CSCI 2033 - Assignment #4 - Key

#1 - 5.1.2) Find the minimizer of the function $f(x,y)=(3x-2y+1)^2+(2x+y+2)^2$

$$f(x,y)=13x^2+5y^2-8xy+14x+5=(x\ y)\begin{pmatrix}13 & -4\\ -4 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}-2(x\ y)\begin{pmatrix}-7\\ 0\end{pmatrix}+5$$

$$K=\begin{pmatrix}13 & -4\\ -4 & 5\end{pmatrix},\ f=\begin{pmatrix}-7\\ 0\end{pmatrix},\ c=5$$

$$\left(\begin{array}{cc|c}13 & -4 & -7\\ -4 & 5 & 0\end{array}\right)\rightarrow\left(\begin{array}{cc|c}13 & -4 & -7\\ 0 & 49/13 & -28/13\end{array}\right)\rightarrow\left(\begin{array}{cc|c}13 & -4 & -7\\ 0 & 1 & -4/7\end{array}\right)\rightarrow\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-5/7\\ -4/7\end{pmatrix}$$

verifies positive definite

#2 - 5.2.1) Find the minimum value of the function $f(x,y,z)=x^2+2xy+3y^2+2yz+z^2-2x+3z+2$

$$K=\begin{pmatrix}1 & 1 & 0\\ 1 & 3 & 1\\ 0 & 1 & 1\end{pmatrix},\ f=\begin{pmatrix}1\\ 0\\ -3/2\end{pmatrix},\ c=2$$

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & 1\\ 1 & 3 & 1 & 0\\ 0 & 1 & 1 & -3/2\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1 & 1 & 0 & 1\\ 0 & 2 & 1 & -1\\ 0 & 1 & 1 & -3/2\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1 & 1 & 0 & 1\\ 0 & 2 & 1 & -1\\ 0 & 0 & 1/2 & -1\end{array}\right)\rightarrow\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1/2\\ 1/2\\ -2\end{pmatrix}\Rightarrow\boxed{f\left(\tfrac{1}{2},\tfrac{1}{2},-2\right)=-3/2}$$

#3 - 5.2.5.a) Write of p(x) for $K=\begin{pmatrix}4 & -12\\ -12 & 45\end{pmatrix}$, $f=\begin{pmatrix}-1/2\\ 2\end{pmatrix}$, $c=3$; then find the minimizer x^* and $p(x^*)$

$$\boxed{p(x)=x^TKx-2x^Tf+c=4x^2-24xy+45y^2+x-4y+3}$$

$$\left(\begin{array}{cc|c}4 & -12 & -1/2\\ -12 & 45 & 2\end{array}\right)\rightarrow\left(\begin{array}{cc|c}4 & -12 & -1/2\\ 0 & 9 & 1/2\end{array}\right)\rightarrow\boxed{\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1/24\\ 1/18\end{pmatrix}=x^*}\ \boxed{p(x^*)=\frac{419}{144}}$$

#4 - 5.3.1) find the closest point in the plane spanned by $\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}$ and $\begin{pmatrix}0\\ -1\\ 3\end{pmatrix}$ to the point $\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$. What is the distance between these points?

$$A=\begin{pmatrix}1 & 0\\ 2 & -1\\ -1 & 3\end{pmatrix}\quad K=A^TA=\begin{pmatrix}6 & -5\\ -5 & 10\end{pmatrix},\quad f=A^Tb=\begin{pmatrix}2\\ 2\end{pmatrix}$$

$$\begin{pmatrix}6 & -5 & 2\\ -5 & 10 & 2\end{pmatrix}\rightarrow\begin{pmatrix}6 & -5 & 2\\ 0 & 35/6 & 22/6\end{pmatrix}\rightarrow\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}30/35\\ 22/35\end{pmatrix}\Rightarrow \text{closest point}=\frac{30}{35}\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}+\frac{22}{35}\begin{pmatrix}0\\ -1\\ 3\end{pmatrix}=\boxed{\begin{pmatrix}30/35\\ 38/35\\ 36/35\end{pmatrix}}$$

$$\text{distance}=\left\|\frac{1}{35}\begin{pmatrix}30\\ 38\\ 36\end{pmatrix}-\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}\right\|=\left\|\left(\frac{1}{35}\right)\begin{pmatrix}-5\\ 3\\ 1\end{pmatrix}\right\|=\sqrt{\frac{25}{35^2}+\frac{9}{35^2}+\frac{1}{35^2}}=\boxed{\frac{1}{\sqrt{35}}\cong 0.1690}$$

#5-5.4.1.b) Find the least squares solution to $\begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 5 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$

$$K = A^T A = \begin{pmatrix} 14 & 13 \\ 13 & 26 \end{pmatrix}, \quad f = A^T b = \begin{pmatrix} 28 \\ 32 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 13 & 28 \\ 13 & 26 & 32 \end{pmatrix} \rightarrow \begin{pmatrix} 14 & 13 & 28 \\ 0 & 195/14 & 84/14 \end{pmatrix} \rightarrow \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 104/65 \\ 28/65 \end{pmatrix}}$$

#6-5.4.4.b) Find the least squares solution to $\begin{pmatrix} 2 & 1 & 4 \\ 1 & -2 & 1 \\ 1 & 0 & -3 \\ 5 & 2 & -2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$K = A^T A = \begin{pmatrix} 31 & 10 & -4 \\ 10 & 9 & -2 \\ -4 & -2 & 30 \end{pmatrix} \quad f = A^T b = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 31 & 10 & -4 & 1 \\ 10 & 9 & -2 & 0 \\ -4 & -2 & 30 & -3 \end{pmatrix} \rightarrow \boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 218/5262 \\ -358/5262 \\ -521/5262 \end{pmatrix}}$$

#7-5.5.1.a) Find the line $y = \alpha + \beta t$ that best fits the data in the least squares sense

t_i	-2	0	1	3
y_i	0	1	2	5

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix}, \quad A^T A = \begin{pmatrix} 4 & 2 \\ 2 & 14 \end{pmatrix}, \quad A^T y = \begin{pmatrix} 8 \\ 17 \end{pmatrix}$$

this gives normal equations: $4\alpha + 2\beta = 8$ and $2\alpha + 14\beta = 17$

$$\begin{pmatrix} 4 & 2 & 8 \\ 2 & 14 & 17 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 2 & 8 \\ 0 & 13 & 13 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}, \text{ so } \boxed{y = \frac{3}{2} + t}$$

#8-5.5.2)

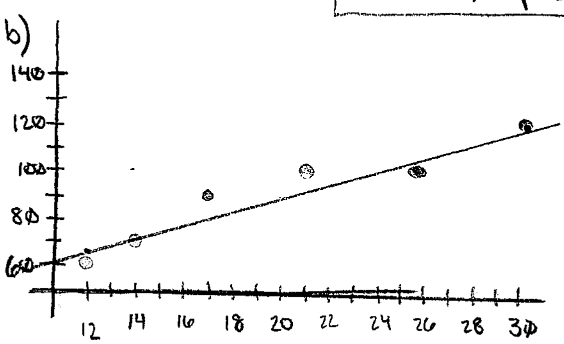
expenditure	12	14	17	21	26	30
profit	60	70	90	100	100	120

$$A^T A = \begin{pmatrix} 6 & 120 \\ 120 & 2646 \end{pmatrix}$$

$$A^T y = \begin{pmatrix} 540 \\ 11530 \end{pmatrix}$$

a) least squares line: $\boxed{y = \left(\frac{3770}{123}\right) + \left(\frac{365}{123}\right) t}$

b)



c) Estimate profit when expenditures are 50k

$$\left(\frac{3770}{123}\right) + \left(\frac{365}{123}\right) 50 = \boxed{179.02}$$

d) Estimate profit when expenditures are 100k

$$\left(\frac{3770}{123}\right) + \left(\frac{365}{123}\right) 100 = \boxed{327,40}$$