ECE 580 Fun Work #5

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Answer 1

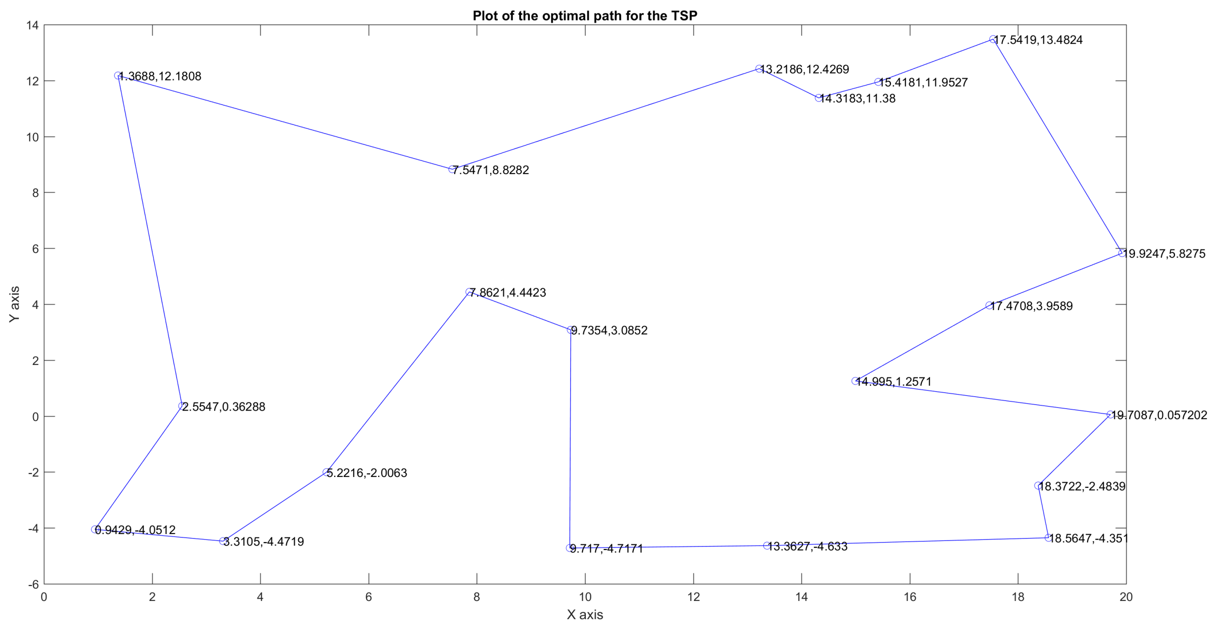
The MATLAB Code, is shown after the answer below:-

The genetic algorithm mentioned in the problem was implemented. For 20 cities there are 20! combinations of paths available. The phenotype space was generated as mentioned in the question. We implemented the single parent crossover operator and use elitism while implementing the GA. The number of elite chromosomes chosen in each iteration was 10. The fitness function used has $value = 1/distancetravelled(currentpath)$ for evaluvating the fitness of the chromosomes in the mating pool.
The MATLAB code is commented, explaining the genetic algorithm implementation.

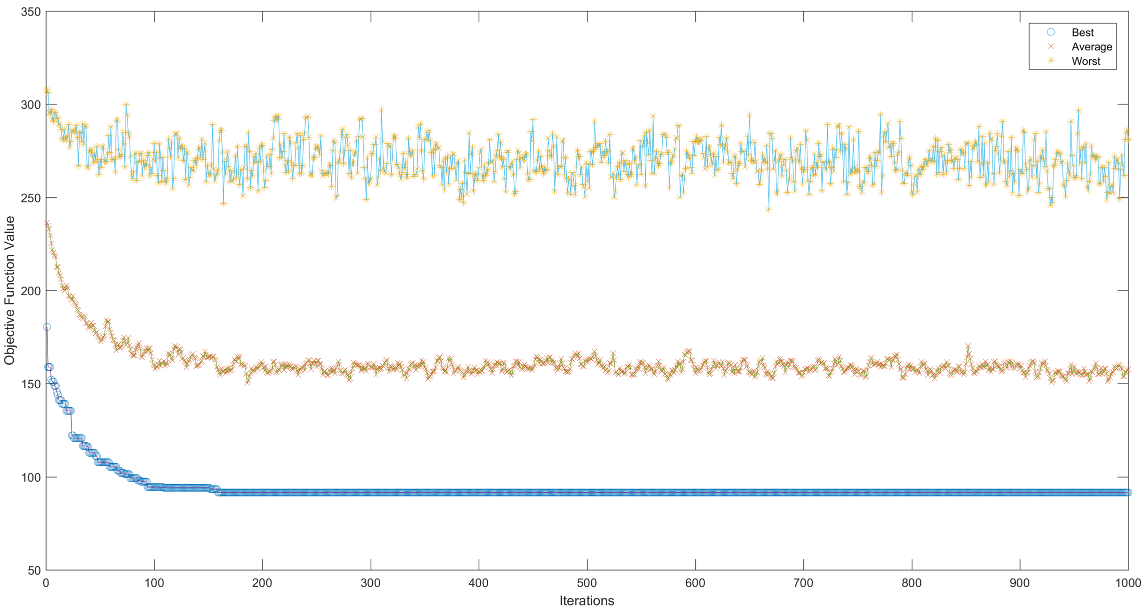
The shortest path is plotted in MATLAB below:-

The length of the optimal path traversed for the TSP is **91.5409**

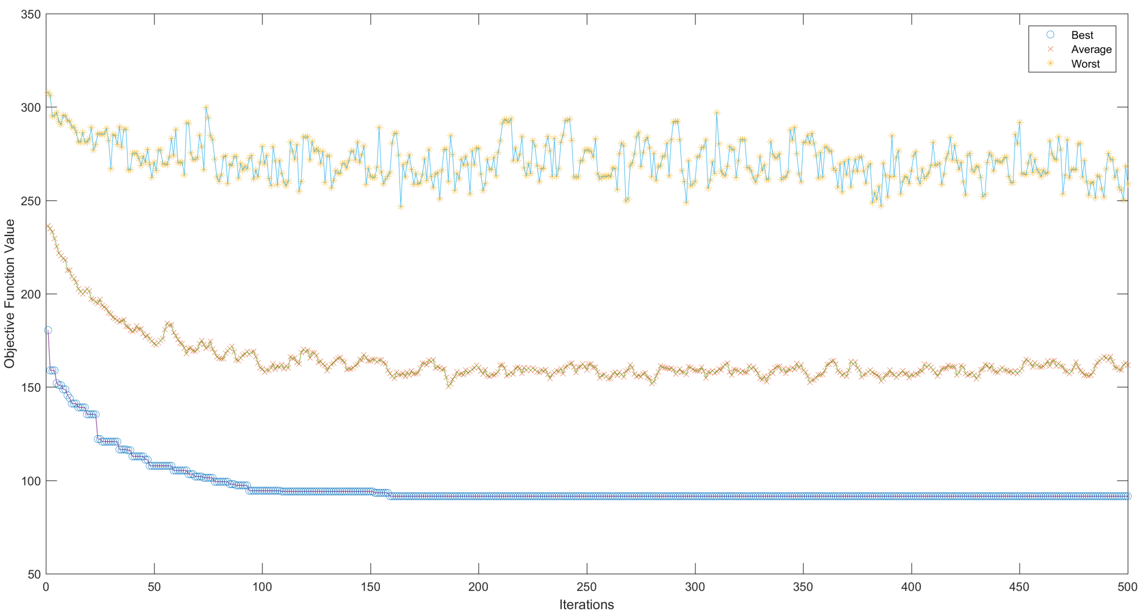
Plot of the shortest path :-



The Plot of the best, worst and average objective function values. As we can see, We don't need to run the GA for 1000 iterations. The shortest path is computed within 500 iterations, as shown below:-



The shortest path is found, and the value as seen from the graph below converges to **91.5409**.



MATLAB CODE

```
%% The fitness functions is the function which is called every iteration to compute the fitnes
% The fitness function for the genetic algorithm

function [result]=fitness(cities,index)

num_cities=20;                          % Total Number of cities

result=norm(cities(:,index(1))-cities(:,index(num_cities)));

for i=2:num_cities                      % Computing the Distance between consecutive Points
    result=result+norm(cities(:,index(i))-cities(:,index(i-1)));
end

result=1/result;                        % fitness function is the inverse of the total distance

end

%% Main function for implementing the GA algorithm for solving the TSP
```

```
% Travelling Salesman Problem using Genetic Algorithm

num_cities=20;                            % Total Number of Cities
                                          % Totallty 20! combinations
                                          % possible
x_min=0;
x_max=20;                                 % Space of generating city coordinates
y_min=-5;
y_max=15;

cities=zeros(2,num_cities);                                % Phenotype space
cities(1,:)=x_min + (x_max-x_min).*rand(1,num_cities);   % X coordinate of cities
cities(2,:)=y_min + (y_max-y_min).*rand(1,num_cities);   % Y coordinate of cities

N_population=500;                     % Population / Number of Chromosomes
N_iterations=1000;                    % Number of iterations of the genetic algorithm
N_elites=10;                          % Number of Elite Chromosones, passed to next generatio
pc=0.75;                              % CrossOver probability
pm=0.0075;                            % Mutation probability

best=zeros(N_iterations);
worst=zeros(N_iterations);
average=zeros(N_iterations);
x=zeros(N_population,num_cities);     % Index/Order of the cities for every chromosone
m=zeros(N_population,num_cities);     % Mating Pool
f=zeros(N_population,1);              % Fitness Computation for chromosome

for i=1:N_population                  % Initialzing the Chromosomes
    x(i,:)=randperm(num_cities);
    f(i)=fitness(cities,x(i,:));
end

for k=1:N_iterations                    % Running 1000 iterations of the GA
    [Sort_Values,Sort_Index]=sort(f(:),'descend');

    for i=1:N_elites
        m(i,:)=x(Sort_Index(i),:);          % Elitism of Chromosomes
    end

    for i=(N_elites+1):N_population

        P_RW=cumsum(f)/sum(f);              % Roulette-Wheel Method of selecting chromosomes
        j=find((P_RW-rand())>0,1);
        m(i,:)=x(j,:);                  % Choosing fit chromosomes to mating pool

        if(pc>rand())                     % Applying CrossOver operator
            p=randi([1 num_cities]);
            q=randi([1 num_cities]);
            temp=m(i,p);
            m(i,p)=m(i,q);
            m(i,q)=temp;
        end
    end
    best(k)=1/(max(f));                    % Computing Best, Worst , Average Objective Values-Di
    worst(k)=1/(min(f));
    average(k)=1/(mean(f));

    x=m;                              % Next generation of Chromosomes
    for i=1:N_population
    f(i)=fitness(cities,x(i,:));      % Evaluvating fitness
    end
end

[value,index]=max(f);                 % Choosing BestFit chromosome
```

```
Shortest_Route=x(index,:);                  % Optimal Route
Shortest_Distance=1/fitness(cities,Shortest_Route);  % Optimal distance

X=cities(1,Shortest_Route);
X(num_cities+1)=cities(1,Shortest_Route(1));
Y=cities(2,Shortest_Route);
Y(num_cities+1)=cities(2,Shortest_Route(1));

disp('The Shortest Route PATH for the TSP is');
disp(Shortest_Route);
disp('The length of the optimal path traversed for the TSP is');
disp(Shortest_Distance);

plot(X,Y,'b-o');                        % Plotting the optimal path for TSP

for i=1:20
    text(X(i),Y(i),[num2str(X(i)) ',' num2str(Y(i))]);
end

xlabel('X axis');
ylabel('Y axis');
title('Plot of the optimal path for the TSP');

b=best(:,1);
a=average(:,1);
w=worst(:,1);
figure ;                                % Plot for best,worst and average objective function
x=1:N_iterations;
plot(x,b,'o',x,a,'x',x,w,'*');
hold on;
plot(x,[b a w]);
hold off;
legend('Best','Average','Worst');
xlabel('Iterations');
ylabel('Objective Function Value');
```

Answer 2

We first convert the problem to standard form:-

$$\begin{gathered}\text{minimize } -2x_1 - x_2\\ \text{subject to}\\ x_1 + x_3 = 5\\ x_2 + x_4 = 7\\ x_1 + x_2 + x_5 = 9\\ x_1, ..., x_5 \geqslant 0\end{gathered}$$

We now use the Simplex Method in Tabular form for easier presentation. c' is used for computing z_j

c^T	-2	-1	0	0	0		
r_j	x_1	x_2	x_3	x_4	x_5	b	c'
	1	0	1	0	0	5	0
	0	1	0	1	0	7	0
	1	1	0	0	1	9	0
r_j	-2	-1	0	0	0	0	

We Pivot the $(1, 1)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	-2	-1	0	0	0		
r_j	x_1	x_2	x_3	x_4	x_5	b	c'
	1	0	1	0	0	5	-2
	0	1	0	1	0	7	0
	0	1	-1	0	1	4	0
r_j	0	-1	2	0	0	10	

We Pivot the $(3, 2)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	-2	-1	0	0	0		
r_j	x_1	x_2	x_3	x_4	x_5	b	c'
	1	0	1	0	0	5	-2
	0	0	1	1	-1	3	0
	0	1	-1	0	1	4	-1
r_j	0	0	1	0	1	14	

The reduced cost coefficients are all non-negative. Hence the simplex algorithm terminates. The optimal solution in standard form is $[5, 4, 0, 3, 0]^T$. The optimal value is -14.

Answer:
$x_1 = 5$
$x_2 = 4$
Optimal Value of the original Objective Function, (which is a maximization problem is) :-
$2 * 5 + 1 * 4 = 14$

Answer 3

We first convert the problem to standard form:-
(a)

$$\text{minimize } 4x_1 + 3x_2$$
$$\text{subject to}$$
$$5x_1 + x_2 - x_3 + a_1 = 11$$
$$2x_1 + x_2 - x_4 + a_2 = 8$$
$$x_1 + 2x_2 - x_5 + a_3 = 7$$
$$x_1, ..., x_5, a_1, a_2, a_3 \geqslant 0$$

We apply the two phase simplex method:-

Phase 1 : minimize $a_1 + a_2 + a_3$

We now use the Simplex Method in Tabular form for easier presentation. c' is used for computing z_j

c^T	0	0	0	0	0	1	1	1		
r_j	x_1	x_2	x_3	x_4	x_5	a_1	a_2	a_3	b	c'
	5	1	-1	0	0	1	0	0	11	1
	2	1	0	-1	0	0	1	0	8	1
	1	2	0	0	-1	0	0	1	7	1
r_j	-8	-4	1	1	1	0	0	0	-26	

We Pivot the $(1, 1)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	0	0	0	0	0	1	1	1		
r_j	x_1	x_2	x_3	x_4	x_5	a_1	a_2	a_3	b	c'
	1	$1/5$	$-1/5$	0	0	$1/5$	0	0	$11/5$	0
	0	$3/5$	$2/5$	-1	0	$-2/5$	1	0	$18/5$	1
	0	$9/5$	$1/5$	0	-1	$-1/5$	0	1	$24/5$	1
r_j	0	$-12/5$	$-3/5$	1	1	$3/5$	0	0	$-42/5$	

We Pivot the $(3, 2)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	0	0	0	0	0	1	1	1		
r_j	x_1	x_2	x_3	x_4	x_5	a_1	a_2	a_3	b	c'
	1	0	$-2/9$	0	$1/9$	$2/9$	0	$-1/9$	$5/3$	0
	0	0	$1/3$	-1	$1/3$	$-1/3$	1	$-1/3$	2	1
	0	1	$1/9$	0	$-5/9$	$-1/9$	0	$5/9$	$8/3$	0
r_j	0	0	$-1/3$	1	$-1/3$	$4/3$	0	$4/3$	-2	

We Pivot the $(2, 3)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	0	0	0	0	0	1	1	1		
r_j	x_1	x_2	x_3	x_4	x_5	a_1	a_2	a_3	b	c'
	1	0	0	$-2/3$	$1/3$	0	$2/3$	$-1/3$	3	0
	0	0	1	-3	1	-1	3	-1	6	0
	0	1	0	$1/3$	$-2/3$	0	$-1/3$	$2/3$	2	0
r_j	0	0	0	0	0	1	1	1	0	

The reduced cost coefficients are all non-negative. Hence the Phase 1 of the simplex algorithm is over and we begin Phase 2.

We now use the Simplex Method in Tabular form for easier presentation. c' is used for computing z_j

Phase 2 : minimize $4x_1 + 3x_2$

c^T	-2	-1	0	0	0		
r_j	x_1	x_2	x_3	x_4	x_5	b	c'
	1	0	0	$-2/3$	$1/3$	3	4
	0	0	1	-3	1	6	0
	0	1	0	$1/3$	$-2/3$	2	3
r_j	0	0	0	$5/3$	$2/3$	-18	

The reduced cost coefficients are all non-negative. Hence the simplex algorithm terminates. The optimal solution in standard form is $[3, 2, 6, 0, 0]^T$. The optimal value is -18.

Answer:

$x_1 = 3$

$x_2 = 2$

Optimal Value of the original Objective Function, (which is a maximization problem is) :-

$-4 * 3 - 3 * 2 = -18$

(b)

We first convert the problem to standard form:-

$$\text{minimize } -6x_1 - 4x_2 - 7x_3 - 5x_4$$
$$\text{subject to}$$
$$x_1 + 2x_2 + x_3 + 2x_4 + x_5 = 20$$
$$6x_1 + 5x_2 + 3x_3 + 2x_4 + x_6 = 100$$
$$3x_1 + 4x_2 + 9x_3 + 12x_4 + x_7 = 75$$
$$x_1, ..., x_7 \geqslant 0$$

We now use the Simplex Method in Tabular form for easier presentation. c' is used for computing z_j

c^T	-6	-4	-7	-5	0	0	0		
r_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b	c'
	1	2	1	2	1	0	0	20	0
	6	5	3	2	0	1	0	100	0
	3	4	9	12	0	0	1	75	0
r_j	-6	-4	-7	-5	0	0	0	0	

We Pivot the $(3, 3)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	-6	-4	-7	-5	0	0	0		
r_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b	c'
	$2/3$	$14/9$	0	$2/3$	1	0	$-1/9$	$35/3$	0
	5	$11/3$	0	-2	0	1	$-1/3$	75	0
	$3/9$	$4/9$	1	$4/3$	0	0	$1/9$	$25/3$	-7
r_j	$-11/3$	$-8/9$	0	$13/3$	0	0	$7/9$	$175/3$	

We Pivot the $(2, 1)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	-6	-4	-7	-5	0	0	0		
r_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b	c'
	0	$16/15$	0	$14/15$	1	$-2/15$	$-1/15$	$5/3$	0
	1	$11/15$	0	$-2/15$	0	$1/15$	$-1/15$	15	-6
	0	$1/5$	1	$22/15$	0	$-1/15$	$2/15$	$10/3$	-7
r_j	0	$9/5$	0	$43/15$	0	$11/15$	$8/15$	$340/3$	

The reduced cost coefficients are all non-negative. Hence the simplex algorithm terminates. The optimal solution in standard form is $[15, 0, 10/3, 0, 0, 0, 0]^T$. The optimal value is $340/3$.

Answer:

$x_1 = 15$

$x_2 = 0$

$x_3 = 10/3$

$x_4 = 0$

Optimal Value of the original Objective Function, (which is a maximization problem is) :-

$6 * 15 + 7 * 10/3 = 340/3$

Answer 4

We first convert the problem to standard form:-

$$\begin{gathered}\text{minimize } -2x_1 - 3x_2\\ \text{subject to}\\ x_1 + 2x_2 + x_3 = 4\\ 2x_1 + x_2 + x_4 = 5\\ x_1, ..., x_4 \geqslant 0\end{gathered}$$

We now use the Simplex Method in Tabular form for easier presentation. c' is used for computing z_j

c^T	-2	-3	0	0		
r_j	x_1	x_2	x_3	x_4	b	c'
	1	2	1	0	4	0
	2	1	0	1	5	0
r_j	-2	-3	0	0	0	

We Pivot the $(1,2)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	-2	-3	0	0		
r_j	x_1	x_2	x_3	x_4	b	c'
	$1/2$	1	$1/2$	0	2	-3
	$3/2$	0	$-1/2$	1	3	0
r_j	$-1/2$	0	$3/2$	0	6	

We Pivot the $(2,1)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	-2	-3	0	0		
r_j	x_1	x_2	x_3	x_4	b	c'
	0	1	$2/3$	$-1/3$	1	-3
	1	0	$-1/3$	$2/3$	2	-2
r_j	0	0	$4/3$	$1/3$	7	

The reduced cost coefficients are all non-negative. Hence the simplex algorithm terminates.
The optimal solution in standard form is $[2, 1, 0, 0]^T$. The optimal value is 7.

Answer:
$x_1 = 2$
$x_2 = 1$
Optimal Value of the original Objective Function, (which is a maximization problem is) :-
$2 * 2 + 3 * 1 = 7$

(b)

When the Primal Problem is of the form:-

$$\begin{gathered}\text{minimize } c^T x\\ \text{subject to } Ax \geqslant b\\ x \geqslant 0\end{gathered}$$

The Dual of the problem is :-

$$\begin{gathered}\text{maximize } \lambda^T b\\ \text{subject to } \lambda^T A \leqslant c^T\\ \lambda \geqslant 0\end{gathered}$$

Now, the Primal is of the form of a maximization problem, So dual is a minimization problem.

We first convert the dual problem to standard form:-

For ease of notation (dual variable) λ = y .

$$\begin{gathered}\text{minimize } 4y_1 + 5y_2\\ \text{subject to}\\ y_1 + 2y_2 - y_3 + a_1 = 2\\ 2y_1 + x_2 - y_4 + a_2 = 3\\ y_1, ..., y_4, a_1, a_2 \geqslant 0\end{gathered}$$

We apply the two phase simplex method:-

Phase 1 : minimize $a_1 + a_2$

We now use the Simplex Method in Tabular form for easier presentation. c' is used for computing z_j

c^T	0	0	0	0	1	1		
r_j	y_1	y_2	y_3	y_4	a_1	a_2	b	c'
	1	2	-1	0	1	0	2	1
	2	1	0	-1	0	1	3	1
r_j	-3	-3	1	1	0	0	-5	

We Pivot the $(2, 1)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	0	0	0	0	1	1		
r_j	y_1	y_2	y_3	y_4	a_1	a_2	b	c'
	0	3/2	-1	1/2	1	$-1/2$	1/2	1
	1	1/2	0	$-1/2$	0	1/2	3/2	0
r_j	-3	-3	1	1	0	0	-5	

We Pivot the $(1, 2)$ element in the table, for the next iteration of the simplex algorithm to get

c^T	0	0	0	0	1	1		
r_j	y_1	y_2	y_3	y_4	a_1	a_2	b	c'
	0	1	$-2/3$	1/3	2/3	$-1/3$	1/3	0
	1	0	1/3	$-2/3$	$-1/3$	2/3	4/3	0
r_j	0	0	0	0	1	1	0	

The reduced cost coefficients are all non-negative. Hence the Phase 1 of the simplex algorithm is over and we begin Phase 2.

We now use the Simplex Method in Tabular form for easier presentation. c' is used for computing z_j

Phase 2 : minimize $4y_1 + 5y_2$

c^T	-4	-5	0	0		
r_j	y_1	y_2	y_3	y_4	b	c'
	0	1	$-2/3$	1/3	1/3	5
	1	0	1/3	$-2/3$	4/3	4
r_j	0	0	2	1	7	

The reduced cost coefficients are all non-negative. Hence the simplex algorithm terminates.
The optimal solution in standard form is $[4/3, 1/3, 0, 0]^T$. The optimal value is 7.

Answer:

$y_1 = 4/3$

$y_2 = 1/3$

Optimal Value of the original Objective Function, (which is a minimization(dual) problem is) :-

$4 * 4/3 + 5 * 1/3 = 7$

Answer 5

(a)

$$f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3$$

$$\text{subject to } x_1^2 + x_2^2 + x_3^2 = 16$$

We form the Lagrange function $L' = x_1^2 + 3x_2^2 + x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 16)$

We compute the critical points, by applying the Lagrange conditions $\nabla_{(x,\lambda)} L(x, \lambda) = 0$:-

$$2x_1 + 2\lambda x_1 = 0$$
$$6x_1 + 2\lambda x_2 = 0$$
$$1 + 2\lambda x_3 = 0$$
$$x_1^2 + x_2^2 + x_3^2 - 16 = 0$$

The critical points, which satisfy the above equations are:-

$$x^{(1)} = [\sqrt{63}/2, 0, 1/2]^T,\ \lambda^{(1)} = -1$$
$$x^{(2)} = [-\sqrt{63}/2, 0, 1/2]^T,\ \lambda^{(2)} = -1$$
$$x^{(3)} = [0, 0, 4]^T,\ \lambda^{(3)} = -1/8$$
$$x^{(4)} = [0, 0, -4]^T,\ \lambda^{(4)} = 1/8$$

$$x^{(5)} = [0,\sqrt{575}/6, 1/6]^T,\ \lambda^{(5)} = -3$$
$$x^{(6)} = [0, -\sqrt{575}/6, 1/6]^T,\ \lambda^{(6)} = -3$$

The above points are all regular. We now apply SONC to check if they are extremizers. Computing the Hessian Matrix we get :-

$$F(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } H(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$T(x*) = \{\ y \in R^3 : [2x_1, 2x_2, 2x_3]y = 0\ \}$$

For $x^{(1)} = (\sqrt{63}/2, 0, 1/2)^T$ we have

$$L(x^{(1)}, \lambda^{(1)}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$T(x^{(1)}) = \{\ \text{y} = [-a/\sqrt{63}, b, a]^T : a, b \in \mathbb{R}\}$$

Let $y \in T(x^{(1)}) =$, where a,b are not both $= 0$. Then

$$y^T L(x^{(1)}, \lambda^{(1)})y = 4b^2 - 2a^2$$

$$\text{which is } \begin{pmatrix} \geqslant 0; |a| < b\sqrt{2} \\ = 0; |a| = b\sqrt{2} \\ \leqslant 0; |a| > b\sqrt{2} \end{pmatrix}$$

Hence, from the above, we can see that $x^{(1)}$ does not satisfy the SONC. Hence $x^{(1)}$ cannot be an extremizer. Thus the Point is indefinite.

Similarly for

$$x^{(2)} = (-\sqrt{63}/2, 0, 1/2)$$

$$T(x*) = \{y : [2x_1 \;\; 2x_2 \;\; 2x_3]y = 0\}$$

$$T(x*) = \{y : [-\sqrt{63} \;\; 0 \;\; 1]y = 0\}$$

$$y = a \begin{bmatrix} 1/\sqrt{63} \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a/\sqrt{63} \\ b \\ a \end{bmatrix}$$

wherein a,b$\in \mathbb{R}$

$$\text{Now, } y^T L y = \begin{bmatrix} a/\sqrt{63} & b & a \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} a/\sqrt{63} \\ b \\ a \end{bmatrix} = 4b^2 - 2a^2.$$

Let $y' \in T(x^{(2)}) =$, where a,b are not both $= 0$. Then

Similarly, performing the same calculations with $x^{(2)}$, we get that $y'^T L(x^{(2)}, \lambda^{(2)})y'$ is not always $\geqslant 0$. Hence $x^{(2)}$ cannot be an extremizer as well.

Thus, this point is indefinite.

For $x^{(3)} = (0, 0, 4)^T$ we have

$$L(x^{(3)}, \lambda^{(3)}) = \begin{bmatrix} 7/4 & 0 & 0 \\ 0 & 23/4 & 0 \\ 0 & 0 & -1/4 \end{bmatrix}$$

$$T(x^{(3)}) = \{\ \text{y} = [a, b, 0]^T : a, b \in \mathbb{R}\}$$

Let $y \in T(x^{(3)}) =$, where a,b are not both $= 0$. Then

$$y^T L(x^{(3)}, \lambda^{(3)})y = (7/4)a^2 + (23/4)b^2$$

which is > 0 always.

Hence, from the above, we can see that $x^{(3)}$ does satisfy the SOSC. Hence $x^{(3)}$ is a strict local minimizer. Thus, this point is a minimizer.

Similarly for

$$x^{(4)} = (0, 0, -4)$$

$$T(x*) = \{y : [2x_1 \;\; 2x_2 \;\; 2x_3]y = 0\}$$

$$T(x*) = \{y : [0 \;\; 0 \;\; -8]y = 0\}$$

$$y = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}a\\b\\0\end{bmatrix}$$

wherein a,b$\in \mathbb{R}$

$$\text{Now, } y^T Ly = \begin{bmatrix}a & b & 0\end{bmatrix}\begin{bmatrix}9/4 & 0 & 0\\0 & 25/4 & 0\\0 & 0 & 1/4\end{bmatrix}\begin{bmatrix}a\\b\\0\end{bmatrix} = (9/4)a^2 + (25/4)b^2 > 0.$$

Let $y' \in T(x^{(4)}) =$, where a,b are not both = 0. Then

Similarly, performing the same calculations with $x^{(4)}$, we get that $y'^T L(x^{(4)}, \lambda^{(4)})y'$ is always > 0 . Hence $x^{(4)}$ is a strict local minimizer as well.

Thus, this point is a minimizer.

For

$$x^{(5)} = (0, \sqrt{575}/6, 1/6)$$

$$T(x*) = \{y : [2x_1 \;\; 2x_2 \;\; 2x_3]y = 0\}$$

$$T(x*) = \{y : [0 \;\; \sqrt{575}/3 \;\; 1/3]y = 0\}$$

$$y = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\-1/\sqrt{575}\\1\end{bmatrix} = \begin{bmatrix}a\\-b/\sqrt{575}\\b\end{bmatrix}$$

wherein a,b$\in \mathbb{R}$

$$\text{Now, } y^T Ly = \begin{bmatrix}a & -b/\sqrt{575} & b\end{bmatrix}\begin{bmatrix}-4 & 0 & 0\\0 & 0 & 0\\0 & 0 & -6\end{bmatrix}\begin{bmatrix}a\\-b/\sqrt{575}\\b\end{bmatrix} = -4a^2 - 6b^2 < 0.$$

Thus, this point is a maximizer.

Similarly for

$$x^{(6)} = (0, -\sqrt{575}/6, 1/6)$$

$$T(x*) = \{y : [2x_1 \;\; 2x_2 \;\; 2x_3]y = 0\}$$

$$T(x*) = \{y : [0 \;\; -\sqrt{575}/3 \;\; 1/3]y = 0\}$$

$$y = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1/\sqrt{575}\\1\end{bmatrix} = \begin{bmatrix}a\\b/\sqrt{575}\\b\end{bmatrix}$$

wherein a,b$\in \mathbb{R}$

$$\text{Now, } y^T Ly = \begin{bmatrix}a & b/\sqrt{575} & b\end{bmatrix}\begin{bmatrix}-4 & 0 & 0\\0 & 0 & 0\\0 & 0 & -6\end{bmatrix}\begin{bmatrix}a\\b/\sqrt{575}\\b\end{bmatrix} = -4a^2 - 6b^2 < 0.$$

Thus, this point is a maximizer.

Hence $x^{(5)}$ and $x^{(6)}$ are strict local maximizers.

Thus, the extremizers for this problem are $(0, \sqrt{575}/6, 1/6), (0, -\sqrt{575}/6, 1/6), (0, 0, 4)$ and $(0, 0, -4)$.

(b)

$$f(x_1, x_2) = x_1^2 + x_2^2$$

$$\text{subject to } 3x_1^2 + 4x_1x_2 + 6x_2^2 = 140$$

We form the Lagrange function $L' = x_1^2 + x_2^2 + \lambda(3x_1^2 + 4x_1x_2 + 6x_2^2 - 140)$
We compute the critical points, by applying the Lagrange conditions $\nabla_{(x,\lambda)}L(x, \lambda) = 0$:-

$$2x_1 + \lambda(6x_1 + 4x_2) = 0$$
$$2x_2 + \lambda(4x_1 + 12x_2) = 0$$
$$3x_1^2 + 4x_1x_2 + 6x_2^2 - 140 = 0$$

We find the critical points by solving the matrix equation below:-

$$\begin{bmatrix}2+6\lambda & 4\lambda\\4\lambda & 2+12\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$X = [0, 0]^T$, does not satisfy one of the Lagrange conditions, Hence we compute the determinant and equate it to 0, to find λ. Hence we get λ = -1/7 and λ = -1/2 .

The critical points, which satisfy the above equations are:-

$$x^{(1)} = [2, 4]^T,\ \lambda^{(1)} = -1/7$$
$$x^{(2)} = [-2, -4]^T,\ \lambda^{(2)} = -1/7$$
$$x^{(3)} = [-2\sqrt{14}, \sqrt{14}]^T,\ \lambda^{(3)} = -1/2$$
$$x^{(4)} = [2\sqrt{14}, -\sqrt{14}]^T,\ \lambda^{(4)} = -1/2$$

The above points are all regular. We now apply SONC to check if they are extremizers. Computing the Hessian Matrix and repeating the same procedure as in (a) we get :-

For

$$x^{(1)} = (2, 4)$$

$$T(x*) = \{y : [6x_1 + 4x_2 \;\; 4x_1 + 12x_2]y = 0\}$$
$$T(x*) = \{y : [28 \;\; 56]y = 0\}$$
$$y = a\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}-2a\\a\end{bmatrix}$$

wherein a$\in \mathbb{R}$

Now, $y^T L y = a^2 \begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}8/7 & -4/7\\-4/7 & 2/7\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = (50/7)a^2 > 0.$

Thus, this point is a minimizer.

Similarly for

$$x^{(2)} = (-2, -4)$$

$$T(x*) = \{y : [6x_1 + 4x_2 \;\; 4x_1 + 12x_2]y = 0\}$$
$$T(x*) = \{y : [-28 \;\; -56]y = 0\}$$
$$y = a\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}-2a\\a\end{bmatrix}$$

wherein a$\in \mathbb{R}$

Now, $y^T L y = a^2 \begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}8/7 & -4/7\\-4/7 & 2/7\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = (50/7)a^2 > 0.$

Thus, this point is a minimizer.

For

$$x^{(3)} = (-2\sqrt{14}, \sqrt{14})$$

$$T(x*) = \{y : [6x_1 + 4x_2 \;\; 4x_1 + 12x_2]y = 0\}$$
$$T(x*) = \{y : [-8\sqrt{14} \;\; 4\sqrt{14}]y = 0\}$$
$$y = a\begin{bmatrix}1/2\\1\end{bmatrix} = \begin{bmatrix}a/2\\a\end{bmatrix}$$

wherein a$\in \mathbb{R}$

Now, $y^T L y = a^2 \begin{bmatrix}1/2 & 1\end{bmatrix}\begin{bmatrix}-1 & -2\\-2 & -4\end{bmatrix}\begin{bmatrix}1/2\\1\end{bmatrix} = -(25/4)a^2 < 0.$

Thus, this point is a maximizer.

Similarly for

$$x^{(4)} = (2\sqrt{14}, -\sqrt{14})$$

$$T(x*) = \{y : [6x_1 + 4x_2 \;\; 4x_1 + 12x_2]y = 0\}$$
$$T(x*) = \{y : [8\sqrt{14} \;\; -4\sqrt{14}]y = 0\}$$
$$y = a\begin{bmatrix}1/2\\1\end{bmatrix} = \begin{bmatrix}a/2\\a\end{bmatrix}$$

wherein a$\in \mathbb{R}$

Now, $y^T L y = a^2 \begin{bmatrix}1/2 & 1\end{bmatrix}\begin{bmatrix}-1 & -2\\-2 & -4\end{bmatrix}\begin{bmatrix}1/2\\1\end{bmatrix} = -(25/4)a^2 < 0.$

Thus, this point is a maximizer.

Applying the SOSC, we conclude that $x^{(1)}$ and $x^{(2)}$ are strict local minimizers, and $x^{(3)}$ and $x^{(4)}$ are strict local maximizers .

Thus, the extremizers for this problem are $(-2\sqrt{14}, \sqrt{14}), (2\sqrt{14}, -\sqrt{14}), (2, 4)$ and $(-2, -4)$.

THE END