HW#4

5.1.2. Find the minimizer of the function $f(x, y) = (3x - 2y + 1)^2 + (2x + y + 2)^2$.

$\frac{\partial f}{\partial x} = \frac{2(3x-2y+1)(3)}{2(2x+y+2)(2)}$

$= \frac{6(3x-2y+1)}{4(2x+y+2)}$

$= 18x - 12y + 6 + 8x + 4y + 8$

$= 26x - 8y + 14$

$\frac{\partial f}{\partial y} = 2(3x-2y+1)(-2) + 2(2x+y+2)(1)$

$\frac{\partial f}{\partial y} = -4(3x-2y+1) + 2(2x+y+2)$

$= -12x + 8y - 4 + 4x + 2y + 4$

$= -8x + 10y$

① $26x - 8y + 14 = 0$

② $-8x + 10y = 0$

$(13/4)(-8x+10y) + (26x - 8y + 14) = 0$

$-26x + 32y + 13x - 10y + 14 = 0$

$-13x + 22y + 14 = 0$

③ $13x - 22y - 14 = 0$

$13x = 22y + 14$

$x = \frac{(22y+14)}{13}$

$26\left[\frac{(22y+14)}{13}\right] - 8y + 14 = 0$

$572y + 364 - 104y + 182 = 0$

$468y + 546 = 0$

$y = -546/468$

$y = -39/34$

$-8x + 10(-39/34) = 0$

$-8x - 390/34 = 0$

$-8x = 390/34$

$x = (390/34)(-1/8)$

$x = -195/272$

$y = -39/34$

5.2.1. Find the minimum value of the function $f(x, y, z) = x^2+2xy+3y^2+2yz+z^2-2x+3z+2$. How do you know that your answer is really the global minimum?

$f(x) = 2x + 2y - 2$
$= 0$
$\Rightarrow x + y \pm 1$

$f(y) = 2x + 6y + 2z$
$= 0$
$\Rightarrow x + 3y + z = 0$

$f(z) = 2y + 2z + 3$
$\Rightarrow 2y + 2z = -3$

$1 - y + 3y + z = 0 \quad \Rightarrow$

$$\begin{array}{r} 2y + z = -1 \\ 2y + 2z = -3 \\ \hline -z = 2 \end{array}$$

$2y + z = -1$

$2y = -1 - z = -1 + 2$

$y = \frac{1}{2}$

$x + y = 1, \quad x = 1 - y$

$x = 1 - 1/2 \quad x = 1/2$

$z = -2$

so the stationary point $(1/2, 1/2, -2)$

$f_{xx} = 2 \quad f_{xy} = 2 \quad f_{xz} = 0$

$f_{yy} = 6 \quad f_{yz} = 2 \quad f_{yx} = 2$

$f_{zn} = 0 \quad f_{zy} = 2 \quad f_{zz} = 2$

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$|2| = 2 > 0$

$\begin{vmatrix} 2 & 2 \\ 2 & 6 \end{vmatrix} = 12 - 6$
$= 8 > 0$

$\begin{vmatrix} 2 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 2 \end{vmatrix} = 2[12-4] - [24 - 0] + 0$
$= 16 - 8$
$= 8 > 0$

$f(1/2, 1/2, -2) = 1/4 + 3/4 + 2/4 - 2 + 3 - 4$
$= 6/4 - 4/2 - 2$
$= -1/2 - 2 \quad = -3/2$

$f(1/2, 1/2, -2)$

5.3.4. Let $\mathbf{b} = (3, 1, 2, 1)^T$. Find the closest point and the distance from $\mathbf{b}$ to the following subspaces: (a) the line in the direction $(1, 1, 1, 1)^T$;

$b = (3, 1, 2, 1)^T \quad b = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

$a = (1, 1, 1, 1)^T \quad a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$|b_\parallel| = \frac{a \cdot b}{|a|^2}|a| = \frac{1\times 3 + 1\times 1 + 1\times 2 + 1\times 1}{2^2} \cdot 2$$

$$= \frac{3+1+2+1}{4} \cdot 2$$

$$= 7/4 \times 2$$

$$= 7/2$$

$$= 3.5$$

$$\sqrt{|b|^2 - |b_\parallel|^2} = \sqrt{15 - (7/2)^2}$$

$$= \sqrt{15 - 49/4}$$

$$= \sqrt{11/4}$$

$$= \sqrt{2.75}$$

5.3.1. Find the closest point in the plane spanned by $(1,2,-1)^T, (0,-1,3)^T$ to the point $(1,1,1)^T$. What is the distance between the point and the plane?

$V_1 = 1,2,-1$

$V_2 = 0,-1,3$

$P = 1,1,1$

$n = V_1 \cdot V_2$

$n = (1,2,-1)\cdot(0,-1,3)$

$n = (7,3-1)$

$r = P - P_0$

$= (1,1,1) - (0,0,0)$

$= (1,1,1)$

$$\text{proj} = \frac{r \cdot n}{\|n\|^2} \cdot n$$

$$= (1,1,1)\cdot(7,3,-1)\,\|7\ 3\ -1\|^2 \cdot (7,3,-1)$$

$$= \frac{7+3-1}{7^2+3^2+(-1)^2} \cdot (7,3,-1)$$

$$= \frac{9}{59} \cdot (7,3,-1)$$

$$= \left(\frac{63}{59}, \frac{27}{59}, -\frac{9}{59}\right)$$

$$\text{distance} = \left\| p - \left(\frac{63}{59}, \frac{27}{59}, -\frac{9}{59}\right)\right\|$$

$$= \left\| \frac{59}{59} - \frac{63}{59}\quad \frac{59}{59} - \frac{27}{59}\quad \frac{59}{59} + \frac{9}{59}\right\|$$

$$= \left\| -\frac{4}{59}\quad \frac{32}{59}\quad \frac{68}{59}\right\|$$

$$\sqrt{(-4/59)^2 + (32/59)^2 + (68/59)^2}$$

Note: Unless otherwise indicated, use the Euclidean norm to measure the least squares error.

5.4.1. Find the least squares solution to the linear system $A\mathbf{x} = \mathbf{b}$ when

$$(b)\ A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix},$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 14 & 13 \\ 13 & 26 \end{bmatrix}$$

inverse of $A^T \cdot A$:

$$(A^T A)^{-1} = \begin{bmatrix} 14 & 13 \\ 13 & 26 \end{bmatrix}^{-1}$$

$$(A^T A)^{-1} = \begin{bmatrix} 2/15 & -1/15 \end{bmatrix}$$

$$X = (A^T A)^{-1} A^T b$$

$$= \begin{bmatrix} 2/15 & 1/3 & 1/15 \\ -1/15 & -8/39 & 31/195 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$

$$X = \begin{bmatrix} 8/5 \\ 28/65 \end{bmatrix}$$

$$\begin{bmatrix} & \\ -1/15 & 14/195 \end{bmatrix}$$

$(A^T A)^{-1} A^T$:

$$(A^TA)^{-1}A^T = \begin{bmatrix} 2/15 & -1/15 \\ -1/15 & 14/195 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \end{bmatrix}$$

$$(A^TA)^{-1}A^T = \begin{bmatrix} 2/15 & 1/3 & 1/15 \\ -1/15 & -8/39 & 31/195 \end{bmatrix}$$

5.4.4. Find the least squares solution to the linear system $A\mathbf{x} = \mathbf{b}$ when

$$(b)\ A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & -2 & 1 \\ 1 & 0 & -3 \\ 5 & 2 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

0.07326
-0.150183
-0.09524

$$A^TA = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 1 & -2 & 0 & 2 \\ 4 & 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 1 & -2 & 1 \\ 1 & 0 & -3 \\ 5 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 10 & -4 \\ 10 & 9 & -2 \\ -4 & -2 & 30 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 1 & -2 & 0 & 2 \\ 4 & 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 31 & 10 & -4 \\ 10 & 9 & -2 \\ -4 & -2 & 30 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$R_1 \quad \frac{R_1}{31}$
$R_2 \quad R_2 - 10\frac{R_1}{31}$
$R_3 = R_3 + 4\frac{R_1}{31}$

$$\begin{bmatrix} 1 & 10/31 & -4/31 & 1/31 \\ 0 & 179/31 & -22/31 & -10/31 \\ 0 & -22/31 & 914/31 & -89/31 \end{bmatrix} \quad R_2 = R_2 \cdot 31/179 \quad \begin{bmatrix} 1 & 10/31 & -4/31 & 1/31 \\ 0 & 1 & -22/179 & -10/179 \\ 0 & -22/31 & 914/31 & -89/31 \end{bmatrix}$$

$R_1 = R_1 - 10/31\, R_2$
$R_3 = R_3 + 22/31\, R_2$

$$\begin{bmatrix} 1 & 0 & -16/179 & 9/179 \\ 0 & 1 & -22/179 & -10/179 \\ 0 & 0 & 5262/179 & -512/179 \end{bmatrix} \quad R_3 = 179/5262 \cdot R_3 \quad \begin{bmatrix} 1 & 0 & -16/179 & 9/179 \\ 0 & 1 & -22/179 & -10/179 \\ 0 & 0 & 1 & -152/5262 \end{bmatrix}$$

$R_1 = R_1 + 16/179\, R_3$
$R_2 = R_2 + 22/179\, R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 109/2631 \\ 0 & 1 & 0 & -179/2631 \\ 0 & 0 & 1 & -521/5262 \end{bmatrix}$$

$X = 109/2631 \quad Y = -179/2631 \quad Z = -521/5262$

5.5.1. Find the straight line $y = \alpha + \beta t$ that best fits the following data in the least squares sense: (a)

t_i	-2	0	1	3
y_i	0	1	2	5

t	y	ty	t^2
-2	0	0	4
0	1	0	0
1	2	2	1
3	5	15	9
2	8	17	14

$\sum y = \beta \sum t + na$

$\sum ty = \beta \sum t^2 + a \sum t$

$8 = 2\beta + 4a$
$17 = 14\beta + 8a$

$10\beta = 1 \Rightarrow \beta = \frac{1}{10} = 0.1$
$4a = 8 - 0.2 = 7.8$
$a = 1.95$

$y = 0.1t + 1.95$

5.5.2. The proprietor of an internet travel company compiled the following data relating the annual profit of the firm to its annual advertising expenditure (both measured in thousands of dollars):

Annual advertising expenditure	12	14	17	21	26	30
Annual profit	60	70	90	100	100	120

(*a*) Determine the equation of the least squares line. (*b*) Plot the data and the least squares line. (*c*) Estimate the profit when the annual advertising budget is \$50,000. (*d*) What about a \$100,000 budget?

x	y	x^2	xy
12	60	144	720
14	70	196	980
17	90	289	1530
21	100	441	2100
26	100	676	2600
30	120	900	3600
120	540	2646	11530

$n = 6$

A)

$540 = 6a + b120$

$= 6a + 120b$

$= 540$

$= 30.65$

$11530 = 120a + 2646b$

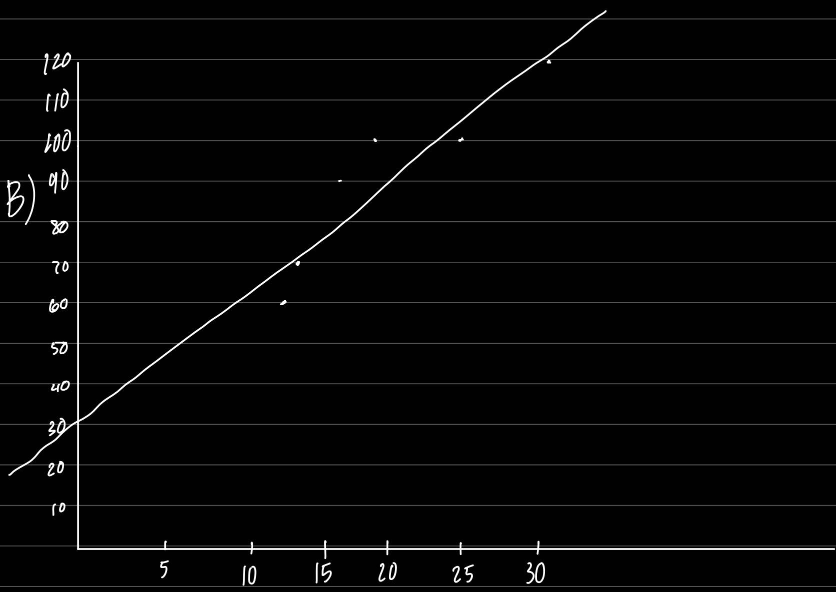
$= 120a + 2646b$

$= 11530$

$= 2.96$

$y = 30.65 + 2.96t$

B)



C) $y = 30.65 + 2.96t$

$y = 30.65 + 2.96(50)$
$= 179$
$179,000

D) $y = 30.65 + 2.96t$
$y = 30.65 + 2.96(100)$
$= 327.4$
$327,400