3 Spectral Analysis Review

1. Let $x(t)$ be a continuous time signal $2cos(2000\pi t)$ defined for all t. Let $x[n]$ be a discrete time signal obtained by sampling $x(t)$ with sampling frequency 16000 Hz for $t \in [0, 8)$ ms (note that 8 ms is not included).
    a) If N is the length of $x[n]$, plot the N-point DFT of $x[n]$.
    b) What is the CTFT of $x(t)$?
    c) We will use $x[n]$'s DFT to think about $x(t)$'s CTFT. Plot the DFT magnitude of $x[n]$. What is the difference between this plot and the CTFT?
    d) Change the frequency axis to the continuous time frequency Ω domain. Does your DFT plot look like a frequency sampled CTFT? Do you need to change the magnitude axis? What should you scale the magnitude axis with? Why?
2. Let $x(t)$ be a continuous time signal $2cos(2000\pi t) + 5cos(500\pi t)$ defined for all t. We define the following sampling rates: $f_{s,1} = 500$, $f_{s,2} = 2000$, $f_{s,3} = 4000$, where all sampling rates have units of Hertz. Let $x_i[n]$ be the discrete time signal obtained by sampling $x(t)$ at sampling rate $f_{s,i}$ for $t \in [0, 8)$ ms (note that 8 ms is not included)
    a) What is the CTFT of $x(t)$?
    b) Plot the magnitude of DFTs of each $x_i[n]$. Compare the magnitude plots? Are they different? Why do the differences arise?
    c) We note that it is possible to obtain a continuous time signal $x_i(t)$ from each $x_i[n]$ by using sinc interpolation which we had studied in class. As in question 1, use the DFT of each $x_i[n]$ as the frequency sampled CTFT of $x_i(t)$, i.e., make appropriate changes to the frequency axis and magnitude axis. What frequency components do you get in each $x_i(t)$? How are they different from those in $x(t)$? Why? (Please note that you are not asked to do the interpolation!)
3. Let $x[n]$ be a sequence defined as $(1, 1, 1, 1)$ for $n = (0, 1, 2, 3)$. Obtain
    a) the 8-point DFT of $x[n]$ obtained by padding zeros to $x[n]$ in order to make a 8 length sequence,
    b) the 16-point DFT of $x[n]$ obtained by padding zeros to $x[n]$ in order to make a 16 length sequence,
    c) the 32-point DFT of $x[n]$ obtained by padding zeros to $x[n]$ in order to make a 32 length sequence,
    d) the 1024-point DFT of $x[n]$ obtained by padding zeros to $x[n]$ in order to make a 1024 length sequence.

    Plot and compare the DFTs that you have obtained above. Suppose the above 8, 16, 32, and 1024 length sequences are thought of as being obtained by sampling continous time signals at a sampling rate of 1 Hz, but for different time durations. Interpret each of the above DFT magnitude plots as frequency sampled CTFTs of the above continuous time signals. What is the significance of the time duration in your interpretation?
4. Let $x(t)$ be $2cos(2000\pi t)$. We define the following sampling rates: $f_{s,1} = 8000$, $f_{s,2} = 16000$, $f_{s,3} = 8888$, $f_{s,4} = 17776$, where all sampling rates have units of Hertz.
    a) Let $x_i[n]$ be the discrete time signal obtained by sampling $x(t)$ at sampling rate $f_{s,i}$ for $t \in [0, 8)$ ms.

b) Let $y_j[n]$ be the discrete time signal obtained by sampling $x(t)$ at sampling rate $f_{s,i}$ for $t \in [0, 8]$ ms.

c) In each case plot the DFT magnitude and use that to think about the CTFT of $x(t)$.

d) How do each one of the sampled CTFTs that you have "computed using DFT" compare with the actual CTFT?

e) Let $z[n]$ be the discrete time sequence obtained by sampling $x(t)$ at $f_{s,1}$ for $[0, 4)$ ms. Extend the length of $z[n]$ using zero padding so that $z[n]$ has the same length as $x_1[n]$. Compare the DFT magnitude plot of $z[n]$ with $x_1[n]$. Is there a difference? If there is, then explain why?

f) If $z(t)$ is the continuous time signal obtained from $z[n]$, what is the relationship between $z(t)$ and $x(t)$?

5. Use internet resources to find out what dual tone multi-frequency (DTMF) signalling is. For example, look at

   a) https://en.wikipedia.org/wiki/Dual-tone_multi-frequency_signaling

   b) http://onlinetonegenerator.com/dtmf.html

   When a key is pressed (e.g., on your landline phone) the tone generator would produce a signal for the duration of the key press which is the sum of a high frequency tone and a low frequency tone. The high and low frequencies corresponding to different keys are shown in the following table:

1209 Hz	1336 Hz	1477 Hz	1633 Hz	
1	2	3	A	697 Hz
4	5	6	B	770 Hz
7	8	9	C	852 Hz
*	0	#	D	941 Hz

   We will first make a DTMF tone simulator. Write a Matlab function "makeDTMFsignal" that takes as input

   a) The sequence of key's pressed (e.g., this input can be the array [1,2]),

   b) The duration of each key press in seconds (all key presses are assumed to be of the same duration, and there is no time taken between two consecutive key presses),

   and returns the sampled DTMF signal. Please choose an appropriate sampling rate; justify your choice. Note that the sampling rate could be another input into your function or hard-coded in your function. Play the sampled DTMF signal through your computer speakers.

   In the second part of this question, you will make a DTMF signal decoder. Note that the DTMF signal can be decoded by looking at its frequency content, which can be computed via the DFT. Write a Matlab function "decodeDTMFsignal" that takes as input

   a) the sampled DTMF signal that you have produced using "makeDTMFsignal"

   b) the duration of each keypress (note that the duration should be the same as that used for making the sampled DTMF signal you are using as input)

   c) the sampling rate used to make the sampled DTMF signal

   and returns the sequence of keys that corresponds to the DTMF signal that you have used as input. Design the function "decodeDTMFsignal" using a flowchart. Note that the decoder needs to decide during a key press duration whether two frequencies are present in a signal. Since all key presses have the same duration, an approach would be to

   a) segment the sampled DTMF signal into smaller duration signals, each duration being that of a keypress duration

   b) apply DFT to find out the frequency content

   c) decide if there is a tone at a particular frequency (how will you make this decision?)

6. Review z-transforms.

7. For the tasks set out in this labsheet, we will use rational z-transforms. Note down your answers to the following questions.
   - Write down the definition of rational z-transforms.
   - We note that rational z-transforms can be represented using their poles and zeros. Think about and note down how you will represent rational z-transforms on a computer.
   - Can rational z-transforms be represented using matlab vectors? Do you need to represent their multiplicities separately? Does there exist a case where separate representation of multiplicities is useful?
   - Does the ordering of poles and zeros (e.g., a magnitude-wise sorted vector) in your representation matter? Does there exist a case where an ordered representation is useful?

8. In this task, you will visualize the magnitude $|X(z)|$ of the z-transform $X(z)$ of a sequence $x[n]$. Assume that $X(z)$ is a rational z-transform which is represented using the representation that you have obtained above in (Task 2). Obtain a 3D plot of $|X(z)|$ as a function of the real and imaginary part of z for three choices of rational $X(z)$. You are free to make these three choices. Comment on how $|X(z)|$ behaves near poles and zeros. How does the behaviour near poles and zeros change with the multiplicity of poles and zeros? Comment on the region of convergence of the rational z-transforms you have chosen.

9. In this task, you will obtain the DTFT of a sequence $x[n]$ from its z-transform $X(z)$. Assume that the sequence you are looking at has a rational z-transform $X(z)$ represented using the representation in (Task 2).
   a) Obtain the magnitude plot of the DTFT from $X(z)$ as discussed in class (hint: use the distance from $e^{j\omega}$ method). Plot the magnitude plot from 0 to 4π radians/sample and comment on the periodicity of the DTFT (use an appropriately sampled ω so that the magnitude plot is smooth).
   b) Obtain the phase plot of the DTFT from $X(z)$ as discussed in class (hint: use the angle from $e^{j\omega}$ method). Plot the phase plot from 0 to 4π radians/sample and comment on the periodicity of the DTFT (use an appropriately sampled ω so that the phase plot is smooth).

   Obtain the DTFT of two rational z-transforms. You are free to choose these two z-transforms (we had learned how to plot the approximate DTFT from the poles and zeros, try to choose two z-transforms which will give you a high pass and a low pass filter response respectively).

10. In this task, you will write a function that will "expand out" a rational z-transform as a partial fractions expansion.
    a) Write a matlab function "partialFractionExpander" that has as input the rational z-transform represented as in Task (2).
    b) The function should first check if the number of zeros is strictly less than the number of poles.
    c) If it is so, then the function should use the method that we had discussed in class in order to obtain the coefficients of the different terms that arise in the partial fraction expansion. For example, the function should expand rational $X(z)$ as

    $$\frac{P(z)}{Q(z)} = \sum_{k=1}^{N} \sum_{s=1}^{s_k} \frac{A_{k,s}}{(z - p_k)^s},$$

    where $P(z)$ and $Q(z)$ are polynomials in z, N is the number of poles, p_k represents the k^{th} pole, and s_k is the multiplicity of the k^{th} pole.
    d) The function should put out an output which represents the RHS above. What representation would you choose? Would a list of 3-tuples, with the tuple containing p_k, s_k, and $A_{k,s}$ be sufficient?
    e) Implement this function under the assumption that $s_k \leq 2$ for all k.

f) Test your function with five different $X(z)$. You are free to choose $X(z)$ (choose $X(z)$ with different number of poles and zeros and multiplicities).

g) In each case, obtain the partial fractions expansion manually. Compare the expansion that you obtain with the function with the expansion that you obtain manually.

11. Let $x[n]$ be a sequence of length N defined for $n \in \{0, \ldots, N-1\}$.

a) Write a matlab function "ztransform" which takes as input $x[n]$ and the value of z, say z_0, that we are interested in and computes the $X(z)|_{z=z_0}$.

b) Modify your function so that it also prints the pole and zero locations, along with their multiplicities.

c) Suppose we fix $z_0 = re^{-j\frac{2\pi k}{N}}$ for some $r > 0$ and $k \in \{0, \ldots, N-1\}$. Modify your function "ztransform" to compute the z transform of $x[n]$ for z_0 of this form by using FFT. Your modified function should use r and k as input instead of z_0.