CSCI 2033 - Assignment #3 - Key

#1 - 4.1.1) a) $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\|v_1\| \neq 1 \Rightarrow$ not orthonormal

$\langle v_1, v_2 \rangle = -2+2 = 0 \Rightarrow$ orthogonal

$\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix} \Rightarrow$ basis of $\mathbb{R}^2$

$\therefore$ orthogonal basis

b) $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\|v_1\| = 1$, $\|v_2\| = 1 \Rightarrow$ unit length

$\langle v_1, v_2 \rangle = 0 \Rightarrow$ orthogonal

$(v_1 \ v_2) \to \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 2/\sqrt{2} \end{pmatrix} \Rightarrow$ basis of $\mathbb{R}^2$

$\therefore$ orthonormal basis

c) $v_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\|v_1\| \neq 1 \Rightarrow$ not orthonormal

$\langle v_1, v_2 \rangle = -4 \Rightarrow$ not orthogonal

$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow$ line in $\mathbb{R}^2$

$\therefore$ basis

#2 - 4.1.3) a) $\langle v_1, v_2 \rangle = (-1)(2) + \frac{1}{9}(2)(1) = -2 + \frac{2}{9} \neq 0$ $\therefore$ basis

b) $\langle v_1, v_2 \rangle = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) + \frac{1}{9}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-1}{2} + \frac{1}{9 \cdot 2} \neq 0$ $\therefore$ basis

c) $\langle v_1, v_2 \rangle = (-1)(2) + \frac{1}{9}(-1)(2) = -2 - \frac{2}{9} \neq 0$ $\therefore$ basis

---

#3 - 4.2.1.c) Use Gram-Schmidt process to get orthonormal basis for $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} - \frac{14}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} \right\|^2} \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - \frac{5}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{18}{27} \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} -5/14 \\ 4/14 \\ -1/14 \end{pmatrix}$

$u_1 = \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}$ $u_2 = \begin{pmatrix} 3/\sqrt{27} \\ 3/\sqrt{27} \\ -3/\sqrt{27} \end{pmatrix}$ $u_3 = \begin{pmatrix} -70/\sqrt{42} \\ 56/\sqrt{42} \\ -14/\sqrt{42} \end{pmatrix}$

#4 - 4.2.17.b) Use modified Gram-Schmidt process to get orthonormal basis for $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$r_{11} = \sqrt{\left\| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\|^2} = \sqrt{2}$, $u_1 = \frac{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{2}} = \boxed{\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}$, $r_{12} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, u_1 \right\rangle = 0 \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 + \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}$

$r_{22} = \sqrt{\left\| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\|^2 - \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{2 - \frac{1}{2}} = \sqrt{3/2}$, $u_2 = \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - r_{12} u_1}{r_{22}} = \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}}{\sqrt{3/2}} = \boxed{\begin{pmatrix} \sqrt{2/3} \\ -\sqrt{2/12} \\ \sqrt{2/12} \end{pmatrix}}$, $r_{13} = \left\langle \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, u_1 \right\rangle = 2 \cdot 0 + 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$r_{23} = \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, u_2 \right\rangle = 2 \cdot \sqrt{\frac{2}{3}} + 1 \cdot -\sqrt{\frac{2}{12}} + 0 \cdot \sqrt{\frac{2}{12}} = 3\sqrt{\frac{2}{12}} = \sqrt{\frac{3}{2}}$

$r_{33} = \sqrt{\left\| \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\|^2 - \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\sqrt{\frac{3}{2}}\right)^2} = \sqrt{5 - \frac{1}{2} - \frac{3}{2}} = \sqrt{3}$, $u_3 = \frac{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - r_{13} u_1 - r_{23} u_2}{r_{33}} = \frac{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}}{\sqrt{3}} = \boxed{\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{\ldots} \end{pmatrix}}$

#5-4.3.27) Find the QR factorization for

b) $\begin{pmatrix}4&3\\3&2\end{pmatrix}$ $\quad r_{11}=\sqrt{4^2+3^2}=\sqrt{25}=5$, $\begin{pmatrix}4&3\\3&2\end{pmatrix}\to\begin{pmatrix}4/5&3\\3/5&2\end{pmatrix}$, $r_{12}=(\frac{4}{5})(3)+(\frac{3}{5})(2)=\frac{18}{5}$, $\begin{pmatrix}4/5&3\\3/5&2\end{pmatrix}\to\begin{pmatrix}4/5&3/25\\3/5&-4/25\end{pmatrix}$

$r_{22}=\sqrt{(\frac{3}{25})^2+(\frac{-4}{25})^2}=\sqrt{\frac{25}{25^2}}=\frac{1}{5}$, $\begin{pmatrix}4/5&3/5\\3/5&-4/5\end{pmatrix}$ $\quad\therefore \boxed{\begin{pmatrix}4&3\\3&2\end{pmatrix}=\begin{pmatrix}4/5&3/5\\3/5&-4/5\end{pmatrix}\begin{pmatrix}5&18/5\\0&1/5\end{pmatrix}}$

---

c) $\begin{pmatrix}2&1&-1\\0&1&3\\-1&-1&1\end{pmatrix}$, $r_{11}=\sqrt{2^2+0^2+(-1)^2}=\sqrt{5}$, $A\Rightarrow\begin{pmatrix}2/\sqrt5&1&-1\\0&1&3\\-1/\sqrt5&-1&1\end{pmatrix}$, $r_{12}=(\frac{2}{\sqrt5})\cdot1+0\cdot1+(\frac{-1}{\sqrt5})\cdot(-1)=\frac{3}{\sqrt5}$, $r_{13}=(\frac{2}{\sqrt5})(-1)+0\cdot3+(\frac{-1}{\sqrt5})(1)=\frac{-3}{\sqrt5}$

$A\Rightarrow\begin{pmatrix}2/\sqrt5&-1/5&1/5\\0&1&3\\-1/\sqrt5&-2/5&4/5\end{pmatrix}$, $r_{22}=\sqrt{(\frac{-1}{5})^2+1^2+(\frac{2}{5})^2}=\sqrt{\frac{30}{25}}=\frac{\sqrt{30}}{5}$, $A\Rightarrow\begin{pmatrix}2/\sqrt5&-1/\sqrt{30}&1/5\\0&5/\sqrt{30}&3\\-1/\sqrt5&-2/\sqrt{30}&2/5\end{pmatrix}$

$r_{23}=(\frac{-1}{\sqrt{30}})(\frac{1}{5})+(\frac{5}{\sqrt{30}})3+(\frac{-2}{\sqrt{30}})(\frac{2}{5})=\frac{14}{\sqrt{30}}$, $A\Rightarrow\begin{pmatrix}2/\sqrt5&-1/\sqrt{30}&2/3\\0&5/\sqrt{30}&2/3\\-1/\sqrt5&-2/\sqrt{30}&4/3\end{pmatrix}$ $\quad r_{33}=\sqrt{(\frac{2}{3})^2+(\frac{2}{3})^2+(\frac{4}{3})^2}=\frac{2\sqrt6}{3}$

$A\Rightarrow\begin{pmatrix}2/\sqrt5&-1/\sqrt{30}&1/\sqrt6\\0&5/\sqrt{30}&1/\sqrt6\\-1/\sqrt5&-2/\sqrt{30}&2/\sqrt6\end{pmatrix}$, $\quad\therefore\boxed{\begin{pmatrix}2&1&-1\\0&1&3\\-1&-1&1\end{pmatrix}=\begin{pmatrix}2/\sqrt5&-1/\sqrt{30}&1/\sqrt6\\0&5/\sqrt{30}&1/\sqrt6\\-1/\sqrt5&-2/\sqrt{30}&2/\sqrt6\end{pmatrix}\begin{pmatrix}\sqrt5&3/\sqrt5&-3/\sqrt5\\0&\sqrt{30}/5&14/\sqrt{30}\\0&0&2\sqrt6/3\end{pmatrix}}$

---

d) $\begin{pmatrix}0&1&2\\-1&1&1\\-1&1&3\end{pmatrix}$, $r_{11}=\sqrt{0^2+(-1)^2+(-1)^2}=\sqrt2$, $A\Rightarrow\begin{pmatrix}0&1&2\\-1/\sqrt2&1&1\\-1/\sqrt2&1&3\end{pmatrix}$, $r_{12}=\frac{-2}{\sqrt2}$, $r_{13}=\frac{-4}{\sqrt2}$, $A\Rightarrow\begin{pmatrix}0&1&2\\-1/\sqrt2&0&-1\\-1/\sqrt2&0&1\end{pmatrix}$

$r_{22}=\sqrt{1^2+0^2+0^2}=1$, no change for A, $r_{23}=2$, $A\Rightarrow\begin{pmatrix}0&1&0\\-1/\sqrt2&0&-1\\-1/\sqrt2&0&1\end{pmatrix}$, $r_{33}=\sqrt{0^2+(-1)^2+1^2}=\sqrt2$, $A=\begin{pmatrix}0&1&0\\-1/\sqrt2&0&-1/\sqrt2\\-1/\sqrt2&0&1/\sqrt2\end{pmatrix}$

$\therefore\boxed{\begin{pmatrix}0&1&2\\-1&1&1\\-1&1&3\end{pmatrix}=\begin{pmatrix}0&1&0\\-1/\sqrt2&0&-1/\sqrt2\\-1/\sqrt2&0&1/\sqrt2\end{pmatrix}\begin{pmatrix}\sqrt2&-2/\sqrt2&-4/\sqrt2\\0&1&2\\0&0&\sqrt2\end{pmatrix}}$

#6-4.4.4) Find the orthogonal projection of $\begin{pmatrix}1\\2\\3\end{pmatrix}$ onto $\begin{pmatrix}3&2\\2&-2\\1&-2\end{pmatrix}$

$$W=\frac{\langle\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}3\\2\\1\end{pmatrix}\rangle}{\|\begin{pmatrix}3\\2\\1\end{pmatrix}\|^2}\begin{pmatrix}3\\2\\1\end{pmatrix}+\frac{\langle\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\-2\\-2\end{pmatrix}\rangle}{\|\begin{pmatrix}2\\-2\\-2\end{pmatrix}\|^2}\begin{pmatrix}2\\-2\\-2\end{pmatrix}=\frac{10}{14}\begin{pmatrix}3\\2\\1\end{pmatrix}+\frac{-8}{12}\begin{pmatrix}2\\-2\\-2\end{pmatrix}=\boxed{\begin{pmatrix}17/21\\58/21\\43/21\end{pmatrix}}$$

#7-4.4.12) Find the orthogonal complement $W^\perp$ of the subspace $W \subset \mathbb{R}^3$. What is $\dim W^\perp$

a) W spanned by $\begin{pmatrix}3\\-1\\1\end{pmatrix}$, $W^\perp$ is set of vectors $z=\begin{pmatrix}x\\y\\z\end{pmatrix}$ such that $z\cdot w_i=\emptyset\ \forall i$

$z\cdot w_1 = 3x-y+z=\emptyset \Rightarrow z=\begin{pmatrix}+1/3\\1\\0\end{pmatrix}a+\begin{pmatrix}-1/3\\0\\1\end{pmatrix}b$. Therefore $W^\perp$ spanned by $\begin{pmatrix}1/3\\1\\0\end{pmatrix},\begin{pmatrix}-1/3\\0\\1\end{pmatrix}$ with $\dim W^\perp = 2$

---

b) W spanned by $\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\\emptyset\\1\end{pmatrix}$ $\quad z\cdot w_1=\emptyset$ and $z\cdot w_2=\emptyset \Rightarrow \begin{pmatrix}1&2&3\\2&\emptyset&1\end{pmatrix}\to\begin{pmatrix}1&2&3\\\emptyset&-4&-5\end{pmatrix}\to z=\begin{pmatrix}-1/2\\-5/4\\1\end{pmatrix}a$

Therefore $W^\perp$ is spanned by $\begin{pmatrix}-1/2\\-5/4\\1\end{pmatrix}$ with $\dim W^\perp = 1$

---

c) W spanned by $\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\4\\6\end{pmatrix}$ $\quad z\cdot w_1=\emptyset$ and $z\cdot w_2=\emptyset \Rightarrow \begin{pmatrix}1&2&3\\2&4&6\end{pmatrix}\to\begin{pmatrix}1&2&3\\0&0&0\end{pmatrix}\to z=\begin{pmatrix}-2\\1\\0\end{pmatrix}a+\begin{pmatrix}-3\\0\\1\end{pmatrix}b$

Therefore $W^\perp$ is spanned by $\begin{pmatrix}-2\\1\\\emptyset\end{pmatrix},\begin{pmatrix}-3\\\emptyset\\1\end{pmatrix}$ with $\dim W^\perp = 2$