AV312 - Lecture 5

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Figures from “Communication Systems” by Haykin and “An Intro. to Analog and Digital Commn.” by Haykin and Moher

August 8, 2016

Review of last class

- Frequency modulation
  - Relationship between frequency and phase
  - Relationship between frequency modulation and phase modulation
  - FM is a non-linear transformation $\Rightarrow$ finding bandwidth of the modulated signal $s(t)$ is not straightforward
  - Frequency deviation Δf and modulation index β (phase deviation)

Today's plan

- Frequency modulation and its bandwidth
- Carson's rule
- Today's scribes are Bojja Venkata Hemambhar and Chanumuru Mallikarjuna

Bandwidth of FM signals

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- The phase $\theta(t) =$

$$2\pi f_c t + \frac{\Delta f}{f_m} sin(2\pi f_m t)$$

- Let $\beta = \frac{\Delta f}{f_m}$ (modulation index, measure of phase deviation)

Bandwidth of FM signals

- The FM signal $s(t)$ for single tone $m(t)$ is therefore

$$s(t) = A_c cos(2\pi f_c t + \beta sin(2\pi f_m t))$$

- Intuitively, as Δf and β should affect the bandwidth of the signal
- If $\beta < 1$, we have narrowband modulation
- If $\beta > 1$, we have wideband modulation

Bandwidth of narrowband FM signals

- An approximate approach
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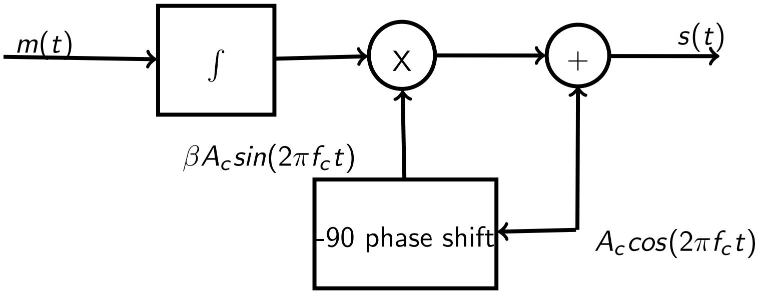
- How to interpret this ?
- The second term is a DSB signal

Narrowband FM modulator



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Narrowband FM modulator

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- What is the distortion?
- What is the envelope for a FM signal $s(t)$ (using an ideal modulator)?
- Multiply and divide by $A_c \sqrt{1 + \beta^2 sin^2(2\pi f_m t)}$.
- This is a very useful mathematical tool in comm. sys. analysis

General analysis

- The modulating signal $m(t) = A_m cos(2\pi f_m t)$
- Is $s(t) = A_c cos(2\pi f_c t + \beta sin(2\pi f_m t))$ periodic ?
- What is the condition for periodicity?

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- $s(t) = Re[\tilde{s}(t) e^{j(2\pi f_c t)}]$
- Here $\tilde{s}(t) = A_c e^{j\beta sin(2\pi f_m t)}$. Is $\tilde{s}(t)$ periodic ?

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- What is c_n?
- $c_n = f_m \int_{-\frac{1}{2f_m}}^{\frac{1}{2f_m}} \tilde{s}(t)e^{-j2\pi n f_m t}dt$

Fourier series exp. for $\tilde{s}(t)$

- $c_n = f_m \int_{-\frac{1}{2f_m}}^{\frac{1}{2f_m}} \tilde{s}(t) e^{-j2\pi n f_m t} dt$
- With $\tilde{s}(t) = A_c e^{j\beta sin(2\pi f_m t)}$, we have

$$c_n = f_m A_c \int_{-\frac{1}{2f_m}}^{\frac{1}{2f_m}} e^{j\beta sin(2\pi f_m t) - j2\pi n f_m t} dt.$$

- Suppose $x = 2\pi f_m t$

$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta sin(x) - \pi x)} dx.$$

Fourier series exp. for $\tilde{s}(t)$

- We have
$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta sin(x) - \pi x)} dx.$$
- nth order Bessel function of the first kind and argument β
$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta sin(x) - \pi x)} dx.$$
- Therefore, $c_n = A_c J_n(\beta)$.

Fourier series exp. for $\tilde{s}(t)$

- We have
$$c_n = \frac{A_c}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta sin(x)-\pi x)}dx.$$
- nth order Bessel function of the first kind and argument β
$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta sin(x)-\pi x)}dx.$$
- Therefore, $c_n = A_c J_n(\beta)$.
- Recall the Fourier series expansion $\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t}$
- Then $\tilde{s}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{j2\pi n f_m t}$
- Also recall that $s(t) = Re[\tilde{s}(t)e^{j2\pi f_c t}]$

An expansion for $s(t)$

- Then $\tilde{s}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{j2\pi n f_m t}$
- Also recall that $s(t) = Re[\tilde{s}(t) e^{j2\pi f_c t}]$
- $s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) cos(2\pi(f_c + nf_m)t)$

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- What is $S(f)$?

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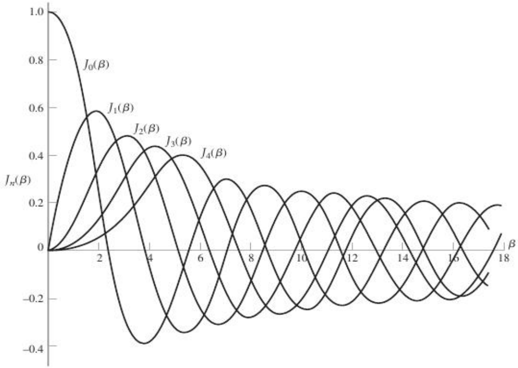
- $s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) cos(2\pi(f_c + nf_m)t)$
- What is $S(f)$?
- $S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right]$

The Bessel coefficients $J_n(\beta)$



- $J_n(\beta) = (-1)^n J_{-n}(\beta)$
- For β small, $J_0(\beta) = 1$, $J_1(\beta) = \frac{\beta}{2}$, $J_n(\beta) \approx 0, n \geq 2$
- $\sum_n J_n^2(\beta) = 1$. What is the power in an FM signal?

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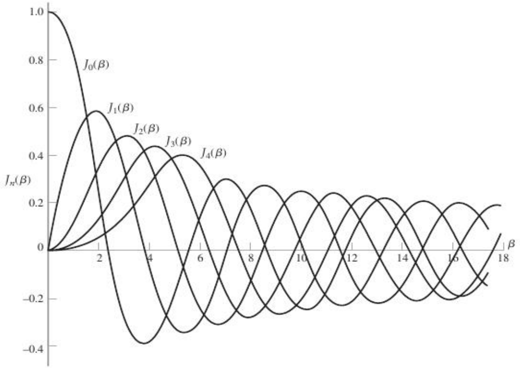

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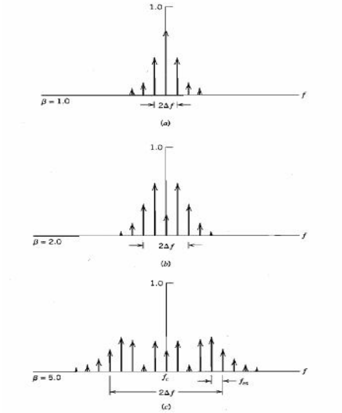
- Power $= \frac{A_c^2}{2}$

The effect of β and Δf

- Recall $\Delta f = k_f A_m$, $\beta = \frac{\Delta f}{f_m}$, and $S(f)$.

f_m fixed, A_m is varied



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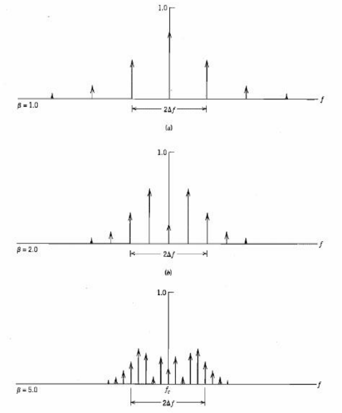

Carson's rule

- For wideband FM, the actual bandwidth is ∞
- Empirically, the bandwidth is $2\Delta f$ for wideband FM
- For narrowband FM, the bandwidth is $2f_m$
- Carson's rule: Bandwidth is $2\Delta f + 2f_m$
- Read the textbook for an alternative method using $J_n(\beta)$

Carson's rule for general $m(t)$

- Let the maximum freq. component in $m(t)$ be W
- Let $\Delta f = k_f max|m(t)|$
- Use Carson's rule (is an underestimate!)