20/7

UNIT - V

STOCHASTIC PROCESS

Stochastic process is a set of random variables depending on some parameter T. The values assumed by the random variable are called states. The set of all possible values of a single random variable X_n of a stochastic process $\{X_n, n \geq 1\}$ is called state space.

→ Classification of stochastic process:

1) Continuous stochastic process
2) Discrete stochastic process
3) Deterministic stochastic process
4) Non-deterministic stochastic process

→ Markov's process: If the given value of $X(t)$ depends only on its preceeding value and not in the past values.

$$P\left[X_{n+1}(t) = x_{n+1} / X_n(t) = x_n, X_{n-1}(t) = n-1, \cdots X_0(t) = x_0\right]$$

$$= P\left[X_{n+1}(t) = X_{n+1} / X_n(t) = x_n\right]$$

→ Markov chain: A sequence of states X_n is a markov chain if each X_n is a random variable and satisfies

$P_{ij} = P(X_{n+1} = j / X_n = i)$ {Transition probabilities}

→ Transition matrix: The transition probabilities P_{ij} will be arranged in a matrix of transition probabilities.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & P_{n3} & \cdots & P_{nn} \end{bmatrix}$$

→ Stochastic Matrix ⇛

A transition probability matrix (TPM) which is square, having non-negative elements with unit row sums is called a stochastic matrix. A stochastic matrix P is said to be regular if all the entries of some power of P are positive. A stochastic matrix P is not regular if 1 occurs in the principle diagonal.

Problems:

1) Which of the following are stochastic matrices?

i) $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ {Not square ⇒ Not stochastic}

ii) $P = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ {Not unit row sum ⇒ ✗}

iii) $P = \begin{bmatrix} 15/16 & 1/16 \\ 2/3 & 1/3 \end{bmatrix}$ {All satisfied ⇒ ✓}

2) Which of the following is regular?

i) $\begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

(Stochastic But not regular)

ii) $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$ (Not regular, contains 0 in any power)

→ Unit Step Transition probability:

A particle performs a random walk with 4 states: 0 1 2 3 4 wherever it is at any positions r $(0<r<4)$, it moves to $r+1$ with probability P_r to $r-1$ and Q with $P+Q$. But as soon as it reaches 0 or 4 itself will have probability 1. Find TPM.

A) Let X_n be the different positions of the particle at different states. X_n follows markov chain.

$P\{X_{n+1}=r+1 \mid X_n=r\}=p$, $P\{X_{n+1}=r-1 \mid X_n=r\}=q$

$P\{X_{n+1}=0 \mid X_n=0\}=1$, $P\{X_{n+1}=4 \mid X_n=4\}=1$

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \Rightarrow P^2 = [\quad] = \text{two step TPM}$$

m step TPM

If $P = P_{ij}$ is a unit step TPM then $P^m = (P_{ij})^m$ denote

We can see that elements of $\underline{\underline{P^{(2)}}}$ in P^2

2 step PTM

Similary, $P^{(m+1)} = P^{(m)} \times P$

Problems:

1) A raining process is considered as a 2-state markov chain. If it rains it is considered to be in state 0 and if it does not rain, the chain is in state 1. The TPM of markov chain is $\begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$. Find prob. that it will rain for 3 days, assuming that it is raining today.

A) Given TPM $\Rightarrow$ $P = \begin{array}{c|cc} & 0 & 1 \\ 0 & 0.6 & 0.4 \\ 1 & 0.2 & 0.8 \end{array}$ $P_{00} = 0.6$, $P_{01} = 0.4$
$P_{10} = 0.2$, $P_{11} = 0.8$

$P^{(3)} = P^3 = \begin{array}{c|cc} & 0 & 1 \\ 0 & 0.376 & 0.624 \\ 1 & 0.312 & 0.688 \end{array}$ $\Rightarrow P_{00}$ = probability that it will rain for 3 days if it is raining today = 0.376

2) An urn initially contains 5 black balls and 5 white balls. The following experiment is repeated indefinitely. A ball is drawn from the urn. If ball is white, put it back in urn otherwise it is left out. Let X_n be no of black balls remaining in the urn after 'n' draws from the urn.

Find: a) Find the appropriate transition probabilities

b) Find one step TPM for X_n. c) Find 2 step TPM.

A) Let X_n denote number of black balls in the urn.

$P[X_{n+1}=4 \mid X_n=5] = 5/10 = 1/2 \Rightarrow 1 - P[X_{n+1}=5 \mid X_n=5]$

$P[X_{n+1}=3 \mid X_n=4] = 4/9 \Rightarrow 1 - 5/9 = 1 - P[X_{n+1}=4 \mid X_n=4]$

$P[X_{n+1}=2 \mid X_n=3] = 3/8 \Rightarrow 1 - 5/8 = 1 - P[X_{n+1}=3 \mid X_n=3]$

$P[X_{n+1}=1 \mid X_n=2] = 2/7 \Rightarrow 1 - 5/7 = 1 - P(X_{n+1}=2 \mid X_n=2]$

$P[X_{n+1}=0 \mid X_n=1] = 1/6 \Rightarrow 1 - 5/6 = 1 - P[X_{n+1}=0 \mid X_n=1]$

TPM:

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 5/6 & 0 & 0 & 0 & 0 \\ 0 & 2/7 & 5/7 & 0 & 0 & 0 \\ 0 & 0 & 3/8 & 5/8 & 0 & 0 \\ 0 & 0 & 0 & 4/9 & 5/9 & 0 \\ 0 & 0 & 0 & 0 & 5/10 & 5/10 \end{bmatrix} \end{array}$$