ECE 580 Fun Work #1

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MATLAB Code for solutions written after all the Answers.

Answer 1

The determinant was computed through MATLAB. Code is shown after the Answers.
Determinant of Matrix is $-\mathbf{3}$.

Answer 2

The MATLAB Code is shown after the Answers.

The Matrix is invertible, since the determinant of matrix A is equal to $11a^2b^2c^2$, which is $\neq 0$, when $abc \neq 0$.
Hence the solution to the system of linear equations is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ 3c \end{pmatrix}$$

Answer 3

The MATLAB Code is shown after the Answers.

$V^{-1}AV = J$
$U^{-1}BU = K$
Yes, the Matrices A and B are similar, Since $J = K$.
i.e.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$T = VU^{-1}$

The Matrix T is =

$$\begin{bmatrix} 1 & 1.8333 & -3.5 \\ 0 & 0.3333 & 0 \\ 0 & -0.5 & 1 \end{bmatrix}$$

The Similarity Transform is $\mathbf{T^{-1}AT = B}$

$$\begin{bmatrix} 1 & -0.25 & 3.5 \\ 0 & 3 & 0 \\ 0 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -5 \\ 2 & 6 & -10 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1.8333 & -3.5 \\ 0 & 0.3333 & 0 \\ 0 & -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 20 & -34 \\ 6 & 32 & -51 \\ 4 & 20 & -32 \end{bmatrix}$$

Hence Found.

Q.E.D

Answer 4

The MATLAB Code is shown after the Answers.

$$x_1 + x_2 - 2x_3 + 3x_4 = 1 \tag{1}$$
$$x_1 + x_2 - x_3 + 2x_4 = 3 \tag{2}$$
$$x_1 - x_2 - 4x_3 + 5x_4 = -3 \tag{3}$$

The Matrix Equation is of the form $Ax = b$.
Reducing the Matrix A to Row Echelon Form we get:-

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & -4 & 5 \end{bmatrix} \xrightarrow{RowEchelonForm} \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are **two non-zero Rows**, Hence the Rank of the matrix is **2**.

So, We need to shift 2 columns to the right hand side, to make the Matrix B (from the theorem) invertible.

Solving in MATLAB using equations 1 and 2 for x_1 and x_2 we get :-

$$\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} 3x_3 - 4x_4 - 1 \\ x_4 - x_3 + 2 \end{pmatrix}$$

Substituting for x_1 and x_2, in equation 3 we get:-
$(3x_3 - 4x_4 - 1) - (x_4 - x_3 + 2) - 4x_3 + 5x_4 = -3$
$\implies$ -3 = -3

Hence LHS = RHS is satisfied for all equations of the form $Ax = b$.

$\therefore$ The general solution for the system of linear equations is of the form:-

$$\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} 3x_3 - 4x_4 - 1 \\ x_4 - x_3 + 2 \end{pmatrix}$$

Hence Found.

Q.E.D

Answer 5

The MATLAB Code is shown after the Answers.

The Matrix Equation is of the form $Ax = 0$
The general solution for such a Matrix Equation is the Null-space of A.
i.e. $N(A)$
Using MATLAB for finding $N(A)$, and verifying it by hand we get that the general solution is:-

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.4082 \\ -0.8165 \\ 0.4082 \end{pmatrix} (X)$$

The above form is the normalized solution. The below form gives us the unnormalized solution.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} (X)$$

(X) is just a scale factor (i.e. a constant), Which scales the Vector $(1, -2, 1)^T$.
Hence Found.

Q.E.D

Answer 6

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} (X) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Solving the above equation

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

We get

$$\begin{pmatrix} 2a+c & 2b+d \\ 2a+c & 2b+d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Hence we have:-
$2a + c = 1$
$2b + d = 1$
$\implies c = 1 - 2a$
$\implies d = 1 - 2b$

Hence the General solution to the matrix equation is of the form:-

$$\begin{pmatrix} a & b \\ 1-2a & 1-2b \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \end{pmatrix}$$

$$(V1) = (V_2) = \begin{pmatrix} X \\ 1-2X \end{pmatrix}$$

$\implies$

$$(V1) = (V_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} (X)$$

Where X is just a scale factor (i.e. a constant).
Hence we have found the general solution.

Q.E.D

Answer 7

(a)

$f(x) = x^T \mathbf{Q} x - x^T b + c$
where

$Q = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}$ *and* $b = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\mathbf{Df(x) = \quad x^T(Q + Q^T) - b^T}$
$\implies$
$(x^T) \begin{pmatrix} 2 & 8 \\ 8 & 4 \end{pmatrix} - \begin{pmatrix} -2 & 3 \end{pmatrix}$
$\implies$

$Df(x) = \begin{pmatrix} 2x_1 + 8x_2 + 2 & 8x_1 + 4x_2 - 3 \end{pmatrix}$

Hence Found

(b)

$f(x) = \frac{1}{2} x^T \mathbf{Q} x + x^T b + c$
where

$Q = \begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}$ *and* $b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\mathbf{Df(x) = \quad \frac{1}{2} x^T(Q + Q^T) - b^T}$

$Hessian = F(x) = D(D(f(x))$

$\implies$ $\mathbf{F(x) = \frac{1}{2}(Q + Q^T)}$
$\implies$
$F(x) = \frac{1}{2}\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$

$\implies$
The Hessian $F(x)$ is $= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Hence Found.

Answer 8

$f = f(x_1, x_2) = 5e^{x_1^3 x_2} + \frac{1}{x_1 x_2^2}$

(a)

The gradient of f is :-

$\nabla f(x) = \begin{pmatrix} 15x_1^2 x_2 e^{x_1^3 x_2} - \frac{1}{x_1^2 x_2^2} \\ 5x_1^3 e^{x_1^3 x_2} - \frac{2}{x_1 x_2^3} \end{pmatrix}$
$\implies$

$\nabla f(x)$ at $x = [1,1]^T is = \begin{pmatrix} \mathbf{15e - 1} \\ \mathbf{5e - 2} \end{pmatrix}$

(b)

We first normalize d to get

$$\frac{d}{||d||} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

The rate of increase of f at the point $x = [1,1]^T$ in the direction $d = [-3,4]^T$ is $=$

$$\nabla f^T \frac{d}{||d||}$$

$\implies$

$\nabla f^T \frac{d}{||d||} = \mathbf{-5e - 1}$

(c)

The direction of maximum rate of increase at $x = [1,1]^T$ is in the direction of the gradient of f at this point. The rate of increase is $=$

$\nabla f^T \frac{\nabla f}{||\nabla f||} = ||\nabla f||$
$\implies$

$||\nabla f|| = \sqrt[2]{\mathbf{250e^2 - 50e + 5}}$

Hence Found.

Answer 9

$f = f(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_1 x_2 + \frac{1}{3}x_2^3 + x_2 + 5$

(a)

To find the points which satisfy FONC condition for the extremum, we solve the equation
$\nabla f(x) = 0$
$\implies$

$$\begin{pmatrix} x_1 + 2x_2 \\ x_2 + 2x_1 + x_2^2 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\implies$

$$x_1 + 2x_2 = 0 \tag{4}$$
$$x_2 + 2x_1 + x_2^2 + 1 = 0 \tag{5}$$

$\implies$

$$x_1 = -2x_2 \tag{6}$$

Substituting equation 6 in equation 5 we get

$$x_2^2 - 3x_2 + 1 = 0 \tag{7}$$

$\implies$

$x_2 = \frac{3 \pm \sqrt[2]{5}}{2}$

Hence the two points which satisfy the FONC condition for the extremum are:-

$$x^{(1)} = \begin{pmatrix} -3 - \sqrt[2]{5} \\ \frac{3+\sqrt[2]{5}}{2} \end{pmatrix} \quad x^{(2)} = \begin{pmatrix} -3 + \sqrt[2]{5} \\ \frac{3-\sqrt[2]{5}}{2} \end{pmatrix}$$

(b)

The Hessian of f is =

$$F(x) = \begin{pmatrix} 1 & 2 \\ 2 & 1 + 2x_2 \end{pmatrix}$$

The Hessian evaluated at $x^{(2)}$ is

$$F(x^{(1)}) = \begin{pmatrix} 1 & 2 \\ 2 & 4 - \sqrt[2]{5} \end{pmatrix}$$

This matrix is indefinite, Since the determinant is $\leqslant 0$ and the Eigen values are:-
$\lambda 1 = -0.6542$
$\lambda 2 = 3.4181$
Hence, the Hessian evaluated at this point is indefinite.

So, $\mathbf{x^{(2)}}$ is neither a minimizer nor a maximizer of f.

The Hessian evaluated at $x^{(1)}$ is

$$F(x^{(1)}) = \begin{pmatrix} 1 & 2 \\ 2 & 4 + \sqrt[2]{5} \end{pmatrix}$$

This matrix is positive definite, Since the Eigen values are:-
$\lambda 1 = 0.3235$
$\lambda 2 = 6.9126$
Hence, the Hessian evaluated at this point is positive definite.

So, $\mathbf{x^{(1)}}$ is a strict local minimizer of f.
$\therefore$ The point

$$x^{(1)} = \begin{pmatrix} -3 - \sqrt[2]{5} \\ \frac{3+\sqrt[2]{5}}{2} \end{pmatrix}$$

is a strict local minimizer.
Hence Found.
The MATLAB Code is given after the Answers.

Answer 10

(a)

$$Q = Q^T = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

(b)

$$Q = Q^T = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

(c)

$$Q = Q^T = \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix}$$

Hence Found

Q.E.D

MATLAB Code

```
%% FunWork Question 1
e=exp(i*2*pi/3);
A=[1 1 e;1 1 e^2;e^2 e 1];
answer=det(A);
anscheck=(e^2-e)^2;
disp(answer);

%% FunWork Question 2
syms a b c
A=[4*b*c a*c -2*a*b;5*b*c 3*a*c -4*a*b;3*b*c 2*a*c -a*b];
B=transpose([0 -a*b*c 4*a*b*c]);
X=linsolve(A,B);
Ycheck=A\B;
disp(X);
disp(det(A));

%% FunWork Question 3
A=[3 2 -5;2 6 -10; 1 2 -3];
B=[6 20 -34;6 32 -51;4 20 -32];
[V,J]=jordan(A);
[U,K]=jordan(B);
%Since J = K, (Jordan formas) => The matrices A and B are similar
T=V/U;
disp(T);
disp(inv(T));
AnsCheck=T\A*T;
disp(AnsCheck);
% Hence Similarity transform is found.

%% FunWork Question 4
% From Row echelon form we get RANK = 2
syms x3 x4
A=[1 1;1 2];
B=transpose([1+2*x3-3*x4 3+x3-2*x4]);
X=linsolve(A,B);
Ycheck=A\B;
disp(X);
```

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%% FunWork Question 5
A=[3 2 1;5 4 3;4 3 2];
Z=null(A);
disp(Z);

%% FunWork Question 10
A1=[1 2;2 4+sqrt(5)];
A2=[1 2;2 4-sqrt(5)];
disp(eig(A1));
disp(eig(A2));
```

THE END