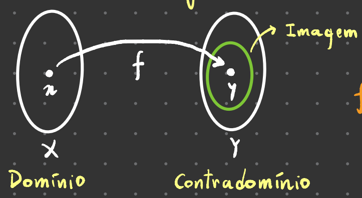
Cálculo Variacional

→ sistema evolui de forma "econômica"

* **Função**: mapeamento entre dois conjuntos numéricos



Dados $x \in X$, $y \in Y$:

$f: X \mapsto Y$ ou $X \xmapsto{f} Y$

tal que $y = f(x)$

1) $y = e^x$ → Domínio: $X \in \mathbb{R}$ | Contradomínio: $Y = \mathbb{R}^+$

2) $y = \operatorname{sen} x$ → Domínio: $X \in \mathbb{R}$ | Contradomínio: $Y = [-1, 1]$

3) $y = \frac{1}{x}$ → Domínio: $X = \mathbb{R}^*$ ou $X = \mathbb{R} \setminus \{0\}$

4) $y = \sqrt{x}$ → Domínio: $X = \mathbb{R}^+ \cup \{0\}$

⋮

* **Classe funcional**: Se X e Y são conjuntos numéricos, então o conjunto de todas as funções $f: X \mapsto Y$, denotado por $\mathcal{M}$ ou Y^X, forma uma classe funcional

↳ Ex.: $y = a + bx + cx^2$, $a, b, c \in \mathbb{R}$, $X = Y = \mathbb{R}$

* Exemplos:

- $\mathcal{C}[a,b]$: conjunto de todas as funções $f: \mathbb{R} \mapsto \mathbb{R}$ contínuas no intervalo fechado $[a, b]$
  ↳ $\mathcal{C}(a,b)$, $\mathcal{C}(a,b]$ etc.

- $\mathcal{C}^r[a,b]$: conjunto das funções cuja derivada de ordem $r \geq 0$ é contínua em $[a,b]$

- $\mathcal{C}^\infty(\mathbb{R})$: funções suaves

* **Funcional**: mapeamento J que associa cada função $y(x) \in \mathcal{M}$ a um elemento do conjunto numérico X, denotado por $J\{y(x)\}$

* Exemplo: Dada a classe $\mathcal{C}[0,1]$ e o conjunto $\mathbb{R}$

$$y(x) \in \mathcal{C}[0,1] \longrightarrow J\{y(x)\} \in \mathbb{R}$$

Pela operação $J\{y(x)\} = \int_0^1 y(x)\,dx$

a) $y(x) = 1 \Rightarrow J\{1\} = \int_0^1 1\,dx = 1$

b) $y(x) = e^x \Rightarrow J\{e^x\} = \int_0^1 e^x\,dx = e - 1$

* Exemplo 1.3: $y(x) \in \mathcal{C}[-1, 1]$ outro funcional

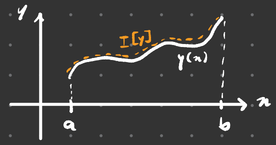
$$I[y] = \int_{-1}^{1} G\{y(x); x\}\, dx$$

↳ $G\{y(x); x\} = \dfrac{x}{1+y^2} \longrightarrow I[y] = \displaystyle\int_{-1}^{1} \frac{x}{1+y^2}\, dx$

↳ Se $y(x) = 1 + x$: $I = \displaystyle\int_{-1}^{1} \frac{x}{1+(1+x)^2}\, dx = \ln\sqrt{5} - \text{tg}^{-1} 2$

* Exemplo 1.4 (extensão de uma curva):

$y(x) \in \mathcal{C}^1[a, b] \longrightarrow I[y] = \displaystyle\int_a^b \sqrt{1 + (y')^2}\, dx$



Problema básico do Cálculo Variacional

→ Qual a curva de mínima extensão entre a e b?

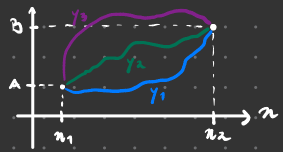
→ Dados $y(x) \in \mathcal{C}^2[x_1, x_2]$ e o funcional $J[y] = \int_{x_1}^{x_2} G\{y(x); y'(x); x\}\, dx$ (↱ var. dep.; ↱ var. indep.), para $G\{y(x); y'(x); x\}$ dado, e de tal forma que $J[y]$ seja um extremum — ou seja, que resulte em um valor mínimo ou máximo frente a qualquer outra função $\tilde{y}(x) \in \mathcal{C}^2[x_1, x_2]$

→ Condições:

1) $G = G\{y; y'; x\}$ (condição "fraca": maioria dos problemas da Mecânica Clássica)
2) Limites de integração são fixos, i.e.

$$\tilde{y}(x_1) = A \in \mathbb{R}$$
$$\tilde{y}(x_2) = B \in \mathbb{R}$$

para quaisquer $\tilde{y} \in \mathcal{C}^2[x_1, x_2]$



* Solução fraca:

→ Escrevendo $y(x,\alpha) = y(x) + \alpha\eta(x)$, onde
$$\begin{cases} y(x) \text{ é o extremum} \\ \alpha \in \mathbb{R} \\ \eta(x) \in \mathcal{C}^2[x_1, x_2] \text{ arbitrário, mas tal que } \eta(x_1) = \eta(x_2) = 0 \end{cases}$$

→ $J(\alpha) = \int_{x_1}^{x_2} G\{y(x,\alpha); \underbrace{y'(x,\alpha)}_{y'(x,\alpha) = \frac{dy(x,\alpha)}{dx}}; x\}\, dx$

→ Portanto, o extremum será dado por $\frac{\partial J}{\partial \alpha} = 0$

→ Regra integral de Leibniz: $g(c) = \int_{a(c)}^{b(c)} f(x,c)\, dx$

$$\frac{d}{dc}\int_{a(c)}^{b(c)} f(x,c)\, dx = \int_a^b \frac{\partial f}{\partial c}\, dx + f(b,c)\frac{db}{dc} - f(a,c)\frac{da}{dc}$$

→ Equação de Euler: $\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} G\, dx = \int_{x_1}^{x_2} \frac{\partial G}{\partial \alpha}\, dx = 0$

$$\frac{\partial G}{\partial \alpha} = \frac{\partial}{\partial \alpha} G\{y(x,\alpha); y'(x,\alpha); x\} = \frac{\partial G}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial G}{\partial y'}\frac{\partial y'}{\partial \alpha} + \frac{\partial G}{\partial x}\overset{0}{\cancel{\frac{\partial x}{\partial \alpha}}}$$

$$\begin{cases} y(x) = y(x,\alpha) + \alpha\eta(x) \\ y'(x) = y'(x,\alpha) + \alpha\eta'(x) \end{cases}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial G}{\partial y}\overbrace{\frac{\partial y}{\partial \alpha}}^{\eta(x)} + \frac{\partial G}{\partial y'}\overbrace{\frac{\partial y'}{\partial \alpha}}^{\eta'(x)}\right)dx = \int_{x_1}^{x_2}\left(\frac{\partial G}{\partial y}\eta(x) + \frac{\partial G}{\partial y'}\eta'(x)\right)dx$$

Por partes: $\int_{x_1}^{x_2} \frac{\partial G}{\partial y'}\eta'(x)\, dx = \overset{0}{\cancel{\frac{\partial G}{\partial y'}\eta\Big|_{x_1}^{x_2}}} - \int_{x_1}^{x_2}\frac{d}{dx}\frac{\partial G}{\partial y'}\eta\, dx$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'}\right)\eta(x)\, dx = 0$$

Mas, como $\eta(x)$ é arbitrária em (x_1, x_2) ⇒ $\boxed{\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0}$ Equação de Euler

↳ Resulta em uma EDO de 2ª ordem para o extremum $y(x)$

11/10/23

* Exemplo: menor distância

→ Comprimento de arco: $I[y] = \int_a^b \sqrt{1+y'}\, dx$

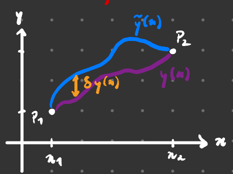
→ $G\{y(x); y'(x); x\} = \sqrt{1+y'^2}$

→ $\frac{\partial G}{\partial y} = 0 \;\Big|\; \frac{\partial G}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \;\Big|\; \frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0$

→ $\frac{d}{dx}\left[\frac{1}{\sqrt{1+y'^2}}\right] = 0 \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \alpha = cte \Rightarrow y'^2 = \alpha^2(1+y'^2) \Rightarrow y'^2 = \frac{\alpha^2}{1-\alpha^2} \equiv \beta^2 \Rightarrow y' = \beta$

$\therefore \boxed{y(x) = \beta x + \delta}$

* Solução forte:



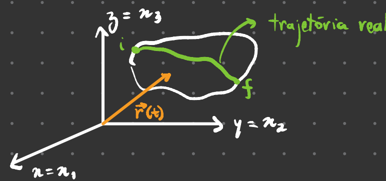
$\delta y(x) = \tilde{y}(x) - y(x)$ (variação)

$\Delta J[y, \delta y] \equiv J[\tilde{y}] - J[y] \Rightarrow \Delta J = J[y+\delta y] - J[y]$

16/10/23

Mecânica Lagrangiana

* Princípio de Hamilton (1834): Dentre todas as possíveis trajetórias ao longo das quais um sistema dinâmico pode se mover entre dois pontos, dentro de um determinado intervalo de tempo e de forma consistente com os vínculos impostos ao mesmo, a trajetória real seguida é aquela que minimiza a integral no tempo da diferença entre as energias cinética e potencial do sistema.



* Sistema físico (N = 1):
- ↳ Energia cinética: $T = T(\dot{\vec{r}}) = T(\dot{x}, \dot{y}, \dot{z})$
- ↳ Energia potencial: $U = U(\vec{r}) = U(x, y, z)$

* Lagrangiana: $\boxed{L = T(\dot{\vec{r}}) - U(\vec{r}) = L\{\vec{r}, \dot{\vec{r}}\}}$

↳ var. independente

* Integral de ação: $\boxed{S\{\vec{r}, \dot{\vec{r}}\} = \int_{t_1}^{t_2} L\{\vec{r}, \dot{\vec{r}}\}\, dt = \int_{t_1}^{t_2} [T(\dot{\vec{r}}) - U(\vec{r})]\, dt}$

↳ var. dependentes

→ Trajetória real: $\delta S = 0$

$$\delta S = \delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \delta L\, dt \Rightarrow \boxed{\delta L = 0}$$

* Equações de movimento de Euler-Lagrange: $\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \quad (i = 1, 2, 3)}$

* Mecânica Lagrangiana → Mecânica Newtoniana

$$T(\dot{\vec{r}}) = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

$$U(\vec{r}) = U(x_1, x_2, x_3)$$

$$L\{x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3\} = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - U(x_1, x_2, x_3)$$

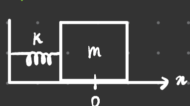
$$\rightarrow \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \quad \Bigg| \quad \frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i} \quad \Bigg| \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{d}{dt}(\overset{\text{cte}}{m}\dot{x}_i) = m\ddot{x}_i$$

$$\rightarrow \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \Rightarrow \boxed{-\frac{\partial U}{\partial x_i} - m\ddot{x}_i = 0 \quad (i = 1, 2, 3)}$$

} sistema de 3 EDOs de 2ª ordem para $x_i = x_i(t)$ $(i = 1, 2, 3)$

$$F_i(\vec{r}) = -\frac{\partial U}{\partial x_i} \Rightarrow -m\ddot{x}_i = F_i(\vec{r}) \Rightarrow \boxed{\vec{F} = m\vec{a}}$$

* Exemplo: Oscilador Harmônico 1D



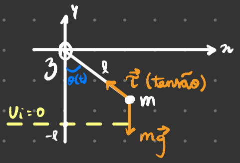
$$\left.\begin{aligned} T &= \frac{1}{2} m \dot{x}^2 \\ U &= \frac{1}{2} K x^2 \end{aligned}\right\} L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} K x^2$$

$$\frac{\partial L}{\partial x} = -Kx$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

$$\rightarrow \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \Rightarrow -Kx - m\ddot{x} = 0 \Rightarrow \boxed{\ddot{x} + \overset{\omega_0^2}{\left(\frac{K}{m}\right)} x = 0} \rightarrow x(t) = x_0 \operatorname{sen}(\omega_0 t)$$

* Exemplo: Pêndulo plano



Energia cinética: $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

Energia potencial: $U_g(y) = mg(\ell+y)$

$\vec{\tau} = ?$ → força de vínculo

↳ Vínculos: $\begin{cases} z = 0 & \text{(pêndulo plano)} \\ x^2(t)+y^2(t) = \ell^2 & \text{(vínculo geométrico / holônomo)} \end{cases}$

↳ Sistema físico: nº partículas: $N = 1$; nº coordenadas: $3N = 3$; nº vínculos: $m = 2$ } nº de graus de liberdade: $\boxed{n = 3N - m = 1}$ → 1 eq. movimento

↳ Transformação de coordenadas: cartesiano → plano-polar

$$\begin{cases} x(t) = \ell \,\text{sen}\,\theta(t) \\ y(t) = -\ell\cos\theta(t) \end{cases} \Rightarrow \begin{matrix} \dot{x}(t) = \ell\dot{\theta}\cos\theta \\ \dot{y}(t) = \ell\dot{\theta}\,\text{sen}\,\theta \end{matrix}$$

Vínculo: $r(t) = \ell = cte$

$$T = \frac{1}{2}m(\ell^2\dot{\theta}^2\cos^2\theta + \ell^2\dot{\theta}^2\text{sen}^2\theta) \Rightarrow T = \frac{1}{2}m\ell^2\dot{\theta}^2$$

$$U = mg(\ell+y) \Rightarrow U(\theta) = mg\ell(1-\cos\theta)$$

$$L\{\theta,\dot{\theta}\} = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta) \quad \begin{cases} \frac{\partial L}{\partial\theta} = -mg\ell\,\text{sen}\,\theta \\ \frac{\partial L}{\partial\dot{\theta}} = m\ell^2\dot{\theta} \end{cases}$$

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 0 \Rightarrow -\cancel{m}g\cancel{\ell}\,\text{sen}\,\theta - \cancel{m}\ell^{\cancel{2}}\ddot{\theta} = 0 \Rightarrow \boxed{\ddot{\theta} + \frac{g}{\ell}\,\text{sen}\,\theta = 0}$$

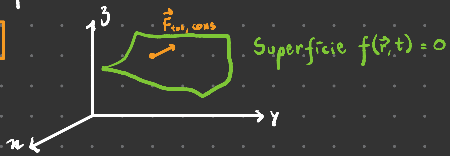
Vínculos na Mecânica Clássica

→ nº de partículas: N
→ nº de coordenadas: $3N$
→ nº de equações de vínculo: $m < 3N$
↳ $f_\ell(\vec{r}_1, \dots, \vec{r}_N; t) = 0 \quad (\ell = 1, \dots, m)$

nº de graus de liberdade: $n = 3N - m$

* Vínculos holônomos ou integráveis: dependem somente das coordenadas e do tempo

$$\boxed{f_\ell(\vec{r}_1, \dots, \vec{r}_N; t) = 0 \quad (\ell = 1, \dots, m)}$$



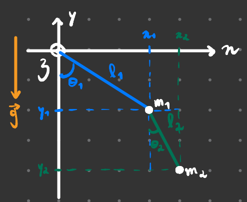
Ex.: $x^2 + y^2 + z^2 - R^2 = 0$ (esfera)

a) Vínculos fixos ou escleronômos: $f_\ell = f_\ell(\vec{r}_1, \dots, \vec{r}_N) = 0 \quad (\ell = 1, \dots, m)$

b) Vínculos reônomos: $f_\ell = f_\ell(\vec{r}_1, \dots, \vec{r}_N; t) = 0$

* Vínculos não holônomos: dependem também das velocidades: $\boxed{f_\ell = f_\ell(\{\vec{r}_i\}, \{\dot{\vec{r}}_i\}; t) = 0 \quad (\ell = 1, \dots, m)}$ → necessariamente se usa multiplicadores de Lagrange

* Ex.: Pêndulo duplo (plano):

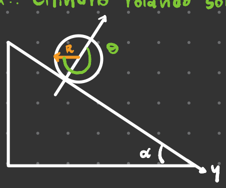


$$\left\{\begin{array}{l} x_1^2 + y_1^2 - \ell_1^2 = 0 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - \ell_2^2 = 0 \\ z_1 = z_2 = 0 \end{array}\right\} \to \text{holônomos, escleronômos}$$

$m = 4$
$N = 2 \to 3N = 6$
$n = 3N - m = 2$

Coordenadas plano polares: $x_1 = \ell_1 \operatorname{sen}\theta_1$, $y_1 = -\ell_1 \cos\theta_1$ | $x_2 = \ell_1 \operatorname{sen}\theta_1 + \ell_2 \operatorname{sen}\theta_2$

* Ex.: Cilindro rolando sobre plano inclinado:



$$g(\dot{y}, \dot{\theta}) = \dot{y} - R\dot{\theta} = 0$$
$$\int (\dot{y} - R\dot{\theta})\, dt = 0$$
$$\boxed{y - R\theta + C = 0} \to \text{vínculo integrável / holônomo}$$

Coordenadas Generalizadas

→ sistema cartesiano $\longmapsto$ sistema de coordenadas generalizadas

→ N partículas, 3N coordenadas

$x_{\alpha,i} = x_{\alpha,i}(t) \rightarrow$ i-ésima coordenada da α-ésima partícula $\begin{cases} i, j, k, \dots : \text{coordenadas } (i, j, k, \dots = 1, 2, 3) \\ \alpha = 1, \dots, N: \text{ partículas} \end{cases}$

$L = T(\dot{\vec{r}}_1, \dots, \dot{\vec{r}}_N) - U(\vec{r}_1, \dots, \vec{r}_N) = T(\{\dot{x}_{\alpha,i}\}) - U(\{x_{\alpha,i}\})$

$L = L\{\vec{r}_1, \dots, \vec{r}_N, \dot{\vec{r}}_1, \dots, \dot{\vec{r}}_N\}$

* Vínculos holônomos: $q_j = q_j(\{\vec{r}_i\}, t) \quad (j = 1, \dots, n)$

* Lei de transformação: $x_{\alpha,i} = x_{\alpha,i}(q_1, \dots, q_n, t) = x_{\alpha,i}(\{q_j\}, t) = x_{\alpha,i}(\vec{q}, t)$

$\vec{q} = (q_1, q_2, \dots, q_n)$

* Transformações inversas: $q_j = q_j(\vec{r}_1, \dots, \vec{r}_N, t) \quad (j = 1, \dots, n)$

* Velocidades generalizadas: $\dot{q}_j = \dfrac{dq_j}{dt}$ (j-ésima velocidade generalizada)

$$\begin{cases} \dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(\vec{q}, \dot{\vec{q}}, t) \rightarrow \alpha = 1, \dots, N;\ i = 1, 2, 3 \\ \dot{q}_j = \dot{q}_j(\{\vec{r}_\alpha\}, \{\dot{\vec{r}}_\alpha\}, t) \end{cases}$$

* Equações de Euler-Lagrange (holônomos):

$$\left.\begin{aligned} T &= T(\{\dot{\vec{r}}_\alpha\}) = T(\vec{q}, \dot{\vec{q}}, t) \\ U &= U(\{\vec{r}_\alpha\}) = U(\vec{q}, t) \end{aligned}\right\} \quad L = L\{\vec{q}, \dot{\vec{q}}, t\} = T\{\vec{q}, \dot{\vec{q}}, t\} - U\{\vec{q}, t\}$$

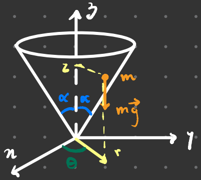
$$S = \int_{t_1}^{t_2} L\,dt \rightarrow \delta S = 0 \Rightarrow \boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0} \quad (j = 1, \dots, n)$$

→ $\left.\begin{array}{l} \text{N partículas} - \text{3N coordenadas} \\ m \text{ vínculos holônomos } (m < 3N) \end{array}\right\}$ $n = 3N - m$ graus de liberdade

→ $\begin{cases} x_{\alpha,i} = x_{\alpha,i}(q_1, \dots, q_n, t) = x_{\alpha,i}(\vec{q}, t) & (\alpha = 1, \dots, N \mid i = 1, 2, 3) \\ q_j = q_j(\vec{r}_1, \dots, \vec{r}_N, t) & (j = 1, \dots, n) \end{cases}$

→ $\begin{cases} \dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(\vec{q}, \dot{\vec{q}}, t) \\ \dot{q}_j = \dot{q}_j(\{\vec{r}_\alpha\}, \{\dot{\vec{r}}_\alpha\}, t) \end{cases} \Rightarrow \boxed{\dfrac{\partial L}{\partial q_j} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} = 0}$

* Exercício 1.6



$$L\{\vec{q},\dot{\vec{q}},t\} = T(\vec{q},\dot{\vec{q}},t) - U(\vec{q},t)$$

$N=1,\ m=1 \rightarrow n=2$ | Eq. vínculo: $z = r\cot g\alpha \rightarrow \dot{z} = \dot{r}\cot g\alpha$

$\{x,y,z\} \rightarrow \{r,\theta,z\}$ | $q_1 = r,\ q_2 = \theta$

↳ $r = \sqrt{x^2+y^2}$

$\theta = \arctan(y/x)$

$z = z$

$$\rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2\operatorname{cosec}^2\alpha + r^2\dot{\theta}^2)$$

$$\rightarrow U = mgz = mgr\cot g\alpha$$

$$\rightarrow L\{r,\dot{r},\dot{\theta}\} = \frac{1}{2}m(\dot{r}^2\operatorname{cosec}^2\alpha + r^2\dot{\theta}^2) - mgr\cot g\alpha$$

$$\begin{cases} \dfrac{\partial L}{\partial r} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} = 0 \\ \dfrac{\partial L}{\partial \theta} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} = 0 \end{cases}$$

↳ $\dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cot g\alpha$ | $\dfrac{\partial L}{\partial \theta} = 0$

$\dfrac{\partial L}{\partial \dot{r}} = m\dot{r}\operatorname{cosec}^2\alpha$ | $\dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

↳ $\cancelto{0}{\dfrac{\partial L}{\partial \theta}} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} = 0 \Rightarrow \boxed{mr^2\dot{\theta} = l_z = cte}$

↳ $\dfrac{\partial L}{\partial r} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} = 0 \Rightarrow \cancel{m}r\dot{\theta}^2 - \cancel{m}g\cot g\alpha - \dfrac{d}{dt}\cancel{m}\dot{r}\operatorname{cosec}^2\alpha = 0$

$r\dot{\theta}^2 - g\cot g\alpha - \ddot{r}\operatorname{cosec}^2\alpha = 0 \quad \times(-\operatorname{sen}^2\alpha)$

$$\boxed{\ddot{r} - r\dot{\theta}^2\operatorname{sen}^2\alpha + g\operatorname{sen}\alpha\cos\alpha = 0}$$

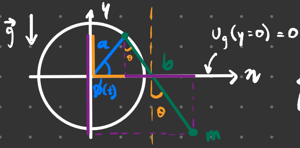
↳ Solução para $r = R = cte$:

$$\left.\begin{matrix} \dot{r} = \ddot{r} = 0 \\ \dot{\theta} = \dfrac{l_z}{mR^2} = cte \end{matrix}\right\} -R\frac{l_z^2}{m^2R^4}\operatorname{sen}^2\alpha + g\operatorname{sen}\alpha\cos\alpha = 0 \Rightarrow R^3 = \frac{l_z^2\operatorname{tg}\alpha}{m^2g}$$

↳ Solução para $l_z = 0 \Rightarrow \dot{\theta} = 0 \Rightarrow \theta = cte$

$$\ddot{r} + g\operatorname{sen}\alpha\cos\alpha = 0 \Rightarrow \ddot{r} = -g\operatorname{sen}\alpha\cos\alpha \Rightarrow r(t) = -\frac{1}{2}g\operatorname{sen}\alpha\cos\alpha\, t^2$$

* Exercício 1.7



$$\phi(t) = \omega t$$

$$\begin{cases} x(t) = a\cos\phi + b\,\text{sen}\,\theta \\ y(t) = a\,\text{sen}\,\phi - b\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x} = -a\omega\,\text{sen}\,\phi + b\dot{\theta}\cos\theta \\ \dot{y} = a\omega\cos\phi + b\dot{\theta}\,\text{sen}\,\theta \end{cases}$$

$$x^2+y^2 = a^2+b^2+2ab\,\text{sen}(\theta-\phi)$$

→ $n=1$ → eq movimento para $\theta=\theta(t)$

→ $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m(a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\omega\dot{\theta}\,\text{sen}(\theta-\phi))$

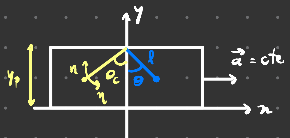
→ $U = mgy = mg(a\,\text{sen}\,\phi - b\cos\theta)$

$$\Rightarrow L\{\theta,\dot{\theta},t\} = \frac{1}{2}m\left[a^2\omega^2+b^2\dot{\theta}^2+2ab\omega\dot{\theta}\,\text{sen}(\theta-\phi)\right] - mg(a\,\text{sen}\,\phi - b\cos\theta)$$

$$\frac{\partial L}{\partial\theta} = -mb\left[g\,\text{sen}\,\theta - a\omega\dot{\theta}\cos(\theta-\phi)\right] \quad \Bigg| \quad \frac{\partial L}{\partial\dot{\theta}} = mb\left[b\dot{\theta} + a\omega\,\text{sen}(\theta-\phi)\right]$$

$$\therefore \boxed{b\ddot{\theta} + g\,\text{sen}\,\theta - a\omega^2\cos(\theta-\omega t) = 0}$$

* Exercício 1.8:



$$\begin{cases} y = y_p - \ell\cos\theta \\ x = x_p(t) + \ell\,\text{sen}\,\theta = x_0 + v_0 t + \frac{1}{2}at^2 + \ell\,\text{sen}\,\theta \end{cases}$$

$$\begin{cases} \dot{y} = \ell\dot{\theta}\,\text{sen}\,\theta \\ \dot{x} = v_0 + at + \ell\dot{\theta}\cos\theta \end{cases}$$

$$U(y) = mg(y-y_p)$$

$$L\{\theta,\dot{\theta},t\} = \frac{1}{2}m\left[(v_0+at+\ell\dot{\theta}\cos\theta)^2 + \ell^2\dot{\theta}^2\,\text{sen}^2\theta\right] + mg\ell\cos\theta$$

$$\therefore \boxed{\ddot{\theta} + \frac{g}{\ell}\,\text{sen}\,\theta + \frac{a}{g}\cos\theta = 0}$$

↳ Ângulo de equilíbrio $\theta = \theta_c = \text{cte} \neq 0$

$$\dot{\theta} = \ddot{\theta} = 0$$

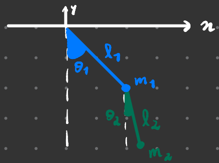
$$\frac{g}{\ell}\,\text{sen}\,\theta_c + \frac{a}{g}\cos\theta_c = 0 \Rightarrow \text{tg}\,\theta_c = -\frac{a}{g} \Rightarrow \theta_c = -\arctan\left(\frac{a}{g}\right)$$

↳ Frequência de pequenas oscilações:

$$\eta(t) = \theta(t) - \theta_c \Rightarrow \theta = \eta + \theta_c \Rightarrow \ddot{\eta} + \frac{1}{\ell}\sqrt{a^2+g^2}\,\text{sen}\,\eta = 0$$
$$\ddot{\eta} = \ddot{\theta}$$

$$|\eta| \ll 1 \Rightarrow \eta \approx \text{sen}\,\eta \Rightarrow \ddot{\eta} + \omega_0^2\eta = 0 \Rightarrow \omega_0^2 = \frac{1}{\ell}\sqrt{a^2+g^2}$$

* Exercício 1.11 (Pêndulo duplo)



$$\begin{cases} x_1^2 + y_1^2 = l_1^2 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2 \end{cases} \longrightarrow \begin{array}{l} N = 2 \rightarrow 2N = 4 \text{ coordenadas} \\ m = 2 \rightarrow n = 2 \text{ g.l.} \end{array}$$

$$\begin{cases} x_1 = l_1 \,\text{sen}\,\theta_1 \\ x_2 = l_1 \,\text{sen}\,\theta_1 + l_2 \,\text{sen}\,\theta_2 \\ y_1 = -l_1 \cos\theta_1 \\ y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \end{cases}$$

$$T = \frac{1}{2}\left[m_1 (\dot{x}_1 + \dot{y}_1)^2 + m_2 (\dot{x}_2 + \dot{y}_2)^2\right] = \frac{1}{2}(m_1 + m_2)\, l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

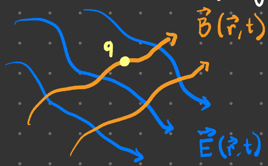
$$U = -(m_1 + m_2)\, g\, l_1 \cos\theta_1 - m_2 g\, l_2 \cos\theta_2$$

$$L = L\{\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2\}$$

$$\begin{cases} (m_1 + m_2)\, l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\, \ddot{\theta}_2 + m_2 l_1 l_2 \,\text{sen}(\theta_1 - \theta_2)\, \dot{\theta}_2^2 + (m_1 + m_2)\, g\, l_1 \,\text{sen}\,\theta_1 = 0 \\ m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\, \ddot{\theta}_1 - m_2 l_1 l_2 \,\text{sen}(\theta_1 - \theta_2)\, \dot{\theta}_1^2 + m_2 g\, l_2 \,\text{sen}\,\theta_2 = 0 \end{cases}$$

Forças e potenciais generalizados em sistemas holonômicos

* Exemplo: Lagrangiana da carga elétrica submetida a um campo eletromagnético (Unidades gaussianas)



$$\vec{F}_L(\vec{r},\vec{v},t) = q\left(\vec{E}(\vec{r},t) + \frac{1}{c}\vec{v}\times\vec{B}(\vec{r},t)\right)$$

Potencial escalar elétrico: $\phi(\vec{r},t)$ → Se $\phi = \phi(\vec{r}) \Rightarrow \vec{E}(\vec{r}) = -\nabla\phi$

Potencial vetor: $\vec{B}(\vec{r},t) = \nabla\times\vec{A}(\vec{r},t)$

→ Lei de Gauss do magnetismo: $\nabla\cdot\vec{B}(\vec{r},t) = 0$

→ Campo elétrico: $\vec{E}(\vec{r},t) = -\nabla\phi(\vec{r},t) - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

→ Transformações de calibre (gauge): $\nabla\times(\nabla\phi(\vec{r},t)) = 0$

$$\vec{A} \xrightarrow[T.C.]{} \vec{A} + \nabla\psi \Rightarrow \phi \to \phi + \frac{\partial\psi}{\partial t}$$

$$\Rightarrow \vec{F}_L = q\left[-\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} + \frac{1}{c}\vec{v}\times(\nabla\times\vec{A})\right]$$

$$\vec{v}\times(\nabla\times\vec{A}) = \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}$$

$$\vec{F}_L = -q\nabla\left[\phi - \frac{1}{c}\vec{v}\cdot\vec{A}\right] - \frac{q}{c}\left[\frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A}\right]$$

→ Ao longo da trajetória de q, $\vec{A}$ varia $\begin{cases}\text{Dependência explícita em } t \\ \text{Inomogeneidade de } \vec{A}\end{cases}$

→ $\vec{A} = \vec{A}(\vec{r}(t), t)$ → $\vec{A} = \vec{A}_0 \operatorname{sen}(kx(t) - \omega t)$

→ $\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \sum_{i=1}^{3}\frac{\partial\vec{A}}{\partial x_i}\frac{dx_i}{dt}$ → dependência implícita em t

↳ dependência explícita em t

$$\Rightarrow \boxed{\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A}}$$

Derivada convectiva

→ Ou seja, a componente k de $\vec{F}_L$ é: $F_k = -q\frac{\partial}{\partial x_k}\left(\phi - \frac{1}{c}\vec{v}\cdot\vec{A}\right) - \frac{q}{c}\frac{dA_k}{dt}$ $k = 1,2,3$

→ Como $\phi = \phi(\vec{r},t)$ e $\vec{A} = \vec{A}(\vec{r},t)$: $A_k = \frac{\partial}{\partial\dot{x}_k}(\dot{\vec{r}}\cdot\vec{A}) = \frac{\partial}{\partial\dot{x}_k}(\dot{\vec{r}}\cdot\vec{A} - c\phi)$

Então $F_k = -\frac{\partial}{\partial x_k}\left(q\phi - \frac{q}{c}\vec{v}\cdot\vec{A}\right) + \frac{d}{dt}\frac{\partial}{\partial\dot{x}_k}\left(q\phi - \frac{q}{c}\dot{\vec{r}}\cdot\vec{A}\right)$

→ Potencial generalizado: $U(\vec{r},\dot{\vec{r}},t) = q\phi(\vec{r},t) - \frac{q}{c}\dot{\vec{r}}\cdot\vec{A}$

$$\therefore \boxed{F_k = -\frac{\partial}{\partial x_k}U(\vec{r},\dot{\vec{r}},t) + \frac{d}{dt}\frac{\partial}{\partial\dot{x}_k}U(\vec{r},\dot{\vec{r}},t)}$$

→ Portanto: $L = T - \underline{U(\vec{r},\dot{\vec{r}},t)}$ → potencial generalizado

→ Lagrangiana da carga elétrica no campo EM:

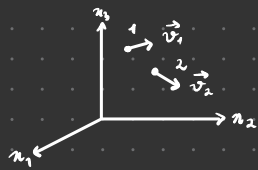
$$L\{\vec{r},\dot{\vec{r}},t\} = \frac{1}{2}m\dot{r}^2 - q\phi(\vec{r},t) + \frac{q}{c}\dot{\vec{r}}\cdot\vec{A}(\vec{r},t)$$

* Sistemas de partículas:

$$\begin{cases} N \text{ partículas} \rightarrow 3N \text{ coordenadas} \\ m \text{ vínculos holônomos} \quad (m < 3N) \\ n = 3N - m \text{ graus de liberdade} \end{cases}$$

→ Coordenadas generalizadas: $\begin{cases} q_j = q_j(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t) = q_j(\{\vec{r}_\alpha\}, t) & \begin{cases} j = 1, \dots, n \\ \alpha = 1, \dots, N \end{cases} \\ x_{\alpha,i} = x_{\alpha,i}(q_1, \dots, q_n, t) = x_{\alpha,i}(\vec{q}, t) & (i = 1, 2, 3) \end{cases}$

→ Velocidades generalizadas: $\begin{cases} \dot{q}_j = \dot{q}_j(\{\vec{r}_\alpha\}, \{\dot{\vec{r}}_\alpha\}, t) \\ \dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(\vec{q}, \dot{\vec{q}}, t) \end{cases}$



→ $T = T(\{\dot{\vec{r}}_\alpha\}) = T(\vec{q}, \dot{\vec{q}}, t)$

→ Sistema conservativo: $U = U(\{\vec{r}_\alpha\}) = U(\vec{q}, t)$

→ Ou seja: $L = T(\vec{q}, \dot{\vec{q}}, t) - U(\vec{q}, t)$

→ Equações de Euler-Lagrange: $\dfrac{\partial L}{\partial q_j} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} = 0$

$$\frac{\partial}{\partial q_j}(T - U) - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = 0$$

$$\boxed{\frac{\partial T}{\partial q_j} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial U}{\partial q_j} \equiv -F_j(\vec{q}, t)}$$

Força generalizada

→ Relação $\vec{F}_\alpha(\vec{r})\ (E^3) \leftrightarrow F_j(\vec{q}, t)\ (E^n)$

$$\vec{F}_\alpha = -\nabla_\alpha U(\{\vec{r}_\alpha\})$$

$$\vec{F}_\alpha = -\sum_{i=1}^{3} \frac{\partial U}{\partial x_{\alpha,i}} \hat{x}_i$$

$$\nabla_\alpha = \sum_{i=1}^{3} \hat{x}_i \frac{\partial}{\partial x_{\alpha,i}}$$

$$\frac{\partial \vec{r}_\alpha}{\partial q_j} \rightarrow \sum_{\alpha=1}^{N} \ ?$$

$$\sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} = -\sum_{\alpha=1}^{N} \nabla_\alpha U \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} = -\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial U}{\partial x_{\alpha,i}} \frac{\partial x_{\alpha,i}}{\partial q_j} = -\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial U}{\partial q_j}$$

$$\therefore \boxed{F_j(\vec{q}, t) = \sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j}} \quad (j = 1, \dots, n)$$

* Forças deriváveis de potenciais dependentes em velocidade e tempo

→ $U = U(\{\vec{r}_\alpha\}, \{\dot{\vec{r}}_\alpha\}, t)$ tais que $F_{\alpha,i}(\vec{r}_\alpha, \dot{\vec{r}}_\alpha, t) = -\frac{\partial U}{\partial x_{\alpha,i}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}_{\alpha,i}}$

→ Neste caso,
$$\sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} = \underbrace{-\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial U}{\partial x_{\alpha,i}}\frac{\partial x_{\alpha,i}}{\partial q_j}}_{①} + \underbrace{\sum_{\alpha=1}^{N}\sum_{i=1}^{3} \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{d}{dt}\frac{\partial U}{\partial \dot{x}_{\alpha,i}}}_{②}$$

↳ Termo ②:
$$\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{d}{dt}\frac{\partial U}{\partial v_{\alpha,i}} = \underbrace{\frac{d}{dt}\left(\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial U}{\partial v_{\alpha,i}}\right)}_{③} - \underbrace{\frac{\partial U}{\partial v_{\alpha,i}}\frac{d}{dt}\frac{\partial x_{\alpha,i}}{\partial q_j}}_{④}$$

↳ Dadas as leis de transformação: $\vec{r}_\alpha = \vec{r}_\alpha(\vec{q}, t)$ e $\vec{v}_\alpha = \vec{v}_\alpha(\vec{q}, \dot{\vec{q}}, t)$

$$\vec{v}_\alpha = \dot{\vec{r}}_\alpha = \frac{d\vec{r}_\alpha}{dt} = \frac{\partial \vec{r}_\alpha}{\partial t} + \sum_{l=1}^{n}\frac{\partial \vec{r}_\alpha}{\partial q_l}\dot{q}_l \rightarrow \boxed{\frac{\partial \vec{v}_\alpha}{\partial \dot{q}_j} = \frac{\partial \dot{\vec{r}}_\alpha}{\partial \dot{q}_j} = \frac{\partial \vec{r}_\alpha}{\partial q_j}} \quad ⑤$$

$$\frac{d}{dt}\left(\frac{\partial x_{\alpha,i}}{\partial q_j}\right) = \frac{\partial}{\partial t}\left(\frac{\partial x_{\alpha,i}}{\partial q_j}\right) + \sum_{l=1}^{n}\dot{q}_l\frac{\partial}{\partial q_l}\left(\frac{\partial x_{\alpha,i}}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{\partial x_{\alpha,i}}{\partial t} + \sum_{l=1}^{n}\dot{q}_l\frac{\partial x_{\alpha,i}}{\partial q_l}\right) \Rightarrow \boxed{\frac{d}{dt}\frac{\partial x_{\alpha,i}}{\partial q_j} = \frac{\partial v_{\alpha,i}}{\partial q_j}}$$

↳ Retornando:
$$\sum_{\alpha=1}^{N}\vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} = ① + ③ + ④ = (① + ④) + ③$$
$$= -\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left(\frac{\partial U}{\partial x_{\alpha,i}}\frac{\partial x_{\alpha,i}}{\partial q_j} + \frac{\partial U}{\partial v_{\alpha,i}}\frac{\partial v_{\alpha,i}}{\partial q_j}\right) + \sum_{\alpha=1}^{N}\sum_{i=1}^{3}\frac{d}{dt}\left(\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial U}{\partial v_{\alpha,i}}\right)$$

$$\sum_{\alpha=1}^{N}\vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j}$$

* Força generalizada: $\boxed{Q_k(\vec{q}, \dot{\vec{q}}, t) = -\frac{\partial U}{\partial q_k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_k}}$

→ Retornando: $F_k \rightarrow Q_k$: $\frac{\partial T}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} = -Q_k \Rightarrow$
$$\boxed{\begin{array}{l} \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0 \\ L\{\vec{q}, \dot{\vec{q}}, t\} = T(\vec{q}, \dot{\vec{q}}, t) - U(\vec{q}, \dot{\vec{q}}, t) \end{array}}$$

23/10/23

Forças e potenciais generalizados

→ N partículas

→ 3N coordenadas $\begin{cases} \vec{r}_\alpha; \quad \alpha = 1, \dots, N \\ \vec{r}_\alpha = \sum_i x_{\alpha,i}\, \hat{n}_i \end{cases}$

→ Coordenadas e velocidades generalizadas $\begin{cases} x_{\alpha,i} = x_{\alpha,i}(\vec{q}, t) \quad \begin{pmatrix} \alpha = 1, \dots, N \\ i = 1, 2, 3 \end{pmatrix} \\ q_j = q_j(\{\vec{r}_\alpha\}, t) \quad (j = 1, \dots, n) \end{cases}$

→ m vínculos holônomos: $f_\ell(\vec{q}, t) = 0 \quad (\ell = 1, \dots, m)$

→ $n = 3N - m$ graus de liberdade

Forças conservativas e/ou potenciais

→ $\vec{F}_\alpha = \vec{F}_\alpha(\{\vec{r}_\alpha\}, \{\vec{v}_\alpha\}, t)$ se $F_{\alpha,i} = -\dfrac{\partial U}{\partial x_{\alpha,i}} + \dfrac{d}{dt}\dfrac{\partial U}{\partial \dot{x}_{\alpha,i}}$

→ Então:

$$\sum_{\alpha=1}^{N} \vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} = Q_j(\vec{q}, \dot{\vec{q}}, t) = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} \quad (j = 1, \dots, n), \quad \text{onde } U = U(\vec{q}, \dot{\vec{q}}, t)$$

e

$$L\{\vec{q}, \dot{\vec{q}}, t\} = T(\vec{q}, \dot{\vec{q}}, t) - U(\vec{q}, \dot{\vec{q}}, t), \quad \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \quad (j = 1, \dots, n)$$

Forças dissipativas ou motrizes

→ $\vec{F}_\alpha^{(tot)} = \underbrace{\vec{F}_\alpha}_{\text{potencial}} + \overbrace{\vec{f}_\alpha}^{\text{não potencial}}$

→ Força generalizada não potencial (dissipativa/motriz): $Q_j'(\vec{q}, \dot{\vec{q}}, t) = \sum_{\alpha=1}^{N} \vec{f}_\alpha \cdot \dfrac{\partial \vec{r}_\alpha}{\partial q_j}$

→ Obtemos a força generalizada total $Q_j = -\dfrac{\partial U}{\partial q_j} + \dfrac{d}{dt}\dfrac{\partial U}{\partial \dot{q}_j} + Q_j'$

→ $\dfrac{\partial T}{\partial q_j} - \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{q}_j} = -Q_j \Rightarrow \boxed{\dfrac{\partial L}{\partial q_j} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} = -Q_j'} \quad (j = 1, \dots, n)$

* Funções dissipativas ou de Rayleigh

* Forças viscosas: $\vec{f}_\alpha = -g_\alpha(\vec{v}_\alpha)\,\vec{v}_\alpha$ $\quad \begin{cases} |\vec{v}_\alpha| \text{ pequeno} \to f_\alpha \propto v_\alpha \\ |\vec{v}_\alpha| \text{ grande} \to f_\alpha \propto v_\alpha^2 \end{cases}$ $\quad \Big|\; g(\vec{v}_\alpha) = g(v_\alpha)$ isotrópico

→ Então, $Q_j' = -\sum_{\alpha=1}^{N} g_\alpha(\vec{v}_\alpha)\,\vec{v}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j}$ $\quad (j = 1, \dots, n)$

→ $\frac{\partial \vec{r}_\alpha}{\partial q_j} = \frac{\partial \vec{v}_\alpha}{\partial \dot{q}_j} \Rightarrow Q_j' = -\sum_{\alpha=1}^{N} g_\alpha \vec{v}_\alpha \cdot \frac{\partial \vec{v}_\alpha}{\partial \dot{q}_j} = -\sum_{\alpha=1}^{N} g_\alpha \cdot v_\alpha \frac{\partial v_\alpha}{\partial \dot{q}_j}$

→ Supondo que o fluido seja isotrópico: $g_\alpha(\vec{v}_\alpha) = g_\alpha(v_\alpha)$

$$Q_j' = -\sum_{\alpha=1}^{N} g_\alpha(v_\alpha)\, v_\alpha \frac{\partial v_\alpha}{\partial \dot{q}_j} \equiv -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

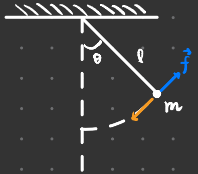
$\mathcal{F}$: função dissipativa → $\mathcal{F} = \sum_{\alpha=1}^{N} \int_0^{v_\alpha} g_\alpha(v')\, v'\, dv'$

→ as equações de Euler-Lagrange ficam $\quad \boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0} \quad (j = 1, \dots, n)$

* Regime de baixa velocidade $\forall\, \alpha$

$g_\alpha(v_\alpha) = b_\alpha = \text{cte} \Rightarrow \boxed{\mathcal{F} = \frac{1}{2}\sum_{\alpha=1}^{N} b_\alpha v_\alpha^2}$ Função de Rayleigh

* Exemplo: Pêndulo viscoso



$x^2 + y^2 = l^2$ $\quad |\quad N = 1 \to 3N = 3$ coordenadas
$z = 0$ $\quad |\quad m = 2 \Rightarrow n = 1 \to q = \theta$

$\vec{f} = -b\vec{v}$

$\mathcal{F} = \frac{1}{2} b v^2 = \frac{1}{2} b l^2 \dot{\theta}^2$

$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta \;\to\; \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial \mathcal{F}}{\partial \dot{\theta}} = 0$

$$\boxed{\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{l}\operatorname{sen}\theta = 0}$$

Tratamento geral para vínculos não holônomos

→ N partículas, 3N coordenadas (cartesianas ou generalizadas)

→ m equações de vínculo: $f_\ell(\vec{q}, \dot{\vec{q}}, t) = 0 \quad (\ell = 1, \dots, m)$

→ $n = 3N - m$ graus de liberdade

→ Multiplicadores de Lagrange

↳ ler seção 1.8.1

→ Equações de Euler-Lagrange para sistemas não holonômicos:

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = 0} \quad (j = 1, \dots, 3N), \quad \text{com} \quad \boxed{\mathcal{L}\{\vec{q}, \dot{\vec{q}}; t\} = L\{\vec{q}, \dot{\vec{q}}; t\} + \sum_{\ell=1}^{m} \underbrace{\lambda_j(t)}_{\text{mult. Lagrange}} f_\ell(\vec{q}, \dot{\vec{q}}; t)}$$

* Vínculos não holônomos: $f_\ell = f_\ell(\vec{q}, \dot{\vec{q}}; t)$

→ $\dfrac{\partial \mathcal{L}}{\partial q_j} = \dfrac{\partial L}{\partial q_j} + \sum_\ell \lambda_j \dfrac{\partial f_\ell}{\partial q_j}$

→ $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{q}_j} \rightsquigarrow \dot{\lambda}_\ell \rightarrow$ sistema de m EDOs de 1ª ordem p/ $\lambda_j(t)$

→ Condições iniciais: $q_j(t=0)$, $\dot{q}_j(t=0)$, $\underline{\lambda_j(t=0)}$ → não existe / desconhecido

* Vínculos holônomos: $f_\ell = f_\ell(\vec{q}, t)$

→ $\mathcal{L} = L\{\vec{q}, \dot{\vec{q}}; t\} + \sum_{\ell=1}^{m} \lambda_j(t) f_\ell(\vec{q}, t)$

→ $\dfrac{\partial \mathcal{L}}{\partial q_j} = \dfrac{\partial L}{\partial q_j} + \sum_\ell \lambda_j \dfrac{\partial f}{\partial q_j}$

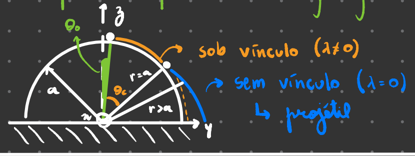
→ $\dfrac{\partial \mathcal{L}}{\partial \dot{q}_j} = \dfrac{\partial L}{\partial \dot{q}_j}$

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{\ell=1}^{m} \lambda_j(t) \frac{\partial f_\ell}{\partial q_j} = 0} \quad (j = 1, \dots, 3N)$$

3N EDOs de 2ª ordem + m equações de vínculo

3N coordenadas $\{q_j\}$ + m $\{\lambda_\ell\}$

* Exemplo: multiplicador de Lagrange com força de vínculo



assumindo $U_g(\theta = \frac{\pi}{2}) = 0$: $L = T - U_g = \frac{1}{2}mv^2 - mgz$ $\quad (N=1)$

vínculos $\begin{cases} x = 0 & (1) \\ r - a = 0 = f(r) & (2) \end{cases}$

Coordenadas esféricas

$$\begin{cases} x = r\,\text{sen}\,\theta\cos\varphi \\ y = r\,\text{sen}\,\theta\,\text{sen}\,\varphi \\ z = r\cos\theta \end{cases}$$

$q_1 = r$
$q_2 = \theta$
$q_3 = \varphi$

$f_1 = x = 0 \Rightarrow r\,\text{sen}\,\theta\cos\varphi = 0$
$f_2 = r - a = 0$

$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2 + r^2\text{sen}^2\theta\,\dot\varphi^2) - mgr\cos\theta$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} + \lambda_1\frac{\partial f_1}{\partial r} + \lambda_2\frac{\partial f_2}{\partial r} = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} + \lambda_1\frac{\partial f_1}{\partial \theta} + \lambda_2\frac{\partial f_2}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} + \lambda_1\frac{\partial f_1}{\partial \varphi} + \lambda_2\frac{\partial f_2}{\partial \varphi} = 0$$

→ assumindo $\lambda_1 = 0$, $\varphi = \text{cte}$:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} + \lambda\frac{\partial f_2}{\partial r} = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} + \lambda\frac{\partial f_2}{\partial \theta} = 0$$

$$m\ddot r - mr\dot\theta^2 + mg\cos\theta + \lambda = 0$$
$$\frac{d}{dt}(r^2\dot\theta) - gr\,\text{sen}\,\theta = 0$$

→ Enquanto $\lambda \neq 0 \Rightarrow r = a$:

$$ma\dot\theta^2 - mg\cos\theta + \lambda = 0$$
$$\ddot\theta = \frac{g}{a}\,\text{sen}\,\theta$$

$$\ddot\theta = \frac{d}{dt}\dot\theta = \frac{d\dot\theta}{d\theta}\frac{d\theta}{dt} = \dot\theta\frac{d\dot\theta}{dt} = \frac{1}{2}\frac{d\dot\theta^2}{d\theta}$$

$$\frac{1}{2}\frac{d\dot\theta^2}{d\theta} = \frac{g}{a}\,\text{sen}\,\theta \Rightarrow \frac{1}{2}\dot\theta^2 = -\frac{g}{a}\cos\theta + \alpha$$

$$t = 0 \Rightarrow \theta = \theta_0 \ll \theta_c \Rightarrow \boxed{\dot\theta^2 = \frac{2g}{a}(\cos\theta_0 - \cos\theta)}$$

$$2mg(\cos\theta_0 - \cos\theta) - mg\cos\theta + \lambda = 0$$

$$\lambda = (3\cos\theta - 2\cos\theta_0)\,mg$$

a força de vínculo se torna nula:

$$\lambda = 0 \Rightarrow \theta > \theta_c = \cos^{-1}\left(\frac{2}{3}\cos\theta_0\right)$$

$$\theta_c \approx 48{,}2°$$

Sistema físico

→ N partículas

→ 3N coordenadas $\{x_{\alpha,i}\}$
3N velocidades $\{\dot{x}_{\alpha,i} = v_{\alpha,i}\}$ $\begin{pmatrix} \alpha = 1, \dots, N \\ i = 1, 2, 3 \end{pmatrix}$

→ m equações de vínculo $f_\ell(\vec{q}, \dot{\vec{q}}, t)$ $(\ell = 1, \dots, m)$

→ $n = 3N - m$ graus de liberdade

→ $\begin{cases} x_{\alpha,i} = x_{\alpha,i}(\vec{q}, t) \\ q_j = q_j(\{\vec{r}_\alpha\}, t) \end{cases}$ $\begin{cases} \dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(\vec{q}, \dot{\vec{q}}, t) \\ \dot{q}_j = \dot{q}_j(\vec{r}_\alpha, \dot{\vec{r}}_\alpha, t) \end{cases}$

Tratamento geral (sistema não holonômico)

→ $\mathcal{L}\{\vec{q}, \dot{\vec{q}}, t\} = L\{\vec{q}, \dot{\vec{q}}, t\} + \sum_{\ell=1}^{m} \lambda_\ell(\ell) f_\ell(\vec{q}, \dot{\vec{q}}, t)$

→ $\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = 0$ $(j = 1, \dots, 3N)$ → EDOs para $\lambda(t)$ ⇝ $\lambda(t=0)$ não conhecidos

Vínculos holônomos

→ $f_\ell(\vec{q}, t) = 0$

→ $\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{\ell=1}^{m} \lambda_\ell(t) \frac{\partial f_\ell}{\partial q_j} = 0$

Lagrangianas equivalentes

* Definição: sejam duas lagrangianas $L_1\{\vec{q},\dot{\vec{q}},t\} \neq L_2\{\vec{q},\dot{\vec{q}},t\}$. Se

$$L_2 - L_1 = \frac{d}{dt} g(\vec{q},t)$$

então L_1 e L_2 são equivalentes

* Teorema: Lagrangianas equivalentes originam as mesmas equações de movimento.

Prova: $$\int_{t_1}^{t_2} L_2\, dt = \int_{t_1}^{t_2} L_1\, dt + \int_{t_1}^{t_2} \frac{dg}{dt}\, dt$$

$$S_2 = S_1 + g(\vec{q}(t_2),t) - g(\vec{q}(t_1),t)$$

Princípio de Hamilton: $\delta S = 0 \rightarrow$ extremum

$$\delta S_2 = \delta S_1 + \delta g(\vec{q}(t_2),t) - \delta g(\vec{q}(t_1),t) \Rightarrow \delta S_1 = \delta S_2$$

$$\delta \vec{q}(t_1) = \delta \vec{q}(t_2) = 0$$

mesmas eqs. Euler-Lagrange

Vínculos não holônomos

Vínculos lineares nas velocidades generalizadas

$\rightarrow$ $$f_\ell(\vec{q},\dot{\vec{q}},t) = \sum_{j=1}^{3N} a_{\ell j}(\vec{q},t)\dot{q}_j + a_{\ell t}(\vec{q},t) = 0 \quad (\ell = 1, \dots, m)$$

$\rightarrow$ Tratamento geral: $$\left.\begin{aligned} \mathcal{I} &= L + \sum_{\ell=1}^{m} \eta_\ell(t) f_\ell \\ \mathcal{L} &= L - \sum_{\ell=1}^{m} \dot{\eta}_\ell \underbrace{\int f_\ell(t)}_{\hat{f}_\ell} \end{aligned}\right\}$$ $\mathcal{I}$ e $\mathcal{L}$ são equivalentes $\rightarrow g = \sum_{\ell=1}^{m} \eta_\ell f_\ell$

$\rightarrow$ Portanto: $\hat{f}_\ell = \int f_\ell\, dt \Rightarrow f_\ell = \frac{d\hat{f}_\ell}{dt}$

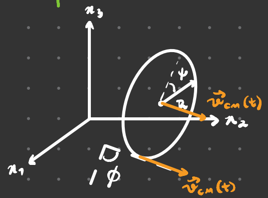
$$\lambda_\ell = -\dot{\eta}_\ell$$

$\rightarrow$ Para $\hat{f}_\ell = \sum_j a_{\ell j}\dot{q}_j + a_{\ell t}$ e para $\mathcal{L}$

$\rightarrow$ As equações de Euler-Lagrange ficam

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{\ell=1}^{m} \lambda_\ell(t) \underbrace{\frac{\partial f_\ell}{\partial \dot{q}_j}}_{a_{\ell j}(\vec{q},t)} = 0}$$

* Exemplo: Disco rolando sobre superfície plana sem tombar



Vínculo: $|\vec{v}_{cm}| = R|\dot{\psi}|$

Vínculos: $f_1 = \dot{x}_1 - R\dot{\psi}\cos\phi = 0$
$f_2 = \dot{x}_2 - R\dot{\psi}\,\text{sen}\,\phi = 0$

* Exercício 1.13: Partícula movendo-se sem atrito sobre o plano, com vínculo $f(\dot{x}, y) = \dot{x} - \overset{\text{cte}}{\omega} y = 0$

a) Mostre que o vínculo é não holônomo

→ vínculo holônomo: $\frac{d}{dt} f(x,y) = 0$

$$\frac{d}{dt} f(x,y) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} = 0 \Rightarrow \tilde{f}(x,\dot{x},y,\dot{y}) = 0$$

Se existir fator integrante $h(x,y,t)$:

$$h\dot{x} - h\omega y = 0 = \frac{d}{dt}H(x,y,t) = 0 \Rightarrow H(x,y,t) - c = 0 \rightarrow \text{holônomo}$$

Mas, se $\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial t} = 0$, então

$$\frac{\partial H}{\partial x} = h, \quad \frac{\partial H}{\partial y} = 0, \quad \frac{\partial H}{\partial t} = -\omega h y \quad \text{e} \quad \frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}, \quad \frac{\partial^2 H}{\partial t \partial x} = \frac{\partial^2 H}{\partial x \partial t}, \quad \frac{\partial^2 H}{\partial t \partial y} = \frac{\partial^2 H}{\partial y \partial t}$$

$$\rightarrow \frac{\partial^2 H}{\partial y \partial x} = \frac{\partial h}{\partial y} \quad \text{e} \quad \frac{\partial^2 H}{\partial x \partial y} = 0 \Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h = h(x,t)$$

$$\left.\begin{array}{l} \rightarrow \frac{\partial H}{\partial t} = -\omega h y = -\omega h(x,t)\, y \Rightarrow \frac{\partial^2 H}{\partial y \partial t} = -\omega h(x,t) \\ \frac{\partial H}{\partial y} = 0 \Rightarrow \frac{\partial^2 H}{\partial t \partial y} = 0 \end{array}\right\} -\omega h(x,t) = 0 \Rightarrow h = 0 \rightarrow \varnothing$$

vínculo não holônomo

b) Equações de Euler-Lagrange:

2 coordenadas: $\{x, y\} = 0 \quad (z = 0)$
1 eq. vínculo: $f = \dot{x} - \omega y = 0$

$$\begin{cases} \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda \frac{\partial f}{\partial \dot{x}} = 0 \\ \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda \frac{\partial f}{\partial \dot{y}} = 0 \end{cases}$$

$$L\{\dot{x},\dot{y}\} = T - \overset{=0}{U} = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$\rightarrow \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$

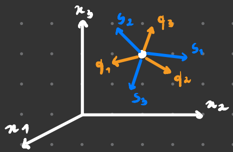
$\rightarrow \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$\rightarrow \frac{\partial f}{\partial \dot{x}} = 1, \quad \frac{\partial f}{\partial \dot{y}} = 0$

$$\Rightarrow \begin{cases} m\ddot{x} - \lambda = 0 \Rightarrow \lambda = m\ddot{x} \Rightarrow \lambda = m\omega v_0 \\ \ddot{y} = 0 \Rightarrow y(t) = y_0 + v_0 t \\ \dot{x} - \omega y = 0 \Rightarrow \dot{x} = \omega y = \omega(y_0 + v_0 t) \Rightarrow x(t) = x_0 + \left(y_0 + \frac{1}{2}v_0 t\right)\omega t \end{cases}$$

Propriedades Matemáticas das Leis de Conservação

* Invariância sob transformações de coordenadas



No sistema Q (coordenadas $\vec{q}$):

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \quad (j=1,\dots,n)$$

Transformação $Q \mapsto S$ ($\{s_j\}$)

Leis de transformação: $\begin{cases} s_k = s_k(\vec{q}, t) & (k=1,\dots,n) \\ q_j = q_j(\vec{s}, t) & (j=1,\dots,n) \end{cases}$

→ Como ficam as equações de Euler-Lagrange em S?

$$\dot{q}_j = \frac{d}{dt} q_j = \sum_{l=1}^{n} \frac{\partial q_j}{\partial s_l}\dot{s}_l + \frac{\partial q_j}{\partial t} \Rightarrow \frac{\partial \dot{q}_j}{\partial \dot{s}_k} = \frac{\partial q_j}{\partial s_k}$$

Em S: $\bar{L}\{\vec{s}, \dot{\vec{s}}, t\} = L\{\vec{s}(\vec{q}, t), \dot{\vec{s}}(\vec{q}, \dot{\vec{q}}, t); t\} \rightarrow \frac{\partial \bar{L}}{\partial s_k} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{s}_k} = ?$

$$\frac{\partial \bar{L}}{\partial s_k} = \sum_{l=1}^{n}\left(\frac{\partial L}{\partial q_l}\frac{\partial q_l}{\partial s_k} + \frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial s_k}\right)$$

$$\frac{\partial \bar{L}}{\partial \dot{s}_k} = \sum_{l=1}^{n} \frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial \dot{s}_k} = \sum_{l=1}^{n} \frac{\partial L}{\partial \dot{q}_l}\frac{\partial q_l}{\partial s_k}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{s}_k}\right) = \sum_{l=1}^{n}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l}\frac{\partial q_l}{\partial s_k} + \frac{\partial L}{\partial \dot{q}_l}\frac{d}{dt}\frac{\partial q_l}{\partial s_k}\right) = \sum_{l=1}^{n}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l}\frac{\partial q_l}{\partial s_k} + \frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial s_k}\right)$$

$$\therefore \left(\frac{\partial \bar{L}}{\partial s_k} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{s}_k}\right) = \sum_{l=1}^{n}\left(\frac{\partial L}{\partial q_l} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l}\right)\frac{\partial q_l}{\partial s_k} = 0 \quad //$$

* Momento canônico conjugado:

→ Coordenada q_j

→ Momento canônico conjugado a q_j: $p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = p_j(\vec{q}, \dot{\vec{q}}, t)$

* Constantes de movimento: $g = g(\vec{q}, \dot{\vec{q}}, t) \rightarrow \dot{g} = 0 \Rightarrow g$ é uma constante de movimento

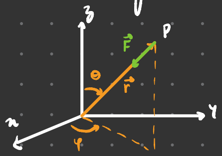
* Coordenada cíclica: Dada q_j, se $\frac{\partial L}{\partial q_j} = 0 \Rightarrow q_j$ é cíclica

Teorema: Se a coordenada q_j for cíclica, então $p_j =$ cte

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \Rightarrow \frac{d}{dt}p_j = \frac{\partial L}{\partial q_j} \overset{q_j \text{ cíclica}}{=} 0 \quad //$$

*** Exemplo:** Partícula sob força central

Sistema esférico $\{r, \theta, \varphi\}$



Força central: $U = U(r) \rightarrow \vec{F} = -\nabla U \propto -\vec{r}$

$$L\{r, \dot{r}, \theta, \dot{\theta}, \dot{\varphi}\} = T - U = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \operatorname{sen}^2\theta\, \dot{\varphi}^2\right) - U(r)$$

$$\left.\begin{aligned} &\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \varphi \text{ é cíclica} \Rightarrow p_\varphi = cte \\ &p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \overbrace{mr^2 \operatorname{sen}^2\theta\, \dot{\varphi}}^{\ell_z} = cte \end{aligned}\right\} \text{conservação do momento angular}$$

Translações e Rotações Virtuais

→ N partículas, sistema holonômico

$$\rightarrow \frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_\alpha} = 0 \quad (\alpha = 1, \dots, N) \quad \Big| \quad \mathcal{L} = L + \sum_{\ell=1}^{m} \lambda_\ell f_\ell(\vec{r}, t)$$

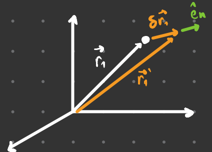
$$\rightarrow \frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha} = \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial x_{\alpha_i}}\, \hat{x}_i \quad \Big| \quad \frac{\partial \mathcal{L}}{\partial \vec{v}_\alpha} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_\alpha} = \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial \dot{x}_{\alpha_i}}\, \hat{x}_i$$

*** Transformações virtuais:**

$$\begin{aligned} \vec{r}_\alpha &\rightarrow \vec{r}_\alpha' = \vec{r}_\alpha + \delta\vec{r}_\alpha \\ \vec{v}_\alpha &\rightarrow \vec{v}_\alpha' = \vec{v}_\alpha + \delta\vec{v}_\alpha \end{aligned} \longrightarrow \begin{aligned} &\mathcal{L}\{\vec{r}_\alpha, \vec{v}_\alpha; t\} \rightarrow \mathcal{L}' = \mathcal{L}\{\vec{r}_\alpha', \vec{v}_\alpha'; t\} \\ &\delta\mathcal{L} = \mathcal{L}\{\vec{r}_\alpha', \vec{v}_\alpha'; t\} - \mathcal{L}\{\vec{r}_\alpha, \vec{v}_\alpha; t\} \end{aligned}$$

Se $\delta\vec{r}$ e $\delta\vec{v}$ forem pequenos: $\delta\mathcal{L} = \sum_{\alpha=1}^{N}\left(\frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha}\cdot\delta\vec{r}_\alpha + \frac{\partial \mathcal{L}}{\partial \vec{v}_\alpha}\cdot\delta\vec{v}_\alpha\right)$

↳ Translação virtual (rígida) → conservação do momentum linear



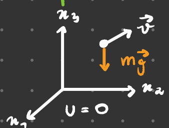
$\delta\vec{r}_\alpha = \varepsilon\,\hat{e}_n \quad (|\varepsilon| \ll 1) \quad \big| \quad \delta\vec{v}_\alpha = 0$

$\hat{e}_n$: $|\hat{e}_n| = 1$ → determina a orientação espacial da translação

Neste caso: $\delta\mathcal{L} = \sum_{\alpha=1}^{N} \frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha}\cdot(\varepsilon\,\hat{e}_n) = \varepsilon\,\hat{e}_n \cdot \sum_{\alpha=1}^{N}\frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha} = 0$ → se $\mathcal{L}$ for invariante à translação

$$\left.\begin{aligned} &\varepsilon \text{ é arbitrária não nula: } \hat{e}_n \cdot \sum_{\alpha=1}^{N}\frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha} = 0 \\ &\text{Mas } \frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \vec{v}_\alpha} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_\alpha} = \frac{d\vec{p}_\alpha}{dt} = m\dot{\vec{r}}_\alpha \end{aligned}\right\} \frac{d}{dt}\Big(\hat{e}_n \cdot \overbrace{\sum_\alpha \vec{p}_\alpha}^{\vec{P}}\Big) = 0 \Rightarrow \frac{d}{dt}(\hat{e}_n\cdot\vec{P}) = 0 \Rightarrow \boxed{\hat{e}_n\cdot\vec{P} = cte}$$

*** Exemplo:** Partícula movendo sob força peso

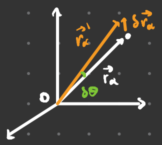


$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - mgx_3$$

L é invariante sob $\delta\vec{r} = \varepsilon(\cos\eta\, \hat{x}_1 + \operatorname{sen}\eta\, \hat{x}_2)$

$$\begin{aligned} p_1 &= m\dot{x}_1 = cte \\ p_2 &= m\dot{x}_2 = cte \end{aligned}$$

↳ Rotação virtual rígida → conservação do momentum angular



Vetor axial: $\delta\vec{\theta} = \delta\theta\, \hat{e}_n$

$$\begin{cases} \delta\vec{r}_\alpha = \delta\vec{\theta} \times \vec{r}_\alpha \\ \delta\vec{v}_\alpha = \delta\vec{\theta} \times \vec{v}_\alpha \end{cases}$$

$\vdots$

$$\delta L = 0 \Rightarrow \frac{d}{dt}(\hat{e}_n \cdot \vec{L}) = 0 \Rightarrow \boxed{\hat{e}_n \cdot \vec{L} = cte} \quad , \text{ onde } \vec{L} = \sum_{\alpha=1}^{N} \vec{l}_\alpha = \sum_{\alpha=1}^{N} (\vec{r}_\alpha \times \vec{p}_\alpha)$$

↳ Conservação da energia total → 1.9.2.3

Formalismo Hamiltoniano

Sistemas holonômicos

→ Formalismo Lagrangiano: N partículas
$n = 3N - m$ graus de liberdade
n coordenadas generalizadas $\vec{q}$
n velocidades generalizadas $\dot{\vec{q}}$

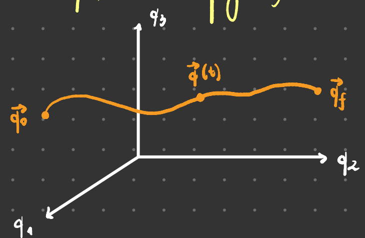
Lagrangiana: $L\{\vec{q}, \dot{\vec{q}}, t\} = T(\vec{q}, \dot{\vec{q}}, t) - U(\vec{q}, \dot{\vec{q}}, t)$

Eq. Euler-Lagrange: $\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \quad (j = 1, \dots, n) \rightarrow$ n EDOs de 2ª ordem

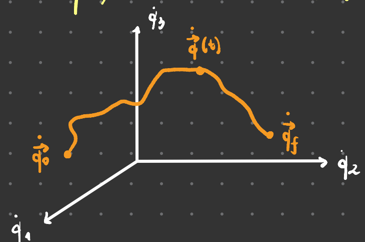
* Formalismo Hamiltoniano: Derivável da Lagrangiana

↳ Condições iniciais: $q_j = q_j(t_0) \equiv q_{j0}$

Espaço de configurações (nD)

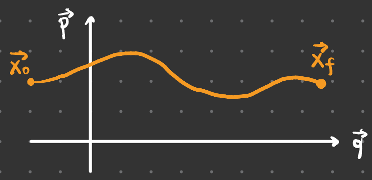


Espaço de velocidades (nD)



Espaço de fase (2nD)

→ $\vec{X} = (\vec{q}, \vec{p})$ → formalmente, $\{q_j\}$ e $\{p_j\}$ são variáveis independentes



→ Transformação Lagrangiana → Hamiltoniana: Transformada de Legendre

1) Dada a Lagrangiana $L\{\vec{q}, \dot{\vec{q}}; t\}$, calculam-se os momentos conjugados

$$P_j = \frac{\partial L}{\partial \dot{q}_j} \quad (j = 1, \cdots, n)$$

Isso fornece $p_j = p_j(\vec{q}, \dot{\vec{q}}, t)$ (2.1a)

2) Inverter as relações, obtendo $\dot{q}_j = \dot{q}_j(\vec{q}, \vec{p}, t)$ (2.1b)

Condições: i) Tanto as relações (2.1a) quanto suas primeiras derivadas

$$\frac{\partial p_j}{\partial \dot{q}_k} = \frac{\partial^2 L}{\partial \dot{q}_k \partial \dot{q}_j} \quad (j, k = 1, \cdots, n)$$

devem ser contínuas

ii) A matriz Hessiana W

$$W = [W_{ij}] \, (n \times n) \Rightarrow W_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

deve ser não singular: $\det(W) \neq 0$

(Quando $\det W = 0$, diz-se que o sistema possui vínculos primários → vínculo não integrável)

3) Aplica-se a transformada de Legendre para obter a Hamiltoniana:

$$\boxed{H(\vec{q}, \vec{p}, t) = \sum_{j=1}^{n} \dot{q}_j p_j - L\{\vec{q}, \dot{\vec{q}}, t\}} \qquad \dot{q}_j = \dot{q}_j(\vec{q}, \vec{p}, t)$$

4) Obter as equações canônicas de Hamilton

$$L\{\vec{q}, \dot{\vec{q}}, t\} \rightarrow \left.\begin{array}{l} \dfrac{\partial L}{\partial q_j} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} = 0 \\ p_j = \dfrac{\partial L}{\partial \dot{q}_j} \end{array}\right\} \quad \dot{p}_j = \frac{\partial L}{\partial q_j}$$

Considera-se dH:

(i) Pelo lado direito,
$$dH = \sum_{j=1}^{n} (\dot{q}_j dp_j + \cancel{p_j d\dot{q}_j}) - \sum_{j=1}^{n} \left(\overbrace{\frac{\partial L}{\partial q_j}}^{\dot{p}_j} dq_j + \cancel{\overbrace{\frac{\partial L}{\partial \dot{q}_j}}^{p_j} d\dot{q}_j} \right) - \frac{\partial L}{\partial t} dt$$

$$\boxed{dH = \sum_{j=1}^{n} (\dot{q}_j dp_j - \dot{p}_j dq_j) - \frac{\partial L}{\partial t} dt} \quad (I)$$

(ii) Pelo lado esquerdo, $H = H(\vec{q}, \vec{p}, t) \Rightarrow$
$$\boxed{dH = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt} \quad (II)$$

Sendo (I) = (II), resultam as equações de Hamilton:

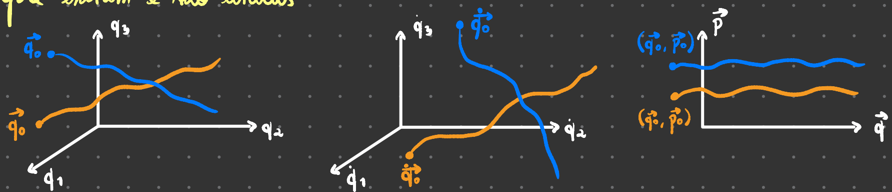
$$(2.3)\quad \begin{cases} \dot{q}_j = \dfrac{\partial H}{\partial p_j} & (j = 1, \ldots, n) \\ \dot{p}_j = -\dfrac{\partial H}{\partial q_j} \\ \dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t} \end{cases}$$

$2n$ EDOs de 1ª ordem para os qs e ps

* Observações:

1) As variáveis $\vec{q}$ e $\vec{p}$ são ditas canônicas conjugadas

2) Teorema de existência e unicidade: Dado o espaço de fase $(\vec{q}, \vec{p})$ do sistema e especificando as condições iniciais $(\vec{q}_0, \vec{p}_0) = (\vec{q}(t_0), \vec{p}(t_0))$, as soluções das equações (2.3) para $t > t_0$ sempre existem e são únicas



↳ Nos espaços de configurações e de velocidades, as soluções podem não ser únicas em um certo ponto (mas não no mesmo instante de tempo). A solução é única no espaço de fase

* Exemplo: Partícula sob força central: $L\{r, \theta, \dot{r}, \dot{\theta}, \dot{\varphi}\} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \operatorname{sen}^2\theta\, \dot{\varphi}^2) - U(r)$

1) Obter p_r, p_θ, p_φ

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2 \operatorname{sen}^2\theta\, \dot{\varphi}$$

2) Inverter

$$\dot{r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{\varphi} = \frac{p_\varphi}{mr^2 \operatorname{sen}^2\theta}$$

3) Derivar $H(r, \theta, \varphi, p_r, p_\theta, p_\varphi, t)$:

$$H = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\varphi}p_\varphi - L\{r, \theta, \dot{r}, \dot{\theta}, \dot{\varphi}\}$$

$$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\varphi^2}{mr^2 \operatorname{sen}^2\theta} - \frac{1}{2}m\left(\frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2 r^2} + \frac{p_\varphi^2}{m^2 r^2 \operatorname{sen}^2\theta}\right) + U(r)$$

$$H(r, \theta, p_r, p_\theta, p_\varphi) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \operatorname{sen}^2\theta}\right) + U(r)$$

4) Equações de Hamilton

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{\varphi} = \frac{\partial H}{\partial \varphi} = \frac{p_\varphi}{mr^2 \operatorname{sen}^2\theta}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\varphi^2}{mr^3 \operatorname{sen}^2\theta} - \frac{dU}{dr}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\varphi^2 \cos\theta}{mr^2 \operatorname{sen}^3\theta}$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \Rightarrow p_\varphi = cte$$

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→ Lagrangiana: $L = L\{\vec{q}, \dot{\vec{q}}, t\}$, $\vec{q} = (q_1, \dots, q_n)$

→ Momentos: $p_j = \frac{\partial L}{\partial \dot{q}_j} = p_j(\vec{q}, \dot{\vec{q}}, t) \quad (j = 1, \dots, n)$

→ Inversão: $\dot{q}_j = \dot{q}_j(\vec{q}, \vec{p}, t) \Leftrightarrow \det\left(\left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right]\right) \neq 0$

→ Transformada de Legendre: $H(\vec{q}, \vec{p}, t) = \sum_{j=1}^{n} \dot{q}_j p_j - L$

→ Equações de Hamilton: $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$, $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

* Exemplo: Carga em campo EM: $\vec{r} = (x, y, z)$

Lagrangiana: $L\{\vec{r}, \dot{\vec{r}}, t\} = \frac{1}{2} m \dot{r}^2 - q\phi(\vec{r}, t) + \frac{q}{c} \dot{\vec{r}} \cdot \vec{A}(\vec{r}, t)$

Momentos conjugados: $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{q}{c} A_x \;\Big|\; p_y = m\dot{y} + \frac{q}{c} A_y \;\Big|\; p_z = m\dot{z} + \frac{q}{c} A_z$

Inversão: $\dot{\vec{r}} = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)$

Hamiltoniana: $H(\vec{r}, \vec{p}, t) = \dot{\vec{r}} \cdot \vec{p} - L = \frac{1}{m}\left(p^2 - \frac{q}{c}\vec{p}\cdot\vec{A}\right) - \frac{1}{2} m \frac{1}{m^2}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi + \frac{q}{c}\frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right) \cdot \vec{A}$

$\vdots$

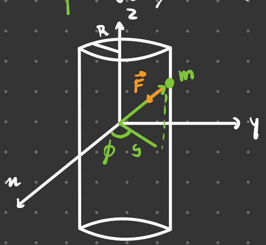
$$\boxed{H(\vec{r}, \vec{p}, t) = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi}$$

Eqs. Hamilton: $\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \nabla_{\vec{p}} H = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)$

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = \frac{q}{mc}\left\{\left[\left(\vec{p} - \frac{q}{c}\vec{A}\right) \cdot \nabla\right]\vec{A} - \frac{q}{c}\left(\vec{p} - \frac{q}{c}\vec{A}\right) \times \left(\nabla \times \vec{A}\right)\right\} - q\nabla\phi$$

→ Identidade: $\frac{1}{2}\nabla(c^2) = (\vec{c} \cdot \nabla)\vec{c} + \vec{c} \times (\nabla \times \vec{c})$

* Exemplo: força central $\vec{F}(\vec{r}) = -k\vec{r}$ ($k = cte$) sobre superfície interna de um cilindro de raio R



Coordenadas cilíndricas: $\{s, \phi, z\} \rightarrow r^2 = s^2 + z^2$

Vínculo: $x^2 + y^2 = R^2 \rightarrow s = R$

$$L = \frac{1}{2} m \dot{r}^2 - \frac{1}{2} k (s^2 + z^2) \rightarrow L\{z, \dot{z}, \phi\} = \frac{1}{2} m (R^2 \dot{\phi}^2 + \dot{z}^2) - \frac{1}{2} k (R^2 + z^2)$$

Lag. equival.

$$L\{z, \dot{z}, \phi\} = \frac{1}{2} m (R^2 \dot{\phi}^2 + \dot{z}^2) - \frac{1}{2} k z^2$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = \frac{p_z}{m}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi} \rightarrow \dot{\phi} = \frac{p_\phi}{mR^2}$$

$$H = \dot{z} p_z + \dot{\phi} p_\phi - L = \frac{p_z^2}{m} + \frac{p_\phi^2}{mR^2} - \frac{1}{2} m \left(R^2 \frac{p_\phi^2}{m^2 R^4} + \frac{p_z^2}{m^2} \right) + \frac{1}{2} k (R^2 + z^2)$$

$$\boxed{H = \frac{1}{2m} \left(p_z^2 + \frac{p_\phi^2}{R^2} \right) + \frac{1}{2} k (R^2 + z^2)}$$

$$\rightarrow \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2} \qquad \rightarrow \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \Rightarrow p_\phi = cte$$

$$\rightarrow \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \rightarrow \dot{p}_z = -\frac{\partial H}{\partial z} = -kz$$

$$\boxed{\phi(t) = \phi_0 + \frac{p_\phi}{mR^2} t}$$

$$\ddot{z} = \frac{\dot{p}_z}{m} = -\frac{k}{m} z \Rightarrow \boxed{\ddot{z} + \omega_0^2 z = 0}$$

Leis de Conservação

* Coordenadas cíclicas:

q_j é cíclica se $\frac{\partial L}{\partial q_j} = 0 \rightarrow p_j = \frac{\partial L}{\partial \dot{q}_j}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \dot{p}_j = -\frac{\partial H}{\partial q_j} = 0 \Rightarrow \boxed{\frac{\partial H}{\partial q_j} = 0 \Rightarrow p_j = cte}$$

* Conservação da Hamiltoniana e da Energia Total

* Teorema:

$$H = H(\vec{q}, \vec{p}, t) \Rightarrow \frac{dH}{dt} = \sum_{j=1}^{n}\left(\frac{\partial H}{\partial q_j}\dot{q}_j + \frac{\partial H}{\partial p_j}\dot{p}_j\right) + \frac{\partial H}{\partial t} \overset{\text{eqs. Hamilton}}{=} \sum_{j=1}^{n}\left(\frac{\partial^2 H}{\partial q_j \partial p_j} - \frac{\partial^2 H}{\partial p_j \partial q_j}\right) + \frac{\partial H}{\partial t}$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}} \longrightarrow \boxed{\text{se } H = H(\vec{q}, \vec{p}) \text{ então } H = cte}$$

* Relação $H \leftrightarrow E = T + U$

→ sistema cartesiano, N partículas
n graus de liberdade

$$T = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha v_\alpha^2 = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha$$

→ Coordenadas generalizadas: $\vec{r}_\alpha = \vec{r}_\alpha(\vec{q}, t)$ onde $\vec{q} = (q_1, \dots, q_n)$

$$\frac{d\vec{r}_\alpha}{dt} = \dot{\vec{r}}_\alpha = \sum_{j=1}^{n}\frac{\partial \vec{r}_\alpha}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_\alpha}{\partial t}$$

→ Então: $\boxed{T(\vec{q}, \dot{\vec{q}}, t) = M_0(\vec{q}, t) + \sum_{j=1}^{n} M_j(\vec{q}, t)\dot{q}_j + \frac{1}{2}\sum_{j,k=1}^{n} M_{jk}(\vec{q}, t)\dot{q}_j\dot{q}_k}$

$$M_0 = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha\left(\frac{\partial \vec{r}_\alpha}{\partial t}\right)^2 \qquad M_j = \sum_\alpha m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_j}\cdot\frac{\partial \vec{r}_\alpha}{\partial t} \qquad M_{jk} = \sum_\alpha m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_j}\cdot\frac{\partial \vec{r}_\alpha}{\partial q_k}$$

(i) A energia cinética é puramente quadrática nas velocidades: $M_0 = M_j = 0$

$$\boxed{T(\vec{q}, \dot{\vec{q}}) = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k}$$

T é função homogênea de grau 2 nas velocidades: $T(\dot{q}_1, \dots, \dot{q}_n) \rightarrow T(\lambda\dot{\vec{q}}) = \lambda^2 T(\dot{\vec{q}})$

Teorema de Euler: $\sum_{j=1}^{n}\dot{q}_j\frac{\partial T}{\partial \dot{q}_j} = 2T$

(ii) Energia potencial não depende das velocidades: $\boxed{U = U(\vec{q}, t)}$

Então,

$$L\{\vec{q}, \dot{\vec{q}}, t\} = T(\vec{q}, \dot{\vec{q}}) - U(\vec{q}, t)$$
$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}} = \frac{\partial T}{\partial \dot{\vec{q}}}$$

condição suficiente

$$H = \sum_{j=1}^{n}\dot{q}_j p_j - L$$
$$\sum_{j=1}^{n}\dot{q}_j p_j = \sum_{j=1}^{n}\dot{q}_j\frac{\partial T}{\partial \dot{q}_j} = 2T$$

$$H = 2T - (T - U)$$
$$\boxed{H(\vec{q}, \vec{p}, t) = T + U = E(\vec{q}, \vec{p}, t)}$$

$H = H(\vec{q}, \vec{p}) = cte$
$H = E$ se (i) e (ii)

* Exemplo:

1) Força central

(i) $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \operatorname{sen}^2\theta\, \dot{\varphi}^2)$: quadrática em $\{\dot{r}, \dot{\theta}, \dot{\varphi}\}$

(ii) $U = U(r)$

$$\Rightarrow H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \operatorname{sen}^2\theta}\right) + U(r) = E = cte$$

2) Carga em campo EM

(i) $T = \frac{1}{2}m\dot{r}^2$ ✓

(ii) $U(\vec{r}, \dot{\vec{r}}, t) = q\phi(\vec{r}, t) - \frac{q}{c}\dot{\vec{r}} \cdot \vec{A}(\vec{r}, t)$ ✗

$\Rightarrow H \neq E$

$$\rightarrow E = T + U = \overbrace{\frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi(\vec{r}, t)}^{H(\vec{r}, \dot{\vec{r}}, t)} - \frac{q}{mc}\left(\vec{p} - \frac{q}{c}\vec{A}\right) \cdot \vec{A}$$

→ além disso, em geral nesses casos $H \neq cte$

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Conservação da Hamiltoniana e da Energia total

1) Conservação de H: $\frac{dH}{dt} = \frac{\partial H}{\partial t} \Rightarrow H(\vec{p},\vec{q}) = cte \rightarrow$ H é constante se não há dependência explícita no tempo

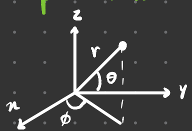
2) Relação $E \leftrightarrow H$: Se:

(i) T for quadrática nas velocidades: $T = \frac{1}{2}\sum_{j,k=1}^{n} M_{jk}(\vec{q})\dot{q}_j\dot{q}_k$

(ii) U não depende de $\dot{\vec{q}}$: $U = U(\vec{q},t)$

Então $H(\vec{q},\vec{p},t) = E = T + U$

* Exemplo: Partícula sob força central



$U = U(r) \rightarrow$ satisfaz (ii)

$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \operatorname{sen}^2\theta\, \dot{\varphi}^2\right) \rightarrow$ satisfaz (i)

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \operatorname{sen}^2\theta}\right) + U(r) = E$$

$\frac{\partial H}{\partial t} = 0 \Rightarrow H = E = cte$ | φ é cíclica $\Rightarrow p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = cte$

* Exemplo: Carga em campo EM

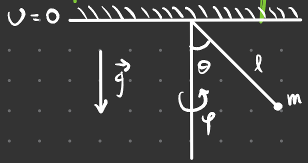
$T = \frac{1}{2}m\dot{r}^2 \rightarrow$ satisfaz (i)

$U(\vec{r},\dot{\vec{r}},t) = q\phi(\vec{r},t) - \frac{q}{c}\dot{\vec{r}}\cdot\vec{A}(\vec{r},t) \rightarrow$ não satisfaz (ii)

$$E = T + U = \underbrace{\frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right) + q\phi}_{H(\vec{r},\vec{p},t)} - \frac{q}{mc}\left(\vec{p} - \frac{q}{c}\vec{A}\right)\cdot\vec{A}$$

$\frac{\partial H}{\partial t} \neq 0$ se $\frac{\partial \vec{A}}{\partial t} = 0$ e/ou $\frac{\partial \phi}{\partial t} \neq 0$

* Exemplo: Pêndulo esférico



Vínculo: $r = \ell = cte$

$$L\{\theta, \dot\theta, \dot\varphi^2\} = \underbrace{\frac{1}{2}m\ell^2(\dot\theta^2 + \dot\varphi^2 sen^2\theta)}_{(i)\ \checkmark} + \underbrace{mg\ell\cos\theta}_{(ii)\ \checkmark}$$

φ é cíclica $\Rightarrow p_\varphi = cte \neq 0$

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = m\ell^2\dot\theta \qquad p_\varphi = m\ell^2\dot\varphi\, sen^2\theta$$

$$H = \dot\theta p_\theta + \dot\varphi p_\varphi - L$$

$$H(\theta, p_\theta, p_\varphi) = \frac{1}{2m\ell^2}\left(p_\theta^2 + \frac{p_\varphi^2}{sen^2\theta}\right) - mg\ell\cos\theta = E = cte$$

→ Eqs. Hamilton:

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} \;\Big|\; \dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m\ell^2 sen^2\theta} \;\Big|\; \dot p_\theta = -\frac{\partial H}{\partial\theta} = \frac{p_\varphi^2 \cot g\theta}{m\ell^2 sen^2\theta} - mg\ell\, sen\,\theta \;\Big|\; \dot p_\varphi = -\frac{\partial H}{\partial\varphi} = 0 \Rightarrow p_\varphi = cte$$

↳ Condição de equilíbrio $\dot\theta = 0 \to \theta = \theta_0$:

→ $\theta = cte$ qualquer: $\dot p_\theta = \alpha \neq 0 \Rightarrow p_\theta = \alpha t + p_{\theta_0} \Rightarrow \dot\theta \neq 0$

→ Para satisfazer a condição, $\dot p_\theta = 0$:

$$\dot p_\theta = 0 \Leftrightarrow \frac{p_\varphi^2 \cot g\,\theta_0}{m\ell^2 sen^2\theta_0} - mg\ell\, sen\,\theta_0 = 0 \Rightarrow p_\varphi = m\sqrt{g\ell^3}\, sen^2\theta_0\sqrt{sec\,\theta_0}$$

$$\dot\varphi = \sqrt{\frac{g}{\ell}sec\,\theta_0} \Rightarrow \varphi(t) = \varphi_0 + \sqrt{\frac{g}{\ell}sec\,\theta_0}\; t$$

↳ Solução para $p_{\varphi_0} = p_\varphi(t=0) = 0$: $H = \dfrac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta = E \Rightarrow E = \dfrac{1}{2}m\ell^2\dot\theta^2 - mg\ell\cos\theta$

$$C.I.: \begin{cases} \theta_0 = \theta(t=0) > 0 \\ \dot\theta_0 = \dot\theta(t=0) = 0 \end{cases} \to E = -mg\ell\cos\theta_0$$

$$\dot\theta = \pm\sqrt{2\omega_0^2(\cos\theta - \cos\theta_0)} \quad (\omega_0^2 = g/\ell) \to \int \frac{d\theta}{\pm\sqrt{\cdots}} = \int dt$$

Novas condições iniciais: $\theta(t=0) = 0$, $\dot\theta(t=0) > 0$

$$\int_0^t \omega\, dt' = \int_0^\theta \frac{d\theta'}{\sqrt{2(\cos\theta' - \cos\theta_0)}} \quad \text{tal que } |\theta(t)| \leq \theta_0$$

Usando $\cos(2\alpha) = \cos^2\alpha - sen^2\alpha = 1 - 2sen^2\alpha$: $\cos\theta = 1 - 2sen^2\left(\frac{\theta}{2}\right)$ e definindo $\chi = \theta/2$

$$\omega_0 t = C_0\int_0^{\frac{\theta}{2}} \frac{d\chi}{\sqrt{1 - C_0^2 sen^2\chi}} \quad \left[C_0 = cossec\left(\frac{\theta_0}{2}\right)\right] \to \text{Integral elíptica incompleta do 1º Tipo:} \quad F(\phi, k) = \int_0^\phi \frac{d\chi}{\sqrt{1 - k^2 sen^2\chi}}$$

$$\therefore \omega_0 t = C_0 F\left(\frac{\theta}{2}, C_0\right)$$

$F(\phi, k_1) = k F(\beta, k)$, $k_1 = \frac{1}{k}$ e $sen\,\beta = k_1\, sen\,\phi$

Ou seja, $\omega_0 t = F(\beta, C_0^{-1})$, $sen\,\beta = C_0\, sen\left(\frac{\theta}{2}\right) = \dfrac{sen\left(\frac{\theta}{2}\right)}{sen\left(\frac{\theta_0}{2}\right)}$

Período: $\frac{1}{4}\omega_0 T = F\left(\frac{\pi}{2}, c_0^{-1}\right) = K(c_0^{-1})$

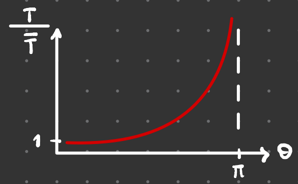
$K(x) = F\left(\frac{\pi}{2}, x\right)$: integral elíptica completa

$$\left.\begin{array}{l} \frac{1}{4}\omega_0 T = F\left(\frac{\pi}{2}, c_0^{-1}\right) = K(c_0^{-1}) \\ K(x) = F\left(\frac{\pi}{2}, x\right) \end{array}\right\} \quad T(\theta_0) = \frac{4}{\omega_0} K\left(\text{sen}\,\frac{\theta_0}{2}\right)$$

Mas: $K(x) = \frac{\pi}{2} + \frac{\pi}{8}x + \frac{9\pi}{128}x^2 + \dots$

Então: $T \approx \frac{2\pi}{\omega_0}\left(1 + \frac{\theta_0}{8}\right) \xrightarrow{\theta_0 \ll 1} \bar{T} = \frac{2\pi}{\omega}$

$0 \leq \theta_0 \leq \pi$



$\to t = t(\theta) \to \theta = \theta(t)$?

Dica: $F^{-1}(t)$: função elíptica de Jacobi $\text{sn}(x, k)$

$F(\phi, k) = \text{sn}^{-1}(\text{sen}\,\phi, k) \to \theta(\theta_0, t) = 2\,\text{sen}^{-1}\left[\text{sen}\left(\frac{\theta_0}{2}\right)\text{sn}\left(\omega t, \text{sen}\,\frac{\theta_0}{2}\right)\right]$