能量函数: $\mathcal{E}_{g,\lambda,\nu}(\phi) = \lambda\mathcal{L}_g(\phi) + \nu\mathcal{A}_g(\phi)$

$\mathcal{L}_g(\phi) = \int_\Omega g\delta(\phi)\left|\nabla\phi\right| dxdy$ $\qquad \mathcal{A}_g(\phi) = \int_\Omega gH(-\phi)dxdy \qquad g = \frac{1}{1+\left|\nabla G_\sigma * I\right|^2}$

δ is the univariate Dirac function.

H is the Heaviside function.

演化方程: $\frac{\partial\phi}{\partial t} = \lambda\delta(\phi)div(g\frac{\nabla\phi}{|\nabla\phi|}) + \nu g\delta(\phi)$

My Proof:

$$\begin{aligned}\mathcal{E}_{g,\lambda,\nu}(\phi) &= \lambda\mathcal{L}_g(\phi) + \nu\mathcal{A}_g(\phi) \\ &= \lambda\int_\Omega g\delta(\phi)\left|\nabla\phi\right| dxdy + \nu\int_\Omega gH(-\phi)dxdy \\ &= \int_\Omega \left[\lambda g\delta(\phi)\left|\nabla\phi\right| + \nu gH(-\phi)\right] dxdy\end{aligned} \tag{1}$$

Let:

$$F(\phi) = \lambda g\delta(\phi)\left|\nabla\phi\right| + \nu gH(-\phi) \tag{2}$$

in terms of a small variable α ans an arbitrary function h which satisfies : $h|_{\partial\Omega} = 0$, we can get:

$$\begin{aligned}F(\phi+\alpha h) &= \lambda g\delta(\phi+\alpha h)\left|\nabla(\phi+\alpha h)\right| + \nu gH(-(\phi+\alpha h)) \\ &= \lambda g\delta(\phi+\alpha h)\sqrt{(\phi+\alpha h)_x^2 + (\phi+\alpha h)_y^2} + \nu gH(-(\phi+\alpha h))\end{aligned} \tag{3}$$

with:

$$\begin{aligned}\delta(\phi) &= H'(\phi) \\ \delta(\phi) &= \delta(-\phi)\end{aligned} \tag{4}$$

then:

$$\begin{aligned}\frac{\partial F(\phi+\alpha h)}{\partial\alpha} &= \lambda g\left[\delta(\phi+\alpha h)\frac{h_x(\phi+\alpha h)_x + h_y(\phi+\alpha h)_y}{\sqrt{(\phi+\alpha h)_x^2 + (\phi+\alpha h)_y^2}} + \right. \\ &\left. h\delta'(\phi+\alpha h)\sqrt{(\phi+\alpha h)_x^2 + (\phi+\alpha h)_y^2}\right] - \nu gH'(-(\phi+\alpha h)) \\ &= \lambda g\left[\delta(\phi+\alpha h)\frac{h_x(\phi+\alpha h)_x + h_y(\phi+\alpha h)_y}{\sqrt{(\phi+\alpha h)_x^2 + (\phi+\alpha h)_y^2}} + \right. \\ &\left. h\delta'(\phi+\alpha h)\sqrt{(\phi+\alpha h)_x^2 + (\phi+\alpha h)_y^2}\right] - \nu gh\delta(\phi+\alpha h)\end{aligned} \tag{5}$$

$$Then\frac{\partial F(\phi+\alpha h)}{\partial\alpha}|_{\alpha\to 0} = \lambda g\left[\delta(\phi)\frac{h_x\phi_x + h_y\phi_y}{\sqrt{\phi_x^2+\phi_y^2}} + h\delta'(\phi)\sqrt{\phi_x^2+\phi_y^2}\right] - \nu gh\delta(\phi) \tag{6}$$

Then

$$
\begin{aligned}
\frac{\partial \mathcal{E}(\phi+\alpha h)}{\partial \alpha}|_{\alpha\to 0} &= \int_\Omega \left\{ \lambda g \left[\delta(\phi)\frac{h_x\phi_x + h_y\phi_y}{\sqrt{\phi_x^2+\phi_y^2}} + h\delta'(\phi)\sqrt{\phi_x^2+\phi_y^2} \right] - \nu g h \delta(\phi) \right\} dxdy \\
&= \int_\Omega \left\{ \lambda g \left[\delta(\phi)\frac{\nabla\phi\nabla h}{|\nabla\phi|} + h\delta'(\phi)|\nabla\phi| \right] \right\} dxdy - \int_\Omega \nu g h \delta(\phi) dxdy
\end{aligned} \tag{7}
$$

According to Green Equation:

$$
\oint_{\partial\Omega} Rdy + Sdx = \iint_\Omega (\frac{dS}{dx} - \frac{dR}{dy})dxdy, h|_{\partial\Omega=0} \tag{8}
$$

we can get

$$
\begin{aligned}
\int_\Omega g\delta(\phi)\frac{\nabla\phi\nabla h}{|\nabla\phi|}dxdy &= \int_{\partial\Omega} hg \left[\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}dy - \delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}dx \right] - \int_\Omega div(g\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}dxdy \\
&= -\int_\Omega div(g\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}dxdy
\end{aligned} \tag{9}
$$

Then

$$
\begin{aligned}
\frac{\partial \mathcal{E}(\phi+\alpha h)}{\partial \alpha}|_{\alpha\to 0} &= \int_\Omega \left\{ \lambda g \left[\delta(\phi)\frac{\nabla\phi\nabla h}{|\nabla\phi|} + h\delta'(\phi)|\nabla\phi| \right] \right\} dxdy - \int_\Omega \nu g h \delta(\phi) dxdy \\
&= \int_\Omega \lambda h \left[g\delta'(\phi)|\nabla\phi| - div(g\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|} \right] dxdy - \int_\Omega \nu g h \delta(\phi) dxdy \\
&= \int_\Omega \lambda h \delta(\phi)(-div(g\frac{\nabla\phi}{|\nabla\phi|})) - \nu g h \delta(\phi) dxdy \\
&= -\int_\Omega h \left[\lambda\delta(\phi) div(g\frac{\nabla\phi}{|\nabla\phi|}) + \nu g \delta(\phi) \right] dxdy
\end{aligned} \tag{10}
$$

When $\mathcal{E}(\phi)$ reach the minimal

$$
\frac{\partial \mathcal{E}(\phi+\alpha h)}{\partial \alpha}|_{\alpha\to 0} = -\int_\Omega \left[\lambda\delta(\phi) div(g\frac{\nabla\phi}{|\nabla\phi|}) + \nu g \delta(\phi) \right] dxdy = 0 \tag{11}
$$

Since function h is arbitrary, we obtain :

$$
\lambda\delta(\phi) div(g\frac{\nabla\phi}{|\nabla\phi|}) + \nu g \delta(\phi) = 0 \tag{12}
$$

Generally, the gradient of functional $\mathcal{E}_{g,\lambda,\nu}(\phi)$ is denoted as $\{\lambda\delta(\phi) div(g\frac{\nabla\phi}{|\nabla\phi|}) + \nu g \delta(\phi)\}$

Now we get: $\frac{\partial\phi}{\partial t} = \lambda\delta(\phi) div(g\frac{\nabla\phi}{|\nabla\phi|}) + \nu g \delta(\phi)$