HW # 2

2.3.1. Show that $\begin{pmatrix}-1\\2\\3\end{pmatrix}$ belongs to the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix}2\\-1\\2\end{pmatrix}$, $\begin{pmatrix}5\\-4\\1\end{pmatrix}$ by writing it as a linear combination of the spanning vectors.

$C_1V_1 + C_2V_2 = X$

$$\begin{bmatrix}2 & 5 & -1\\-1 & -4 & 2\\2 & 1 & 3\end{bmatrix} \xrightarrow{R_1 = R_1/2} \begin{bmatrix}1 & 5/2 & -1/2\\-1 & -4 & 2\\2 & 1 & 3\end{bmatrix} \xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix}1 & 5/2 & -1/2\\0 & -3/2 & 3/2\\2 & 1 & 3\end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix}1 & 5/2 & -1/2\\0 & -3/2 & 3/2\\0 & -4 & 4\end{bmatrix} \xrightarrow{R_2 = (-2R_2)/3} \begin{bmatrix}1 & 5/2 & -1/2\\0 & 1 & -1\\0 & -4 & 4\end{bmatrix}$$

$R_1 = R_1 - (5R_2)/2$

$$\begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & -4 & 4\end{bmatrix} \xrightarrow{R_3 = R_3 + 4R_2} \begin{array}{c}\begin{array}{ccc}C_1 & C_2 & C_3\end{array}\\ \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}\end{array}$$

$C_1 = 2$

$C_2 = -1$

$2\vec{V_1} - \vec{V_2} = \vec{X}$

$$2\begin{bmatrix}2\\-1\\2\end{bmatrix} - \begin{bmatrix}5\\-4\\1\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix}$$

$4 - 5 = -1$ ✓

$-2 + 4 = 2$ ✓

$4 - 1 = 3$ ✓

2.3.2. Show that $\begin{pmatrix}-3\\7\\6\\1\end{pmatrix}$ is in the subspace of $\mathbb{R}^4$ spanned by $\begin{pmatrix}1\\-3\\-2\\0\end{pmatrix}$, $\begin{pmatrix}-2\\6\\3\\4\end{pmatrix}$ and $\begin{pmatrix}-2\\4\\6\\-7\end{pmatrix}$.

$C_1V_1 + C_2V_2 + C_3V_3 + C_4V_4 = X$

$$\begin{bmatrix}1 & -2 & -2 & -3\\-3 & 6 & 4 & 7\\-2 & 3 & 6 & 6\\0 & 4 & -7 & 1\end{bmatrix} \xrightarrow{R_2 = R_2 + 3R_1} \begin{bmatrix}1 & -2 & -2 & -3\\0 & 0 & -2 & -2\\-2 & 3 & 6 & 6\\0 & 4 & -7 & 1\end{bmatrix} \xrightarrow{R_3 = R_3 + 2R_1} \begin{bmatrix}1 & -2 & -2 & -3\\0 & 0 & -2 & -2\\0 & -1 & 2 & 0\\0 & 4 & -7 & 1\end{bmatrix} \xrightarrow{\text{swap by rows 2 \& 3}} \begin{bmatrix}1 & -2 & -2 & -3\\0 & -1 & 2 & 0\\0 & 0 & -2 & -2\\0 & 4 & -7 & 1\end{bmatrix} \xrightarrow{R_2 = -R_2} \begin{bmatrix}1 & -2 & -2 & -3\\0 & 1 & -2 & 0\\0 & 0 & -2 & -2\\0 & 4 & -7 & 1\end{bmatrix}$$

$R_1 = R_1 + 2R_2$

$$\begin{bmatrix}1 & 0 & -6 & -3\\0 & 1 & -2 & 0\\0 & 0 & -2 & -2\\0 & 4 & -7 & 1\end{bmatrix} \xrightarrow{R_4 = R_4 - 4R_2} \begin{bmatrix}1 & 0 & -6 & -3\\0 & 1 & -2 & 0\\0 & 0 & -2 & -2\\0 & 0 & 1 & 1\end{bmatrix} \xrightarrow{R_3 = -R_3/2} \begin{bmatrix}1 & 0 & -6 & -3\\0 & 1 & -2 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{bmatrix} \xrightarrow{R_1 = R_1 + 6R_3} \begin{bmatrix}1 & 0 & 0 & 3\\0 & 1 & -2 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{bmatrix} \xrightarrow{R_2 = R_2 + 2R_3} \begin{bmatrix}1 & 0 & 0 & 3\\0 & 1 & 0 & 2\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$

$R_4 = R_4 - R_3$

$$\begin{array}{c}\begin{bmatrix}1 & 0 & 0 & 3\\0 & 1 & 0 & 2\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}\\ \begin{array}{cccc}C_1 & C_2 & C_3 & C_4\end{array}\end{array}$$

$C_1 = 3$

$C_2 = 2$

$C_3 = 1$

$C_4 = X$

$3C_1 + 2C_2 + 1C_3 = X$

$$3\begin{bmatrix}1\\-3\\-2\\0\end{bmatrix} + 2\begin{bmatrix}-2\\6\\3\\4\end{bmatrix} + \begin{bmatrix}-2\\4\\6\\-7\end{bmatrix} = \begin{bmatrix}-3\\7\\6\\1\end{bmatrix}$$

$3 - 4 - 2 = -3$ ✓

$-9 + 12 + 4 = 7$ ✓

$-6 + 6 + 6 = 6$ ✓

$0 + 8 + -7 = 1$ ✓

yes

2.3.21. Determine whether the given vectors are linearly independent or linearly dependent:

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

the dimension is 2, the basis is 2
the set is linearly independent

(c) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix},$

dimension is 2
since the dimension of the basis of the set is less than the dimension of the set, its dependent

(e) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

dimension is 2

since the dimension of the basis of the set is less than the dimension of the set, its dependent

2.4.1. Determine which of the following sets of vectors are bases of $\mathbb{R}^2$:

(c) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$;

$a(1,2) + b(2,1) = 0$

$$\left.\begin{aligned} a+2b &= 0 \\ 2a+b &= 0 \end{aligned}\right\}$$ solve these equations

$a = -2b$

$2(-2b)+b = 0$

$-4b+b = 0$

$-3b = 0$

$b = 0 \quad a = 0$

since a and b are 0, its linearly independent, so its a basis of $\mathbb{R}^2$

2.5.1. Characterize the image and kernel of the following matrices:

(b) $\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$

$Ax = 0$

$R_2 = R_2 + 2R_1$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 + 2x_3 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} x_3$$ where x_2 and x_3 are free variables

the basis of the kernel of A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

and the basis for the image of A is $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

since only the first column of rref(A) has pivot position

2.5.12. Find the solution $\mathbf{x}_1^\star$ to the system $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the solution $\mathbf{x}_2^\star$ to $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Express the solution to $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ as a linear combination of $\mathbf{x}_1^\star$ and $\mathbf{x}_2^\star$.

$$A = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix} \text{ and } x = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A^{-1} = \left(\frac{1}{2}\right)\begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1^* \\ y_1^* \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3/2 \end{bmatrix}$$

$x_1 = -2 \quad y_1 = 3/2$

$$A^{-1} = \left(\frac{1}{2}\right)\begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix}$$

$x = A^{-1}B$

$$\begin{bmatrix} x_2^* \\ y_2^* \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$$

$x_1 = -1 \quad y_2 = 1/2$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} -12 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 7/2 \end{bmatrix}$$

$$\begin{bmatrix} -6 \\ 7/2 \end{bmatrix} = a \times \begin{bmatrix} -2 \\ 3/2 \end{bmatrix} + B \times \begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6 \\ 7/2 \end{bmatrix} = \begin{bmatrix} -2a \\ 3/2a \end{bmatrix} + \begin{bmatrix} -B \\ \frac{B}{2} \end{bmatrix}$$

$-6 = -2a - B$

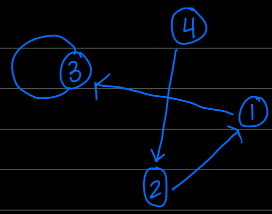
$7 = 3a + B$

$a = 1 \quad B = 4$

$a = 1 \quad B = 4$

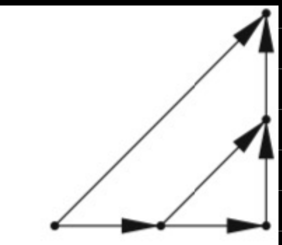
2.6.2. Draw the digraph represented by the following incidence matrices:

(a) $\begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$,



2.6.3. Write out the incidence matrix of the following digraphs.

(c)



$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

3.1.2. Which of the following formulas for $\langle \mathbf{v}, \mathbf{w} \rangle$ define inner products on $\mathbb{R}^2$?

(a) $2 v_1 w_1 + 3 v_2 w_2$,

$\langle U + V, W \rangle = \langle \ , W \rangle + \langle V, W \rangle$

By definition for $U = (U_1, U_2)$, $V = (V_1, V_2)$ & $W = (W_1, W_2) \in \mathbb{R}^2$

$\langle U + V, W \rangle = 2(U_1 + V_1) W_1 + 3(U_2 + V_2) W_2$

$= (2U_1 W_1 + 3U_2 W_2) + (2V_1 W_1 + 3V_2 W_2)$

$= \langle U, W \rangle + \langle V, W \rangle$

(b) $v_1 w_2 + v_2 w_1$

$\langle \alpha V, W \rangle = 2\alpha V_1 W_1 + 3\alpha V_2 W_2$

$= \alpha(2V_1 W_1 + 3V_2 W_2)$

$= \alpha \langle V, W \rangle,$

$= \forall \alpha \in \mathbb{R}^2$

(c) $(v_1 + v_2)(w_1 + w_2)$

$\langle V, W \rangle = 2V_1 W_1 + 3V_2 W_2$

$= 2W_1 V_1 + 3W_2 V_2$

$= \langle W, V \rangle$

(d) $v_1^2 w_1^2 + v_2^2 w_2^2$,

$\langle V, W \rangle = 2V_1^2 + 3V_2^2 \geq 0$ $\quad V_1^2 < V_2^2 \geq 0$

and $= 0 \iff 2v_1^2 + 3v_2^2 = 0$

$\iff v_1^2 = 0$ & $v_2^2 = 0$

$\iff v_1 = 0$ & $v_2 = 0$

$v = 0$

Inner product on $\mathbb{R}^2$

3.2.1. Verify the Cauchy–Schwarz inequality for each of the following pairs of vectors $\mathbf{v}, \mathbf{w}$, using the standard dot product, and then determine the angle between them:

(a) $(1,2)^T$, $(-1,2)^T$,

$V \cdot W = (1)(-1) + (2)(2)$

$= -1 + 4$

$= 3$

$\|V\| = \sqrt{(1^2+2^2)}$

$= \sqrt{1+4}$

$= \sqrt{5}$

$\|W\| = \sqrt{((-1)^2+2^2)}$

$= \sqrt{1+4}$

$= \sqrt{5}$

$|V \cdot W| = |3| = 3$

$\|V\| \, \|W\| = \sqrt{5} * \sqrt{5} = 5$

$\cos\theta = \dfrac{(V \cdot W)}{(\|V\| \, \|W\|)}$

$= \frac{3}{5}$

$\theta = \arccos(3/5)$

$53.13°$

(b) $(1,-1,0)^T$, $(-1,0,1)^T$,

$|\langle V, W \rangle| \leq \|V\| \, \|W\|$

$\langle V, W \rangle = 1 \times -1 + -1 \times 0 + 0 \times 1 = -1$

$|\langle V, W \rangle| = |-1| = 1$

$\|V\| = \sqrt{\langle V, V \rangle} = \sqrt{1 \times 1 + -1 \times -1 + 0 \times 0} = \sqrt{2}$

$\|W\| = \sqrt{\langle W, W \rangle} = \sqrt{-1 \times -1 + 0 \times 0 + 1 \times 1} = \sqrt{2}$

$\|V\| \cdot \|W\| = \sqrt{2} \times \sqrt{2} = 2$

$\cos\theta = \frac{\langle v, w \rangle}{\|v\| \, \|w\|} = \frac{-1}{2}$

$= \cos^{-1}\left(\frac{-1}{2}\right) = \theta$

$= \theta = \frac{2\pi}{3} = 120°$

(c) $(1,-1,0)^T$, $(2,2,2)^T$,

$V \cdot W = (1)(2) + (-1)(2) + (0)(2)$

$= 2 - 2 + 0$

$= 0$

$\|V\| = \sqrt{1^2 + (-1)^2 + 0^2}$

$= \sqrt{1+1+0}$

$= \sqrt{2}$

$\|W\| = \sqrt{2^2 + 2^2 + 2^2}$

$= \sqrt{4+4+4}$

$= 2\sqrt{3}$

$|V \cdot W| = |0| = 0$

$\|V\| \, \|W\| = \sqrt{2} * 2\sqrt{3}$

$= 2\sqrt{6}$

$\cos\theta = \frac{(V \cdot W)}{(\|V\| \, \|W\|)}$

$\cos\theta = \frac{0}{2\sqrt{6}}$

$\theta = \arccos(0/(2\sqrt{6}))$

53.13 degrees

90 degrees

3.2.16. Find all vectors in $\mathbb{R}^3$ that are orthogonal to both $(1,2,3)^T$ and $(-2,0,1)^T$.

$U \times V = (V_2 \times V_3 - U_3 \times V_2,\ U_3 \times V_1 - U_1 \times V_3,\ U_1 \times V_2 - U_2 \times V_1)$

$= (2 \times 1 - 3 \times 0,\ 3 \times -2 - 1 \times 1,\ 1 \times 0 - 2 \times -2)$

$= (2, -7, 4)$

3.3.2. Answer Exercise 3.3.1 for

(c) $\begin{pmatrix}1\\-2\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\-3\end{pmatrix}$.

$(2,-1)$ and $(1,-2)$

1-norm $\|(2,-1)\|_1 = |2| + |-1| = 2+1 = 3$

2-norm $\|(2,-1)\|_2 = \sqrt{2^2+(-1)^2}$

$= \sqrt{5}$

3-norm $\|(2,-1)\|_3 = (2^3+(-1)^3)^3$

$= 7^3$

$\|(2,-1)\| + \|(1,-2)\| = \sqrt{5} + \sqrt{5}$

≈ 4.47

$(1,0,-1)$ and $(-1,1,0)$

1norm $\|(1,0,-1)\|_1 = |1| + |0| + |-1|$

$= 1+0+1$

$= 2$

2 norm $\|(1,0,-1)\|_2 = \sqrt{(1^2+0^2+(-1)^2)}$

$= \sqrt{2}$

3 norm $\|(1,0,-1)\|_3 = (1^3+0^3+(-1)^3)^3$

$= 0$

$\|(1,0,-1)\| + \|(-1,1,0)\|$

$= \sqrt{2} + \sqrt{2}$

≈ 2.83

$(1,-2,-1)$ and $(2,-1,-3)$

1 norm $\|(1,-2,-1)\|_1 = |1| + |-2| + |-1|$

$= 1+2+1$

$= 4$

2 norm $\|(1,-2,-1)\|_2 = \sqrt{(1^2+(-2)^2+(-1)^2}$

$= \sqrt{6}$

3 norm $\|(1,-2,-1)\|_3 = (1^3+(-2)^3+(-1)^3)^3$

≈ -1.26

2

3.4.1. Which of the following 2×2 matrices are positive definite?

(a) $\begin{pmatrix}1&0\\0&2\end{pmatrix}$, (b) $\begin{pmatrix}0&1\\2&0\end{pmatrix}$, (c) $\begin{pmatrix}1&2\\2&1\end{pmatrix}$, (d) $\begin{pmatrix}5&3\\3&-2\end{pmatrix}$, (e) $\begin{pmatrix}1&-1\\-1&3\end{pmatrix}$, (f) $\begin{pmatrix}1&1\\-1&2\end{pmatrix}$.

In the positive definite cases, write down the formula for the associated inner product.

A) $A = \begin{bmatrix}1&0\\0&2\end{bmatrix}$

$\det(A-\lambda I) = \det\left(\begin{bmatrix}1&0\\0&2\end{bmatrix}\right) - \lambda\begin{bmatrix}1&0\\0&2\end{bmatrix}$

$0 = (1-\lambda)(2-\lambda) - 0$

$\lambda = 1, 2$

E) $A = \begin{bmatrix}1&-1\\-1&3\end{bmatrix}$

$\det(A-\lambda I)=\det\left(\begin{bmatrix}1&-1\\-1&3\end{bmatrix}\right)-\lambda\begin{bmatrix}1&0\\0&2\end{bmatrix}$

$0=(1-\lambda)(3-\lambda)-1$

$\lambda=2\pm\sqrt{2}$

positive

3.4.22. (*a*) Find the Gram matrix corresponding to each of the following sets of vectors using the Euclidean dot product on $\mathbb{R}^n$.

(*iii*) $\begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}-3\\0\\2\end{pmatrix}$.

(*b*) Which are positive definite? (*c*) If the matrix is positive semi-definite, find all its null directions.

$V_1=\begin{pmatrix}2\\1\\-1\end{pmatrix} \qquad V_2=\begin{pmatrix}-3\\0\\2\end{pmatrix}$

gram matrix of V_1, V_2 is $\begin{pmatrix}\langle V_1,V_1\rangle & \langle V_1,V_2\rangle\\ \langle V_2,V_1\rangle & \langle V_2,V_2\rangle\end{pmatrix}$

NOW, $\langle V_1,V_1\rangle = V_1V_1^T = 2\times2+1\times1+(-1)(-1)$

$=6$

$\langle V_1,V_2\rangle = V_1V_2^T = 2\times-3+1\times0+(-1)\times2$

$=-6-2$

$=-8$

$\langle V_2,V_1\rangle = V_2V_1^T = -3\times2+0\times1+2\times-1$

$=-8$

$\langle V_2,V_2\rangle = V_2V_2^T = -3\times-3+0+2\times2$

$=13$

$\underset{A}{G}=\begin{pmatrix}6&-8\\-8&13\end{pmatrix} > 0$

positive definite

A is positive if and only if

det A = positive and $a>0$

3.5.1. Are the following matrices are positive definite?

(*d*) $\begin{pmatrix}1&1&1\\1&2&-2\\1&-2&4\end{pmatrix}$,

NO, is not a positive definite matrix

3.5.1. Are the following matrices are positive definite?

$$(e)\quad \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$

yes, its a positive definite matrices

3.5.2. Find an LDL^T factorization of the following symmetric matrices. Which are positive definite?

$$(f)\quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

$l_{11} = \sqrt{a_{11}}$
$= \sqrt{1}$
$= 1$

$l_{21} = \frac{a_{21}}{l_{11}}$
$= 1$

$l_{31} = \frac{a_{31}}{l_{11}}$
$= 1$

$l_{41} = \frac{a_{41}}{l_{11}}$
$= 0$

$l_{22} = \sqrt{a_{22} - l_{21}^2}$
$= 1$

$l_{32} = \frac{a_{32} - l_{31}l_{21}}{l_{22}}$
$= \frac{0 - 1 \times 1}{1}$
$= -1$

$l_{42} = \frac{a_{42} - l_{41}l_{21}}{l_{22}}$
$= \frac{1 - 0 \times 1}{1}$
$= 1$

$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$
$= \sqrt{1 - 1 - (-1)^2}$
$= \sqrt{-1}$

$l_{43} = \frac{a_{43} - l_{41}l_{31} - l_{42}l_{32}}{l_{33}}$
$= \frac{2}{\sqrt{-1}}$

$l_{44} = \sqrt{a_{44} - l_{41}^2 - l_{42}^2 - l_{43}^2}$
$= \sqrt{2 - 0 - 1 - \left(\frac{2}{\sqrt{-1}}\right)^2}$
$= \sqrt{5}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & \sqrt{-1} & 0 \\ 0 & 1 & \frac{2}{\sqrt{-1}} & \sqrt{5} \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & 0 \\ 0 & 0 & 0 & \sqrt{5} \end{bmatrix} \quad L^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & \sqrt{-1} & \frac{2}{\sqrt{-1}} \\ 0 & 0 & 0 & \sqrt{5} \end{bmatrix}$$

3.5.19. Find the Cholesky factorizations of the following matrices:

$$(d) \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1.4142 \\ 0.7071 \\ 0.7071 \end{pmatrix} \begin{pmatrix} 1.4142 & 0.7071 & 0.7071 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1.5 & 0.5 \\ 0 & 0.5 & 1.5 \end{pmatrix}$$