Let's Start

Transportation Problems

Introduction

Transportation problem is a special kind of LP problem in which objective is to transport various quantities of a single homogenous commodity, to different destinations in such a way that the total transportation cost is minimized.

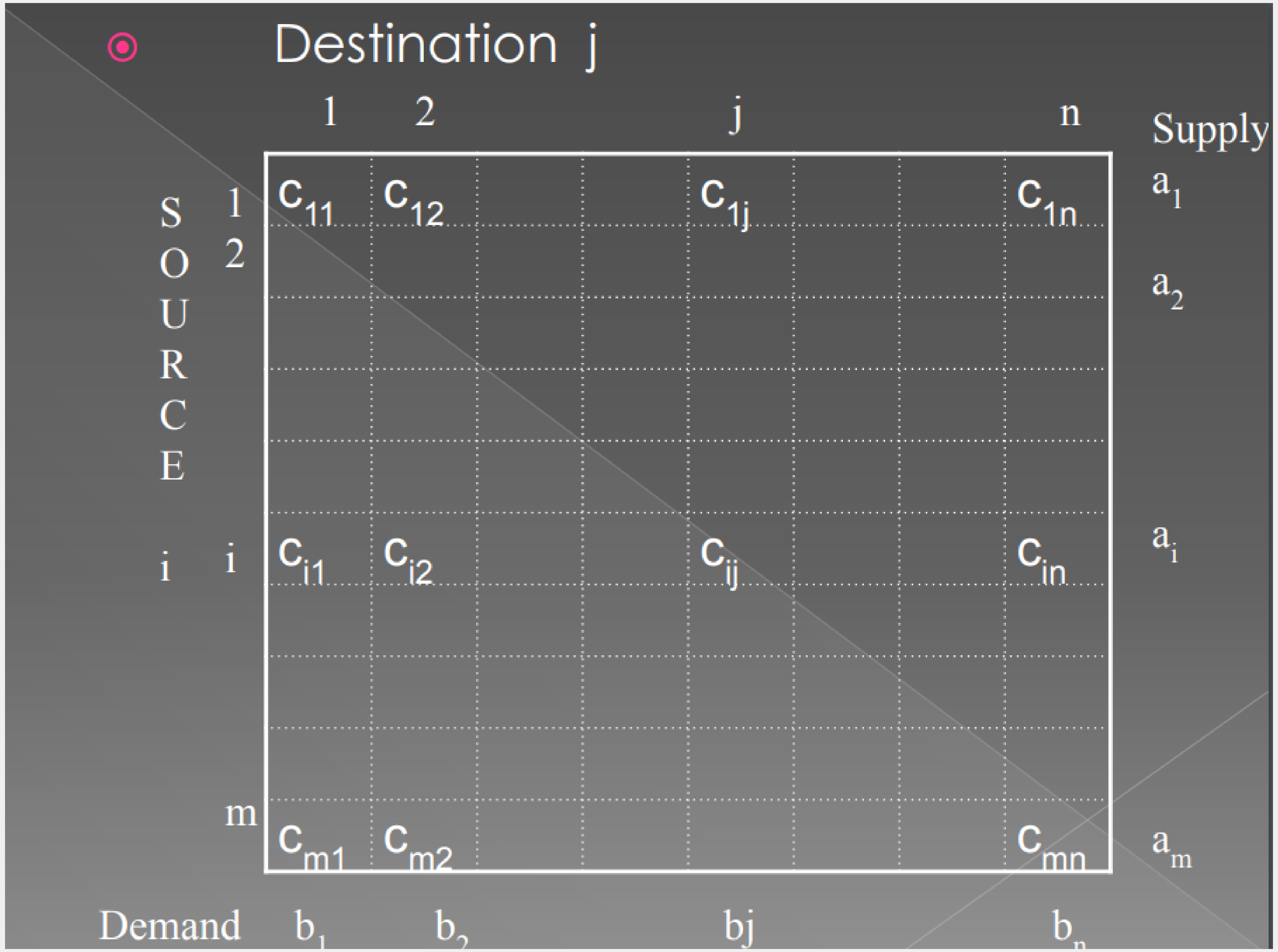
Steps to solve a Transportation Model

1. Formulate the problem and setup in the matrix form.
2. Obtain the Initial basic Feasible Solution.
3. Test the initial solution for optimality.
4. Updating the solution.

Terminology Used

1. Feasible Solution
2. Basic Feasible Solution
3. Optimal Feasible Solution

Matrix Terminology



Matrix Terminology

Where,

- m- number of sources
- n- number of destinations
- a_i- supply at source I
- b_j _ demand at destination j

c_{ij} _ cost of transportation per unit from source i to destination j

X_{ij} _ number of units to be transported from the source i to destination j

Methods for Transportation Problems

1. Initial Basic Feasible Solution
   - Lowest Cost Entry Method (LCEM)
   - Vogel's Approximation Method (VAM)
2. Optimality Tests
   - MODI Method

Lowest Cost Entry Method OR Matrix Minima Method

1. Select the cell with the lowest transportation cost among all the rows or columns of the transportation table. If the minimum cost is not unique the select arbitrarily any cell with the lowest cost.
2. Allocate as many units as possible to the cell determined in step 1 and eliminate that row in which either capacity or requirement is exhausted.
3. Adjust the capacity and requirement for the next allocations.
4. Repeat steps 1 to 3 for the reduced table until the entire capacities are exhausted to fill the requirement at different destinations.

Example

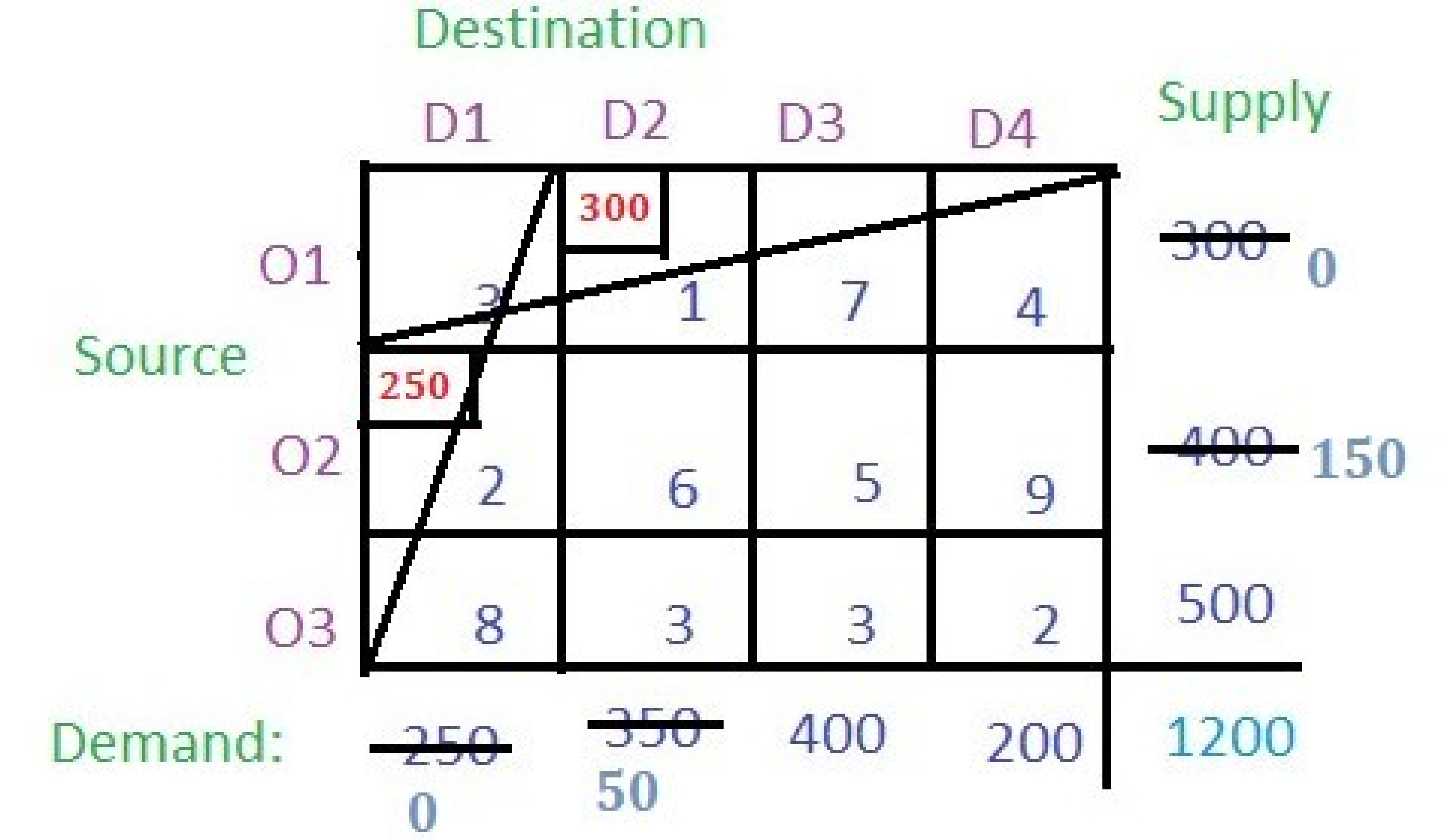
	Destination				
Source	D1	D2	D3	D4	Supply
O1	3	1	7	4	300
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand:	250	350	400	200	1200

- According to the Least Cost Cell method, the least cost among all the cells in the table has to be found which is 1 (i.e. cell (O1, D2)).
- Now check the supply from the row O1 and demand for column D2 and allocate the smaller value to the cell. The smaller value is 300 so allocate this to the cell. The supply from O1 is completed so cancel this row and the remaining demand for the column D2 is 350 – 300 = 50.

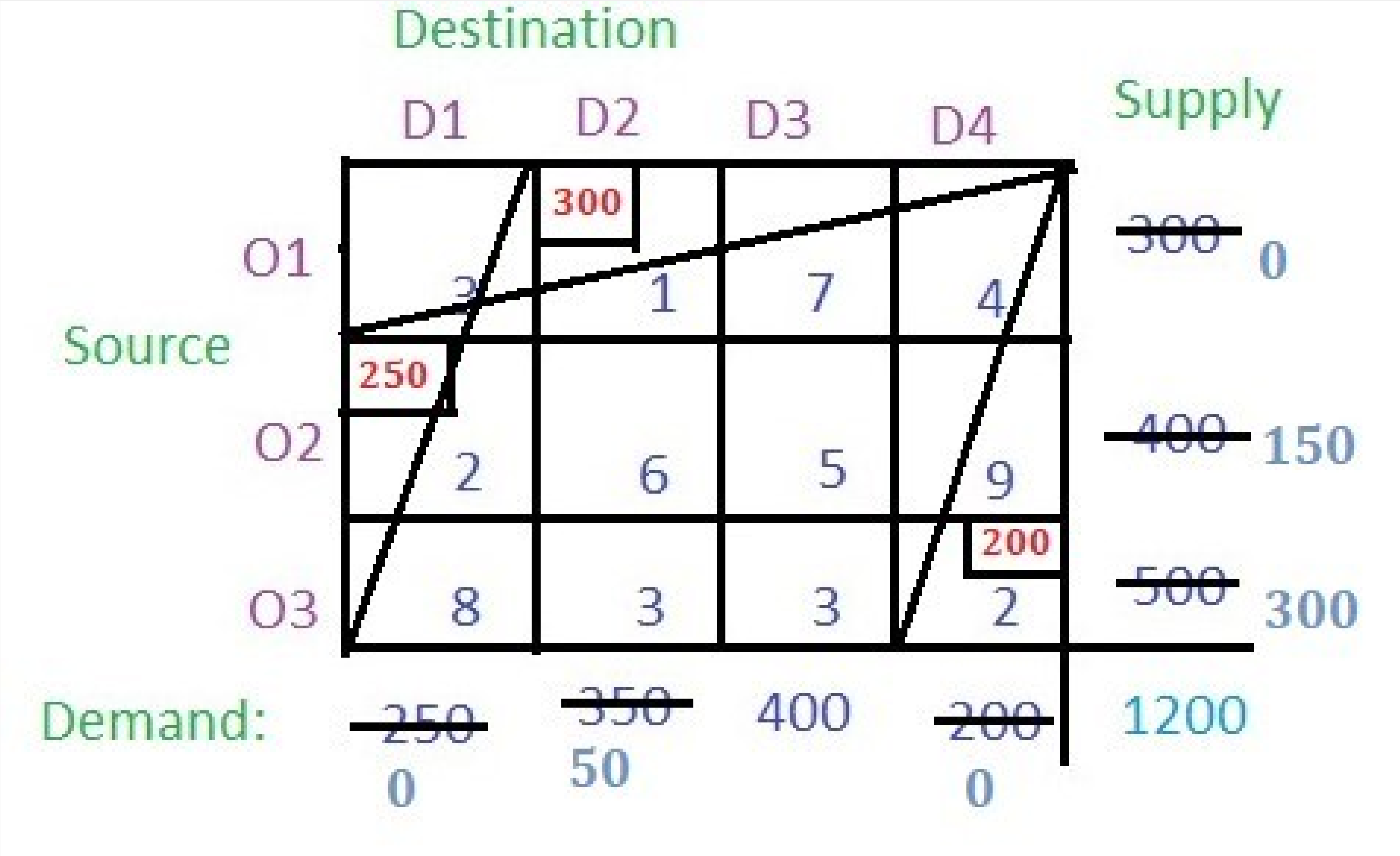
Destination

Source	D1	D2	D3	D4	Supply
O1	3	$\boxed{300}$ 1	7	4	~~300~~ 0
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand:	250	~~350~~ 50	400	200	1200

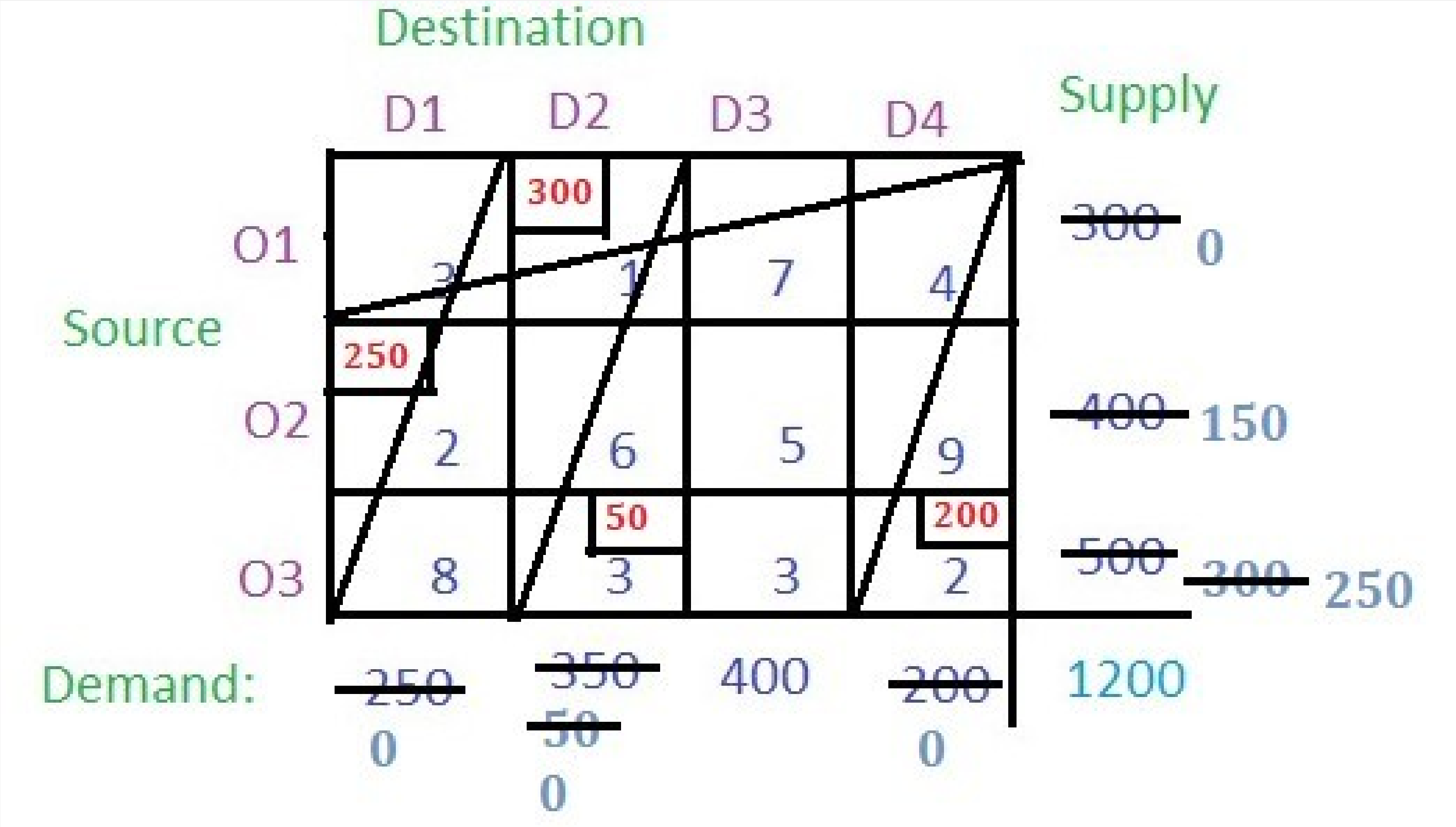
- Now find the cell with the least cost among the remaining cells. There are two cells with the least cost i.e. (O2, D1) and (O3, D4) with cost 2. Lets select (O2, D1). Now find the demand and supply for the respective cell and allocate the minimum among them to the cell and cancel the row or column whose supply or demand becomes 0 after allocation.



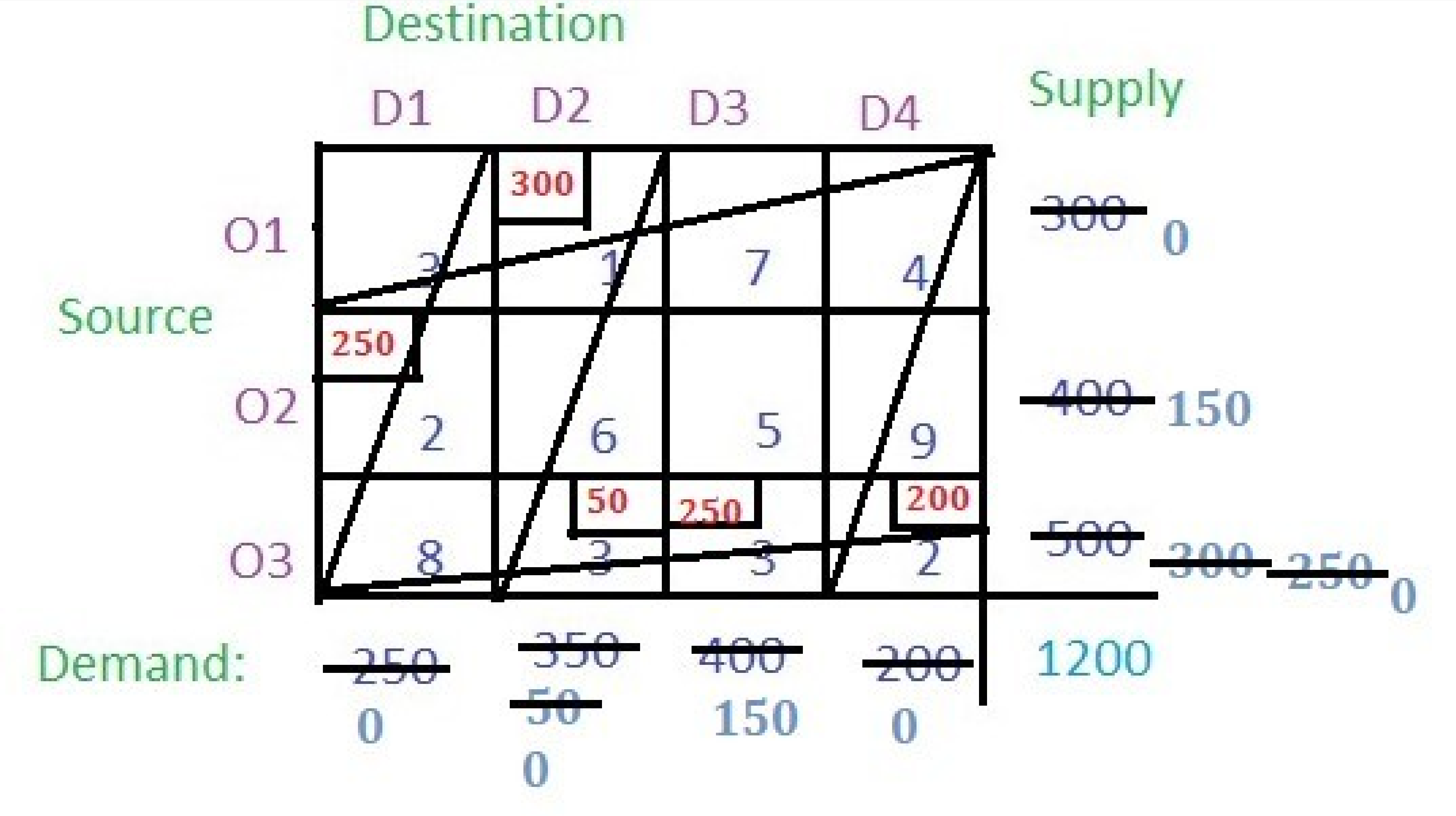
- Now the cell with the least cost is (O3, D4) with cost 2. Allocate this cell with 200 as the demand is smaller than the supply. So the column gets cancelled.



- There are two cells among the unallocated cells that have the least cost. Choose any at random say (O3, D2). Allocate this cell with a minimum among the supply from the respective row and the demand of the respective column. Cancel the row or column with zero value.



- Now the cell with the least cost is (O3, D3). Allocate the minimum of supply and demand and cancel the row or column with zero value.

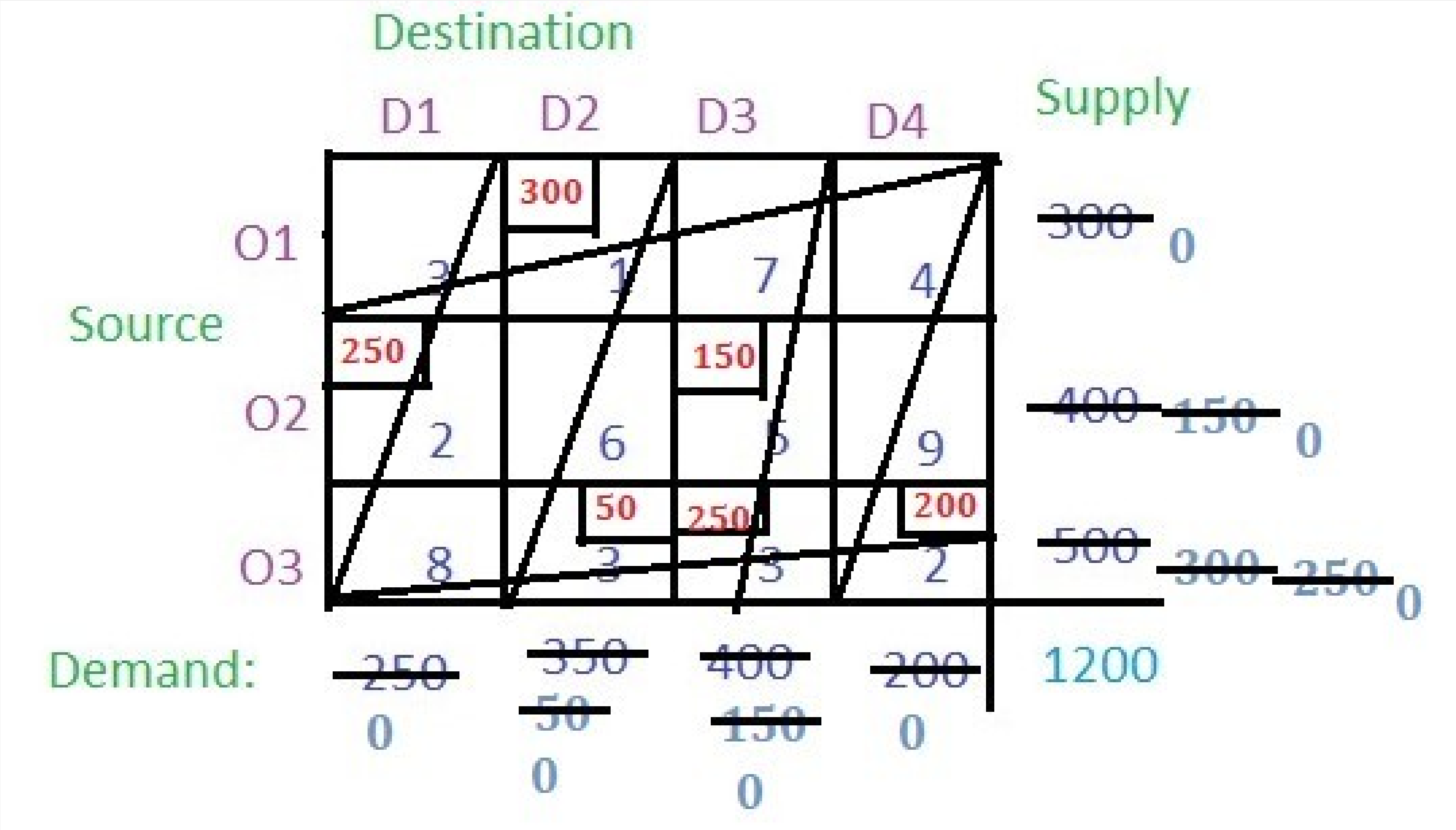


- The only remaining cell is (O2, D3) with cost 5 and its supply is 150 and demand is 150 i.e. demand and supply both are equal. Allocate it to this cell.

Destination

Source	D1	D2	D3	D4	Supply
O1	3	300 1	7	4	~~300~~ 0
O2	250 2	6	150 5	9	~~400~~ ~~150~~ 0
O3	8	50 3	250 3	200 2	~~500~~ ~~300~~ ~~250~~ 0
Demand:	~~250~~ 0	~~350~~ ~~50~~ 0	~~400~~ ~~150~~ 0	~~200~~ 0	1200

- Now just multiply the cost of the cell with their respective allocated values and add all of them to get the basic solution i.e. (300 * 1) + (250 * 2) + (150 * 5) + (50 * 3) + (250 * 3) + (200 * 2) = 2850



Vogel’s Approximation Method (VAM)

1. For each row of the table identify the lowest and the next lowest cost cell. Find their differences and place it to the right of that row. In case two cells contain the same least cost then the difference shall be zero.
2. Similarly find the difference of each column and place it below each column. These differences found in step 2 & 3 are also called penalties.
3. Looking at all the penalties, identify highest of them and the row or column relative to that penalty. Allocate the maximum possible units to the least cost cell in the selected row or column. Ties should be broken in this order Maximum difference least cost cell

Vogel’s Approximation Method (VAM)

4. Adjust the supply & demand and cross the satisfied row or column.
5. Recompute the column and row differences ignoring deleted row/columns and go to step 3. repeat the procedure until all the column and row totals are satisfied.

Example

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	

- The maximum penalty, 22, occurs in column D_2.
- The minimum c_{ij} in this column is $c_{32} = 8$.
- The maximum allocation in this cell is min(18,8) = 8.
- It satisfy demand of D_2 and adjust the supply of S_3 from 18 to 10 (18 - 8 = 10).

	D_1	D_2	D_3	D_4	Supply	Row Penalty
S_1	19	30	50	10	7	9 = 19 - 10
S_2	70	30	40	60	9	10 = 40 - 30
S_3	40	8	70	20	18	12 = 20 - 8
Demand	5	8	7	14		
Column Penalty	21 = 40 - 19	22 = 30 - 8	10 = 50 - 40	10 = 20 - 10		

- The maximum penalty, 21, occurs in column D_1.
- The minimum c_{ij} in this column is $c_{11} = 19$.
- The maximum allocation in this cell is min(7,5) = 5.
- It satisfy demand of D_1 and adjust the supply of S_1 from 7 to 2 (7 - 5 = 2).

	D_1	D_2	D_3	D_4	Supply	Row Penalty
S_1	19	30	50	10	7	9 = 19 - 10
S_2	70	30	40	60	9	20 = 60 - 40
S_3	40	8(8)	70	20	10	20 = 40 - 20
Demand	5	0	7	14		
Column Penalty	21 = 40 - 19	--	10 = 50 - 40	10 = 20 - 10		

- The maximum penalty, 50, occurs in row S_3.
- The minimum c_{ij} in this row is c_{34} = 20.
- The maximum allocation in this cell is min(10,14) = 10.
- It satisfy supply of S_3 and adjust the demand of D_4 from 14 to 4 (14 - 10 = 4).

	D_1	D_2	D_3	D_4	Supply	Row Penalty
S_1	19(5)	30	50	10	2	40 = 50 - 10
S_2	70	30	40	60	9	20 = 60 - 40
S_3	40	8(8)	70	20	10	50 = 70 - 20
Demand	0	0	7	14		
Column Penalty	--	--	10 = 50 - 40	10 = 20 - 10		

- The maximum penalty, 50, occurs in column D_4.
- The minimum c_{ij} in this column is $c_{14} = 10$.
- The maximum allocation in this cell is min(2,4) = 2.
- It satisfy supply of S_1 and adjust the demand of D_4 from 4 to 2 (4 - 2 = 2).

	D_1	D_2	D_3	D_4	Supply	Row Penalty
S_1	19(5)	30	50	10	2	40 = 50 - 10
S_2	70	30	40	60	9	20 = 60 - 40
S_3	40	8(8)	70	20(10)	0	--
Demand	0	0	7	4		
Column Penalty	--	--	10 = 50 - 40	50 = 60 - 10		

- The maximum penalty, 60, occurs in column D_4.
- The minimum c_{ij} in this column is $c_{24} = 60$.
- The maximum allocation in this cell is min(9,2) = 2.
- It satisfy demand of D_4 and adjust the supply of S_2 from 9 to 7 (9 - 2 = 7).

	D_1	D_2	D_3	D_4	Supply	Row Penalty
~~S_1~~	~~19(5)~~	~~30~~	~~50~~	~~10(2)~~	0	--
S_2	70	30	40	60	9	20 = 60 - 40
~~S_3~~	~~40~~	~~8(8)~~	~~70~~	~~20(10)~~	0	--
Demand	0	0	7	2		
Column Penalty	--	--	40	60		

- The maximum penalty, 40, occurs in row S_2.
- The minimum c_{ij} in this row is c_{23} = 40.
- The maximum allocation in this cell is min(7,7) = 7.
- It satisfy supply of S_2 and demand of D_3.

	D_1	D_2	D_3	D_4	Supply	Row Penalty
S_1	19(5)	30	50	10(2)	0	--
S_2	70	30	40	60(2)	7	40
S_3	40	8(8)	70	20(10)	0	--
Demand	0	0	7	0		
Column Penalty	--	--	40	--		

- The minimum total transportation cost $=19\times5+10\times2+40\times7+60\times2+8\times8+20\times10=779$

Initial feasible solution is

	D_1	D_2	D_3	D_4	Supply	Row Penalty
S_1	19(5)	30	50	10(2)	7	9 | 9 | 40 | 40 | -- | -- |
S_2	70	30	40(7)	60(2)	9	10 | 20 | 20 | 20 | 20 | 40 |
S_3	40	8(8)	70	20(10)	18	12 | 20 | 50 | -- | -- | -- |
Demand	5	8	7	14		
Column Penalty	21 21 -- -- -- --	22 -- -- -- -- --	10 10 10 10 40 40	10 10 10 50 60 --		

Optimality Test by MODI Method

1. Find an initial basic feasible solution using any one of the two methods LCM or VAM.
2. Find u_i and v_j for rows and columns. To start
   - Assign 0 to u_i or v_j arbitrary.
   - Calculate other u_i's and v_j's using $c_{ij}=u_i+v_j$, for all occupied cells.
3. For all unoccupied cells, calculate $d_{ij}=c_{ij}-(u_i+v_j)$.

Optimality Test by MODI Method

4. Check the sign of d_{ij}
   - If $d_{ij}>0$, then current basic feasible solution is optimal and stop this procedure.
   - If $d_{ij}=0$ then alternative soluion exists, with different set allocation and same transportation cost. Now stop this procedure.
   - If $d_{ij}<0$, then the given solution is not an optimal solution and further improvement in the solution is possible.

Optimality Test by MODI Method

5. Select the unoccupied cell with the largest negative value of d_{ij}, and included in the next solution.
6. Draw a closed path (or loop) from the unoccupied cell (selected in the previous step). The right angle turn in this path is allowed only at occupied cells and at the original unoccupied cell. Mark (+) and (-) sign alternatively at each corner, starting from the original unoccupied cell.

Optimality Test by MODI Method

7.
- Select the minimum value from cells marked with (-) sign of the closed path.
- Assign this value to selected unoccupied cell (So unoccupied cell becomes occupied cell).
- Add this value to the other occupied cells marked with (+) sign.
- Subtract this value to the other occupied cells marked with (-) sign.

8. Repeat Step-2 to step-7 until optimal solution is obtained. This procedure stops when all $d_{ij} \geq 0$ for unoccupied cells.

Iteration-1 of optimality test

1. Find u_i and v_j for all occupied cells(i,j), where $c_{ij} = u_i + v_j$

1. Substituting, $v_4 = 0$, we get

2. $c_{14} = u_1 + v_4 \Rightarrow u_1 = c_{14} - v_4 \Rightarrow u_1 = 10 - 0 \Rightarrow u_1 = 10$

3. $c_{11} = u_1 + v_1 \Rightarrow v_1 = c_{11} - u_1 \Rightarrow v_1 = 19 - 10 \Rightarrow v_1 = 9$

4. $c_{24} = u_2 + v_4 \Rightarrow u_2 = c_{24} - v_4 \Rightarrow u_2 = 60 - 0 \Rightarrow u_2 = 60$

5. $c_{23} = u_2 + v_3 \Rightarrow v_3 = c_{23} - u_2 \Rightarrow v_3 = 40 - 60 \Rightarrow v_3 = -20$

6. $c_{34} = u_3 + v_4 \Rightarrow u_3 = c_{34} - v_4 \Rightarrow u_3 = 20 - 0 \Rightarrow u_3 = 20$

7. $c_{32} = u_3 + v_2 \Rightarrow v_2 = c_{32} - u_3 \Rightarrow v_2 = 8 - 20 \Rightarrow v_2 = -12$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30	50	10 **(2)**	7	$u_1 = 10$
S_2	70	30	40 **(7)**	60 **(2)**	9	$u_2 = 60$
S_3	40	8 **(8)**	70	20 **(10)**	18	$u_3 = 20$
Demand	5	8	7	14		
v_j	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

2. Find d_{ij} for all unoccupied cells(i,j), where $d_{ij} = c_{ij} - \left(u_i + v_j\right)$

1. $d_{12} = c_{12} - \left(u_1 + v_2\right) = 30 - (10 - 12) = 32$

2. $d_{13} = c_{13} - \left(u_1 + v_3\right) = 50 - (10 - 20) = 60$

3. $d_{21} = c_{21} - \left(u_2 + v_1\right) = 70 - (60 + 9) = 1$

4. $d_{22} = c_{22} - \left(u_2 + v_2\right) = 30 - (60 - 12) = -18$

5. $d_{31} = c_{31} - \left(u_3 + v_1\right) = 40 - (20 + 9) = 11$

6. $d_{33} = c_{33} - \left(u_3 + v_3\right) = 70 - (20 - 20) = 70$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30 [32]	50 [60]	10 **(2)**	7	$u_1 = 10$
S_2	70 [1]	30 [-18]	40 **(7)**	60 **(2)**	9	$u_2 = 60$
S_3	40 [11]	8 **(8)**	70 [70]	20 **(10)**	18	$u_3 = 20$
Demand	5	8	7	14		
v_j	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

3. Now choose the largest negative value from all d_{ij} (opportunity cost) = d_{22} = [-18]

and draw a closed path from S_2D_2.

Closed path is $S_2D_2 \to S_2D_4 \to S_3D_4 \to S_3D_2$

Closed path and plus/minus sign allocation...

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30 [32]	50 [60]	10 **(2)**	7	$u_1 = 10$
S_2	70 [1]	30 [-18] **(+)**	40 **(7)**	60 **(2)** **(-)**	9	$u_2 = 60$
S_3	40 [11]	8 **(8)** **(-)**	70 [70]	20 **(10)** **(+)**	18	$u_3 = 20$
Demand	5	8	7	14		
v_j	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

4. Minimum allocated value among all negative position **(-)** on closed path = 2
Substract 2 from all **(-)** and Add it to all **(+)**

	D_1	D_2	D_3	D_4	Supply
S_1	19 **(5)**	30	50	10 **(2)**	7
S_2	70	30 **(2)**	40 **(7)**	60	9
S_3	40	8 **(6)**	70	20 **(12)**	18
Demand	5	8	7	14	

Iteration-2 of optimality test

1. Find u_i and v_j for all occupied cells(i,j), where $c_{ij} = u_i + v_j$

1. Substituting, $u_1 = 0$, we get

2. $c_{11} = u_1 + v_1 \Rightarrow v_1 = c_{11} - u_1 \Rightarrow v_1 = 19 - 0 \Rightarrow v_1 = 19$

3. $c_{14} = u_1 + v_4 \Rightarrow v_4 = c_{14} - u_1 \Rightarrow v_4 = 10 - 0 \Rightarrow v_4 = 10$

4. $c_{34} = u_3 + v_4 \Rightarrow u_3 = c_{34} - v_4 \Rightarrow u_3 = 20 - 10 \Rightarrow u_3 = 10$

5. $c_{32} = u_3 + v_2 \Rightarrow v_2 = c_{32} - u_3 \Rightarrow v_2 = 8 - 10 \Rightarrow v_2 = -2$

6. $c_{22} = u_2 + v_2 \Rightarrow u_2 = c_{22} - v_2 \Rightarrow u_2 = 30 + 2 \Rightarrow u_2 = 32$

7. $c_{23} = u_2 + v_3 \Rightarrow v_3 = c_{23} - u_2 \Rightarrow v_3 = 40 - 32 \Rightarrow v_3 = 8$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30	50	10 **(2)**	7	$u_1 = 0$
S_2	70	30 **(2)**	40 **(7)**	60	9	$u_2 = 32$
S_3	40	8 **(6)**	70	20 **(12)**	18	$u_3 = 10$
Demand	5	8	7	14		
v_j	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$		

2. Find d_{ij} for all unoccupied cells(i,j), where $d_{ij} = c_{ij} - \left(u_i + v_j\right)$

1. $d_{12} = c_{12} - \left(u_1 + v_2\right) = 30 - (0 - 2) = 32$

2. $d_{13} = c_{13} - \left(u_1 + v_3\right) = 50 - (0 + 8) = 42$

3. $d_{21} = c_{21} - \left(u_2 + v_1\right) = 70 - (32 + 19) = 19$

4. $d_{24} = c_{24} - \left(u_2 + v_4\right) = 60 - (32 + 10) = 18$

5. $d_{31} = c_{31} - \left(u_3 + v_1\right) = 40 - (10 + 19) = 11$

6. $d_{33} = c_{33} - \left(u_3 + v_3\right) = 70 - (10 + 8) = 52$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 **(5)**	30 [32]	50 [42]	10 **(2)**	7	$u_1 = 0$
S_2	70 [19]	30 **(2)**	40 **(7)**	60 [18]	9	$u_2 = 32$
S_3	40 [11]	8 **(6)**	70 [52]	20 **(12)**	18	$u_3 = 10$
Demand	5	8	7	14		
v_j	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$		

Since all $d_{ij} \geq 0$.

So final optimal solution is arrived.

	D_1	D_2	D_3	D_4	Supply
S_1	19 **(5)**	30	50	10 **(2)**	7
S_2	70	30 **(2)**	40 **(7)**	60	9
S_3	40	8 **(6)**	70	20 **(12)**	18
Demand	5	8	7	14	

The minimum total transportation cost $= 19 \times 5 + 10 \times 2 + 30 \times 2 + 40 \times 7 + 8 \times 6 + 20 \times 12 = 743$

End

Thank you

Do you have any questions?