HW #5

7.1.1. Which of the following functions $F\colon \mathbb{R}^3 \to \mathbb{R}$ are linear? (a) $F(x,y,z) = x$, (b) $F(x,y,z) = y - 2$, (c) $F(x,y,z) = x + y + 3$, (d) $F(x,y,z) = x - y - z$, (e) $F(x,y,z) = x\,y\,z$, (f) $F(x,y,z) = x^2 - y^2 + z^2$, (g) $F(x,y,z) = e^{x-y+z}$.

A D

7.2.7. Find a linear transformation that maps the unit circle $x^2 + y^2 = 1$ to the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$. Is your answer unique?

$T(x,y) = \left(\frac{x}{2}, \frac{y}{3}\right)$

$x \to \frac{x}{2},\ y \to \frac{y}{3} \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\Rightarrow \frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$

not unique

$a^2 + c^2 = \frac{1}{4}$

$b^2 + d^2 = \frac{1}{9}$

$ab = -cd$

$T(x,y) = \left(\frac{1}{2\sqrt{2}}x + \frac{1}{3\sqrt{2}}y,\ \frac{1}{2\sqrt{2}}x - \frac{1}{3\sqrt{2}}y\right)$

$= \left(\frac{1}{2\sqrt{2}}x + \frac{1}{3\sqrt{2}}y\right) + \left(\frac{1}{2\sqrt{2}}x - \frac{1}{3\sqrt{2}}y\right) = 1$

$= \frac{1}{8}x^2 + \frac{1}{18}y^2 + \frac{1}{6}xy + \frac{1}{8}x^2 + \frac{1}{18}y^2 - \frac{1}{6}xy = 1$

$= \frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$

8.2.1. Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$,

$Av = \lambda v$

$Av - \lambda v = (A - \lambda I) \cdot V = 0$

$\det(A - \lambda I) = 0$

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix}$

$= \lambda^2 - 2\lambda - 3$

$= (\lambda+1) \cdot (\lambda - 3)$

$= 0$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

$\begin{vmatrix} -\lambda+1 & -2 \\ -2 & -\lambda+1 \end{vmatrix} = (-\lambda+1) \cdot (-\lambda+1) - (-2) \cdot (-2)$

$= \lambda^2 - 2\lambda - 3$

eigenvalue $\lambda_1 = -1$

$\lambda_2 = 3$

$V = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \lambda_1 = -1$

$V = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \lambda_2 = 3$

$\lambda_1 = -1$

$A - \lambda_1 I = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$Av = \lambda v$

$(A - \lambda I) \cdot V = 0$

$\left(\begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array}\right) R_1 \to R_1 \cdot \frac{1}{2}$, $R_1 \to \frac{R_1}{2}$ $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right)$ $R_2 = R_1 \cdot 2$, $R_2 - (-2) \cdot R_1 = R_2$

$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $X_1 - X_2 = 0$ $X_1 = X_2$

$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$A - \lambda_2 I = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$

$Av = \lambda v$ $(A - \lambda I) \cdot V = 0$

$\left(\begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array}\right) R_1 = R_1 \cdot (-\frac{1}{2})$, $R_1 = R_1 \div -2$ $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right) R_2 = R_1 \cdot 2$ $R_2 = R_2 - (-2) \cdot R_1$

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $X_1 = -X_2$ $X_1 = X_2$

eigenvectors : $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$,

$Av = \lambda v$

$Av - \lambda v = (A - \lambda I) \cdot V = 0$

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix}$

$= \lambda^2 - 4\lambda + 4$

$= (\lambda - 2)^2$

$= 0$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

$\begin{vmatrix} -\lambda+3 & 1 \\ -1 & -\lambda+1 \end{vmatrix} = (-\lambda+3) \cdot (-\lambda+1) - 1 \cdot (-1)$

$= \lambda^2 - 4\lambda + 4$

$\lambda_1 = 2$

$\lambda_1 = 2$

$A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

$Av = \lambda V$

$(A - \lambda I) \cdot V = 0$

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right) R_2 = R_1 \cdot 1$ $R_2 = R_2 - (-1) \cdot R_1$ $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$X_1 = -X_2$

$X_2 = X_2$

$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2$

$A - \lambda_2 I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

$Av = \lambda v$

$(A - \lambda I) \cdot V = 0$

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right) R_2 = R_1 \cdot 1$ $R_2 = R_2 - (-1) \cdot R_1$ $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$X_1 = -X_2$ $X_2 = X_2$

$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$,

$Av = \lambda v$

$Av - \lambda v = (A - \lambda I) \cdot v = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 8\lambda^2 - 19\lambda + 12$$

$$= -(\lambda - 1) \cdot (\lambda^2 - 7\lambda + 12)$$

$$= -(\lambda - 1) \cdot (\lambda - 3) \cdot (\lambda - 4)$$

$$= 0$$

$$\begin{vmatrix} -\lambda+3 & -1 & 0 \\ -1 & -\lambda+2 & -1 \\ 0 & -1 & -\lambda+3 \end{vmatrix} = (-\lambda+3)\cdot(-\lambda+2)\cdot(-\lambda+3) + (-1)\cdot(-1)\cdot 0 + 0\cdot(-1)\cdot(-1) - 0\cdot(-\lambda+2)\cdot 0 - (-1)\cdot(-1)\cdot(-\lambda+3) - (-\lambda+3)\cdot(-1)\cdot(-1) = -\lambda^3 + 8\lambda^2 - 19\lambda + 12$$

$\lambda_1 = 1 \quad \lambda_2 = 3 \quad \lambda_3 = 4$

$\lambda_1 = 1$

$$A - \lambda_1 I = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) R_1 = R_1/2 \left(\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) R_2 = R_2 - (-1)\cdot R_1 \left(\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 1/2 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) R_2 = R_2/\tfrac{1}{2} \left(\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) R_3 = R_3 - (-1)\cdot R_2$$

$Av = \lambda v$

$(A - \lambda I)\cdot v = 0$

$$\left(\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_1 = R_1 - (-1/2)\cdot R_2 \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = 2x_3$ $\quad$ $x_1 = x_3$

$x_1 = x_3$ $\quad$ $x_2 = 2x_3$ $\quad$ $x_3 = x_3$

$$x = \begin{pmatrix} x_3 \\ 2x_3 \\ x_3 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda_2 = 3$

$$A - \lambda_2 I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$Av = \lambda v$

$(A - \lambda I)\cdot v = 0$

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) R_2 \leftrightarrow R_1 \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) R_1 = R_1/-1 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) R_2 = R_2/-1 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) R_3 = R_3 - (-1)\cdot R_2 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_1 = R_1 - 1\cdot R_2 \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = 0$ $\quad$ $x_1 = -x_3$ $\quad$ $x_3 = x_3$

$$x = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} \quad x_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_3 = 4$

$$A - \lambda_3 I = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

$Av = \lambda v$

$(A - \lambda I)\cdot v = 0$

$$\left(\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) R_1 = R_1/-1 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) R_2 = R_2 - (-1)\cdot R_1 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) R_2 = R_2/-1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) R_3 = R_3 - (-1)\cdot R_2 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_1 = R_1 - 1\cdot R_2 \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = -x_3$

$x_1 = x_3$

$x_3 = x_3$

$$X = \begin{pmatrix} X_3 \\ -X_3 \\ X_3 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(g) $\begin{pmatrix} 1 & -3 & 11 \\ 2 & -6 & 16 \\ 1 & -3 & 7 \end{pmatrix}$,

$Av = \lambda v$

$Av - \lambda v = (A - \lambda I) \cdot v$

$= 0$

$\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 & 11 \\ 2 & -6-\lambda & 16 \\ 1 & -3 & 7-\lambda \end{vmatrix}$$

$= -\lambda^3 + 2\lambda^2 - 2\lambda$

$= -\lambda \cdot (\lambda^2 - 2\lambda + 2)$

$= -\lambda \cdot (\lambda - (1-i)) \cdot (\lambda - (1+i))$

$= 0$

$$\begin{vmatrix} -\lambda+1 & -3 & 11 \\ 2 & -\lambda-6 & 16 \\ 1 & -3 & -\lambda+7 \end{vmatrix} = (-\lambda+1)\cdot(-\lambda+6)\cdot(-\lambda+7) + (-3)\cdot 16 \cdot 1 + 11 \cdot 2 \cdot (-3) - 1 \cdot (-\lambda-6) \cdot 11 - (-3) \cdot 16 \cdot (-\lambda+1) - (-\lambda+7) \cdot 2 \cdot (-3)$$

$= -\lambda^3 + 2\lambda^2 - 2\lambda$

$\lambda_1 = 0$

$\lambda_2 = 1 - i$

$\lambda_3 = 1 + i$

$\lambda_1 = 0$

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 11 \\ 2 & -6 & 16 \\ 1 & -3 & 7 \end{pmatrix}$$

$Av = \lambda v$

$(A - \lambda I) \cdot v = 0$

$$\left(\begin{array}{ccc|c} 1 & -3 & 11 & 0 \\ 2 & -6 & 16 & 0 \\ 1 & -3 & 7 & 0 \end{array}\right) R_2 = R_2 - 2 \cdot R_1 \left(\begin{array}{ccc|c} 1 & -3 & 11 & 0 \\ 0 & 0 & -6 & 0 \\ 1 & -3 & 7 & 0 \end{array}\right) R_3 = R_3 - R_1 \left(\begin{array}{ccc|c} 1 & -3 & 11 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) R_2 = R_2 / -6 \left(\begin{array}{ccc|c} 1 & -3 & 11 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$R_3 = R_3 - (-4) \cdot R_2 \left(\begin{array}{ccc|c} 1 & -3 & 11 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_1 = R_1 - 11 \cdot R_2 \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$X_3 = 0$

$X_1 = 3X_2$

$X_2 = X_2$

$$X_2 : \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \qquad V_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = 1 - i$

$$A - \lambda_2 I = \begin{pmatrix} i & -3 & 11 \\ 2 & -7+i & 16 \\ 1 & -3 & 6+i \end{pmatrix}$$

$Av = \lambda v \qquad (A - \lambda I) \cdot v = 0$

$$\left(\begin{array}{ccc|c} i & -3 & 11 & 0 \\ 2 & -7+i & 16 & 0 \\ 1 & -3 & 6+i & 0 \end{array}\right) R_1 = R_1 / i \left(\begin{array}{ccc|c} 1 & 3i & -11i & 0 \\ 2 & -7+i & 16 & 0 \\ 1 & -3 & 6+i & 0 \end{array}\right) R_2 = R_2 - 2 \cdot R_1 \left(\begin{array}{ccc|c} 1 & 3i & -11i & 0 \\ 0 & -7-5i & 16+22i & 0 \\ 1 & -3 & 6+i & 0 \end{array}\right) R_3 = R_3 - 1 \cdot R_1$$

$$\left(\begin{array}{ccc|c} 1 & 3i & -11i & 0 \\ 0 & -7-5i & 16+22i & 0 \\ 0 & -3-3i & 6+12i & 0 \end{array}\right) R_2 = R_2 / (-7-5i) \left(\begin{array}{ccc|c} 1 & 3i & -11i & 0 \\ 0 & i & -3-i & 0 \\ 0 & -3-3i & 6+12i & 0 \end{array}\right) R_3 = R_3 - (-3-3i) \cdot R_2 \left(\begin{array}{ccc|c} 1 & 3i & -11i & 0 \\ 0 & 1 & -3-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_1 = R_1 - (3i) \cdot R_2$$

$X_2 = (3+i) \cdot X_3 \qquad X_1 = (3+2i) \cdot X_3$

$x_1 = (3+2i)\cdot x_3$
$x_2 = (3+i)\cdot x_3$
$x_3 = x_3$

$$x_3 = \begin{pmatrix} 3+2i \\ 3+i \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 3+2i \\ 3+i \\ 1 \end{pmatrix}$$

$\lambda_3 = 1+i$

$$A - \lambda_3 I = \begin{pmatrix} -i & -3 & 11 \\ 2 & -7-i & 16 \\ 1 & -3 & 6-i \end{pmatrix}$$

$Av = \lambda v$
$(A-\lambda I)\cdot v = 0$

$$\left(\begin{array}{ccc|c} -i & -3 & 11 & 0 \\ 2 & -7-i & 16 & 0 \\ 1 & -3 & 6-i & 0 \end{array}\right) \xrightarrow{R_1 = R_1/-i} \left(\begin{array}{ccc|c} 1 & -3i & 11i & 0 \\ 2 & -7-i & 16 & 0 \\ 1 & -3 & 6-i & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - 2\cdot R_1} \left(\begin{array}{ccc|c} 1 & -3i & 11i & 0 \\ 0 & -7+5i & 16-22i & 0 \\ 1 & -3 & 6-i & 0 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 1\cdot R_1} \left(\begin{array}{ccc|c} 1 & -3i & 11i & 0 \\ 0 & -7+5i & 16-22i & 0 \\ 0 & -3+3i & 6-12i & 0 \end{array}\right) \xrightarrow{R_2 = R_2/(-7+5i)} \left(\begin{array}{ccc|c} 1 & -3i & 11i & 0 \\ 0 & 1 & -3+i & 0 \\ 0 & -3+3i & 6-12i & 0 \end{array}\right) \xrightarrow{R_3 = R_3 - (-3+3i)\cdot R_2} \left(\begin{array}{ccc|c} 1 & -3i & 11i & 0 \\ 0 & 1 & -3+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - (-3i)\cdot R_2} \left(\begin{array}{ccc|c} 1 & 0 & -3+2i & 0 \\ 0 & 1 & -3+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_2 = (3-i)\cdot x_3$
$x_1 = (3-2i)\cdot x_3$

$x_1 = (3-2i)\cdot x_3$
$x_2 = (3-i)\cdot x_3$
$x_3 = x_3$

$$x_3 = \begin{pmatrix} 3-2i \\ 3-i \\ 1 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 3-2i \\ 3-i \\ 1 \end{pmatrix}$$

8.3.2. Find the eigenvalues and a basis for the each of the eigenspaces of the following matrices. Which are complete?

(a) $\begin{pmatrix} 4 & -4 \\ 1 & 0 \end{pmatrix}$,

$$\det(A - xI) = \begin{vmatrix} 4-x & -4 \\ 1 & -x \end{vmatrix}$$
$= -x(4-x)+4$
$= x^2 - 4x + 4$
$= (x-2)^2$

Eigen values are 2, 2

$(A-2I)\,x = 0$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x - 2y = 0$

Eigen space $= \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

Eigen ≠ multiplicity of eigen value

A isn't complete

8.5.1. Find the eigenvalues and an orthonormal eigenvector basis for the following symmetric matrices:

(a) $\begin{pmatrix} 2 & 6 \\ 6 & -7 \end{pmatrix}$

$Av = \lambda v$
$Av - \lambda v = (A - \lambda I)\cdot v = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 6 \\ 6 & -7-\lambda \end{vmatrix}$$
$= \lambda^2 + 5\lambda - 50$
$= (\lambda+10)(\lambda-5)$
$= 0$

$\lambda_1 = -10$
$\lambda_2 = 5$

$\lambda_1 = -10$

$$A - \lambda_1 I = \begin{pmatrix} 12 & 6 \\ 6 & 3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 12 & 6 & 0 \\ 6 & 3 & 0 \end{array}\right) \xrightarrow{R_1 = R_1/12} \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 6 & 3 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - 6\cdot R_1} \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 = \frac{-1}{2}\cdot x_2 \quad x_2 = x_2$

$$x_2 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \qquad V_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

$\lambda_2 = 5$

$A - \lambda_2 I = \begin{pmatrix} -3 & 6 \\ 6 & 12 \end{pmatrix}$

$\left(\begin{array}{cc|c} -3 & 6 & 0 \\ 6 & \cdot 12 & 0 \end{array}\right) R_1 = R_1/-3 \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 6 & -12 & 0 \end{array}\right) R_2 = R_2 - 6 \cdot R_1 \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\begin{array}{l} X_1 = 2X_2 \\ X_2 = X_2 \end{array}$ $X = \begin{pmatrix} 2X_2 \\ X_2 \end{pmatrix}$ $X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$,

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix}$$
$$= \lambda^2 - 10\lambda + 21$$
$$= (\lambda - 3) \cdot (\lambda - 7)$$
$$= 0$$

$$\begin{vmatrix} -\lambda+5 & -2 \\ -2 & -\lambda+5 \end{vmatrix} = (-\lambda+5) \cdot (-\lambda+5) - (-2) \cdot (-2)$$
$$= \lambda^2 - 10\lambda + 21$$

$\lambda_1 = 3$

$\lambda_2 = 7$

$\lambda_1 = 3$

$A - \lambda_1 I = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$\left(\begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array}\right) R_1 = R_1/2 \left(\begin{array}{cc|c} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right) R_2 = R_2 - (-2) \cdot R_1 \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\begin{array}{l} X_1 = X_2 \\ X_2 = X_2 \end{array}$ $X = \begin{pmatrix} X_2 \\ X_2 \end{pmatrix}$ $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 7$

$A - \lambda_2 I = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$

$\left(\begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array}\right) R_1 = R_1/-2 \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right) R_2 = R_2 - (-2) \cdot R_1 \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\begin{array}{l} X_1 = -X_2 \\ X_2 = X_2 \end{array}$ $X = \begin{pmatrix} -X_2 \\ X_2 \end{pmatrix}$ $X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

8.7.1. Find the singular values of the following matrices:

(c) $\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$,

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

$[1 \quad -2] \cdot [1 \quad -3]$

$W = \lfloor -3 \quad 6 \rfloor \quad \lfloor -2 \quad 6 \rfloor$

$$\begin{matrix} (1)\cdot(1)+(-2)\cdot(-2) & (1)\cdot(-3)+(-2)\cdot(6) \\ (-3)\cdot(1)+(6)\cdot(-2) & (-3)\cdot(-3)+(6)\cdot(6) \end{matrix} = \begin{bmatrix} 5 & -15 \\ -15 & 45 \end{bmatrix}$$

Eigenvalue : 50 , eigenvector $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$

Eigenvalue : 0 , eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\sigma_1 = 5\sqrt{2}$

$$\Sigma = \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} -\sqrt{10}/10 & 3\sqrt{10}/10 \\ 3\sqrt{10}/10 & \sqrt{10}/10 \end{bmatrix}$$

$$\left(\frac{\sqrt{2}}{10}\right)\cdot \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} (\sqrt{2}/10)(1) & (\sqrt{2}/10)(-3) \\ (\sqrt{2}/10)(-2) & (\sqrt{2}/10)(6) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/10 & -3\sqrt{2}/10 \\ -\sqrt{2}/5 & 3\sqrt{2}/5 \end{bmatrix}$$

$$V_1 = \frac{1}{\sigma_1} \cdot \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}^T \cdot U_1 = \frac{1}{5\sqrt{2}} \cdot \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} -\sqrt{10}/10 \\ 3\sqrt{10}/10 \end{bmatrix} = \begin{bmatrix} -\sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2}/10 & -3\sqrt{2}/10 \\ -\sqrt{2}/\sqrt{5} & 3\sqrt{2}/5 \end{bmatrix} \cdot \begin{bmatrix} -\sqrt{10}/10 \\ 3\sqrt{10}/10 \end{bmatrix} = \begin{bmatrix} (\sqrt{2}/10)\cdot(-\sqrt{10}/10) + (-3\sqrt{2}/10)\cdot(3\sqrt{10}/10) \\ (-\sqrt{2}/5)\cdot(-\sqrt{10}/10) + (3\sqrt{2}/5)\cdot(3\sqrt{10}/10) \end{bmatrix} = \begin{bmatrix} -\sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}$$

$$\begin{bmatrix} 2\sqrt{5}/5 \\ \sqrt{5}/5 \end{bmatrix} \qquad V = \begin{bmatrix} -\sqrt{5}/5 & 2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix} \qquad U, \Sigma, V = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} = U\Sigma V^T$$

$$U = \begin{bmatrix} -\sqrt{10}/10 & 3\sqrt{10}/10 \\ 3\sqrt{10}/10 & \sqrt{10}/10 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} -\sqrt{5}/5 & 2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix}$$