Review

Gaussian processes : → DT (X_n) → motivation is noise modelling.
→ CT $(X(t))$.

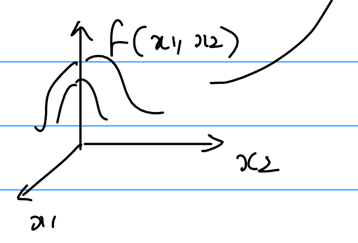
← def. of Gaussian $(X(t))$

1) for any m, $t_1 < t_2 < t_3 \cdots < t_m$

$a_1 X(t_1) + a_2 X(t_2) + \cdots + a_m X(t_m)$ is a Gaussian RV

2) The joint distribution of $(X(t_1) \cdots X(t_m))$ is multivariate Gaussian.

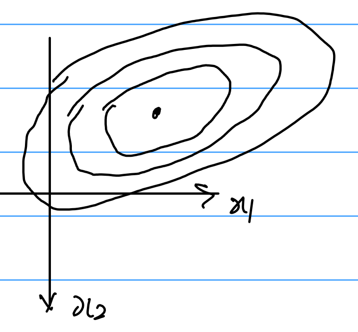
e.g. $f_{X(t_1)\,X(t_2)}(x_1, x_2)$



$$\frac{1}{(\sqrt{2\pi})^2} \frac{1}{\sqrt{\|\Sigma\|}} \times e^{-\frac{1}{2}(\bar{x}-\bar{\mu})^T \Sigma^{-1} (\bar{x}-\bar{\mu})}$$

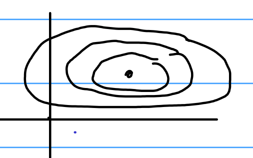
where $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Sigma = \begin{pmatrix} var(x_1) & cov(x_1, x_2) \\ cov(x_1, x_2) & var(x_2) \end{pmatrix} \qquad \bar{\mu} = \begin{pmatrix} \mathbb{E} x_1 \\ \mathbb{E} x_2 \end{pmatrix}$$

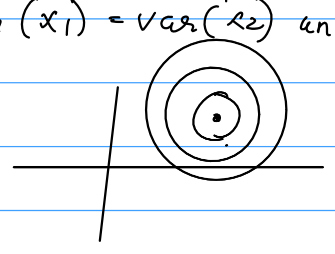


if $X(t_1) \perp X(t_2)$
$cov(x_1, x_2) = 0$

then



if $var(x_1) = var(x_2)$ and $\perp$



suppose we are now looking at $(X(t_1) \cdots X(t_m))$

$$f_{X(t_1)\,X(t_2) \cdots X(t_m)}(x_1, x_2, x_3 \cdots x_m) =$$

$$\frac{1}{(\sqrt{2\pi})^m} \frac{1}{\sqrt{\|\Sigma\|}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu})^T \Sigma^{-1} (\bar{x}-\bar{\mu})}$$

where $\bar{\mu} = \begin{pmatrix} \mathbb{E} X_1 \\ \vdots \\ \mathbb{E} X_m \end{pmatrix}$ and $\Sigma = \big(cov(X(t_i), X(t_j)) \big)$

1) suppose Σ is a diagonal matrix $\Leftarrow$ $cov(X_i, X_j) = 0$ for $i \neq j$

then $(\bar{x}-\bar{\mu})^T \Sigma^{-1} (\bar{x}-\bar{\mu}) = \sum_i (x_i - \mathbb{E} x_i)^2 (\Sigma_{ii})^{-1}$

then
$$\left(\frac{1}{\sqrt{2\pi}}\right)^m \frac{1}{\sqrt{\|\Sigma\|}} \prod_{i=1}^{m} e^{-\frac{1}{2}(x_i - \mathbb{E} x_i)^2 / \Sigma_{ii}}$$

$$= \prod_{i=1}^{m} 1/\sqrt{2\pi} \cdot 1/\sqrt{\Sigma_{ii}} \cdot e^{-\frac{1}{2}(x_i - \mathbb{E} x_i)^2 / \Sigma_{ii}}$$

then we get a independently distributed GP.

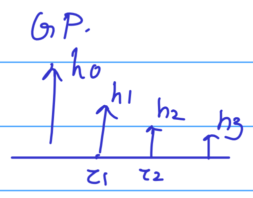
2) A GP is characterized completely by a mean function and a autocorrelation function

3) Suppose we have a WSS GP. then it is strictly stationary!
Why?
for SS $f_{X(t_1) \cdots X(t_m)}(x_1 \cdots x_m) = f_{X(t_1+\tau) \cdot X(t_2+\tau) \cdots X(t_m+\tau)}(x_1 \cdots x_m)$

$$cov(X(t_i), X(t_j)) = R_X(t_i - t_j) \quad = \quad cov(X(t_i+\tau), X(t_j+\tau)) = R_X(t_i - t_j).$$

4) $(X(t))$ → $\boxed{LTI}$ → $(Y(t))$
G.P G.P.

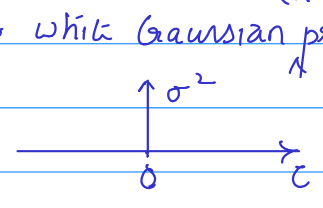
↳ $h(t) =$



$\alpha_1 Y(t_1) + \alpha_2 Y(t_2) + \alpha_3 Y(t_3) + \cdots \alpha_m Y(t_m) \sim \mathcal{N}$
↑
$\sum_i h_i \cdot X(t_1 - \tau_i)$

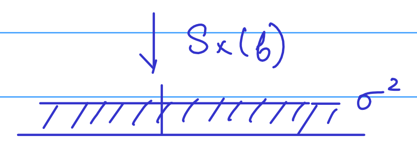
Examples of GPs.

a) IID GP , $X(t) \sim \underline{\mathcal{N}(0, \sigma^2)}$ → white Gaussian (noise) process (WGN)
autocorrelation $R_X(\tau) = \sigma^2 \cdot \delta(\tau)$



↓ $S_X(f)$

white process. ←



b) $N(t) = N_I \cos(2\pi f_c t) - N_Q \sin(2\pi f_c t)$ $\qquad N_I \sim \mathcal{N}(0, \sigma_I^2)$, $N_Q \sim \mathcal{N}(0, \sigma_Q^2)$. $\perp$

Is this Gaussian RP or not ?

$\mathbb{E} N(t) = 0$ $\qquad \mathbb{E} N(t_1) N(t_2) = \mathbb{E} N_I^2 \cdot \cos(2\pi f_c t_1) \cdot \cos(2\pi f_c t_2) + \mathbb{E} N_Q^2 \sin(2\pi f_c t_1) \cdot \sin(2\pi f_c t_2)$

$\mathbb{E} N_I^2 = \sigma_I^2$, $\mathbb{E} N_Q^2 = \sigma_Q^2$

don't get WSS GP.
if $\sigma_I^2 = \sigma_Q^2$ then WSS, SS, GP.