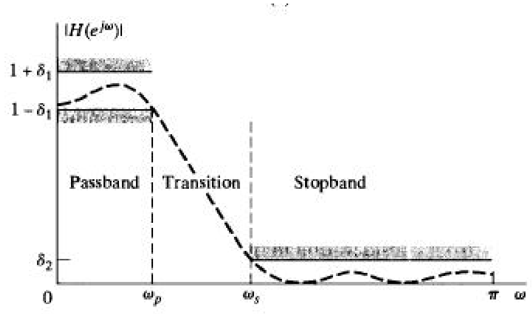


Digital Filter Design

AV316 – Digital Signal Processing

Figures are taken from the textbooks

1. Discrete Time Signal Processing - Oppenheim and Schafer
2. Digital Signal Processing - Oppenheim and Schafer

Filter design – Finite Impulse Response (FIR) filters

- Let h[n] be a sequence of finite extent, i.e., $n \in \{0,1,\ldots, M\}$
- If h[n] is the impulse response of a discrete time LTI or LSI (linear shift invariant) system then h[n] is a finite impulse response

- Design techniques for FIR filters are based on direct approximation of
  - The desired impulse response
  - The desired frequency response

- FIR filters can use FFT for their implementation (FFT with overlap and save)

Filter design – Finite Impulse Response (FIR) filters

- The z-transform of h[n] is $\hat{H}(z) = \sum_{n=0}^{M} h[n] z^{-n}$

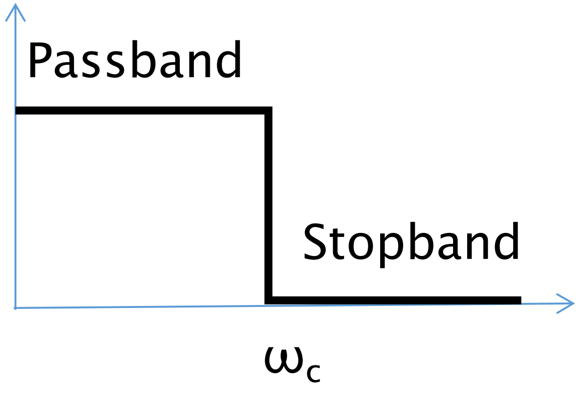
- For an FIR h[n], the RoC is the whole of the z-plane except at z = 0
  - At z = 0, note that the z-transform has M poles
  - The z-transform also has M zeros

- The DTFT of h[n] is $H(\omega) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$

  - In order to make the relationship between the DTFT and the z transform more explicit, we will start using the notation $H(e^{j\omega})$ instead of $H(\omega)$.

  - So the DTFT of h[n] is $H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$

Finite Impulse Response (FIR) filters – Design using Windows (1)



- Suppose $H_d(e^{j\omega})$ is a desired frequency response such as shown
- The corresponding impulse response is $h_d[n]$
  - Usually for brickwall type frequency response (which is desired) $h_d[n]$ would be an infinite impulse response

- The relationship between $H_d(e^{j\omega})$ and $h_d[n]$ is

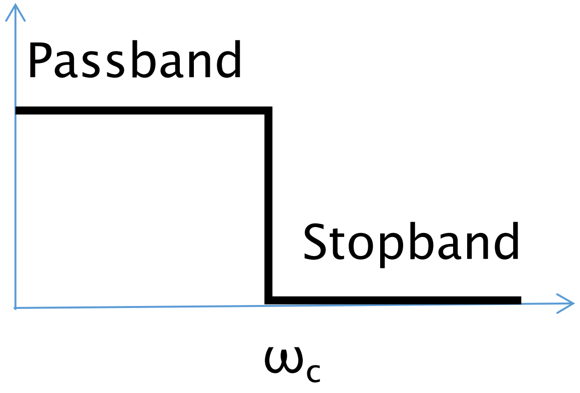
$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n} \qquad h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

- In some cases we can obtain $h_d[n]$ from $H_d(e^{j\omega})$ analytically
  - e.g. on board: $h_d[n]$ for the low pass response shown above.

- In other cases we might use DFT for getting $h_d[n]$ from a sampled version of $H_d(e^{j\omega})$
  - on board

Finite Impulse Response (FIR) filters – Design using Windows (2)

- Suppose $H_d(e^{j\omega})$ is a desired frequency response such as shown
- The corresponding impulse response is $h_d[n]$
  - Usually for brickwall type frequency response (which is desired) $h_d[n]$ would be an infinite impulse response



- An FIR filter should have a finite extent impulse response h[n].
- We define

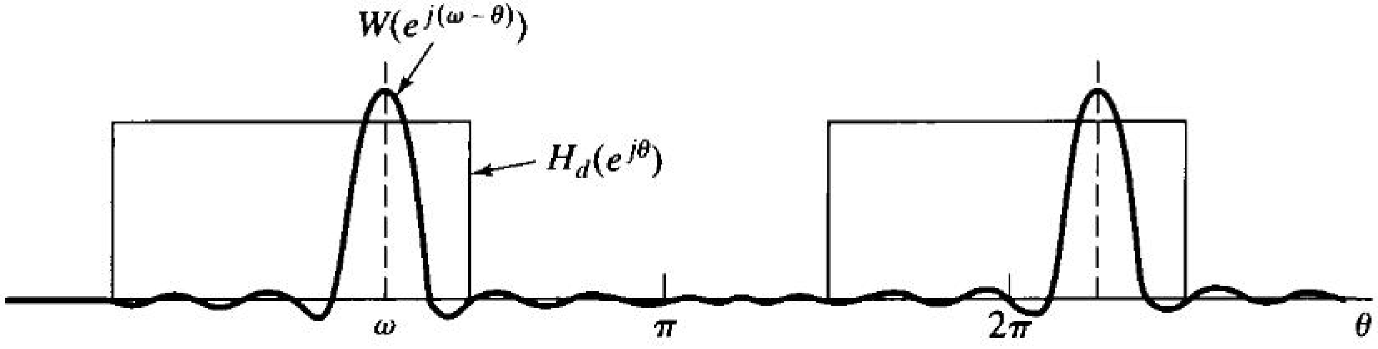
$$h[n] = h_d[n]w[n]$$

- Where w[n] is a window of extent (M + 1), i.e., {0 to M}
  - e.g. w[n] could be a rectangular window
  - on board – how is windowing done?

- What is the difference in $H_d(e^{j\omega})$ and $H(e^{j\omega})$?
  - Desired and what is obtained via windowing.

Finite Impulse Response (FIR) filters – Design using Windows (3)

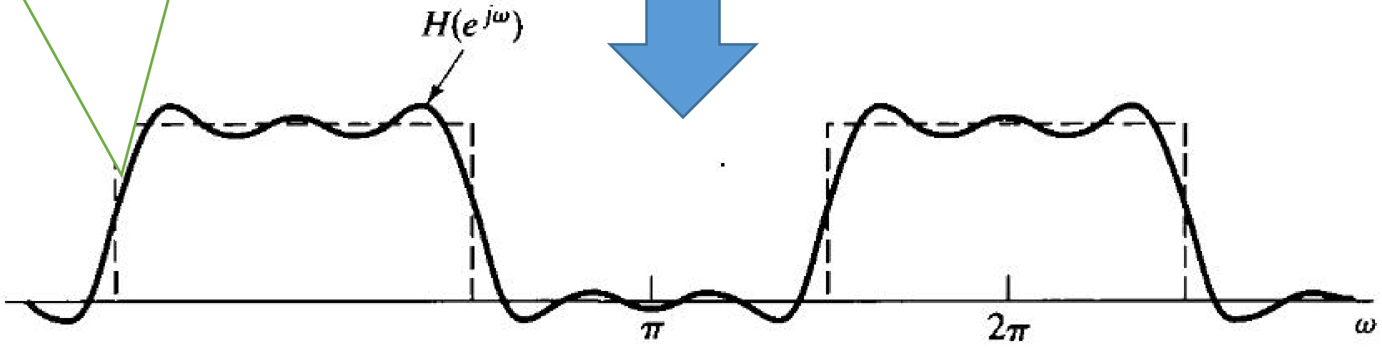
- We have that $h[n] = h_d[n]w[n]$
- From properties of the DTFT we have that



(a)

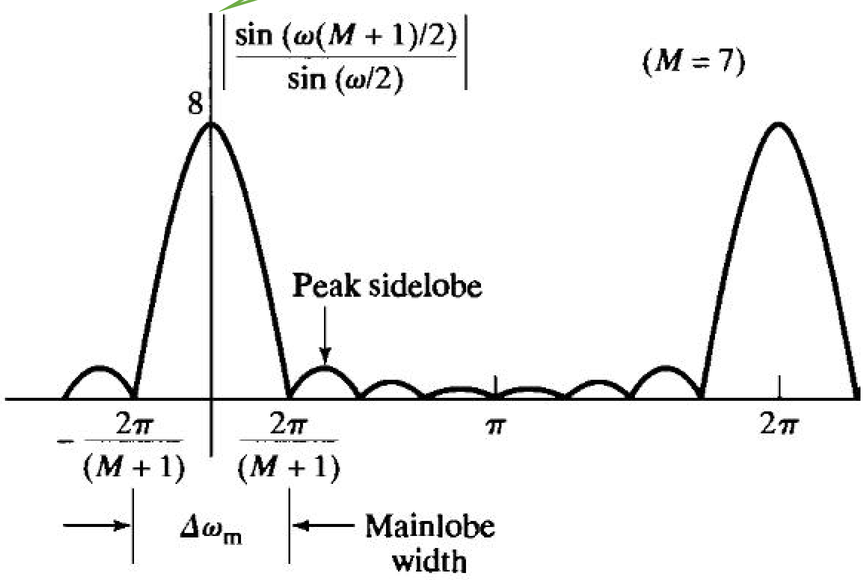
Main lobe width decides the transition band width!

convolved result



(b)

On board:
DTFT of w[n]



Finite Impulse Response (FIR) filters – Design using Windows (4)

- We define

$$h[n] = h_d[n]w[n]$$

- Where w[n] is a window of extent 0 to M

- What is the difference in $H_d(e^{j\omega})$ and $H(e^{j\omega})$?
  - Desired and what is obtained via windowing.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n}$$

Fourier series representation

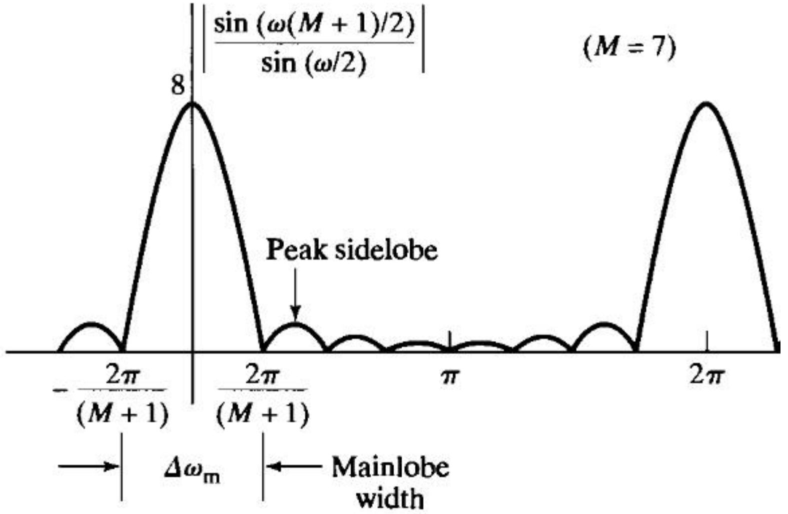
$$H(\omega) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

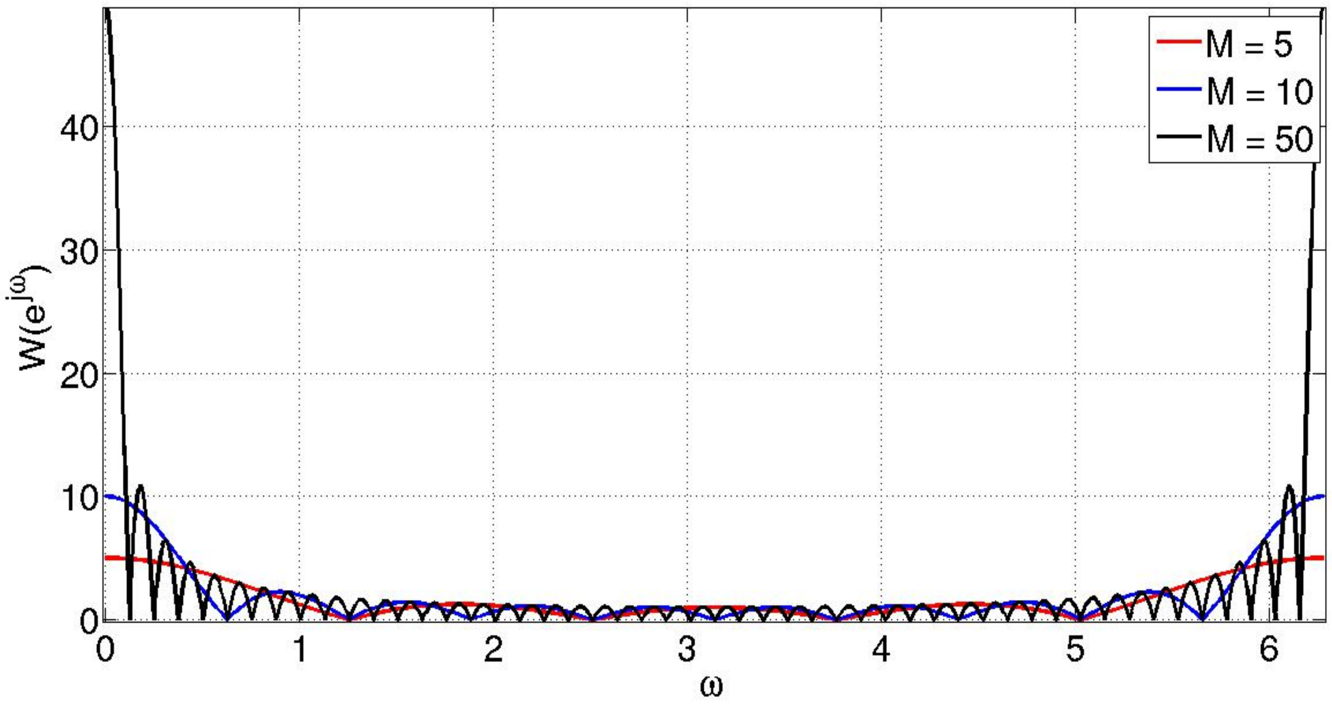
Truncated Fourier series representation

Gibb's phenomenon

DTFT of w[n]

- How does $W(e^{j\omega})$ change with M?



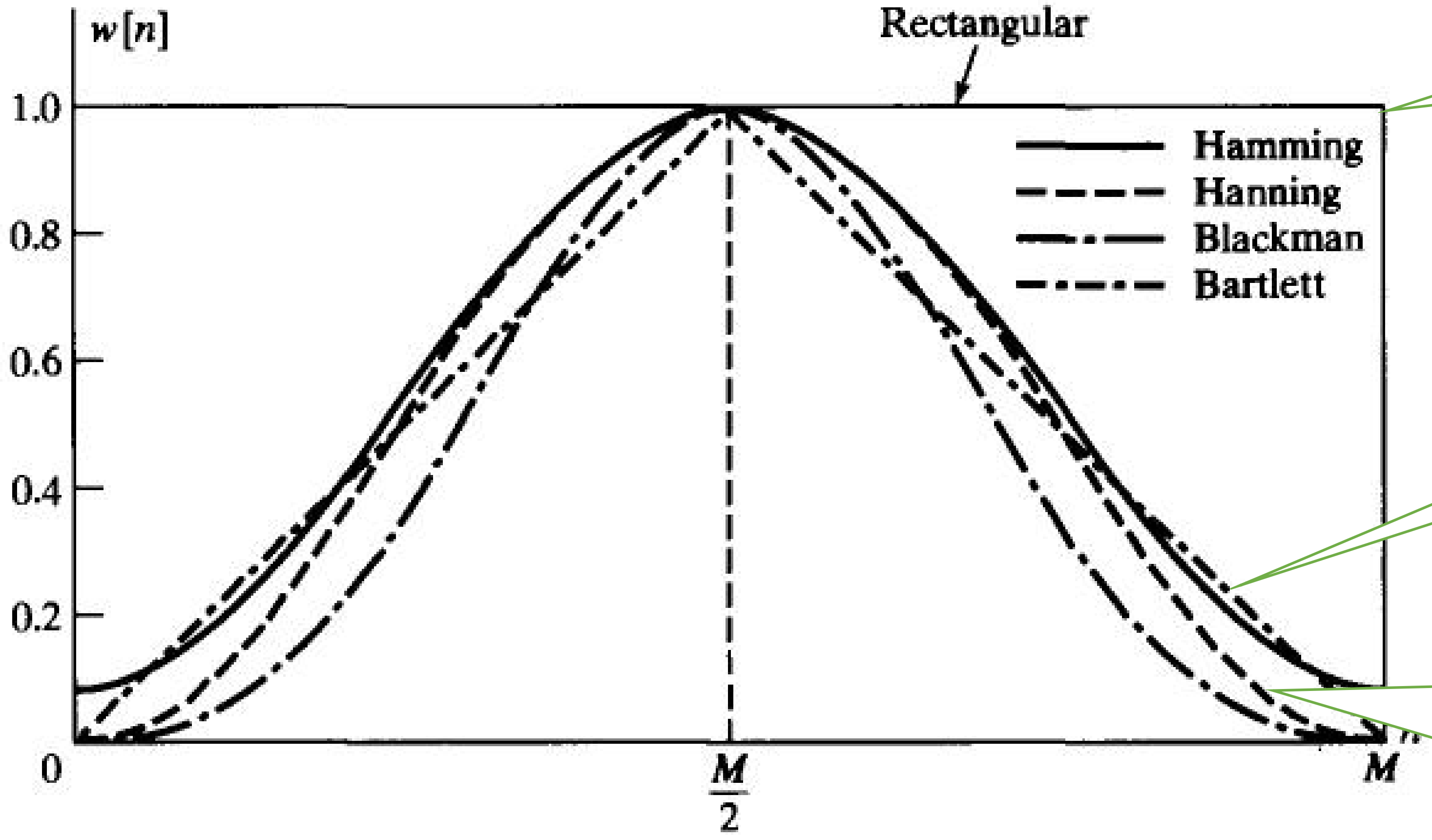


- In time domain: As the window size increases, h[n] = h_d[n] for more and more values of n.
- In freq. domain: As the window size increases, the main lobe size decreases, and the spectrum tends to a delta function.

- However, "ripples" would always be present.
- Area under each side lobe would remain the same as M decreases.

Finite Impulse Response (FIR) filters – Windows

- The rectangular window is one of the possible windows that can be used.
- The side lobe size can be reduced by using windows that "taper" (intuition?)
    - For example, Bartlett (triangular), Blackman, Hanning, Hamming windows
- Other possibilities are



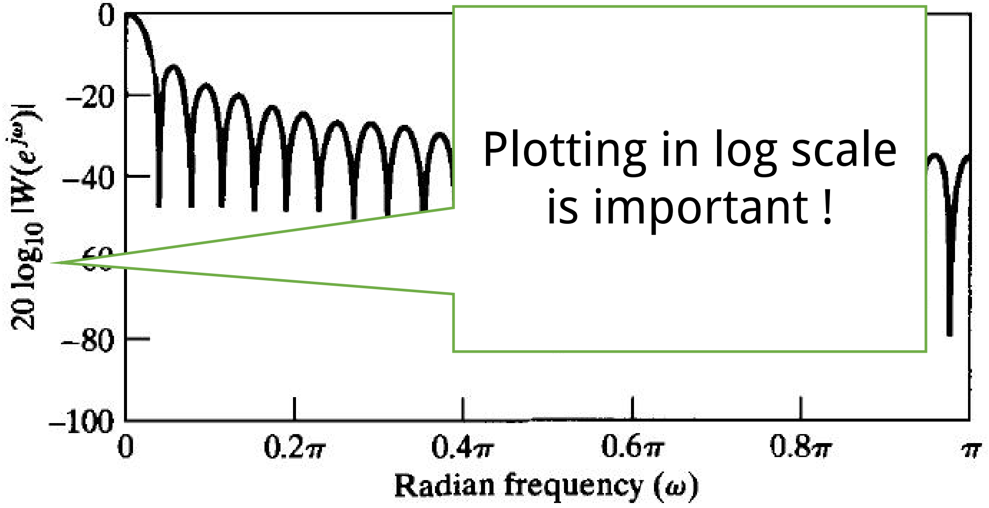
$$w[n] = \begin{cases} 1, 0 \le n \le M, \\ 0, \text{ otherwise} . \end{cases}$$

$$w[n] = \begin{cases} \frac{2n}{M}, 0 \le n \le \frac{M}{2}, \\ 2 - \frac{2n}{M}, \frac{M}{2} \le n \le M, \\ 0, \text{ otherwise} . \end{cases}$$

$$w[n] = \begin{cases} 0.5 - 0.5cos(2\pi n/M), 0 \le n \le M, \\ 0, \text{ otherwise} . \end{cases}$$

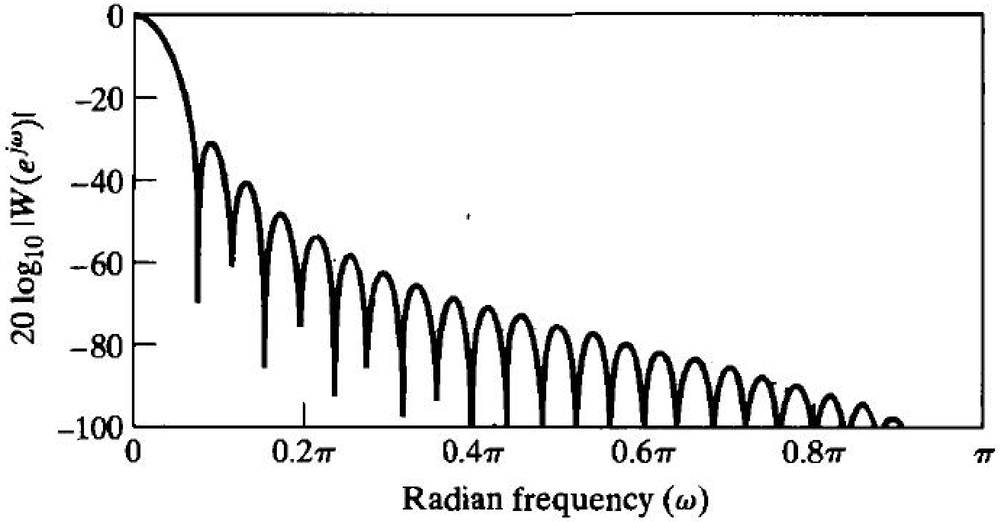
Finite Impulse Response (FIR) filters – Windows

- The spectrum $W(e^{j\omega})$ of these windows (M = 50) are as follows:

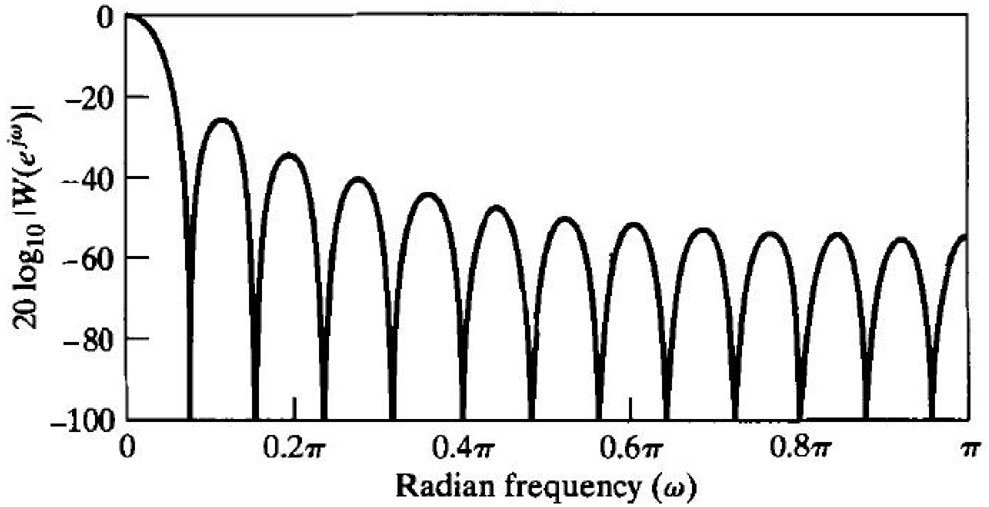


Plotting in log scale is important !

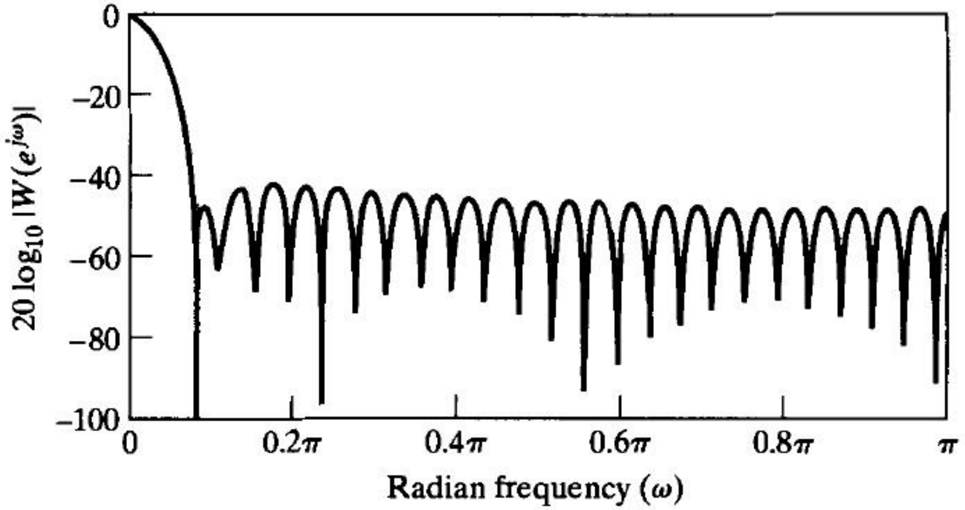
Rectangular Window



Hanning Window



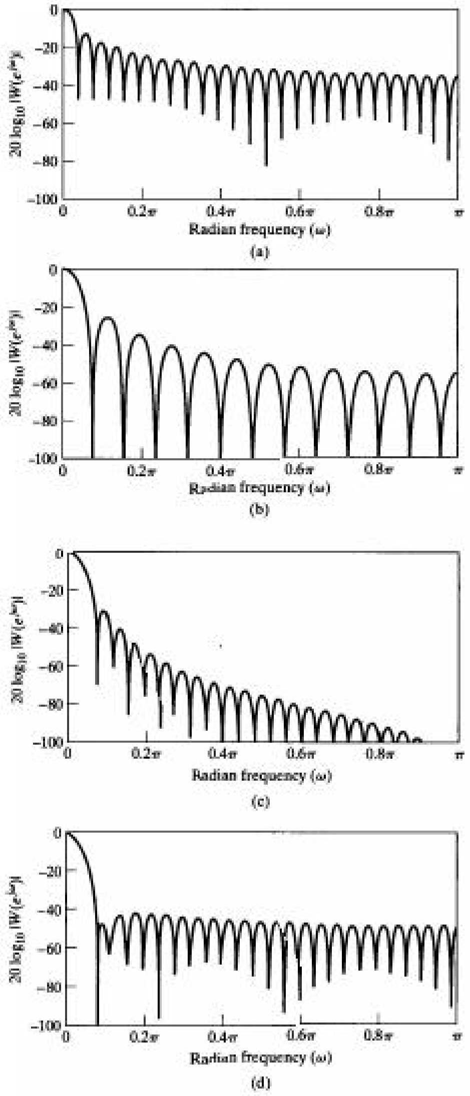
Bartlett Window

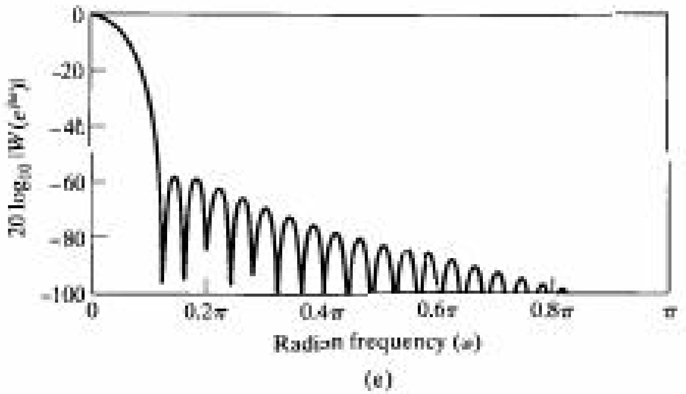


Hamming Window

Finite Impulse Response (FIR) filters – Windows

- The spectrum $W(e^{j\omega})$ of these windows (M = 50) are as follows:



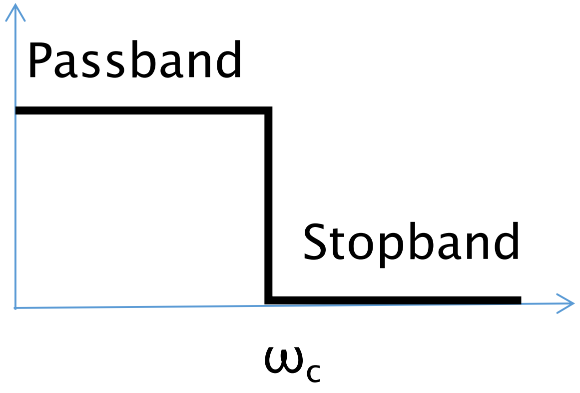


with $M = 50$. (a) Rectangular.
(b) Bartlett. (c) Hanning. (d) Hamming.
(e) Blackman.

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe
Rectangular	−13	$4\pi/(M+1)$
Bartlett	−25	$8\pi/M$
Hanning	−31	$8\pi/M$
Hamming	−41	$8\pi/M$
Blackman	−57	$12\pi/M$

Finite Impulse Response (FIR) filters – Phase response

- Usually filtering is used to selectively pass frequency components
  - we try to approximate the following "brickwall" form



- Within the passband we want the phase response of the filter to be linear
- What this means is that

$$\angle H(e^{j\omega}) = -\alpha\omega, \alpha \in \mathbb{R}$$

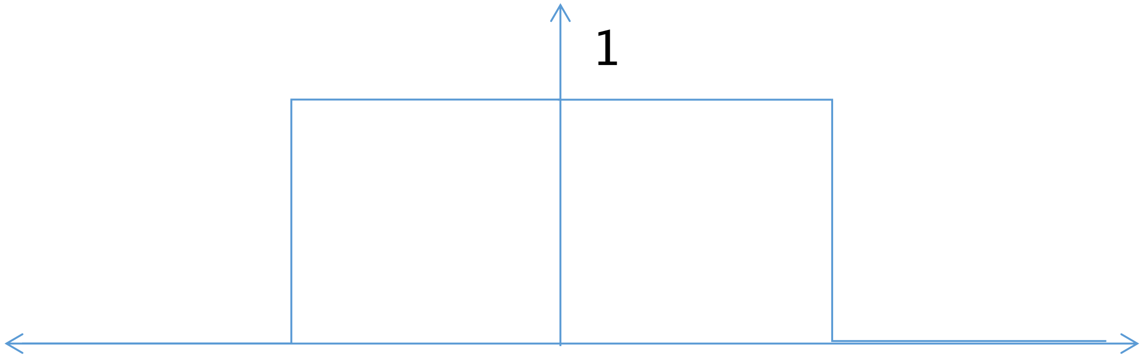
- For FIR filters such a requirement can be easily met by ensuring the following symmetry condition

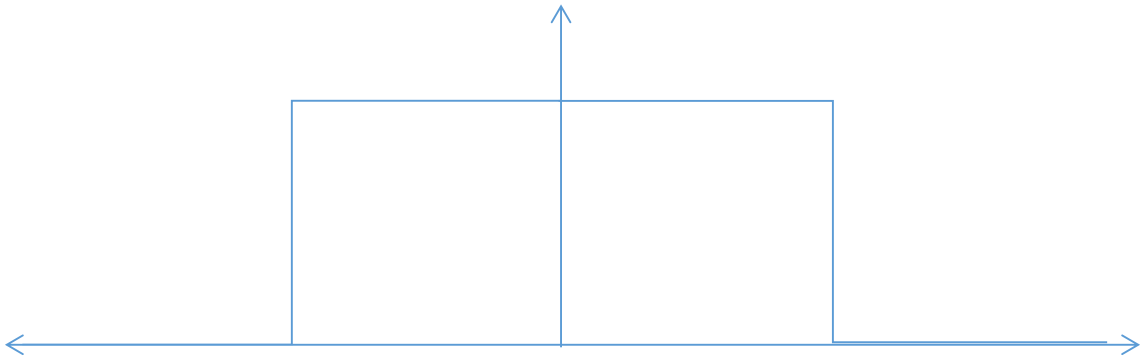
$$h[n] = h[M-n]$$

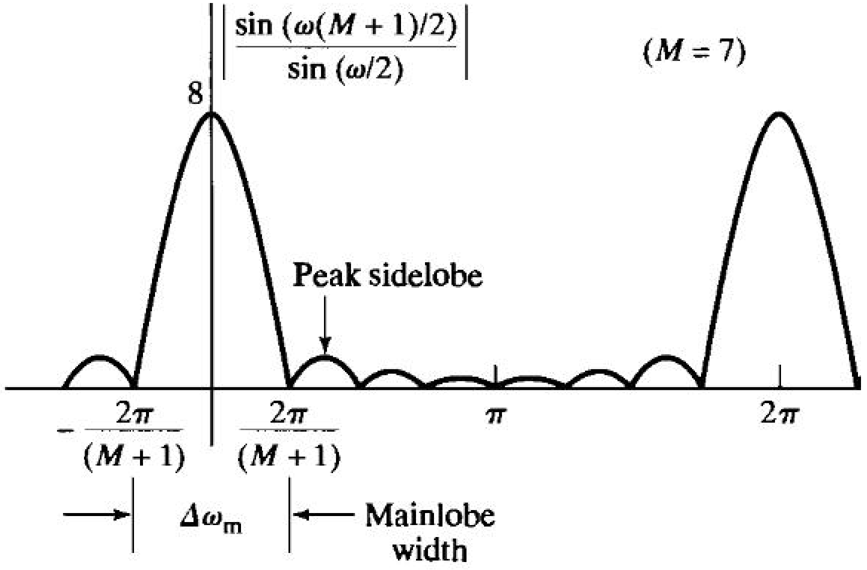
- On board: Example for M odd and M even.
- On board: Proof of linear phase for M odd and M even.
- Windows are defined in such a way so that this symmetry is maintained.
- Major advantages of using FIR filters
  - Use of FFT
  - Easy to obtain linear phase response

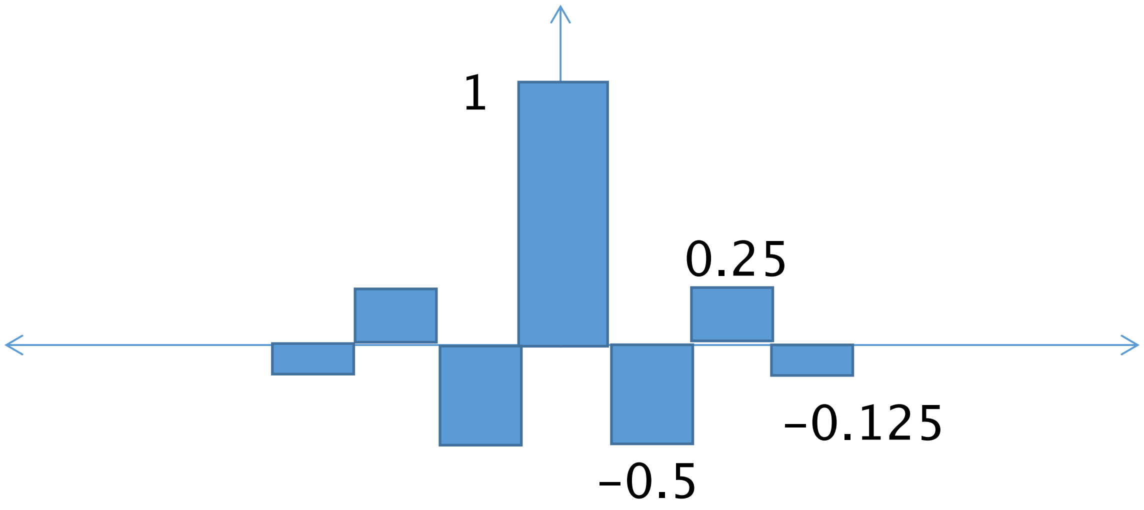
Finite Impulse Response (FIR) filters – Characteristic behaviour when windowing

- In order to understand how $H(e^{j\omega})$ looks like we will use the following simple approximation which captures the windows DTFT $W(e^{j\omega})$ and carry out convolution of $W(e^{j\omega})$ with $H_d(e^{j\omega})$.

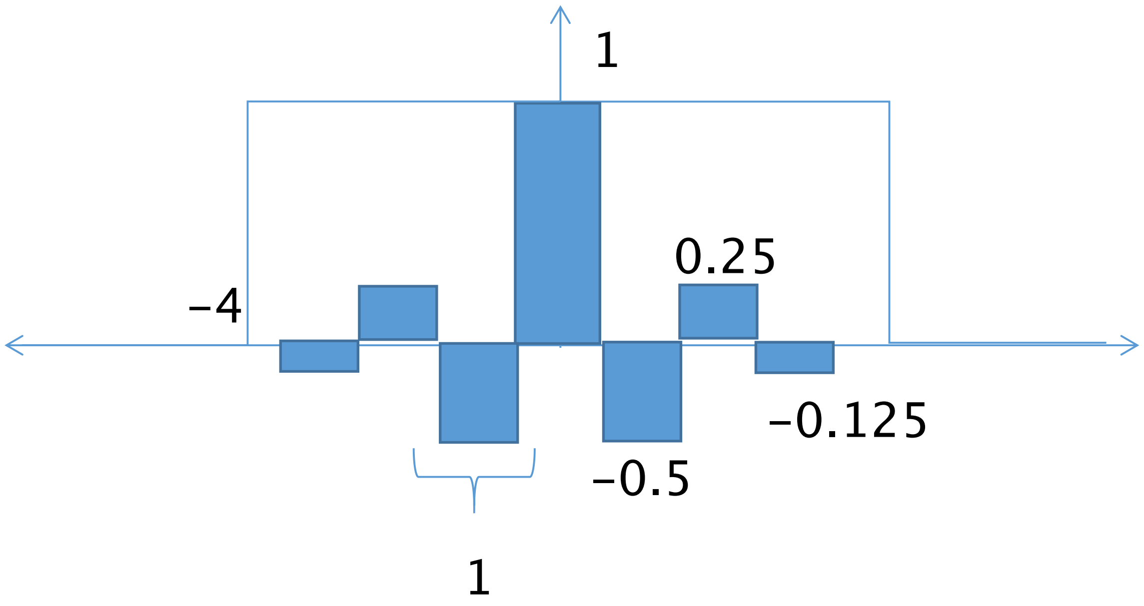


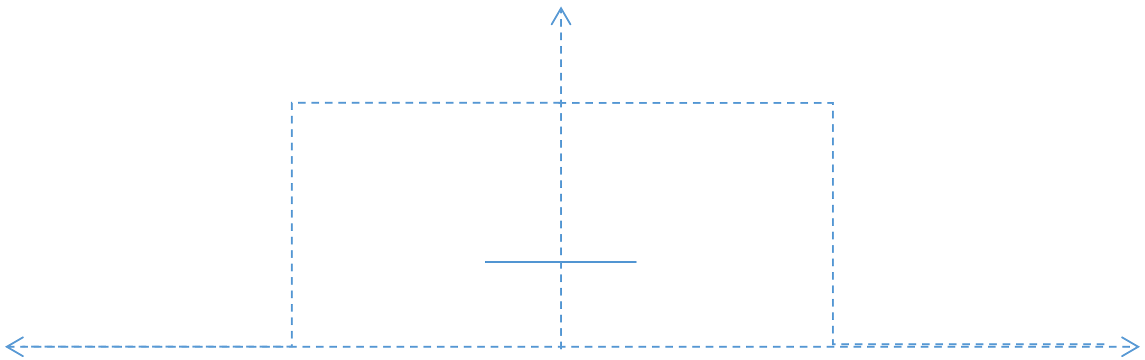




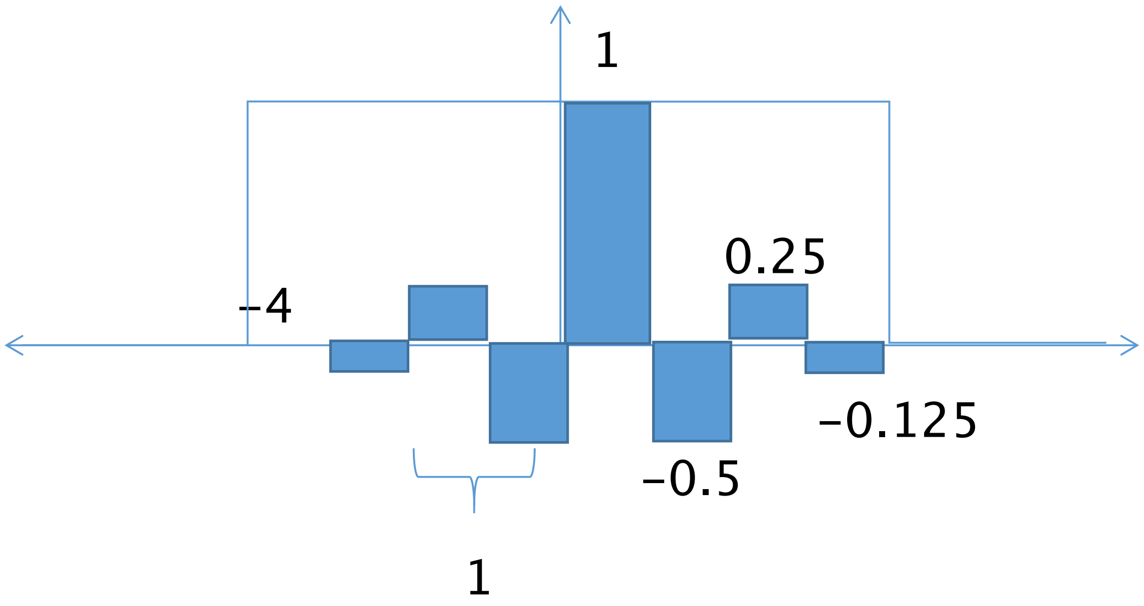


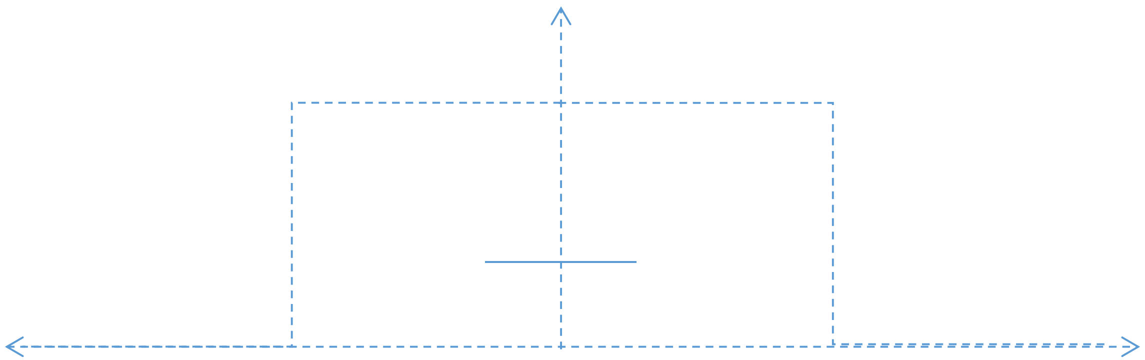
Finite Impulse Response (FIR) filters – Characteristic behaviour when windowing



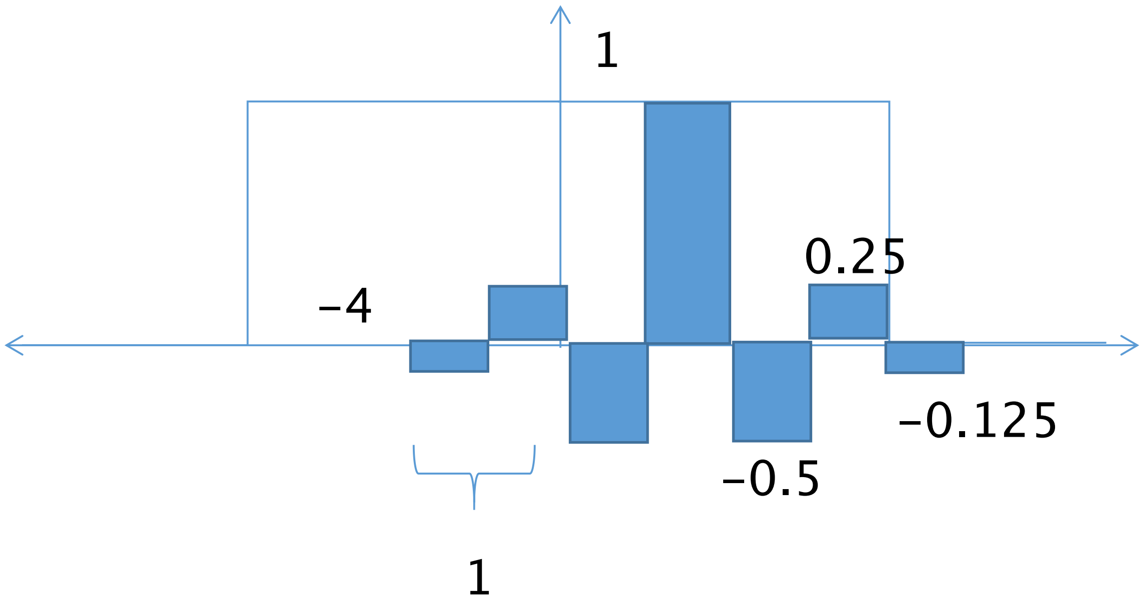


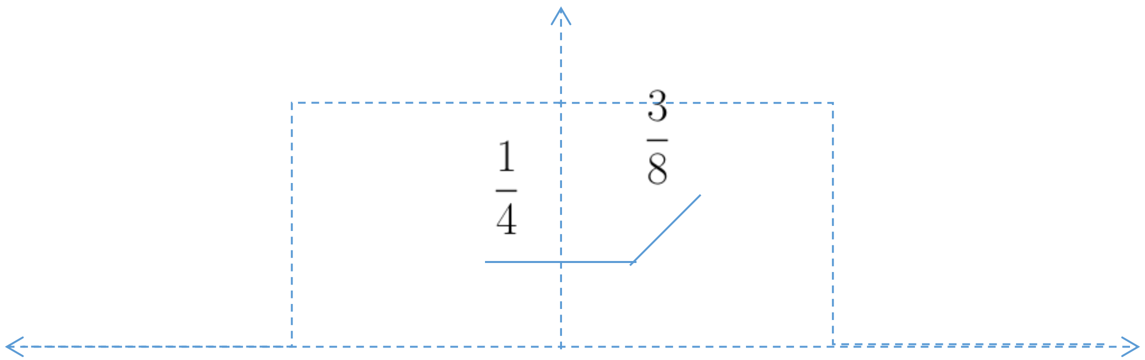
Finite Impulse Response (FIR) filters – Characteristic behaviour when windowing



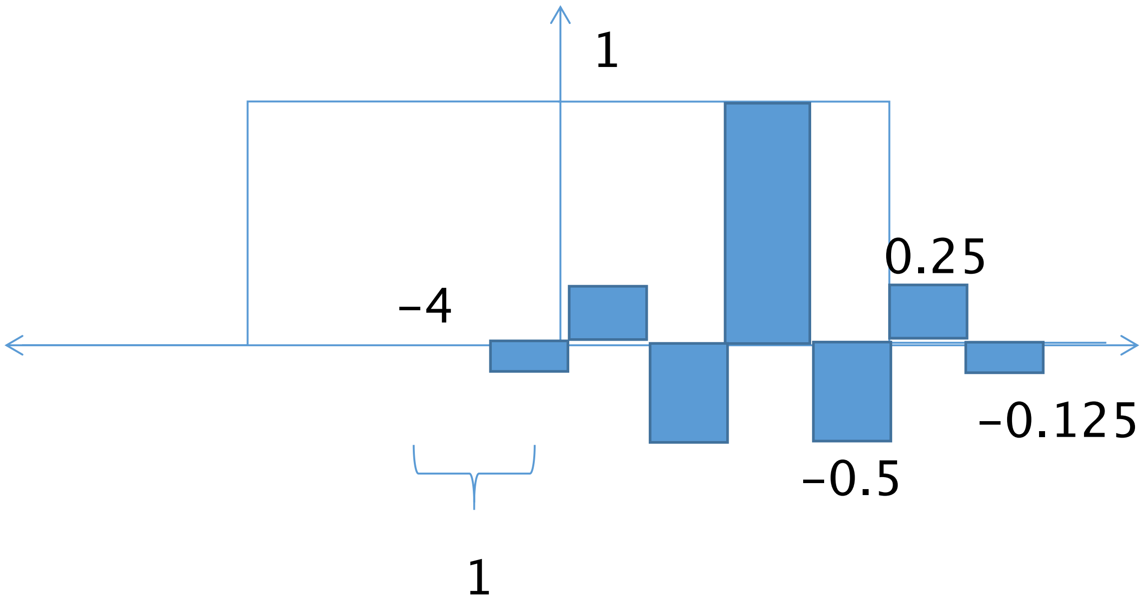


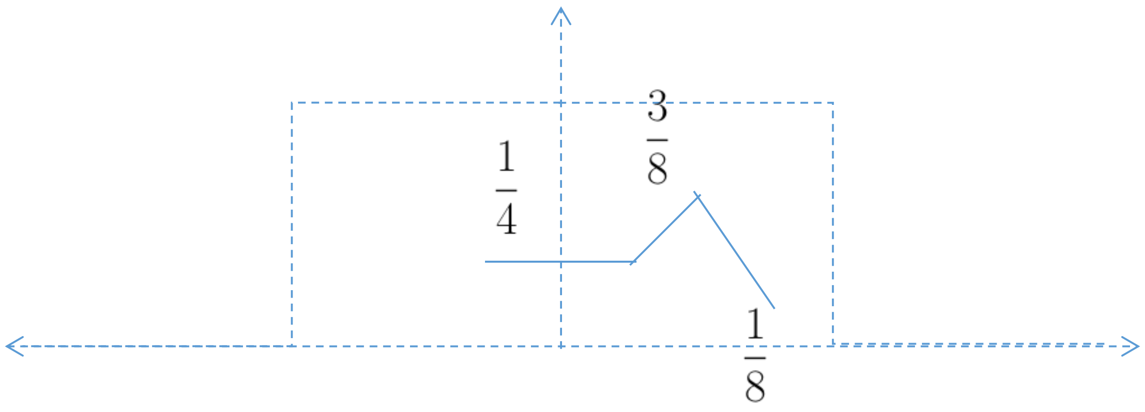
Finite Impulse Response (FIR) filters – Characteristic behaviour when windowing



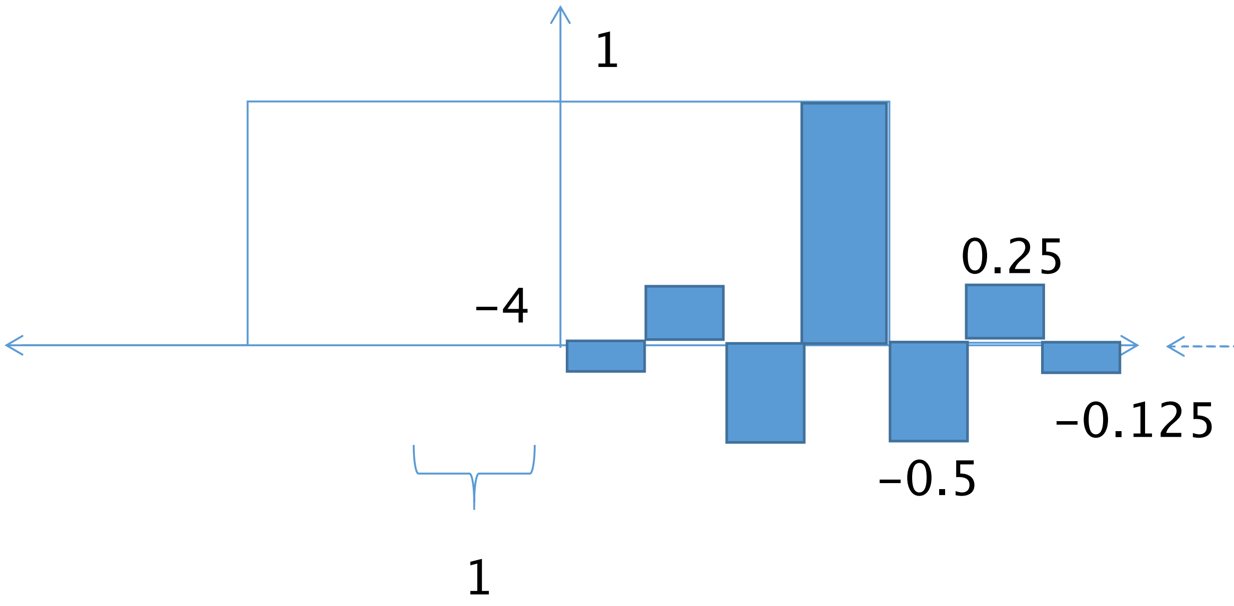


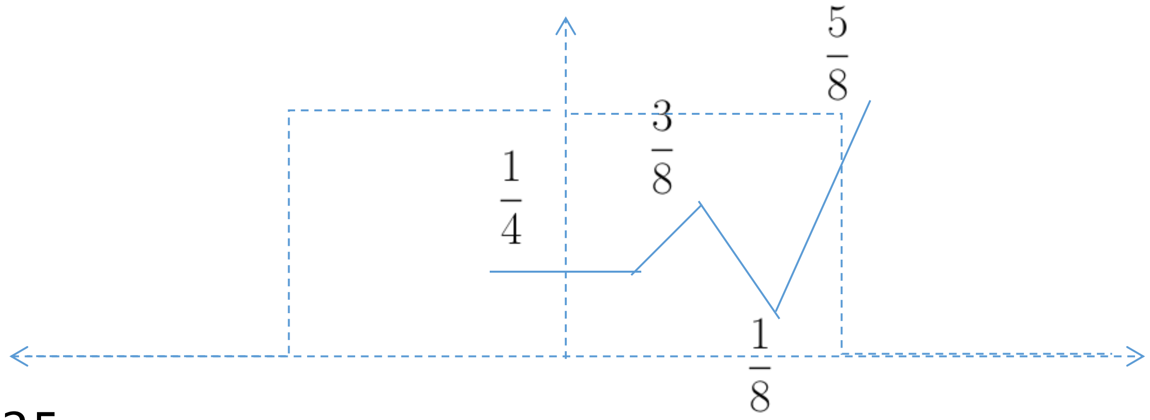
Finite Impulse Response (FIR) filters – Characteristic behaviour when windowing



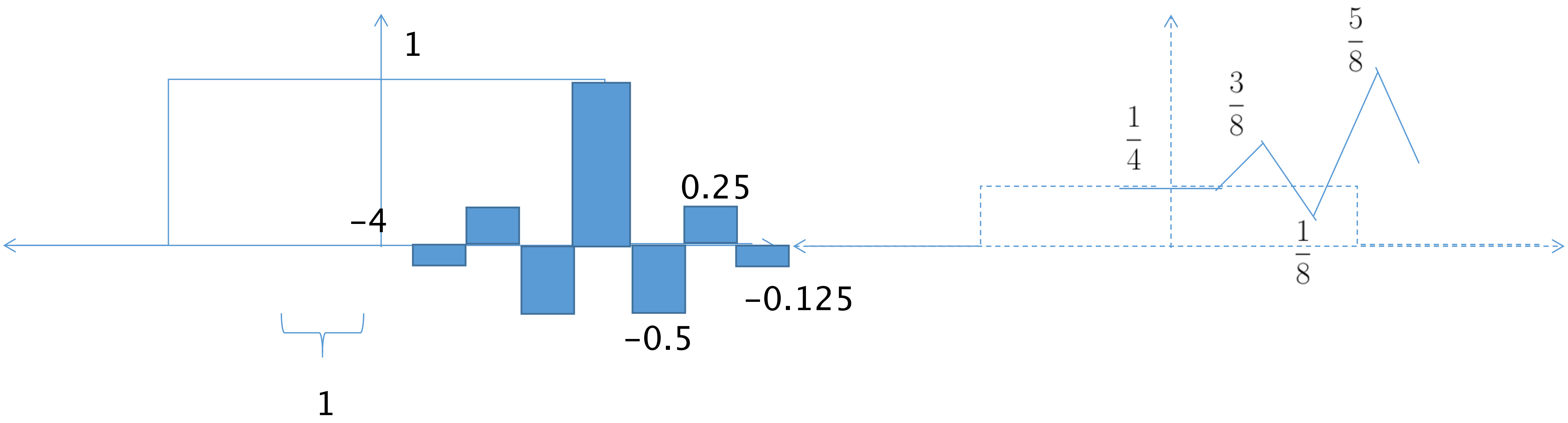


Finite Impulse Response (FIR) filters – Characteristic behaviour when windowing

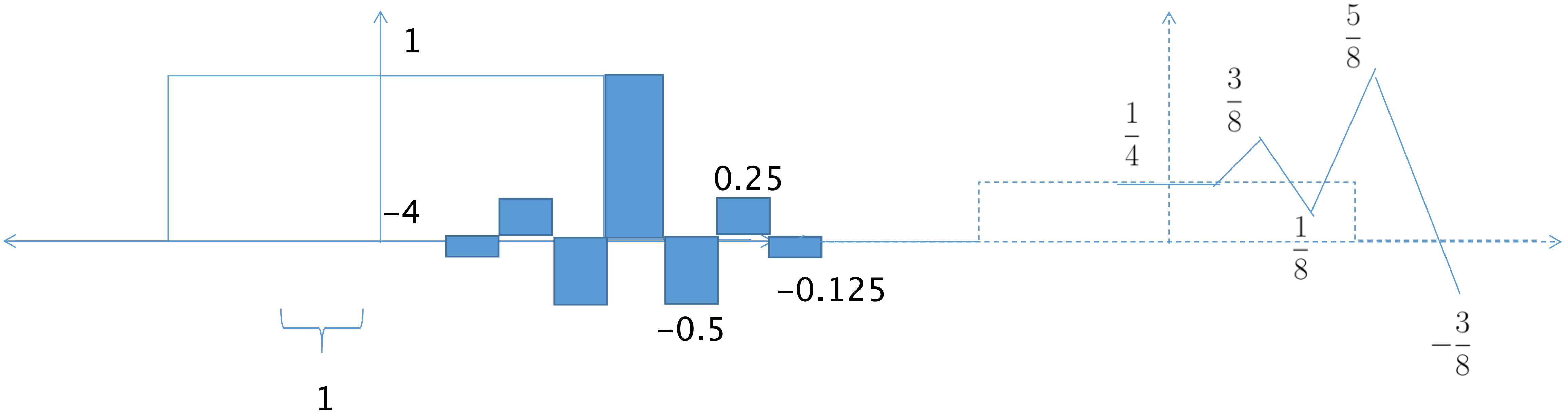




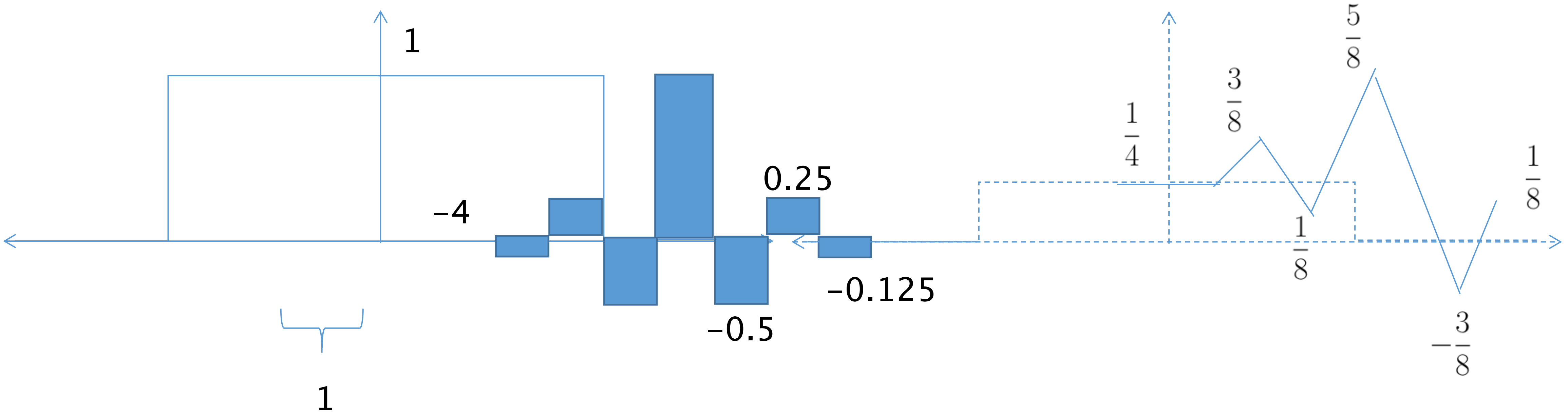
Finite Impulse Response (FIR) filters – Characteristic behaviour when windowing



Finite Impulse Response (FIR) filters – Characteristic behaviour when windowing



Finite Impulse Response (FIR) filters – Characteristic behaviour when windowing

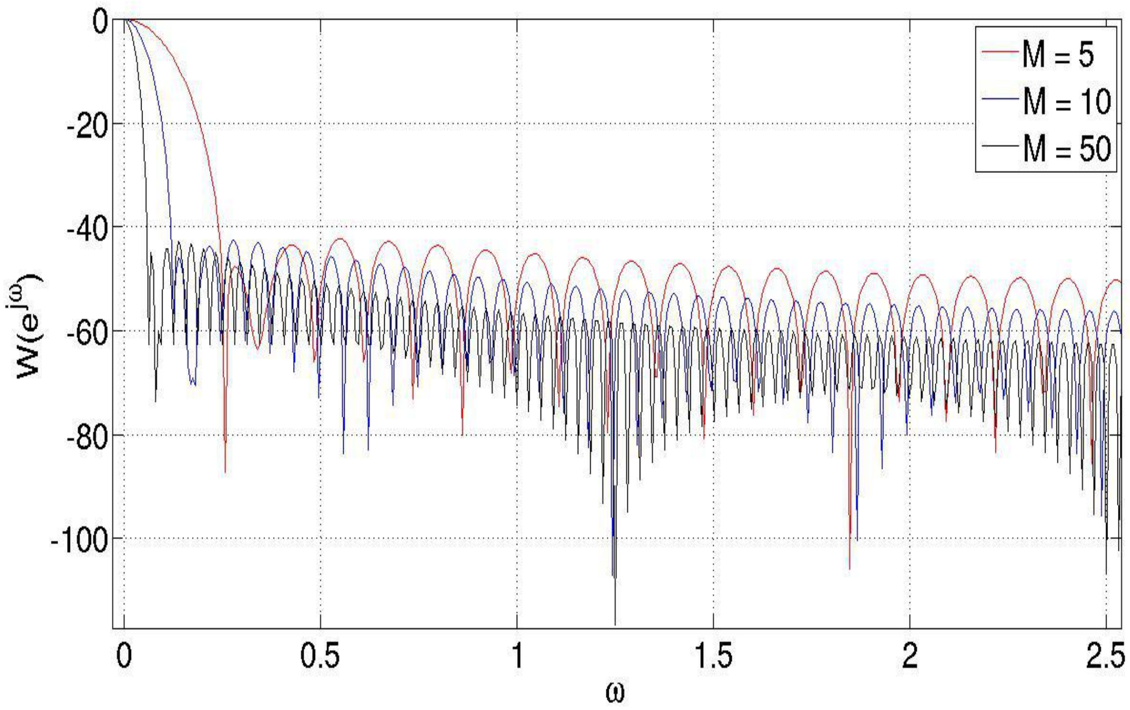


- The transition width depends on the main lobe width
- The overshoot and the undershoot are symmetric – the overshoot and undershoot are measured wrt to the constant gain that we should get (provided the cutoff frequency is more than the main lobe width)

Finite Impulse Response (FIR) filters – Design based on Windows

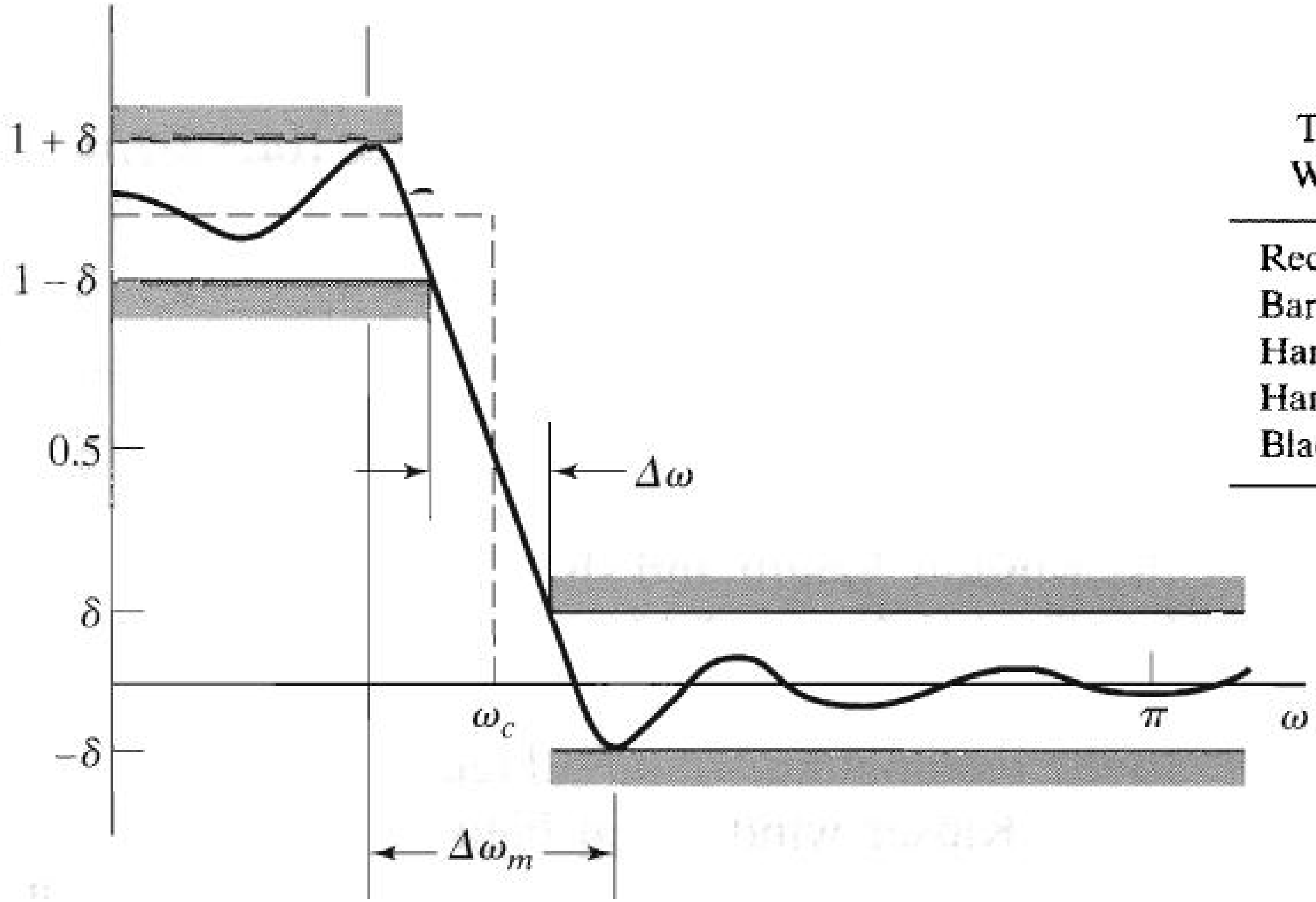
- Windows allow us to control the ripple
  - But they tradeoff ripple with the main lobe width
  - We can then reduce main lobe width by increasing the length M
  - The peak side lobe amplitude is approximately the same even as M is changed!

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe
Rectangular	−13	$4\pi/(M+1)$
Bartlett	−25	$8\pi/M$
Hanning	−31	$8\pi/M$
Hamming	−41	$8\pi/M$
Blackman	−57	$12\pi/M$



Finite Impulse Response (FIR) filters – Design based on Windows

- Why are we looking at the magnitude/amplitude responses only?
  - We are looking at filters with linear phase.



Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)
Rectangular	−13	$4\pi/(M+1)$	−21
Bartlett	−25	$8\pi/M$	−25
Hann	−31	$8\pi/M$	−44
Hamming	−41	$8\pi/M$	−53
Blackman	−57	$12\pi/M$	−74

Finite Impulse Response (FIR) filters – Kaiser Window Method

- We have a tradeoff between mainlobe width and sidelobe area
  - less mainlobe width implies more sidelobe area and more ringing

- For a given mainlobe width which windows has the minimum sidelobe area?
  - Kaiser's window

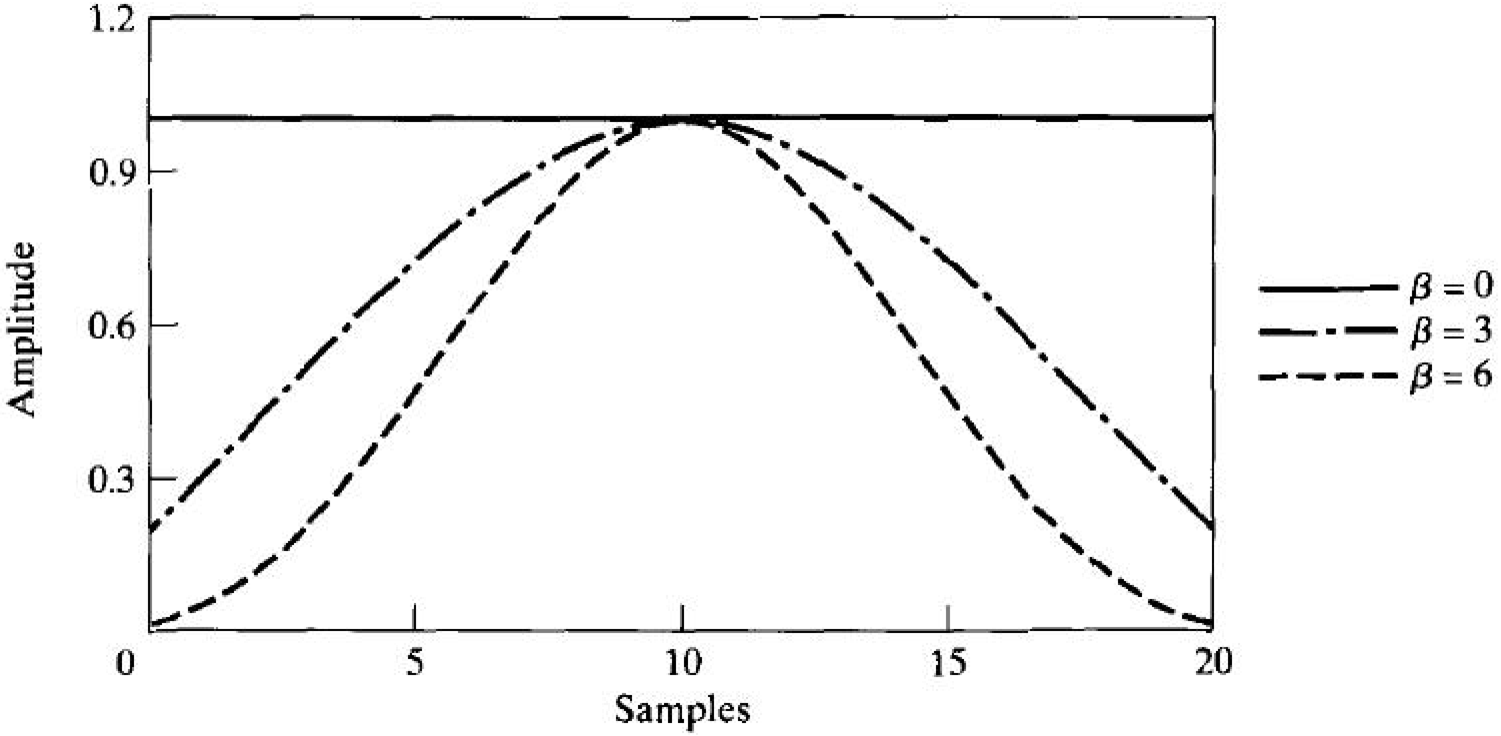
- The Kaiser window is defined as: $w[n] = \begin{cases} \frac{I_0\left(\beta\left(1-\left(\frac{n-\alpha}{\alpha}\right)^2\right)^{\frac{1}{2}}\right)}{I_0(\beta)}, 0 \leq n \leq M, \\ 0, \text{ otherwise.} \end{cases}$

- Here α is chosen as M/2 and $I_0(\,)$ is a special function. $I_0(\,)$ is the zeroth-order modified Bessel function of the first kind.

- The Kaiser window has two parameters – the length (M + 1) and the shape parameter β.

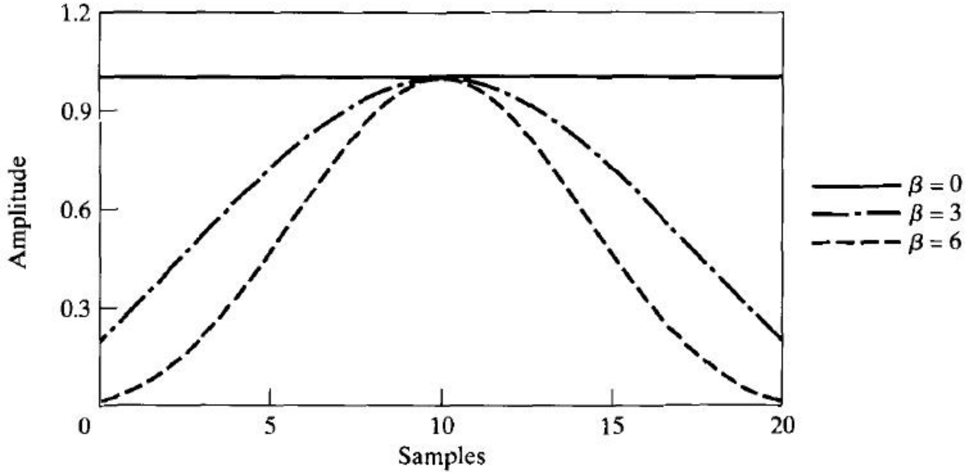
Finite Impulse Response (FIR) filters – Kaiser Window Method

- The Kaiser window has two parameters – the length (M + 1) and the shape parameter β.
- Example for M = 20



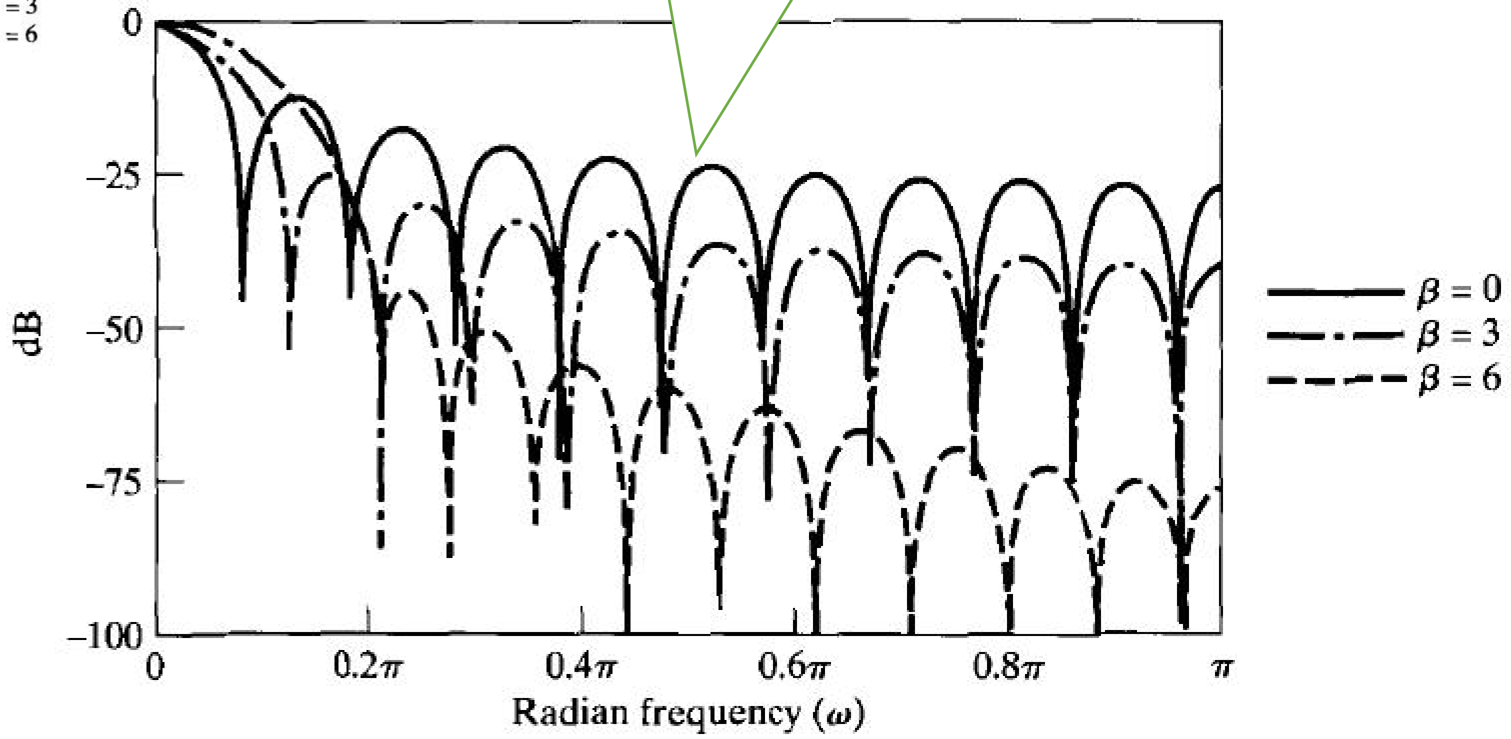
Finite Impulse Response (FIR) filters – Kaiser Window Method

- The Kaiser window has two parameters – the length (M + 1) and the shape parameter β.
- Example for M = 20



Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe
Rectangular	−13	$4\pi/(M+1)$
Bartlett	−25	$8\pi/M$
Hanning	−31	$8\pi/M$
Hamming	−41	$8\pi/M$
Blackman	−57	$12\pi/M$

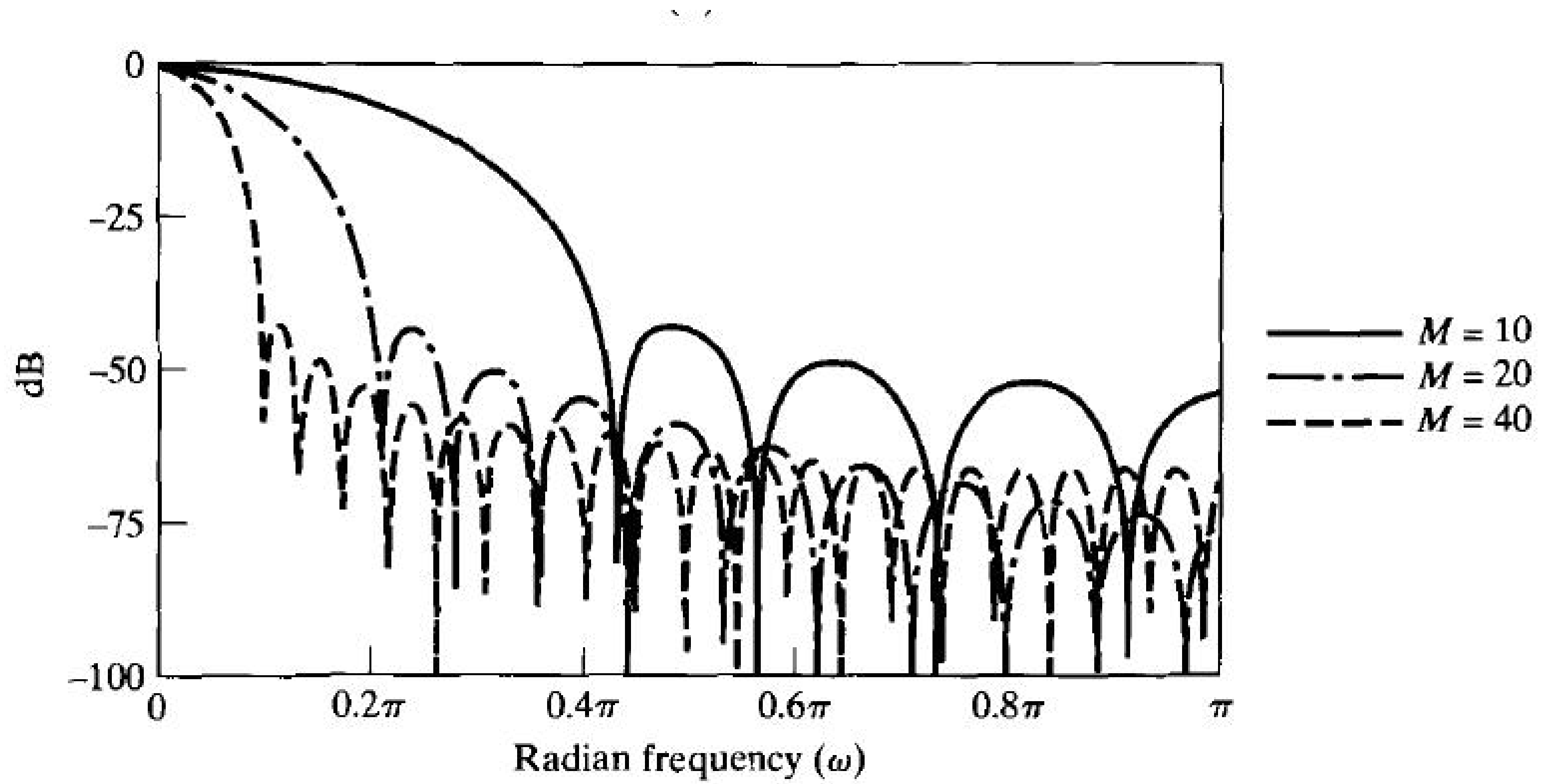
Parameterized spectrum. Important for optimization.



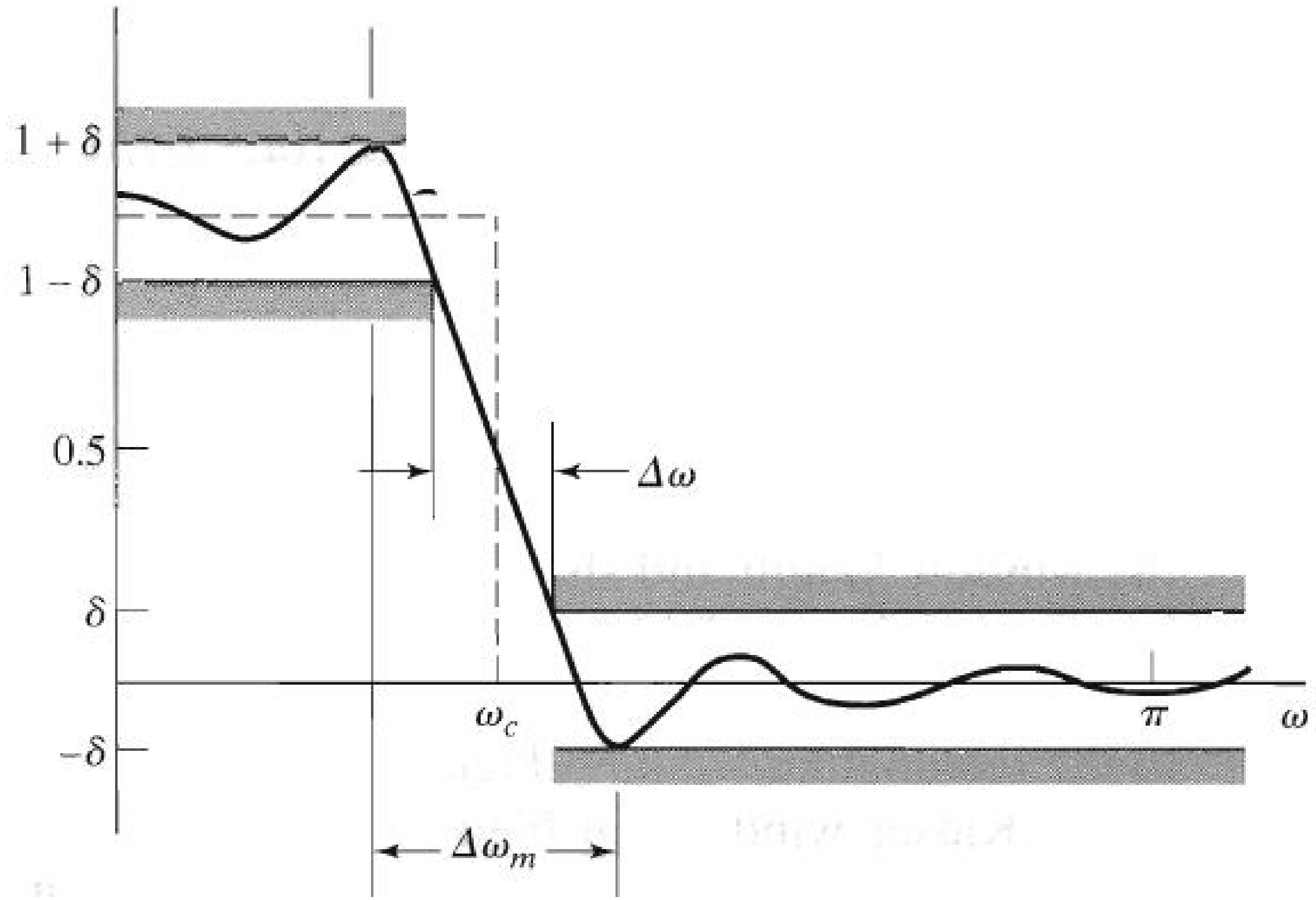
Finite Impulse Response (FIR) filters – Kaiser Window Method

- The Kaiser window has two parameters – the length (M + 1) and the shape parameter β.

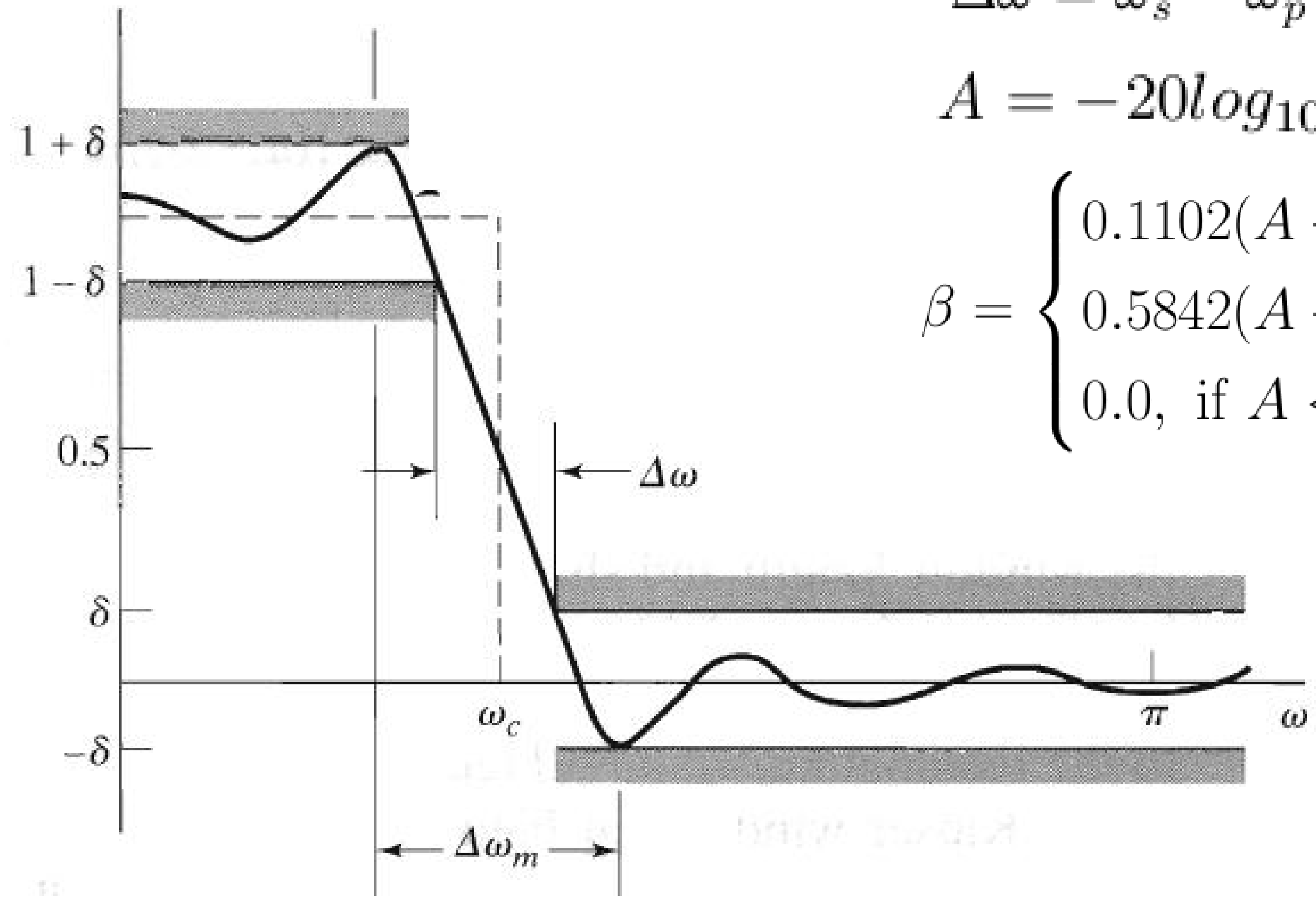
- Example for β = 6



Finite Impulse Response (FIR) filters – Characteristic behaviour of Kaiser windows



Finite Impulse Response (FIR) filters – Design formula



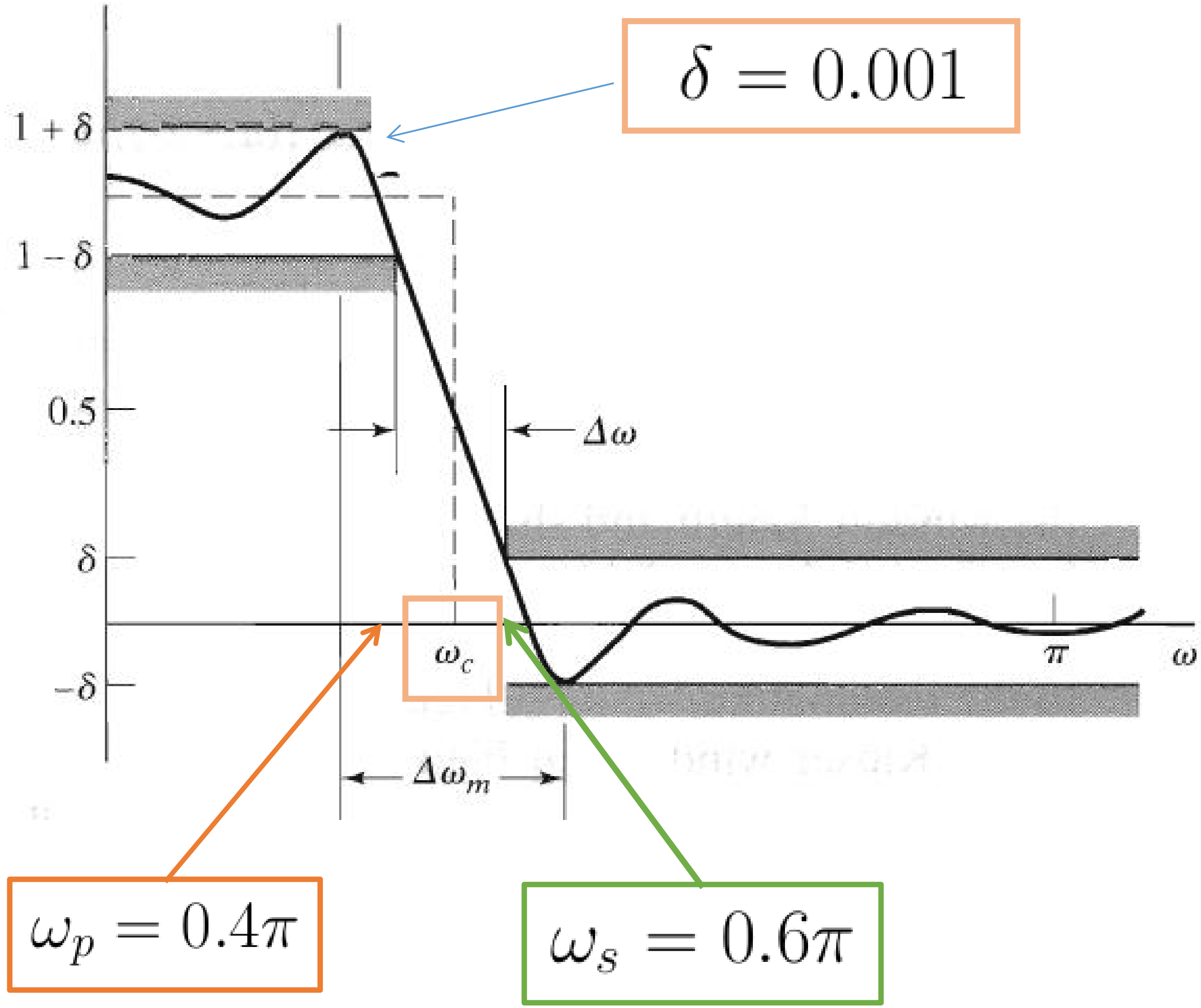
$$\Delta\omega = \omega_s - \omega_p$$

$$A = -20log_{10}\delta$$

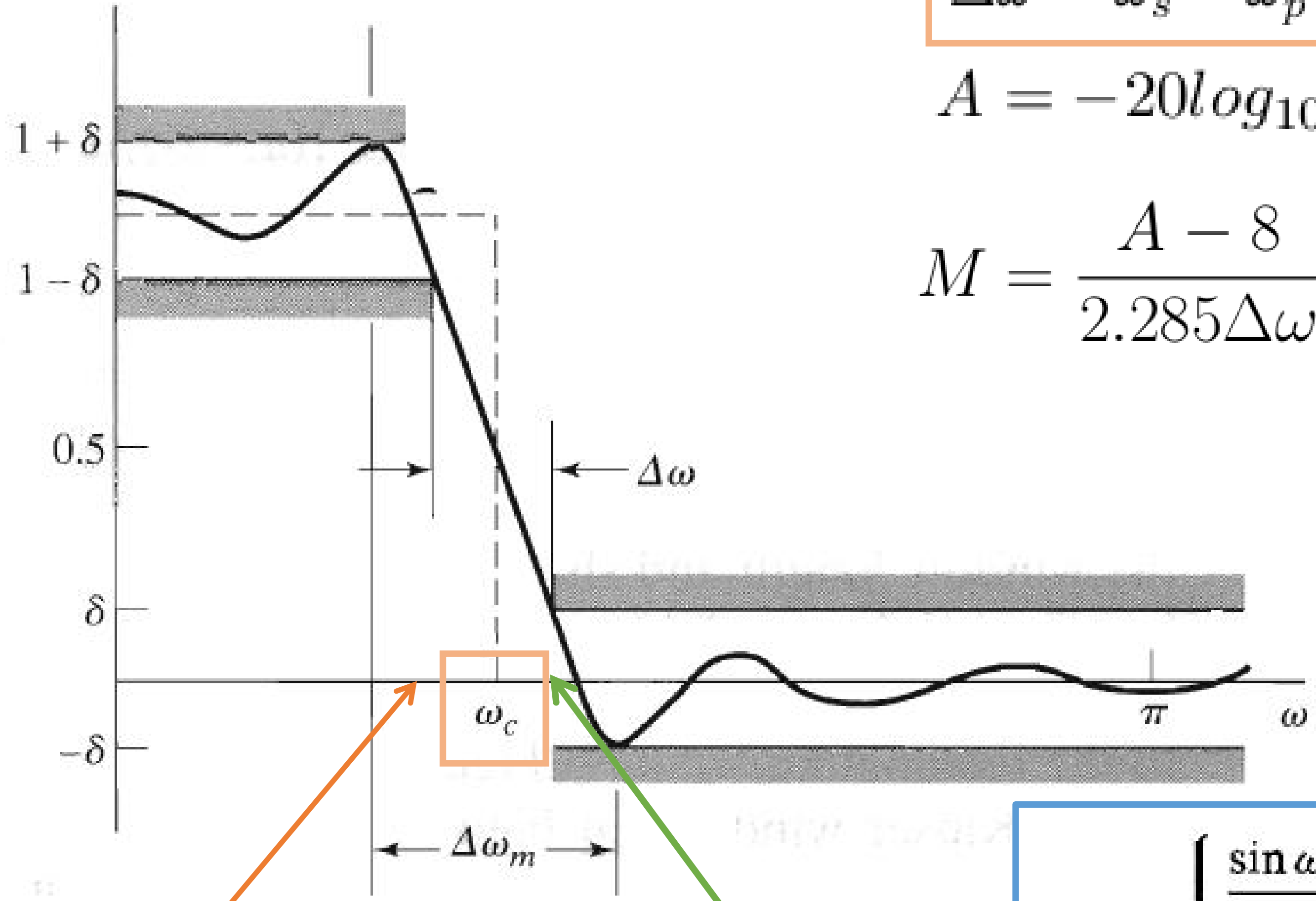
$$\beta = \begin{cases} 0.1102(A-8.7), \text{ if } A > 50, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), \text{ if } 21 \leq A \leq 50, \\ 0.0, \text{ if } A < 21. \end{cases}$$

$$M = \frac{A-8}{2.285\Delta\omega}$$

Finite Impulse Response (FIR) filters – Design example – low pass filter



Finite Impulse Response (FIR) filters – Design example – low pass filter



$$\omega_p = 0.4\pi$$

$$\omega_s = 0.6\pi$$

$$\Delta\omega = \omega_s - \omega_p$$

$$A = -20log_{10}\delta$$

$$\delta = 0.001$$

$$M = \frac{A-8}{2.285\Delta\omega}$$

$$\beta = 5.653, M = 37$$

$$h[n] = \begin{cases} \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \cdot \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases}$$