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FACULTY OF ENGINEERING

Intent detection and somatosensory feedback

05E: Quaternions

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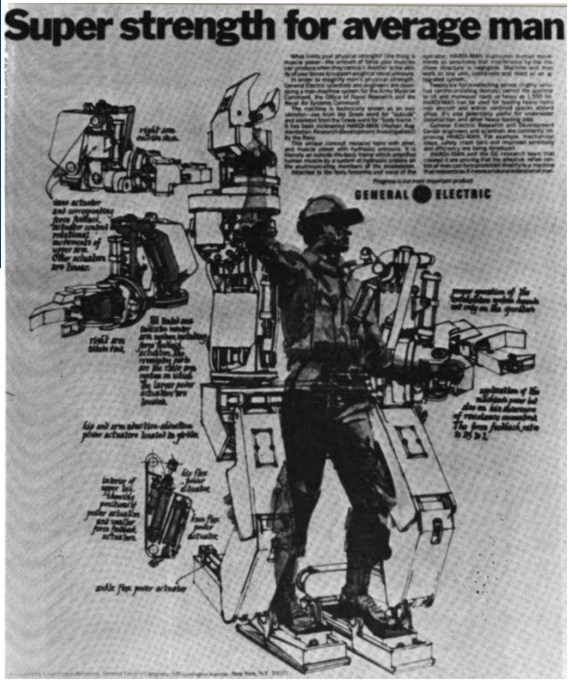


The *Hardiman,* General Electric, 1965/1970.

http://cyberneticzoo.com/man-amplifiers/1966-69-g-e-hardiman-i-ralph-mosher-american/

The *HAL Lumbar* by Cyberdyne, Inc., 2017

https://roboticsandautomationnews.com/2017/10/02/cyberdyne-launches-new-version-of-its-exoskeleton/14360/

Quaternion

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication

$$i^2 = j^2 = k^2 = ijk = -1$$

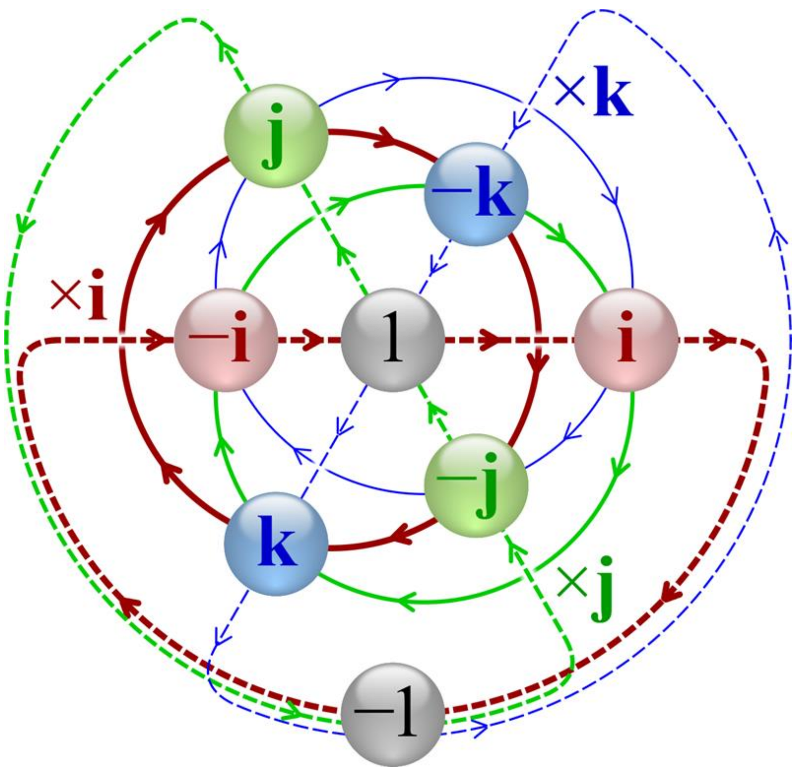
& cut it on a stone of this bridge.

Quaternion

3D Orientation Representations Comparison

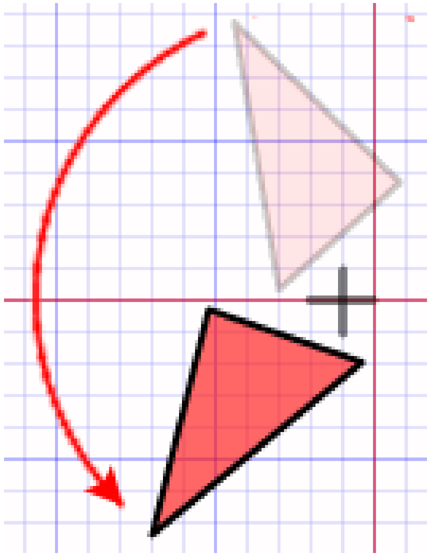
	Singularities	Successive Rotations	Interpolation Extrapolation	Computation Time	Physical Meaning
Quaternion	No	Easy	Easy	Fast	No
Rotation Matrices	No	Easy	Difficult	Slow	Yes
Angle-Axis	No	Difficult	Difficult	Fast	Yes
Euler Angles	Yes	Difficult	Difficult	Fast	Yes

Quaternion



Quaternion

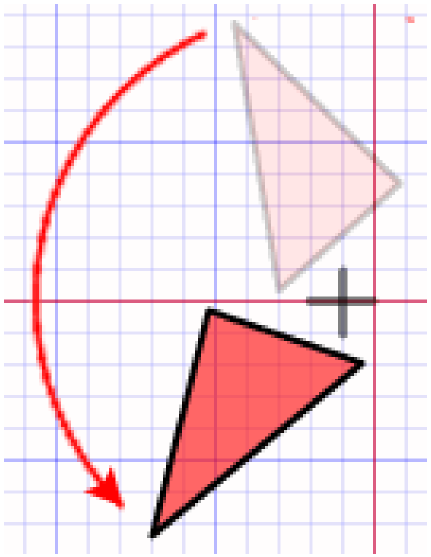
$$\mathbf{q} = (a, b, c, d) \coloneqq a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$



Quaternion

$$\mathbf{q} = (a, b, c, d) \coloneqq a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$

$$\mathbf{q} = s + v_{\mathrm{x}}\,\mathbf{i} + v_{\mathrm{y}}\,\mathbf{j} + v_{\mathrm{z}}\,\mathbf{k} =: s + \vec{v} =: (\underbrace{s}_{\text{Scalar part}}, \underbrace{\vec{v}}_{\text{Vector part}}) \quad \in \mathbb{H}$$



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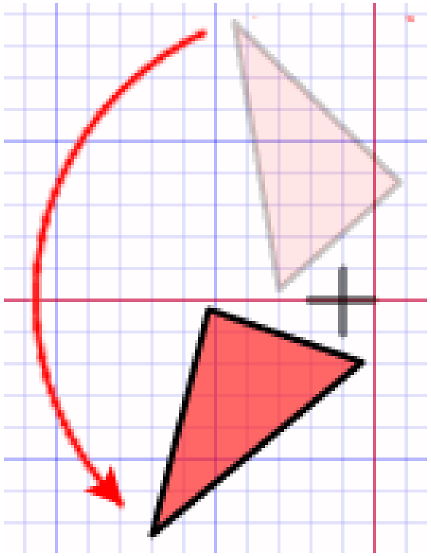

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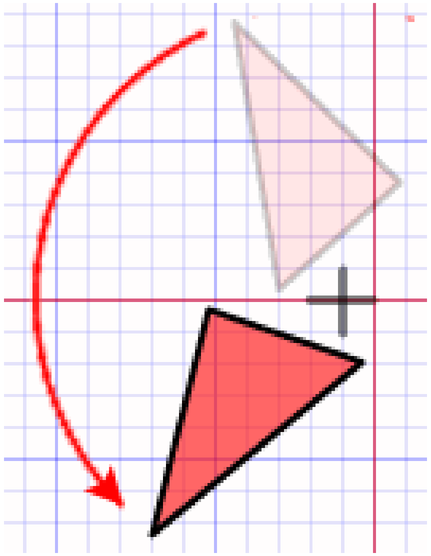
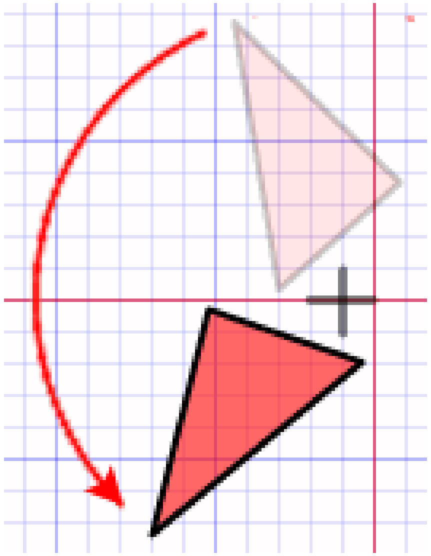


$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

Quaternion

$$\bar{\mathbf{q}} := (s, -\vec{v})$$



Quaternion

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$$\mathbf{q}\bar{\mathbf{q}} = (s, \vec{v})(s, -\vec{v}) = (s^2 + |\vec{v}|^2, 0) =: |\mathbf{q}|^2$$

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$$\mathbf{q}\underbrace{\left(\bar{\mathbf{q}}/|\mathbf{q}|^2\right)}_{\substack{\| \\ \mathbf{q}^{-1}}} = 1$$

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$$\mathbf{q}\underbrace{(\overline{\mathbf{q}}/|\mathbf{q}|^2)}_{\stackrel{!}{=}\,\mathbf{q}^{-1}} = 1$$

$$\mathbf{q}_1\mathbf{q}_2 = (s_1,\ \overrightarrow{v_1})(s_2,\ \overrightarrow{v_2})$$

$$= (s_1 s_2 - \overrightarrow{v_1}\cdot\overrightarrow{v_2},\ s_1\overrightarrow{v_2} + s_2\overrightarrow{v_1} + \underbrace{\overrightarrow{v_1}\times\overrightarrow{v_2}}_{\text{Cross-product makes it non-commutative}})$$

Quaternion

$$\mathbf{q} = (a, b, c, d) := a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$

$$\mathbf{q} := (\cos\theta\,,\ \vec{u}\sin\theta) \qquad \text{where} \quad |\vec{u}| = 1, \quad \text{i.e.} \quad |\mathbf{q}| = 1$$

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https://eater.net/quaternions/video/quatmult

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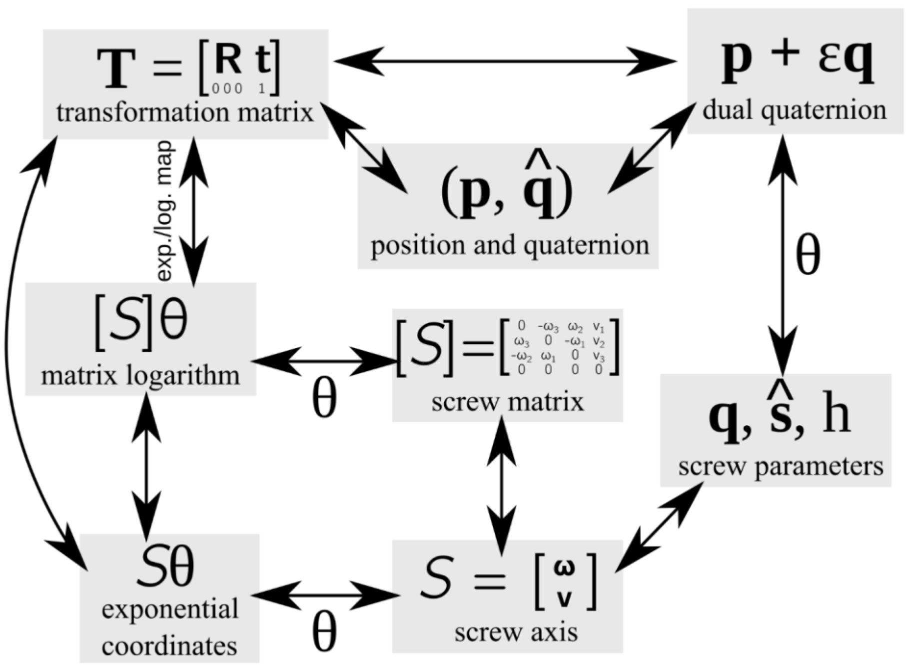
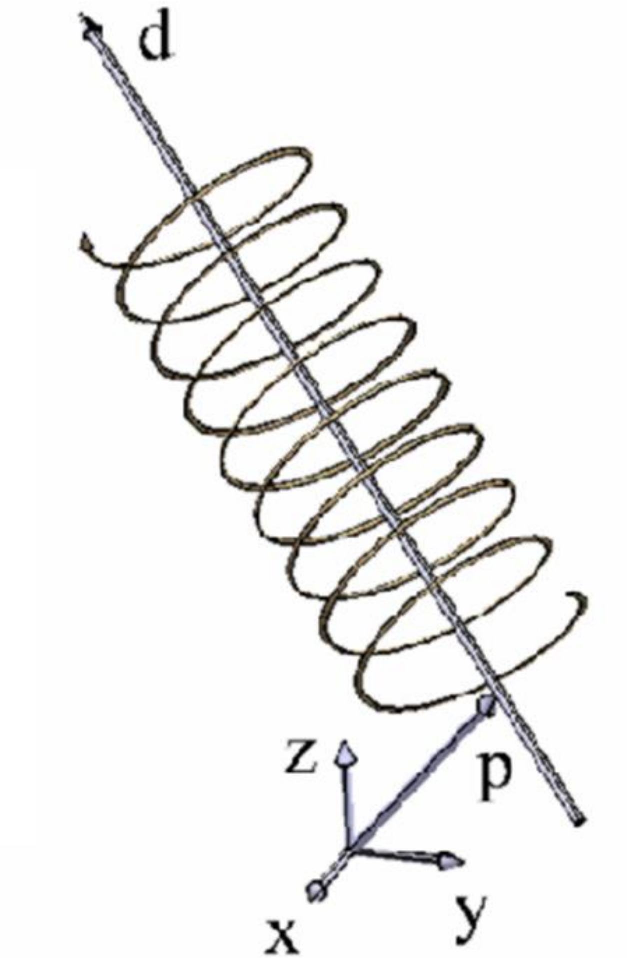
https://quaternions.online/

Dual Quaternion





Dual Quaternion

- Singularity-free
- Un-ambiguous
- Shortest path interpolation
- Most efficient and compact form for representing rigid transforms [SCHI11] - (3x4 matrix 12 floats compared to a dual-quaternion 8 floats)
- Unified representation of translation and rotation
- Can be integrated into a current system with little coding effort
- The individual translation and rotational information is combined to produce a single invariant coordinate frame [GVMC98]

Dual Quaternion

$$\mathbf{q}_r = [\cos(\frac{\theta}{2}), \mathbf{n}_x \sin(\frac{\theta}{2}), \mathbf{n}_y \sin(\frac{\theta}{2}), \mathbf{n}_z \sin(\frac{\theta}{2})][0,0,0,0]$$

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$$\mathbf{q}_t = [1,0,0,0][0, \frac{\mathbf{t}_x}{2}, \frac{\mathbf{t}_y}{2}, \frac{\mathbf{t}_z}{2}]$$

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$$\mathbf{q}_r = [\cos(\frac{\theta}{2}), \mathbf{n}_x \sin(\frac{\theta}{2}), \mathbf{n}_y \sin(\frac{\theta}{2}), \mathbf{n}_z \sin(\frac{\theta}{2})][0,0,0,0]$$

$$\mathbf{q}_t = [1,0,0,0][0, \frac{\mathbf{t}_x}{2}, \frac{\mathbf{t}_y}{2}, \frac{\mathbf{t}_z}{2}]$$

$$\mathbf{q} = \mathbf{q}_t \times \mathbf{q}_r$$

Dual Quaternion

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$$\mathbf{q} = \mathbf{q}_t \times \mathbf{q}_r$$

$$\mathbf{p}' = \mathbf{q}\,\mathbf{p}\,\mathbf{q}^*$$