Assignment # 3

Exercises

4.1.1. Let $\mathbb{R}^2$ have the standard dot product. Classify the following pairs of vectors as (*i*) basis, (*ii*) orthogonal basis, and/or (*iii*) orthonormal basis:

(a) $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$;

$V_1 \cdot V_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= (-1)(2) + (2)(1)$

$= 0$

$\sqrt{(-1)^2 + 2^2}$

$= \sqrt{5}$

$\|V_1\| \neq 1$ orthogonal basis

(b) $\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$;

$V_1 \cdot V_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$= 0$

$\|V_1\| = \sqrt{(1/\sqrt{2})^2 + (1/\sqrt{2})^2}$

$= \sqrt{1}$

$= 1$

$\|V_2\| = \sqrt{(-1/\sqrt{2})^2 + (1/\sqrt{2})^2}$

$= \sqrt{1}$

$= 1$

orthogonal and orthonormal basis

(c) $\mathbf{v}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$;

$V_2 = (-2)\begin{pmatrix}-1\\-1\end{pmatrix}$

$V_2 = (-2)V_1$

V_1, V_2 is linearly dependent

not the basis of $\mathbb{R}^2$

4.1.3. Repeat Exercise 4.1.1, but use the weighted inner product $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + \frac{1}{9} v_2 w_2$ instead of the dot product.

(a) $\mathbf{v}_1 = \begin{pmatrix}-1\\2\end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix}2\\1\end{pmatrix};$

lineary independent, basis of R^2

$\langle V_1, V_2 \rangle = (-1) \times (2) + \frac{1}{9} \times 2 \times 1$

$= \frac{2}{9} - 2$

$= -16/9$

$-16/9 = \neq 0$

not orthogonal basis

(b) $\mathbf{v}_1 = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix};$

$\langle V_1, V_2 \rangle = \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{-1}{\sqrt{2}}\right) + \frac{1}{9} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$

$= \frac{1}{18} - \frac{1}{2}$

$= -4/9$

$-4/9 \neq 0$

not orthonormal basis

(c) $\mathbf{v}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$;

linealy dependent, not basis of $\mathbb{R}^2$
not orthonormal basis

4.2.17. Use the modified Gram-Schmidt process (4.26–27) to produce orthonormal bases for the spaces spanned by the following vectors: (b) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$,

$$\frac{V_1}{\|\vec{V_1}\|} = \frac{1}{\sqrt{0^2+1^2+1^2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{U_2} = \frac{\vec{V_2} - (\vec{V_2} \cdot \vec{U_1})\vec{U_1}}{\| \vec{V_2} - (\vec{V_2} \cdot \vec{U_1}) \vec{U_1} \|}$$

$$\vec{V_2} - (\vec{V_2} \cdot \vec{U_1}) \vec{V_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}(-1) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\| \vec{V_2} - (\vec{V_2} \cdot \vec{U_1}) \vec{U_1} \| = \sqrt{1^2 + (-1/2)^2 + (1/2)^2}$$

$$V_3 = \frac{\vec{V_3} - (\vec{V_3} \cdot \vec{U_1})\vec{U_1} - (\vec{V_3} \cdot \vec{V_2})\vec{U_2}}{\| \text{Numerator term} \|}$$

$$\vec{V_3} - (\vec{V_3} \cdot \vec{V_1}) \vec{U_1} - (\vec{V_3} \cdot \vec{U_2}) \vec{U_2}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right] \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \sqrt{2}/3 \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix} \right] \sqrt{2}/3 \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - 1/2 (1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 2/3 (3/4) \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ 1 & -1/2 & 1/2 \\ 0 & -1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\left\| \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\| = \sqrt{1^2 + 0^2 + (-1)^2}$$

$$= \sqrt{2}$$

$= \sqrt{1+1/2} = \sqrt{3/2}$

$v_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

4.3.27. Find the $Q\,R$ factorization of the following matrices:

(b) $\begin{pmatrix}4 & 3\\3 & 2\end{pmatrix}$,

$\vec{e}_k = \frac{\vec{u}_k}{|\vec{u}_k|}$

$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix}4\\3\end{bmatrix}$ $\quad |\vec{v}| = 5$

$\vec{e}_1 = \frac{\vec{u}_1}{|\vec{u}_1|} = \begin{bmatrix}4/5\\3/5\end{bmatrix}$ $\quad \frac{4}{5}, \frac{3}{5}$

$\vec{u}_2 = \vec{v}_2 - \mathrm{proj}_{\vec{u}_1}(\vec{v}_2) = \begin{bmatrix}3/25\\-4/25\end{bmatrix}$

$\vec{e}_2 = \frac{\vec{u}_2}{|\vec{u}_2|} = \begin{bmatrix}3/5\\-4/5\end{bmatrix}$

$Q = \begin{bmatrix}4/5 & 3/5\\3/5 & -4/5\end{bmatrix}$

$R = \begin{bmatrix}4/5 & 3/5\\3/5 & -4/5\end{bmatrix}\begin{bmatrix}4 & 3\\3 & 2\end{bmatrix}$

$= \begin{bmatrix}5 & 18/5\\0 & 1/5\end{bmatrix}$

(c) $\begin{pmatrix}2 & 1 & -1\\0 & 1 & 3\\-1 & -1 & 1\end{pmatrix}$,

$|\vec{u}| = \sqrt{5}$

$\vec{e} = \langle \frac{2\sqrt{5}}{5}, 0, -\frac{\sqrt{5}}{5} \rangle$

$u_1 = v_1 = \begin{bmatrix}2\\0\\-1\end{bmatrix}$

$\vec{e}_1 = \frac{\vec{u}_1}{|\vec{u}_1|} = \begin{bmatrix}2\sqrt{5}/5\\0\\-\sqrt{5}/5\end{bmatrix}$

$\vec{u}_2 = \begin{bmatrix}-1/5\\1\\-2/5\end{bmatrix}$ $\quad \vec{e}_2 = \frac{\vec{u}_2}{|\vec{u}_2|} = \begin{bmatrix}-\sqrt{30}/30\\\sqrt{30}/6\\-\sqrt{30}/15\end{bmatrix}$

$\vec{u}_3 = \vec{v}_3 = \begin{bmatrix}2/3\\2/3\\4/3\end{bmatrix}$ $\quad \vec{e}_3 = \frac{\vec{u}_3}{|\vec{u}_3|} = \begin{bmatrix}\sqrt{6}/6\\\sqrt{6}/6\\\sqrt{6}/3\end{bmatrix}$

orthonormal vector $\left\{\begin{bmatrix}2\sqrt{5}/5\\0\\-\sqrt{5}/5\end{bmatrix}, \begin{bmatrix}-\sqrt{30}/30\\\sqrt{30}/6\\-\sqrt{30}/15\end{bmatrix}, \begin{bmatrix}\sqrt{6}/6\\\sqrt{6}/6\\\sqrt{6}/3\end{bmatrix}\right\}$

$Q =$

$Q = \begin{bmatrix}2\sqrt{5}/5\\0\\-\sqrt{5}/5\end{bmatrix}, \begin{bmatrix}-\sqrt{30}/30\\\sqrt{30}/6\\-\sqrt{30}/15\end{bmatrix}, \begin{bmatrix}\sqrt{6}/6\\\sqrt{6}/6\\\sqrt{6}/3\end{bmatrix}$

$R = \begin{bmatrix}2\sqrt{5}/5 & 0 & -\sqrt{5}/5\\-\sqrt{30}/30 & \sqrt{30}/6 & -\sqrt{30}/15\\\sqrt{6}/6 & \sqrt{6}/6 & \sqrt{6}/3\end{bmatrix}\begin{bmatrix}2 & 1 & -1\\0 & 1 & 3\\-1 & -1 & 1\end{bmatrix}$

$= \begin{bmatrix}\sqrt{5} & 3\sqrt{5}/5 & -3\sqrt{5}/5\\0 & \sqrt{30}/5 & 7\sqrt{30}/15\\0 & 0 & 2\sqrt{6}/3\end{bmatrix}$

(d) $\begin{pmatrix}0 & 1 & 2\\-1 & 1 & 1\\-1 & 1 & 3\end{pmatrix}$,

$|\vec{v}| = \sqrt{2}$

$\vec{e} = \langle 0, -\sqrt{2}/2, -\sqrt{2}/2 \rangle$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\vec{e}_3 = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

orthonormal vector : $\left\{ \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \right\}$

$Q = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

$$R = \begin{bmatrix} 0 & -\sqrt{2}/2 & -\sqrt{2}/2 \\ 1 & 0 & 0 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & -2\sqrt{2} \\ 0 & 1 & 2 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

4.4.4. Find the orthogonal projection of the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto the image of $\begin{pmatrix} 3 & 2 \\ 2 & -2 \\ 1 & -2 \end{pmatrix}$.

A

$R_1 \leftrightarrow R_3$ $\begin{bmatrix} 1 & -2 \\ 2 & -2 \\ 3 & 2 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$ $\begin{bmatrix} 1 & -2 \\ 0 & 4 \\ 0 & 8 \end{bmatrix}$ $R_3 \to R_3 - 2R_1$ $\begin{bmatrix} 1 & -2 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$ $\begin{bmatrix} 1 & -2 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$

$$\frac{(1,2,3)\cdot(3,2,1)}{(3,2,1)\cdot(3,2,1)}(3,2,1) + \frac{(1,2,3)\cdot(2,-2,-2)}{(2,-2,-2)\cdot(2,-2,-2)}(2,-2,-2)$$

$$= \frac{(3+4+3)}{(9+4+1)}(3,2,1) + \frac{(2-4-6)}{(4+4+4)}(2,-2,-2)$$

$= 5/7(3,2,1) - 2/3(2,-2,-2)$

$= (15/7 - 4/3,\ 10/7 + 4/3,\ 5/7 + 4/3)$

$= (17/21,\ 58/21,\ 43/21)$

4.4.12. Find the orthogonal complement $W^\perp$ of the subspaces $W \subset \mathbb{R}^3$ spanned by the indicated vectors. What is the dimension of $W^\perp$ in each case?

(a) $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$,

x_1

$$\begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = 0$$

$x_2 = r \quad x_3 = s$

$$x_1 = \frac{r}{3} - \frac{s}{3}$$

$$x = \left(\begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} r + \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} s \right)$$

$$W^{\perp} = \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$$

Dimension of $W^{\perp}$ is 2.

(b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix},$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$$

$x + 2y + 3z$
$= 2x + z$
$= 0$

$z = -2x$
$x + 2y + 3z = 0 \Rightarrow y = -\frac{1}{2}[x - 6x]$
$= 5/2x$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 5/2x \\ -2x \end{pmatrix}$$

$$= x \begin{pmatrix} 1 \\ 5/2 \\ -2 \end{pmatrix}$$

Dimension of $W^{\perp} = 1$

(c) $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix},$

$$\begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

let $x_2 = r$ $x_3 = s$

then $x_1 = -2r - 3s$

$$x = \begin{bmatrix}-2r-3s\\r\\s\end{bmatrix}$$

$$= \begin{bmatrix}-2\\1\\0\end{bmatrix}r + \begin{bmatrix}-3\\0\\1\end{bmatrix}s$$

orthogonal complement

$$W^{+} = \begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}$$

Dimension of $W^{\perp}$ is 2.

4.2.1. Use the Gram–Schmidt process to determine an orthonormal basis for $\mathbb{R}^3$ starting with the following sets of vectors:

(c) $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\0\end{pmatrix}, \begin{pmatrix}2\\3\\-1\end{pmatrix}.$

$$V_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$V_2 = U_2 - \frac{(U_2, U_1)}{(U_1, U_1)} U_1$$

$$= \begin{bmatrix}4\\5\\0\end{bmatrix} - \frac{16}{16}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$= \begin{bmatrix}3\\3\\-3\end{bmatrix}$$

$$U_3 = U_3 - \left(\frac{U_3, V_1}{V_1, V_1}\right)V_1 - \frac{(U_3, V_2)}{(V_2, V_2)} V_2$$

$$= \begin{bmatrix}2\\3\\-1\end{bmatrix} - \frac{5}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{18}{27}\begin{bmatrix}3\\3\\-3\end{bmatrix}$$

$$= \begin{bmatrix}-5/14\\2/7\\-1/16\end{bmatrix}$$

$$e_1 = \frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{1^2+2^2+3^2}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$e_2 = \frac{V_2}{\|V_2\|} = \frac{1}{\sqrt{3^2+3^2+(-3)^2}} \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$e_3 = \frac{V_3}{\|V_3\|} = \frac{1}{\sqrt{(-5/14)^2+(2/7)^2+(-1/14)^2}} \begin{bmatrix} -5/14 \\ 2/7 \\ -1/14 \end{bmatrix} = \frac{\sqrt{14}}{\sqrt{3}} \begin{bmatrix} -5/14 \\ 2/7 \\ -1/14 \end{bmatrix}$$

orthonormal basis $\left\{ \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \frac{\sqrt{14}}{\sqrt{3}} \begin{bmatrix} -5/14 \\ 2/7 \\ -1/14 \end{bmatrix} \right\}$