CSCI 2033 - Assignment #2 - Key

1) $\boxed{2.3.1}$ Show $\begin{pmatrix}-1\\2\\3\end{pmatrix}$ as a combination of $\begin{pmatrix}2\\-1\\2\end{pmatrix}, \begin{pmatrix}5\\-4\\1\end{pmatrix}$ (a span of $\mathbb{R}^3$)

$$\left(\begin{array}{cc|c}2&5&-1\\-1&-4&2\\2&1&3\end{array}\right) \to \left(\begin{array}{cc|c}2&5&-1\\-1&-4&2\\0&-4&4\end{array}\right) \to y=-1 \to x=-(2+4(-1))=2, \text{ verify: } 2(2)+5(-1)=-1 \checkmark$$

$$\boxed{\begin{pmatrix}-1\\2\\3\end{pmatrix} = 2\begin{pmatrix}2\\-1\\2\end{pmatrix} + (-1)\begin{pmatrix}5\\-4\\1\end{pmatrix}}$$

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2) $\boxed{2.3.2}$ Show $(-3,7,6,1)^T$ as a combination of $\begin{pmatrix}1\\-3\\-2\\0\end{pmatrix}, \begin{pmatrix}-2\\6\\3\\4\end{pmatrix}, \begin{pmatrix}-2\\4\\6\\-7\end{pmatrix}$ (a span of $\mathbb{R}^4$)

$$\left(\begin{array}{ccc|c}1&-2&-2&-3\\-3&6&4&7\\-2&3&6&6\\0&4&-7&1\end{array}\right) \to \left(\begin{array}{ccc|c}1&-2&-2&-3\\0&0&-2&-2\\0&-1&2&0\\0&4&-7&1\end{array}\right) \to z=1 \to y=2 \to \text{verify } 4(2)-7(1)=1\checkmark \to x=(-3)+2(2)+2(1)=3$$

$$\boxed{\begin{pmatrix}-3\\7\\6\\1\end{pmatrix} = 3\begin{pmatrix}1\\-3\\-2\\0\end{pmatrix} + 2\begin{pmatrix}-2\\6\\3\\4\end{pmatrix} + \begin{pmatrix}-2\\4\\6\\-7\end{pmatrix}}$$

$\boxed{2.3.21}$ Are the following linearly independent or linearly dependent?

#3

a) $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$

$$\begin{pmatrix}1&2\\2&1\end{pmatrix} \to \begin{pmatrix}1&2\\0&-3\end{pmatrix}$$

$\boxed{\text{independent}}$

#4

(c) $\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}$

$$\begin{pmatrix}2&-1&5\\1&3&2\end{pmatrix} \to \begin{pmatrix}0&-7&1\\1&3&2\end{pmatrix}$$

$$(-1)\begin{pmatrix}5\\2\end{pmatrix} + \left(\tfrac{17}{7}\right)\begin{pmatrix}2\\1\end{pmatrix} - \left(\tfrac{1}{7}\right)\begin{pmatrix}-1\\3\end{pmatrix} = \emptyset$$

$\boxed{\text{dependent}}$

#5

(e) $\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}3\\-1\\2\end{pmatrix}$ $\boxed{\text{dependent}}$

$$\begin{pmatrix}0&1&3\\1&-1&-1\\1&0&2\end{pmatrix} \to \begin{pmatrix}0&1&3\\1&-1&-1\\0&1&3\end{pmatrix} \to \begin{pmatrix}1&-1&-1\\0&1&3\\0&0&0\end{pmatrix}$$

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6) $\boxed{2.4.1.c}$ Is $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$ a basis of $\mathbb{R}^2$? $\boxed{\text{Yes}}$ Since $\begin{pmatrix}1&2\\2&1\end{pmatrix}$ is nonsingular as demonstrated in (#3)

7) $\boxed{2.5.1.b}$ Characterize the image and kernel of $\begin{pmatrix}1&-1&2\\-2&2&-4\end{pmatrix}$

$$\begin{pmatrix}1&-1&2\\-2&2&-4\end{pmatrix} \to \begin{pmatrix}1&-1&2\\0&0&0\end{pmatrix}$$

image: any vector made from $\boxed{c_1\begin{pmatrix}1\\-2\end{pmatrix} + c_2\begin{pmatrix}-1\\2\end{pmatrix} + c_3\begin{pmatrix}2\\-4\end{pmatrix}}$ kernel: $\boxed{\begin{pmatrix}x\\y\\z\end{pmatrix} = y\begin{pmatrix}1\\1\\0\end{pmatrix} + z\begin{pmatrix}-2\\0\\1\end{pmatrix}}$

8) $\boxed{2.5.12}$ Find solutions to $\begin{pmatrix}1&2\\-3&-4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}b_1\\b_2\end{pmatrix}$ for $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, write solution for $\begin{pmatrix}1\\4\end{pmatrix}$ as a combination of previous solutions.

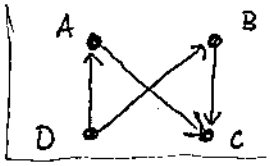
$$\left(\begin{array}{cc|cc}1&2&1&0\\-3&-4&0&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&2&1&0\\0&2&3&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&2&1&0\\0&1&3/2&1/2\end{array}\right) \to \left(\begin{array}{cc|cc}1&0&-4/2&-2/2\\0&1&3/2&1/2\end{array}\right)$$

$$\boxed{x_1^* = \begin{pmatrix}-4/2\\3/2\end{pmatrix},\ x_2^* = \begin{pmatrix}-1\\1/2\end{pmatrix}}$$

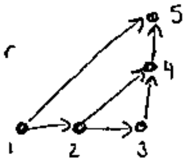
$$\begin{pmatrix}1\\4\end{pmatrix} = 1\begin{pmatrix}1\\0\end{pmatrix} + 4\begin{pmatrix}0\\1\end{pmatrix}$$

$$\boxed{x_3^* = x_1^* + 4x_2^* = \begin{pmatrix}-4/2\\3/2\end{pmatrix} + 4\begin{pmatrix}-1\\1/2\end{pmatrix} = \begin{pmatrix}-6\\7/2\end{pmatrix}}$$

9) $\boxed{2.6.2.a}$ Draw graph represented by $\begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$



10) $\boxed{2.6.3.c}$ Write an incidence matrix for



$$\boxed{\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}}$$

11) $\boxed{3.1.2}$ Which formulas define an inner product on $\mathbb{R}^2$?

(a) $2v_1w_1 + 3v_2w_2$

Symmetric: ✓

Positivity: $\underbrace{2v_1^2}_{\geq 0} + \underbrace{3v_2^2}_{\geq 0}$ ✓ ; $\emptyset$ only when $(v_1, v_2) = (0,0)$ ✓

Bilinearity: $2(cv_1+du_1)w_1 + 3(cv_2+du_2)w_2$
$= c(2v_1w_1 + 3v_2w_2) + d(2u_1w_1 + 3u_2w_2)$

$\boxed{\text{is inner product}}$

(b) $v_1w_2 + v_2w_1$

Symmetric: ✓

Positivity: $v_1v_2 + v_2v_1$
$= 2v_1v_2 \not> \emptyset$

$\boxed{\text{not inner product}}$

(c) $(v_1+v_2)(w_1+w_2)$

Symmetry: ✓

Positivity: $(v_1+v_2)(v_1+v_2)$
$= 2(v_1+v_2) \not> \emptyset$

$\boxed{\text{not inner product}}$

(d) $v_1^2w_1^2 + v_2^2w_2^2$

Symmetry: ✓

Positivity: ✓ ; $\emptyset$ ✓

Bilinearity: $(cv_1+du_1)^2w_1^2 + (cv_2+du_2)^2w_2^2$
$= \underbrace{c^2}_{\text{problem}} v_1^2w_1^2 + \underbrace{2cd}_{\text{problem}} v_1u_1 w_1^2 + \underbrace{d^2}_{\text{problem}} u_1^2 w_1^2 + (cv_2+du_2)^2 w_2^2$

$\boxed{\text{not inner product}}$

12) $\boxed{3.2.1}$ Verify Cauchy-Schwarz inequality and determine angle, using dot product, for:

(a) $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}$

$\left|\left\langle \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix} \right\rangle\right| \leq \left\|\begin{pmatrix}1\\2\end{pmatrix}\right\| \cdot \left\|\begin{pmatrix}-1\\2\end{pmatrix}\right\|$

$|(1)(-1) + (2)(2)| \leq \sqrt{5} \cdot \sqrt{5}$

$3 \leq 5$ ✓

$\theta = \cancel{\sphericalangle} \cos^{-1}\left(\frac{3}{\sqrt{5}\cdot\sqrt{5}}\right) = \boxed{0.2952\pi}$

(b) $\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}$

$|\langle v, w\rangle| \leq \|v\| \cdot \|w\|$

$|-1+0+0| \leq \sqrt{2} \cdot \sqrt{2}$

$1 \leq 2$ ✓

$\theta = \cancel{\sphericalangle} \cos^{-1}\left(\frac{1}{\sqrt{2}\cdot\sqrt{2}}\right) = \boxed{\frac{\pi}{3}}$

(c) $\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix}$

$|\langle v, w\rangle| \leq \|v\| \cdot \|w\|$

$|2+(-2)+0| \leq \sqrt{2} \cdot \sqrt{12}$

$\emptyset \leq 2\sqrt{6}$ ✓

$\theta = \cancel{\sphericalangle} \cos^{-1}\left(\frac{\emptyset}{\sqrt{2}\sqrt{12}}\right) = \boxed{\frac{\pi}{2}}$

#13) $\boxed{3.2.16d}$ Find all vectors in $\mathbb{R}^3$ that are orthogonal to $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}-2\\0\\1\end{pmatrix}$

$$\left\langle \begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rangle = 0 = \left\langle \begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix} \right\rangle \Rightarrow \begin{matrix} 1a+2b+3c=0 \\ -2a+0b+1c=0 \end{matrix} \Rightarrow \begin{pmatrix}1&2&3\\-2&0&1\end{pmatrix} \to \begin{pmatrix}1&2&3\\0&4&7\end{pmatrix} \to \begin{matrix} z=z \\ y=\frac{-7}{4}z \end{matrix} \to x = -2\left(\frac{-7}{4}z\right) - 3(z) = \frac{z}{2}$$

orthogonal vectors: $\begin{pmatrix}1/2\\-7/4\\1\end{pmatrix} z$ (ie: a line through $\mathbb{R}^3$ along $\begin{pmatrix}1/2\\-7/4\\1\end{pmatrix}$

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14) $\boxed{3.3.2.c}$ Compute 1, 2, 3, and ∞ norms for $\begin{pmatrix}1\\-2\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\-3\end{pmatrix}$, verify triangle inequality for each.

norm 1] $\left\|\begin{pmatrix}1\\-2\\-1\end{pmatrix}\right\|_1 = |1|+|-2|+|-1| = 1+2+1 = 4$, $\left\|\begin{pmatrix}2\\-1\\-3\end{pmatrix}\right\|_1 = |2|+|-1|+|-3| = 6$, $\left\|\begin{pmatrix}1\\-2\\-1\end{pmatrix}+\begin{pmatrix}2\\-1\\-3\end{pmatrix}\right\|_1 = \left\|\begin{pmatrix}3\\-3\\-4\end{pmatrix}\right\|_1 = 3+3+4 = 10$, $10 \leq 4+6 = 10$ ✓

norm 2] $\left\|\begin{pmatrix}1\\-2\\-1\end{pmatrix}\right\|_2 = \sqrt{1^2+(-2)^2+(-1)^2} = \sqrt{6}$, $\left\|\begin{pmatrix}2\\-1\\-3\end{pmatrix}\right\|_2 = \sqrt{2^2+(-1)^2+(-3)^2} = \sqrt{14}$, $\left\|\begin{pmatrix}1\\-2\\-1\end{pmatrix}+\begin{pmatrix}2\\-1\\-3\end{pmatrix}\right\|_2 = \sqrt{3^2+(-3)^2+(-4)^2} = \sqrt{34}$, $\sqrt{34} \leq \sqrt{6}+\sqrt{14}$ ✓

norm 3] $\left\|\begin{pmatrix}1\\-2\\-1\end{pmatrix}\right\|_3 = \sqrt[3]{1^3+|-2|^3+|-1|^3} = \sqrt[3]{10}$, $\left\|\begin{pmatrix}2\\-1\\-3\end{pmatrix}\right\|_3 = \sqrt[3]{36}$, $\left\|\begin{pmatrix}3\\-3\\-4\end{pmatrix}\right\|_3 = \sqrt[3]{118}$, $\sqrt[3]{118} \leq \sqrt[3]{10}+\sqrt[3]{36}$ ✓

norm ∞] $\left\|\begin{pmatrix}1\\-2\\-1\end{pmatrix}\right\|_\infty = \max(|1|,|-2|,|-1|) = 2$, $\left\|\begin{pmatrix}2\\-1\\-3\end{pmatrix}\right\|_\infty = 3$, $\left\|\begin{pmatrix}3\\-3\\-4\end{pmatrix}\right\|_\infty = 4$, $4 \leq 2+3 = 5$ ✓

#15) $\boxed{3.4.1}$ Which of the following are positive definite?

(a) $\begin{pmatrix}1&0\\0&2\end{pmatrix}$

$q(x) = 1x_1^2 + 2x_2^2 > 0$,

$\boxed{\begin{pmatrix}1&0\\0&2\end{pmatrix} > 0}$

(b) $\begin{pmatrix}0&1\\2&0\end{pmatrix}$

not symmetric

$\boxed{\begin{pmatrix}0&1\\2&0\end{pmatrix} \not> 0}$

(c) $\begin{pmatrix}1&2\\2&1\end{pmatrix}$

$q(x) = 1x_1^2 + 4x_1x_2 + 1x_2^2$
$= (x_1+x_2)^2 + \underline{2x_1x_2}$

$\boxed{\begin{pmatrix}1&2\\2&1\end{pmatrix} \not> 0}$

(d) $\begin{pmatrix}5&3\\3&-2\end{pmatrix}$

$q(x) = 5x_1^2 + 6x_1x_2 + 2x_2^2$
$= 5\left(x_1 + \frac{3}{5}x_2\right)^2 - \frac{14}{5}x_2^2$

$\boxed{\begin{pmatrix}5&3\\3&-2\end{pmatrix} \not> 0}$

(e) $\begin{pmatrix}1&-1\\-1&3\end{pmatrix}$

$q(x) = x_1^2 - 2x_1x_2 + 3x_2^2$
$= (x_1-x_2)^2 + 2x_2^2$

$\boxed{\begin{pmatrix}1&-1\\-1&3\end{pmatrix} > 0}$

(f) $\begin{pmatrix}1&1\\-1&2\end{pmatrix}$

not symmetric
(if you allow for non-symmetric then it is positive definite)

$\boxed{\begin{pmatrix}1&1\\-1&2\end{pmatrix} \not> 0}$

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16) $\boxed{3.4.22}$ Find the Gram matrix for $\begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}-3\\0\\2\end{pmatrix}$, is it positive definite? if positive semi-definite, what are the null directions?

$$\begin{pmatrix} v_1\cdot v_1 & v_1\cdot v_2 \\ v_2\cdot v_1 & v_2\cdot v_2 \end{pmatrix} = \boxed{\begin{pmatrix}6&-8\\-8&13\end{pmatrix}}, \quad q(x) = 6x_1^2 - 16x_1x_2 + 13x_2^2 = 6\left(x_1 - \frac{8}{6}x_2\right)^2 + \frac{14}{6}x_2^2 > 0$$

$$\boxed{\begin{pmatrix}6&-8\\-8&13\end{pmatrix} > 0}$$

$\boxed{3.5.1}$ Are these matrices positive definite

#17) (d) $\begin{pmatrix}1&1&1\\1&2&-2\\1&-2&4\end{pmatrix} \to \begin{pmatrix}1&1&1\\0&1&-3\\0&-3&3\end{pmatrix} \to \begin{pmatrix}1&1&1\\0&1&-3\\0&0&-6\end{pmatrix} \boxed{\not> 0}$

#18) (e) $\begin{pmatrix}2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{pmatrix} \to \begin{pmatrix}2&1&1&1\\0&3/2&1/2&1/2\\0&1/2&3/2&1/2\\0&1/2&1/2&3/2\end{pmatrix} \to \begin{pmatrix}2&1&1&1\\0&3/2&1/2&1/2\\0&0&8/6&2/6\\0&0&2/6&8/6\end{pmatrix} \to \begin{pmatrix}2&1&1&1\\0&3/2&1/2&1/2\\0&0&8/6&2/6\\0&0&0&30/24\end{pmatrix} \boxed{> 0}$

#19) $\boxed{3.5.2.f}$ Find the LDL^T factorization of $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$. Is it positive definite?

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{L^T}}$$

It is <u>not</u> positive definite

#20) $\boxed{3.5.19.d}$ Find the Cholesky factorization for $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 8/6 \end{pmatrix} \to \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{4/3} \end{pmatrix}}_{\sqrt{D}=S} = \boxed{\underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2}/2 & \sqrt{3/2} & 0 \\ \sqrt{2}/2 & \sqrt{3/18} & \sqrt{4/3} \end{pmatrix}}_{M}}$$

$$MM^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \checkmark$$