Oscilações Acopladas

→ Sistemas com 1 grau de liberdade

→ Vínculos holônomos

→ Lagrangiana: $L\{q,\dot{q}\} = \frac{1}{2}\alpha(q)\,\dot{q}^2 - U(q)$

→ Sistema conservativo

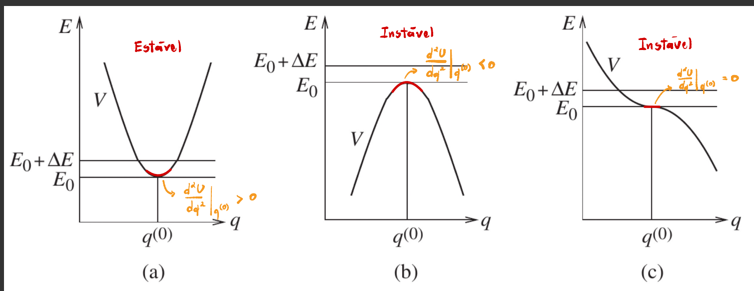
→ Eq. movimento: $\alpha\ddot{q} + \frac{1}{2}\alpha'\dot{q}^2 + \frac{dU}{dq} = 0 \qquad \left(\alpha' = \frac{d\alpha}{dq}\right)$

* Configuração de equilíbrio: $q = q^{(0)} \to \left.\frac{dU}{dq}\right|_{q^{(0)}} = 0$

$$\left.\left(\alpha\ddot{q} + \frac{1}{2}\alpha'\dot{q}^2\right)\right|_{q^{(0)}} = 0$$

Se $\dot{q}|_{q^{(0)}} = 0 \Rightarrow \ddot{q} = 0 \Rightarrow q = q^{(0)}$ → sistema sempre em repouso

→ Pequenas perturbações em torno de $q^{(0)}$: $\begin{cases} q(t) = q^{(0)} + \eta(t), \ |\eta| \ll 1 \\ \dot{q}(t) = \dot{\eta}(t) \end{cases}$



→ Oscilações estáveis em torno de $q^{(0)}$: $\left.\frac{d^2U}{dq^2}\right|_{q^{(0)}} > 0$

→ Desenvolvemos $U(q)$ e $\alpha(q)$ em série de potências em torno de $q(0)$:

↳ $U(q) = U(q^{(0)}) + \overset{0}{\cancel{\left.\frac{dU}{dq}\right|_{q^{(0)}}}}(q-q^{(0)}) + \frac{1}{2}\left.\frac{d^2U}{dq^2}\right|_{q^{(0)}}(q-q^{(0)})^2 + \cdots \Rightarrow U(q) \approx \overset{U(q^{(0)})}{U_0} + \frac{1}{2}\overset{\left.\frac{d^2U}{dq^2}\right|_{q^{(0)}} > 0}{K^{(0)}}\eta^2$

↳ $\alpha(q) = \alpha(q^{(0)}) + \underbrace{\left.\frac{d\alpha}{dq}\right|_{q^{(0)}}\overbrace{(q-q^{(0)})}^{\eta}}_{\to 0} + \cdots \Rightarrow \alpha(q) \approx \overset{\alpha(q^{(0)})}{\alpha^{(0)}}, \quad |\alpha'(q^{(0)})\eta| \ll |\alpha^{(0)}|$

→ Neste caso, $\boxed{L\{q,\dot{q}\} = \frac{1}{2}\alpha^{(0)}\dot{\eta}^2 - \frac{1}{2}K^{(0)}\eta^2}$

→ Eq. movimento: $\boxed{\ddot{\eta} + \omega^2\eta = 0}$

oscilador harmônico com frequência $\boxed{\omega = \sqrt{\frac{K^{(0)}}{\alpha^{(0)}}}}$

Oscilações em sistemas com $n > 1$ g.l

→ Sistema holônomo e conservativo

→ Movimento estacionário:
↳ Existem coordenadas cíclicas
↳ oscilações induzidas por pequenas perturbações sobre as coordenadas não cíclicas

* Exemplo: movimento sob forças centrais (1.19, 2.1)

→ $\{r, \theta, \varphi\}$
→ $U = U(r)$

→ φ cíclico: $p_\varphi = cte \Rightarrow \dot{\varphi} = \dfrac{p_\varphi}{mr^2 \,\text{sen}\,\theta}$

→ $\exists$ solução com $\theta = \frac{\pi}{2} = cte$ → $L\{r, \dot{r}, \dot{\varphi}\} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - U(r)$

↳ $m\ddot{r} - \dfrac{p_\varphi^2}{mr^3} + \dfrac{dU}{dr} = 0$

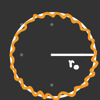
↳ solução estacionária $r = r_0 = cte$, desde que $\left.\dfrac{dU}{dr}\right|_{r_0} = \dfrac{p_\varphi^2}{mr_0^3}$

→ Potencial efetivo: $L = \frac{1}{2} m\dot{r}^2 - U_{ef}(r)$, $E = \frac{1}{2} m\dot{r}^2 + U_{ef}(r)$, $U_{ef} = U(r) + \dfrac{p_\varphi^2}{2mr^2}$ $\Rightarrow$ $m\ddot{r} - \dfrac{dU_{ef}}{dr} = 0$

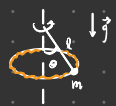
↳ Movimento planetário: $U(r) = -\frac{A}{r}$ $(A > 0)$

$$U_{ef} = -\frac{A}{r} + \frac{p_\varphi^2}{2mr^2} \Rightarrow \left.\frac{dU_{ef}}{dr}\right|_{r_0} = \frac{A}{r_0^2} - \frac{p_\varphi^2}{2mr_0^3} \Rightarrow \boxed{r_0 = \frac{p_\varphi^2}{mA}}$$

Estabilidade: $U_{ef}(r_0) = \dfrac{3p_\varphi^2}{mr_0^4} - \dfrac{2A}{r_0^4} = \dfrac{A}{r_0^3} > 0$



* Exemplo: pêndulo esférico



$L\{\theta, \dot{\theta}, \dot{\varphi}\} = \frac{1}{2} m\ell(\dot{\theta} + \dot{\varphi}^2 \,\text{sen}^2\theta) + mg\ell\cos\theta$

φ cíclica: $p_\varphi = cte \Rightarrow \dot{\varphi} = \dfrac{p_\varphi^2}{m\ell^2 \,\text{sen}^2\theta}$, θ não cíclica

→ soluções estacionárias: $\theta = \theta_0 = cte$
↳ Estabilidade de oscilações em torno de θ_0

→ Equação de movimento: $\ddot{\theta} - \dot{\varphi}^2 \,\text{sen}^2\theta\cos\theta + \frac{g}{\ell}\,\text{sen}\,\theta = 0$

→ Potencial efetivo: $h = \frac{1}{2} m\ell\dot{\theta}^2 + U_{ef}(\theta) = E$

$$U_{ef}(\theta) = \frac{p_\varphi^2}{2m\ell^2 \,\text{sen}^2\theta} - mg\ell\cos\theta \rightarrow \frac{dU_{ef}}{d\theta} = -\frac{p_\varphi^2 \cos\theta}{m\ell^2 \,\text{sen}^3\theta} + mg\ell \,\text{sen}\,\theta = 0 \quad \begin{cases} \cos\theta_0 = \dfrac{g}{\ell\dot{\varphi}^2} \leq 1 \\ |\dot{\varphi}| > \sqrt{\frac{g}{\ell}} \Rightarrow |p_\varphi| > \sqrt{m^2\ell^3 g}\,\text{sen}^2\theta_0 \end{cases}$$

→ $U''_{ef}(\theta_0) = \dfrac{mg\ell}{\cos\theta_0}(1 + 3\cos^2\theta_0) > 0$ → estável

→ $\omega = \sqrt{\dfrac{1 + 3\cos^2\theta_0}{\cos\theta_0}\dfrac{g}{\ell}}$

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Oscilações Acopladas de Pequena Amplitude

→ Sistema com n graus de liberdade

→ Vínculos holônomos

→ Lagrangiana padrão: $\boxed{L\{\vec{q}, \dot{\vec{q}}\} = \frac{1}{2}\sum_{k,l=1}^{n} M_{kl}(\vec{q})\,\dot{q}_k \dot{q}_l - U(\vec{q})}$ (leis de transformação indep. do tempo)

$M_{kl} = M_{lk}$

→ Existe configuração de equilíbrio: $\exists\, \vec{q}^{(0)} = (q_1^{(0)}, \dots, q_n^{(0)})$ tal que $\left.\frac{\partial U}{\partial q_k}\right|_{\vec{q}^{(0)}} = 0 \quad (k = 1, \dots, n)$

→ $U(\vec{q})$ pode ser um potencial efetivo

→ Assumindo pequenas perturbações sobre $\vec{q}^{(0)}$: $q_k = q_k^{(0)} + \eta_k(t)$, $\begin{pmatrix} k = 1, \dots, n \\ |\eta_k| \ll 1 \end{pmatrix}$

→ Desenvolve-se

(aproximação perturbativa)

$$U(\vec{q}) = U(\vec{q}^{(0)}) + \sum_{k=1}^{n}\left(\left.\frac{\partial U}{\partial q_k}\right|_{\vec{q}^{(0)}}\right)\eta_k + \frac{1}{2}\left(\sum_{k,l=1}^{n}\left.\frac{\partial^2 U}{\partial q_k \partial q_l}\right|_{\vec{q}^{(0)}}\right)\eta_k \eta_l + \mathcal{O}(\eta^3)$$

$$\boxed{\begin{aligned} &U(\vec{q}) \approx U_0 + \frac{1}{2}\sum_{k,l=1}^{n} U_{kl}\,\eta_k \eta_l \\ &U_0 = U(\vec{q}^{(0)}), \quad U_{kl} = \left.\frac{\partial^2 U}{\partial q_k \partial q_l}\right|_{\vec{q}^{(0)}} \end{aligned}} \quad (k, l = 1, \dots, n)$$

Matriz potencial

→ Função de Jacobi: $h(\vec{q}, \dot{\vec{q}}) = E = U_0 + \boxed{\frac{1}{2}\sum_{k,l=1}^{n} M_{kl}\,\dot{q}_k \dot{q}_l} + \frac{1}{2}\sum_{k,l=1}^{n} U_{kl}\,\eta_k \eta_l$

↳ afastamento finito → $T = 0$ quando $U = \max$

→ Então, é necessário que $\sum_{k,l=1}^{n} U_{kl}\,\eta_k(t)\,\eta_l(t) \geq 0 \quad (t \geq 0)$

→ Termo cinético: $q_k(t) = q_k^{(0)} + \eta_k(t) \Rightarrow \dot{q}_k = \dot{\eta}_k$

expansão em série de Taylor

$$T = \frac{1}{2}\sum_{k,l=1}^{n} M_{kl}(\vec{q})\,\dot{q}_k \dot{q}_l = \frac{1}{2}\sum_{k,l=1}^{n}\left[T_{kl} + \sum_{j=1}^{n}\left(\left.\frac{\partial M_{kl}}{\partial q_j}\right|_{\vec{q}^{(0)}}\right)\eta_j + \cdots\right]\dot{\eta}_k \dot{\eta}_l$$

$$\boxed{\begin{aligned} &T \approx \frac{1}{2}\sum_{k,l=1}^{n} T_{kl}\,\dot{\eta}_k \dot{\eta}_l \\ &T_{kl} = M_{kl}(\vec{q}^{(0)}) \longrightarrow \text{matriz cinética} \end{aligned}} \quad (k, l = 1, \dots, n)$$

→ A Lagrangiana fica

$$\boxed{L\{\eta, \dot{\eta}\} = \frac{1}{2}\sum_{k,l=1}^{n} T_{kl}\,\dot{\eta}_k \dot{\eta}_l - \frac{1}{2}\sum_{k,l=1}^{n} U_{kl}\,\eta_k \eta_l}$$

→ Introduzindo $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$ e $\tilde{\eta} = (\eta_1 \ \eta_2 \ \cdots \ \eta_n) \Rightarrow \boxed{L = \frac{1}{2}\dot{\tilde{\eta}}\,T\,\dot{\eta} - \frac{1}{2}\tilde{\eta}\,U\,\eta}$

→ Equações de Euler-Lagrange: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_k} - \frac{\partial L}{\partial \eta_k} = 0$

→ Equações de movimento: $\boxed{\sum_{k=1}^{n} T_{jk}\,\ddot{\eta}_k + \sum_{k=1}^{n} U_{jk}\,\eta_k = 0}$, $(j = 1, \dots, n)$ (4.5)

Modos normais de vibração

→ Assumir soluções periódicas para (4.9)

→ Séries de Fourier: $\eta_k(t) = \sum_{s=-\infty}^{\infty} \underbrace{\bar{\eta}_{ks}}_{\text{cte} \in \mathbb{C}} e^{i\omega_s t}$ → frequência angular, $\in \mathbb{C}$

→ $\bar{\eta}_{ks}$ e ω_s podem $\in \mathbb{C}$, mas como $\eta_k(t) \in \mathbb{R} \Rightarrow \eta_k^*(t) = \eta_k(t)$

→ $\ddot{\eta}_k(t) = -\sum_s \omega_s^2 \bar{\eta}_{ks} e^{i\omega_s t}$

→ Introduzindo em (4.9), resulta

$$\sum_{s=-\infty}^{\infty} \left[\sum_{k=1}^{n} (U_{jk} - \omega_s^2 T_{jk})\, \bar{\eta}_{ks}\right] e^{i\omega_s t} = 0 \Rightarrow \sum_{k=1}^{n} (U_{jk} - \omega_s^2 T_{jk})\, \bar{\eta}_{ks} = 0 \quad \forall t \geq 0,\ \forall j = 1, \dots, n$$

→ Mas $\bar{\eta}_{ks} \neq \emptyset$ para solução não trivial. Neste caso: $\det(\underline{U} - \underline{\omega}^2 \underline{T}) = 0$ → obtidos agora; → determinadas pelo sistema, $n \times n$

$$\alpha_n (\omega^2)^n + \alpha_{n-1} (\omega^2)^{n-1} + \dots + \alpha_1 (\omega^2) + \alpha_0 = 0 \rightarrow \text{equação característica}$$

→ $\exists$ no máximo n valores possíveis para ω^2. Ou seja:

$$\eta_k(t) = \sum_{s=-n}^{n} \bar{\eta}_{ks} e^{i\omega_s t}, \quad \omega^2 = \omega_s^2 \Rightarrow \boxed{\begin{array}{l}\omega = \pm \omega_s \\ \omega_{-s} = -\omega_s\end{array}}$$

* Frequências características ou autofrequências: $\{\omega_s\}$

→ Portanto: $\eta_k(t) = \sum_{\substack{s=-n \\ (n \neq 0)}}^{n} \bar{\eta}_{ks} e^{i\omega_s t} \quad (k = 1, \dots, n)$

→ Condição de realidade: $\eta_k(t) = \eta_k^*(t) \Rightarrow \sum_{s=-n}^{n} \bar{\eta}_{ks} e^{i\omega_s t} = \sum_{s=-n}^{n} \bar{\eta}_{ks}^* e^{-i\omega_s^* t} \overset{s \to -s}{=} \sum_{s=-n}^{n} \bar{\eta}_{k(-s)}^* e^{-i\omega_{-s}^* t}$

É suficiente que $\bar{\eta}_{ks} = \bar{\eta}_{k(-s)}$, $\omega_s = -\omega_{-s}^*$. Mas $-\omega_{-s} = \omega_s \Rightarrow \boxed{\omega_s^* = \omega_s}$ Autofrequências são reais

→ Então:

$$\eta_k = \sum_{s=1}^{n} \bar{\eta}_{ks} e^{i\omega_s t} + \sum_{s=-1}^{-n} \bar{\eta}_{ks} e^{i\omega_s t} \quad \} s \to -s$$

$$\eta_k(t) = \sum_{s=1}^{n} (\bar{\eta}_{ks} e^{i\omega_s t} + \bar{\eta}_{ks}^* e^{-i\omega_s t})$$

→ Escrevendo: $\bar{\eta}_{ks} = \frac{1}{2} C_s \rho_{ks}$ $(k, s = 1, \dots, n)$ onde $C_s = \bar{C}_s e^{i\phi_s}$ → fase do s-ésimo modo normal (depende das c.i)

→ Vetores característicos ou autovetores: $\rho_s = \begin{pmatrix} \rho_{1s} \\ \rho_{2s} \\ \vdots \\ \rho_{ns} \end{pmatrix}$ $(s = 1, \dots, n)$

→ Retornando a (4.11): $\sum_{k=1}^{n} (U_{jk} - \omega_s^2 T_{jk}) \rho_{ks} = 0 \quad (s = 1, \dots, n)$

$$\Downarrow$$

$$\boxed{(U - \omega_s^2 T)\rho_s = 0}$$

→ Assim: $\begin{cases} \text{Autovetores indeterminados} \\ \text{Autovetores serão determinados por uma condição de normalização} \end{cases}$

→ Cada valor de s determina um modo normal de oscilação

→ Amplitude de um modo puro: $\eta_s(t) = \rho_s e^{i\omega_s t}$

→ Em geral: $\boxed{\eta_k(t) = \sum_{s=1}^{n} \bar{C}_s \rho_{ks} \cos(\omega_s t + \phi_s)}$

Oscilações Acopladas: Sistemas com n graus de liberdade

→ Lagrangiana: $L\{\vec{q}, \dot{\vec{q}}\} = \frac{1}{2}\sum_{k,l=1}^{n} M_{kl}(\vec{q})\,\dot{q}_k \dot{q}_l - U(\vec{q})$

→ Configuração de equilíbrio: $\vec{q} = \vec{q}^{(0)}$, $\left.\frac{\partial U}{\partial \vec{q}}\right|_{\vec{q}^{(0)}} = 0 \quad (k = 1, \dots, n)$

→ Oscilações de pequena amplitude: $q_k(t) = q_k^{(0)} + \eta_k(t)$

$$U \approx U_0 + \frac{1}{2}\sum_{k,l} U_{kl}\,\eta_k \eta_l \rightarrow U_{kl} = \left.\frac{\partial^2 U}{\partial q_k \partial q_l}\right|_{\vec{q}^{(0)}}$$

$$T \approx \frac{1}{2}\sum_{k,l} T_{kl}\,\dot{\eta}_k \dot{\eta}_l \rightarrow T_{kl} = M_{kl}(\vec{q}^{(0)})$$

→ Oscilações estáveis: $\sum_{k,l} U_{kl}\,\eta_k(t)\,\eta_l(t) \geq 0$

→ Lagrangiana aproximada: $L\{\vec{\eta}, \dot{\vec{\eta}}\} = \frac{1}{2}\sum_{k,l} T_{kl}\,\dot{\eta}_k \dot{\eta}_l - \frac{1}{2}\sum_{k,l} U_{kl}\,\eta_k \eta_l$

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \rightarrow L = \frac{1}{2}\dot{\tilde{\eta}}\,T\,\dot{\eta} - \frac{1}{2}\tilde{\eta}\,U\,\eta$$

→ Eqs. movimento: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_j} - \frac{\partial L}{\partial \eta} = 0 \Rightarrow \sum_k T_{jk}\,\ddot{\eta}_k + \sum_k U_{jk}\,\eta_k = 0$

s-ésimo modo normal

→ Solução eqs. movimento / modos normais: $\eta_k(t) = \sum_s c_s \rho_{ks} \cos(\omega_s t + \phi_s)$

cond. realidade

→ $\{\omega_s\}$: Autofrequências ou frequências características: $\det(U - \omega^2 T) = 0 \Rightarrow \alpha_n (\omega^2)^n + \dots + \alpha_1 \omega^2 + \alpha_0 = 0$, $\omega_s \in \mathbb{R}$

→ $\{\rho_{ks}\}$: Autovetores ou vetores característicos: $\rho_s = \begin{pmatrix} \rho_{1s} \\ \vdots \\ \rho_{ns} \end{pmatrix}$, $(U - \omega_s^2 T)\,\rho_s = 0$

↳ 1 componente indeterminada

→ $\{c_s, \phi_s\}$: Condições iniciais
- c_s: amplitude do s-ésimo modo normal
- ϕ_s: fase do s-ésimo modo normal

*24/01/24

Coordenadas Normais

→ Transformação de coordenadas: $\{\eta_k\} \mapsto \{\xi_k\}$ (coordenadas normais)

→ $\vec{\xi} \mapsto \vec{\eta}$: Dados $\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$ e $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$, $\exists\ A(n \times n)\ /\ \eta = A\xi$

$$\Rightarrow L = \frac{1}{2}\dot{\tilde{\xi}}(\tilde{A}TA)\dot{\xi} - \frac{1}{2}\tilde{\xi}(\tilde{A}UA)\xi$$

→ Se $\tilde{A}TA = I_n$ e $\tilde{A}UA = W$ (diagonal), $W_{rs} = \omega_r^2\delta_{rs}$, então $\boxed{L = \frac{1}{2}\sum_{k=1}^{n}\left(\dot{\xi}_k^2 - \omega_k^2\xi_k^2\right)}$

↳ $\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}_k} - \frac{\partial L}{\partial \xi_k} = 0 \Rightarrow \boxed{\ddot{\xi}_k + \omega_k^2\xi_k = 0}$ $(k = 1, \dots, n)$ → n osciladores desacoplados

→ Soluções: $\boxed{\xi_k(t) = C_k\cos(\omega_k t + \phi_k)}$

→ $\vec{\eta} \mapsto \vec{\xi}$: $\eta = A\xi \overset{\tilde{A}T}{\Rightarrow} \tilde{A}T\eta = \underbrace{(\tilde{A}TA)}_{I}\xi \Rightarrow \boxed{\xi(t) = \tilde{A}T\eta}$

* Matriz modal A:

$$A = \begin{pmatrix} \rho_1 & \cdots & \rho_n \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \rho_{11} & \cdots & \rho_{n1} \\ \vdots & \ddots & \vdots \\ \rho_{1n} & \cdots & \rho_{nn} \end{pmatrix}$$

→ Espaço vetorial $\mathcal{E}$: Matrizes reais $(n\times 1)$ $\quad \psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$, $\{\psi_k\} \subset \mathbb{R}$

→ Espaço vetorial com produto interno: $\psi, \sigma \in \mathcal{E}$: $\langle \psi, \sigma \rangle = \tilde{\psi}T\sigma = \sum_{k,\ell} \psi_k T_{k\ell}\sigma_\ell$ $\quad$ (Normado: $|\psi| = \sqrt{\langle \psi, \psi \rangle}$)

→ Então, como $\{\rho_k\} \subset \mathcal{E}$ e é ortonormal, mostra-se que, se o problema for não degenerado, então

$$\langle \rho_r, \rho_s \rangle = \tilde{\rho}_r T \rho_s = \delta_{rs}, \quad |\rho_r| = \sqrt{\langle \rho_r, \rho_r \rangle}$$

pois $\langle \rho_r, \rho_s \rangle = \sum_{k,\ell}^{n} \rho_{kr} T_{k\ell} \rho_{\ell s} \overset{\rho_{rs} = A_{rs}}{=} \sum_{k,\ell}^{n} A_{kr} T_{k\ell} A_{\ell s} \overset{A_{kr} = (\tilde{A})_{rk}}{=} \sum_{k,\ell} (\tilde{A})_{rk} T_{k\ell} A_{\ell s} = (\tilde{A}TA)_{rs} = \delta_{rs}$

→ $\tilde{A}TA = I_n \rightarrow (U - \omega^2 T)\rho_s = 0 \Rightarrow \overbrace{\tilde{\rho}_r U \rho_s}^{W} - \omega_s^2 \overbrace{\tilde{\rho}_r T \rho_s}^{I_n} = 0$

Então,

$$W = \begin{pmatrix} \omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \omega_n^2 \end{pmatrix} \Rightarrow W_{rs} = \omega_r^2\delta_{rs} \Rightarrow W = \tilde{A}UA$$

* 23/01/24

Oscilações 1D em rede periódica

→ Proposta de solução: $\eta_j(t) = \sum_{r=1}^{r_m} \sum_{s=\pm 1} \left[\alpha_{sr} e^{i(k_r j a - s\omega_r(k_r)t)} + \alpha_{sr}^* e^{-i(k_r j a - s\omega_r(k_r)t)} \right]$ $(j = 0, \ldots, n+1)$

↳ a: parâmetro de rede

↳ K: constante elástica

↳ ω_r: autofrequência do r-ésimo modo

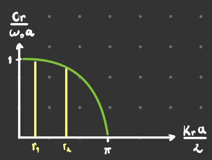
↳ K_r: nº de onda do r-ésimo modo

→ Fase: $\phi_{jrs}(t) = k_r j a - s\omega_r t = (r, j, s)$

→ Relação de dispersão: $\omega_r(k_r) = 2\omega_0 \left| \operatorname{sen}\left(\frac{1}{2} k_r a\right) \right|$, $\omega_0 = \sqrt{\frac{k}{m}}$, $0 \leq k_r \leq \frac{\pi}{a}$, $0 \leq \omega_r \leq 2\omega_0$

$\omega_r(-k_r) = \omega_r(k_r)$

→ Velocidade de fase: $c_r(k_r) = \left| \frac{\omega_r}{k_r} \right| = \omega_0 a \left| \frac{\operatorname{sen}(k_r a/2)}{(k_r a/2)} \right|$



→ Condição de realidade: $\begin{cases} \eta_j^*(t) = \eta_j(t) \\ \alpha_{sr} = \alpha_{s,-r}^* \\ k_r = -k_{-r} \\ \omega_r(k_r) = -\omega_{-r}(-k_r) \end{cases}$

→ Condições de contorno:

(A) Extremos fixos: $\eta_0(t) = \eta_{n+1}(t) = 0$

(B) Extremos livres: $U_0 = U_{n+1} = 0$ → CM livre

(C) Extremos absorventes: não são permitidas reflexões nas bordas → emula rede infinita

(D) Condições de contorno periódicas: $\eta_0(t) = \eta_{n+1}(t) \neq 0$

Considerando o caso A:

$$\eta_j(t) = \sum_{r=1}^{r_m} \sum_{s=\pm 1} \left[\alpha_{sr} e^{i(k_r j a - s\omega_r t)} + \alpha_{sr}^* e^{-i(k_r j a - s\omega_r(k_r)t)} \right] \xrightarrow[s \to -s]{(2)} \eta_j(t) = \sum_{r=1}^{r_m} \sum_{s=\pm 1} \left[\alpha_{sr} e^{i(k_r j a - s\omega_r t)} + \alpha_{-sr}^* e^{-i(k_r j a + s\omega_r(k_r)t)} \right] \quad (j = 0, \ldots, n+1)$$

Então:

$$\eta_0(t) = \sum_{r=1}^{r_m} \sum_{s=\pm 1} \left[(\alpha_{sr} + \alpha_{-sr}^*) e^{-is\omega_0 t} \right] = 0$$

$$\eta_{n+1}(t) = \sum_{r=1}^{r_m} \sum_{s=\pm 1} \left[\alpha_{sr} e^{ik_r(n+1)a} + \alpha_{-sr}^* e^{-ik_r(n+1)a} \right] e^{-is\omega_r t} = 0$$

Ou seja:

1) $\alpha_{sr} = -\alpha^*_{-sr}$

Com isto: $\eta_j(t) = 2i \sum_{r=1}^{n} \sum_{s=\pm 1} \alpha_{sr} \operatorname{sen}(k_r j a)\, e^{-i\omega_r t}$ $\quad (j = 0, \dots, n+1)$ $\rightarrow \eta_0(t) = 0$

2) $\eta_{n+1} = 2i \sum_{r=1}^{n} \sum_{s=\pm 1} \alpha_{sr} \operatorname{sen}\left[k_r (n+1) a\right] e^{-i\omega_r t} = 0$

$$\operatorname{sen}\left[k_r (n+1) a\right] = 0 \Rightarrow \boxed{k_r = \frac{r\pi}{(n+1)a}} \quad (r = 1, \dots, n),\ \eta_j \neq 0$$

→ Condições iniciais: $\alpha_{sr} = \dfrac{\beta_{sr}}{2i}$ $\quad (\beta_{sr} = \beta^*_{-sr})$

Escreve-se (4.31) como

$$\boxed{\begin{aligned} \eta_j(t) &= \sum_{r=1}^{n} \sum_{s=\pm 1} \beta_{sr} \rho_{jr} e^{-is\omega_r t} = \sum_{r=1}^{n} \rho_{jr} \xi_r(t) \\ \xi_r(t) &= \mu_r \cos(\omega_r t) - \nu_r \operatorname{sen}(\omega_r t) \end{aligned}} \quad (j = 1, \dots, n)$$

partícula

autovetor ρ_j: $\quad \rho_j = \begin{pmatrix} \rho_{1j} \\ \vdots \\ \rho_{nj} \end{pmatrix} \Rightarrow \rho_{rj} = F_j \operatorname{sen}\left(\dfrac{r j \pi}{n+1}\right)$ $\quad (j, r = 1, \dots, n)$

modo

Onde: $\langle \rho_j, \rho_k \rangle = \tilde{\rho}_j \rho_k = \sum_{r=1}^{n} \rho_{rj} \rho_{rk} = \delta_{jk} \Rightarrow F_j = \sqrt{\dfrac{2}{n+1}}$

$$\therefore \boxed{\rho_{jr} = \sqrt{\frac{2}{n+1}} \operatorname{sen}\left(\frac{r j \pi}{n+1}\right)} \quad (r, j = 1, \dots, n)$$

Dadas as C.I.: $\eta_{r_0} = \eta_r(t=0)$, $\dot{\eta}_{r_0} = \dot{\eta}_r(t=0)$ $\quad (r = 1, \dots, n)$

$$\boxed{\mu_r = \sum_{\ell=1}^{n} \rho_{\ell r} \eta_{\ell 0}, \quad \nu_r = -\frac{1}{\omega_r} \sum_{\ell=1}^{n} \rho_{\ell r} \dot{\eta}_{\ell 0}}$$

E as autofrequências:

$$\boxed{\omega_r = 2\omega_0 \operatorname{sen}\left(\frac{r\pi}{2(n+1)}\right)}$$

→ Retornando à Lagrangiana:

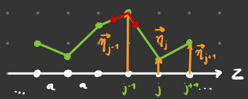
$$L = \frac{1}{2} m \sum_{j=1}^{n} \dot{\eta}_j^2 - \frac{1}{2} K \sum_{j=1}^{n} (\eta_{j+1} - \eta_j)^2 \quad (\eta_{n+1} = \eta_0 = 0)$$

$$\Big\downarrow (4.33)$$

$$\boxed{L = \frac{1}{2} \sum_{j=1}^{n} \left(\dot{\xi}_j^2 - \omega_j^2 \xi_j\right)}$$ $\quad \xi_j$: coordenadas normais

※ 31/01/2024

Oscilações Arbitrárias em Redes Periódicas 1D



Equilíbrio: $\vec{r}_{0j} = (d_1 + ja)\hat{z}$ $\quad (j = 0, \dots, n+1)$

Suposições:
1) Rede 1D com n+2 $(n \geq 1)$ osciladores acoplados
2) Partículas fixas nas bordas $(\vec{\eta}_0 = \vec{\eta}_{n+1} = 0)$
3) Forças internas centrais e restauradoras e entre vizinhos imediatos

Sistema perturbado: $\vec{r}_j(t) = \vec{r}_{0j} + \vec{\eta}_j(t)$ $\quad (j = 0, \dots, n+1)$

sendo

$$\vec{\eta}_j = \sum_{i=1}^{3} \eta_{j,i}\,\hat{x}_i = \eta_{j,x}\hat{x} + \eta_{j,y}\hat{y} + \eta_{j,z}\hat{z}$$
$$\vec{\eta}_0(t) = \vec{\eta}_{n+1}(t) = 0$$

Forças centrais: $\vec{F}_{jk} = f(r_{jk})\,\hat{r}_{jk}$, $\quad \hat{r}_{jk} = \frac{\vec{r}_{jk}}{r_{jk}}$, $\quad \vec{r}_{jk} = \vec{r}_k - \vec{r}_j$

$$\vec{F}_j = \vec{F}_{j,j-1} + \vec{F}_{j,j+1}$$

Forças elásticas: $f(r_{jk}) = k(r_{jk} - a) \Rightarrow \vec{F}_{jk} = k(r_{jk} - a)\,\hat{r}_{jk}$

Para a j-ésima partícula: $\delta_{\pm x_i} = \dfrac{\eta_{j\pm1,x_i} - \eta_{j,x_i}}{a}$

Pequena amplitude: $|\delta_{\pm x_i}| \ll 1$

Neste caso, $\vec{F}_{j,j\pm1} \approx ka\left(\pm\delta_{\pm x}\delta_{\pm z}\,\hat{z} \pm \delta_{\pm y}\delta_{\pm z}\,\hat{y} + \delta_{\pm z}\,\hat{z}\right)$

Modelo da corda carregada: $f(r_{jk}) = \tau_j$ (módulo da força de tensão da corda sobre a j-ésima partícula)

$$\vec{F}_{j,j-1} = \tau_j\,\hat{r}_{j,j-1}, \quad \vec{F}_{j-1,j} = -\vec{F}_{j,j-1}$$
$$\vec{F}_{j,j+1} = \tau_{j+1}\,\hat{r}_{j,j+1}$$

Força total: $\vec{F}_j \approx (\tau_j\delta_{-x} + \tau_{j+1}\delta_{+x})\,\hat{x} + (\tau_j\delta_{-y} + \tau_{j+1}\delta_{+y})\,\hat{y}$

$$\vec{F}_{j,j\pm1} = -\nabla_j U_{j,j\pm1}$$

↳ $U_{j,j-1} = \frac{1}{2a}\tau_j\left[(\eta_{j-1,x} - \eta_{j,x})^2 + (\eta_{j-1,y} - \eta_{j,y})^2\right]$

$U_{j,j+1} = \frac{1}{2a}\tau_{j+1}\left[(\eta_{j+1,x} - \eta_{j,x})^2 + y^2\right]$

↳ $\eta_{0,x_i} = \eta_{n+1,x_i} = 0$

$$U^{(int)} = \sum_{j=0}^{n} U_{j,j+1} = \sum_{j=0}^{n} \frac{\tau_{j+1}}{2a}\left[(\eta_{j+1,x} - \eta_{j,x})^2 + y^2\right]$$

Energia cinética: $T = \frac{1}{2}\sum_{j=1}^{n} m_j(\dot{\eta}_{j,x}^2 + \dot{\eta}_{j,y}^2)$

Lagrangiana: $L = \frac{1}{2}\sum_{j=1}^{n} m_j(\dot{\eta}_{j,x}^2 + \dot{\eta}_{j,y}^2) - \frac{1}{2a}\sum_{j=0}^{n}\tau_{j+1}\left[(\eta_{j+1,x} - \eta_{j,x})^2 + (\eta_{j+1,y} - \eta_{j,x})^2\right]$

→ Se $m_j = m$ e $\tau_j = \tau$, $\forall j$: $L = \frac{m}{2}\sum_{j=1}^{n}(\dot{\eta}_{j,x}^2 + \dot{\eta}_{j,y}^2) - \frac{\tau}{2a}\sum_{j=0}^{n}\left[(\eta_{j+1,x} - \eta_{j,x})^2 + y^2\right]$ (4.35b)

→ Equações de Euler-Lagrange: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\eta}_{j,x}} - \dfrac{\partial L}{\partial \eta_{j,x}} = 0$ (e para y)

$$(4.23) \xrightarrow[k \to \tau]{\eta_{j,y} = 0 \text{ e } \eta_{j,x} \neq 0} (4.35b)$$

Corda Contínua

* Procedimento: $n \to \infty$

$a \to 0,\quad a = \Delta x \xrightarrow{n\to\infty} dx \Rightarrow a\,dx = \ell$ — comprimento da corda

$m \to \Delta m \xrightarrow[\Delta x \to 0]{n\to\infty} dm$

$\lambda = \frac{dm}{dx}$ (densidade linear de massa)

Introdução a uma Teoria de Campos Clássica

Discreto: $L = \frac{1}{2}\sum_{j=1}^{N} m_j \dot{\eta}_j^2 - \frac{1}{2a}\sum_{j=0}^{N} \tau_{j+1}(\eta_{j+1}-\eta_j)^2$, onde $\eta_j = \eta_j(t)$ $(j = 1, \dots, N)$ e $\eta_0 = \eta_{N+1} = 0$

Contínuo: $\eta_j(t) \to \eta(x_j, t)$

$\dot{\eta}_j(t) \to \frac{\partial}{\partial t}\eta(x_j, t)$

$$L = \sum_{j=1}^{N} \Delta x \, \frac{1}{2}\frac{m_j}{\Delta x}\left[\frac{\partial}{\partial t}\eta(x_j,t)\right]^2 - \sum_{j=0}^{N} \Delta x \, \frac{\tau_{j+1}}{2}\left[\frac{\eta(x_j+\Delta x, t) - \eta(x_j, t)}{\Delta x}\right]^2$$

Agora: $N \to \infty$

$\Delta x \to dx \rightsquigarrow x_j\ (j = 1, \dots, N) \to 0 \leq x \leq \ell$

$m_j \to \Delta m_j$

$\lim\limits_{\substack{\Delta m \to 0 \\ \Delta x \to 0}} \frac{\Delta m_j}{\Delta x} = \frac{dm}{dx} \doteq \lambda(x)$

$\tau_{j+1} \to \tau(x)$

$\eta(x_j, t) \to \eta(x, t)$: campo de oscilações da corda no espaço 1+1 (espaço + tempo)

$$\lim_{\Delta x \to 0} \frac{\eta(x_j+\Delta x, t) - \eta(x_j, t)}{\Delta x} = \frac{\partial \eta(x,t)}{\partial x}$$

$$\lim_{\substack{N\to\infty \\ \Delta x \to 0}} \sum_{j=0,1}^{N} \Delta x = \int_0^\ell dx$$

Lagrangiana da corda contínua: $\boxed{L = \int_0^\ell dx\, \mathcal{L}\left\{\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial t}, x\right\}}$

onde $\boxed{\mathcal{L}\left\{\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial t}, x\right\} = \frac{1}{2}\lambda(x)\left(\frac{\partial \eta}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial \eta}{\partial x}\right)^2}$ é a densidade lagrangiana (por comprimento) da corda contínua

⁂ 05/02/24

Integral de Ação

→ $S = \int_{t_1}^{t_2} dt\, L$

→ Em geral: $\mathcal{L} = \mathcal{L}\{\eta, \eta_x, \eta_t; x, t\}$ → $S = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx\, \mathcal{L}\{\eta, \eta_x, \eta_t; x, t\}$

∗ Princípio de Hamilton: $\delta S = 0$ (extremum)

Com limites fixos: $\begin{cases} \delta\eta(x, t_1) = \delta\eta(x, t_2) = 0 \\ \delta\eta(x_1, t) = \delta\eta(x_2, t) = 0 \end{cases}$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\eta}\delta\eta + \frac{\partial\mathcal{L}}{\partial\eta_x}\delta\eta_x + \frac{\partial\mathcal{L}}{\partial\eta_t}\delta\eta_t$$

$$\delta\eta = \eta_2(x,t) - \eta_1(x,t) \Rightarrow \delta\eta_x = \frac{\partial\eta_2}{\partial x} - \frac{\partial\eta_1}{\partial x} = \frac{\partial}{\partial x}(\eta_2 - \eta_1) = \frac{\partial}{\partial x}(\delta\eta) \;\Big|\; \delta\eta_t = \frac{\partial}{\partial t}(\delta\eta)$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\eta}\delta\eta + \frac{\partial\mathcal{L}}{\partial\eta_x}\frac{\partial}{\partial x}(\delta\eta) + \frac{\partial\mathcal{L}}{\partial\eta_t}\frac{\partial}{\partial t}(\delta\eta)$$

Mas $\int_{x_1}^{x_2} dx \frac{\partial\mathcal{L}}{\partial\eta_x}\frac{\partial}{\partial x}(\delta\eta) = \frac{\partial\mathcal{L}}{\partial\eta_x}\cancel{(\delta\eta)}^{0}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\eta_x}\right)(\delta\eta_x)$, e analogamente para t. Portanto,

$$\delta S = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left[\frac{\partial\mathcal{L}}{\partial\eta} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\eta_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\eta_t}\right)\right](\delta\eta) = 0$$

Resulta então a Equação de Campo: $\boxed{\frac{\partial\mathcal{L}}{\partial\eta} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\eta_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\eta_t}\right) = 0}$

Equação da corda contínua homogênea

→ $\lambda = \frac{m}{\ell} = \frac{dm}{dx} = cte$, $\tau = cte$

→ $\mathcal{L} = \frac{1}{2}\lambda\eta_x^2 - \frac{1}{2}\tau\eta_t^2$

→ $\frac{\partial\mathcal{L}}{\partial\eta_x} = \lambda\eta_x$, $\frac{\partial\mathcal{L}}{\partial\eta_t} = -\tau\eta_t$ ⇒

$$\frac{\partial}{\partial x}(-\tau\eta_x) + \frac{\partial}{\partial t}(\lambda\eta_t) = 0$$

$$-\tau\eta_{xx} + \lambda\eta_{tt} = 0$$

$$\boxed{\lambda\frac{\partial^2\eta}{\partial t^2} - \tau\frac{\partial^2\eta}{\partial x^2} = 0}$$

Generalizações

→ N campos em 3+1 D: $\eta_i(\vec{r}, t)$

→ $\mathcal{L} = \mathcal{L}\{\eta_1, \dots, \eta_N; \frac{\partial\eta_1}{\partial t}, \dots, \frac{\partial\eta_N}{\partial t}, \nabla\eta_1, \dots, \nabla\eta_N; \vec{r}, t\}$ ⟶ $\frac{\partial\mathcal{L}}{\partial\eta_\alpha} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\eta_{\alpha,t}} - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial(\nabla\eta_\alpha)} = 0$ $(\alpha = 1, \dots, N)$

Soluções da eq. da onda para a corda homogênea

$$
\begin{cases}
\dfrac{\partial^2 \eta}{\partial t^2} - c^2 \dfrac{\partial^2 \eta}{\partial x^2} = 0 & \left(c = \sqrt{\dfrac{\tau}{\lambda}} \right) \\
\eta(0,t) = \eta(\ell,t) = 0 \\
\eta(x,0) = \phi(x), \quad \eta_t(x,0) = \Psi(x)
\end{cases}
$$

→ Método de Fourier: $\eta(x,t) = X(x)\psi(t)$

$$X\ddot{\psi} - c^2 X''\psi = 0$$

$$\frac{\ddot{\psi}}{c^2\psi} - \frac{X''}{X} = 0 \Rightarrow \frac{\ddot{\psi}}{c^2\psi} = \frac{X''}{X} = -k^2$$

$$
\begin{cases}
\ddot{\psi} + k^2 c^2 \psi = 0 \\
X'' + k^2 X = 0 \rightarrow X(x) = A\cos(kx) + B\,\mathrm{sen}(kx)
\end{cases}
$$

→ C.C.: $X(0) = A = 0 \Rightarrow X(x) = B\,\mathrm{sen}(kx)$

$X(\ell) = B\,\mathrm{sen}(k\ell) = 0 \Rightarrow k_r = \frac{r\pi}{\ell}, \quad r = 0, 1, 2, \dots$

$\therefore X_r(x) = B_r\,\mathrm{sen}(k_r x)$

→ $\psi(t) = C\cos(kct) + D\,\mathrm{sen}(kct) \Rightarrow \psi_r(t) = C_r\cos(k_r ct) + D_r\,\mathrm{sen}(k_r ct)$

→ Portanto: $\eta(x,t) = \sum_{r=1}^{\infty} \left[\mu_r \cos(k_r ct) - \nu_r\,\mathrm{sen}(k_r ct) \right] \mathrm{sen}(k_r x)$

→ C.I.: $\int_0^\ell dx\,\mathrm{sen}(k_r x)\,\mathrm{sen}(k_m x) = \int_0^\ell dx\,\cos(k_r x)\cos(k_m x) = \frac{1}{2}\ell\delta_{rm}$

$$\eta(x,0) = \sum_{r=1}^{\infty} \mu_r\,\mathrm{sen}(k_r x) = \Phi(x)$$

$$\eta_t(x,0) = -\sum_{r=1}^{\infty} c k_r \nu_r\,\mathrm{sen}(k_r x) = \Psi(x)$$

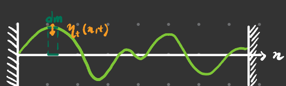
$$\sum_{r=1}^{\infty} \mu_r \int_0^\ell dx\,\mathrm{sen}(k_r x)\,\mathrm{sen}(k_m x) = \int_0^\ell \Phi(x)\,\mathrm{sen}(k_m x)\,dx \Rightarrow \boxed{\begin{aligned} \mu_r &= \frac{2}{\ell}\int_0^\ell dx\,\phi(x)\,\mathrm{sen}(k_r x) \\ \nu_r &= -\frac{2}{\ell\omega_r}\int_0^\ell dx\,\Psi(x)\,\mathrm{sen}(k_r x) \\ \omega_r &= k_r c = \frac{r\pi}{\ell}c \end{aligned}}$$

Energia de uma corda oscilante

→ Oscilações da corda "livre": $\eta(x,t) = \sum_{r=1}^{\infty} \psi_r(t)\,\mathrm{sen}(k_r x)$

$$\psi_r(t) = \mu_r\cos(\omega_r t) - \nu_r\,\mathrm{sen}(\omega_r t)$$

$$E = T_{corda} + U_{corda} = cte$$



→ Energia cinética de um elemento de massa dm:

$$dT = \frac{1}{2}dm\left(\frac{\partial\eta}{\partial t}\right)^2, \quad \frac{dm}{dx} = \lambda \rightarrow dm = \lambda dx$$

$$T = \int dT = \frac{1}{2}\lambda\int_0^\ell dx\left(\frac{\partial\eta}{\partial t}\right)^2 = \frac{1}{2}\lambda\sum_{r,s=1}^{\infty}\dot{\psi}_r\dot{\psi}_s\int_0^\ell dx\,\mathrm{sen}(k_r x)\,\mathrm{sen}(k_s x) \Rightarrow T(t) = \frac{1}{4}\lambda\ell\sum_{r=1}^{\infty}\dot{\psi}_r^2(t)$$

→ Energia potencial da corda carregada:

$$U = \sum_{j=0}^{N}\frac{1}{2}\tau\left(\frac{\eta_{j+1}-\eta_j}{\Delta x}\right)^2\Delta x \rightarrow U = \int_0^\ell dx\,\frac{1}{2}\tau\left(\frac{\partial\eta}{\partial x}\right)^2 \Rightarrow U(t) = \frac{1}{4}\tau\ell\sum_{r=1}^{\infty}k_r^2\psi_r^2(t)$$

→ $E = T(t) + U(t) \Rightarrow \boxed{E = \frac{1}{4}\lambda\ell\sum_{r=1}^{\infty}\omega_r^2\left(\mu_r^2 + \nu_r^2\right)}$

deformações iniciais (U) ; velocidades iniciais (T)

Teorema do Virial:

$$\langle T\rangle = \overbrace{\frac{\omega_1}{2\pi}}^{1/T_1}\int_0^{\frac{2\pi}{\omega_1}} dt\,T(t) = \frac{1}{2}E$$

$$\langle U\rangle = \frac{\omega_1}{2\pi}\int_0^{\frac{2\pi}{\omega_1}} dt\,U(t) = \frac{1}{2}E$$

$$\therefore \boxed{\langle T\rangle = \langle U\rangle = \frac{1}{2}E} \quad (U \propto r^2)$$

※ 09/02/24

Eq. da onda em meios dispersivos

→ Eq. da onda no vácuo: $\frac{\partial^2\eta}{\partial t^2} - c^2\frac{\partial^2\eta}{\partial x^2} = 0$, C.I.: $\begin{cases}\eta(x,0) = \phi(x)\\ \eta_t(x,0) = \psi(x)\end{cases}$

* Método das transformadas integrais:

Transformação de Fourier: $\eta(x,t) \xrightarrow{T.F.} \eta(k,\omega)$

nº de onda

freq. angular

Dada $\eta(x,t)$, a função dual no espaço de Fourier: $\eta(k,\omega) = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dt\ \eta(x,t)\,e^{-i(kx-\omega t)}$, com $(x,t)\in\mathbb{R}$, $(k,\omega)\in\mathbb{C}$

Condição necessária: $\eta(x\to\pm\infty, t\to\pm\infty)\to 0$

Transformação inversa: $\eta(x,t) = \left(\frac{1}{2\pi}\right)^2\iint dk\,d\omega\ \overbrace{\eta(k,\omega)\,e^{i(kx-\omega t)}}^{\eta_{k\omega}(x,t)}$

Relação de dispersão

→ Aplicamos a T.F. sobre a equação da onda:

$$\int_{-\infty}^{+\infty}dt\,e^{i\omega t}\frac{\partial\eta}{\partial t} \overset{(I.\ Partes)}{=} -i\omega\int_{-\infty}^{+\infty}dt\,\eta\,e^{i\omega t} = -i\omega\eta \qquad \frac{\partial}{\partial t}\to -i\omega$$

$$\int_{-\infty}^{+\infty}dx\,e^{-ikx}\frac{\partial\eta}{\partial x} = ik\int_{-\infty}^{+\infty}dx\,\eta\,e^{-ikx} = ik\eta \qquad \frac{\partial}{\partial x}\to ik$$

$$\Rightarrow \left(\frac{\partial^2\eta}{\partial t^2} - c^2\frac{\partial^2\eta}{\partial x^2}\right) = 0 \to (\omega^2 - k^2c^2)\,\eta(k,\omega) = 0$$

Mas $\eta(k,\omega)\neq 0 \Rightarrow \boxed{\omega^2 - k^2c^2 = 0}$ Equação de dispersão

↳ as soluções são as relações de dispersão

→ $k = k(\omega)$ ou $\omega = \omega(k)$ $\Rightarrow$ $\boxed{\omega = \omega_k^\sigma \equiv \omega_\sigma(k) = \sigma kc}$ $(\sigma = \pm 1)$

→ $\eta(k,\omega) = 0\ \ \forall\ \omega\neq\omega_\sigma(k)$

$u_\sigma(k) = \frac{1}{2\pi}\eta(k,\omega_k^\sigma)$

→ Neste caso: $\eta(k,\omega) = \sum_\sigma \eta(k,\omega_k^\sigma)\,\delta(\omega-\omega_k^\sigma) \equiv 2\pi\sum_\sigma u_\sigma(k)\,\delta(\omega-\omega_k^\sigma)$

$$\eta(x,t) = \frac{1}{2\pi}\sum_\sigma\int_{-\infty}^{\infty}dk\,\underbrace{u_\sigma(k)}\,e^{i(kx-\omega t)}$$

↳ amplitude espectral do modo $\sigma\in\mathbb{C}$

→ Condição de realidade: $\eta(x,t) = \eta^*(x,t)$

$$\underbrace{\int_{-\infty}^{+\infty}dt\ u_\sigma^*(k)\,e^{-i(kx-\omega_k^{\sigma*}t)}}_{k\to -k} = \int_{-\infty}^{\infty}dt\ u_\sigma(k)\,e^{i(kx-\omega_k^\sigma t)}$$

Condições suficientes: $u_\sigma(k) = u_\sigma^*(-k)$

$\omega_k^\sigma = -\omega_{-k}^{\sigma*} \Rightarrow \begin{cases}\mathrm{Re}\,\omega_k^\sigma = -\mathrm{Re}\,\omega_{-k}^\sigma & \to \text{propagação}\\ \mathrm{Im}\,\omega_k^\sigma = \mathrm{Im}\,\omega_{-k}^\sigma & \to \text{dissipação/perturbação}\end{cases}$

Condições iniciais e amplitude espectral

$\rightarrow \eta(x,0) = \phi(x), \quad \eta_t(x,0) = \psi(x)$

$\rightarrow \eta_t(x,t) = -\frac{i}{2\pi}\sum_\sigma \int_{-\infty}^{\infty} dk\, \omega_k^\sigma u_\sigma(k) e^{i(kx-\omega_k^\sigma t)}$

Então:

$$\eta(x,0) = \frac{1}{2\pi}\sum_\sigma \int dk' \int dx\, u_\sigma e^{ik'x} e^{-ikx} = \int dx\, e^{-ikx}\phi(x)$$

$$\eta_t(x,0) = -\frac{i}{2\pi}\sum_\sigma \int dk' \int dx\, \omega_k^\sigma u_\sigma e^{ik'x} e^{-ikx} = \int dx\, e^{-ikx}\psi(x)$$

$$\int_{-\infty}^{\infty} e^{i(k'-k)x} = 2\pi\delta(k'-k)$$

Resultando

$$\begin{cases} \sum_\sigma u_\sigma(k) = \int_{-\infty}^{\infty} dx\, \phi(x) e^{-ikx} \\ \sum_\sigma \omega_k^\sigma u_\sigma(k) = i\int_{-\infty}^{+\infty} dx\, \psi(x) e^{-ikx} \end{cases}$$

Em muitas situações, $\sigma = \pm 1$:

$$u_\sigma(k) = \frac{1}{2}\int_{-\infty}^{\infty} dx\, e^{-ikx}\left[\eta(x,0) + \frac{i\sigma}{\omega_k}\eta_t(x,0)\right] \quad (\sigma = \pm 1)$$

* Velocidade de fase:

Um dado modo normal de oscilação: $\eta_{k,\sigma}(x,t) = u_\sigma(k) e^{i(kx-\omega_k^\sigma t)} = \underbrace{u_\sigma(k) e^{\operatorname{Im}\omega_k^\sigma t}}_{\text{Amplitude de modulação}} \underbrace{e^{i(kx - \operatorname{Re}\omega_k^\sigma t)}}_{\text{Parte oscilante}}$

$\operatorname{Max} \eta_{k,\sigma}(t) = u_\sigma(k) e^{\operatorname{Im}\omega_k^\sigma t}, \quad \operatorname{Im}\omega_k^\sigma \leq 0$ (em geral)

$\rightarrow$ Fase: $\phi(x,t) = kx - \operatorname{Re}\omega_k^\sigma t$

Fase cte: $\phi = \text{cte} = kx(t) - \operatorname{Re}\omega_k^\sigma t$

$$\frac{d\phi}{dt} = 0 = k\underbrace{\frac{dx}{dt}}_{v_\phi} - \operatorname{Re}\omega_k^\sigma \Rightarrow v_{\phi,\sigma}(k) = \frac{\operatorname{Re}\omega_k^\sigma}{k}$$

Se $v_{\phi,\sigma} = \text{cte}$ (meio não dispersivo) $\rightarrow$ forma da perturbação é mantida

$v_{\phi,\sigma} = v_{\phi,\sigma}(k)$ (meio dispersivo) $\rightarrow$ forma da perturbação muda dependendo de k

Solução em meio não dispersivo

$\rightarrow$ Corda uniforme no vácuo: $\omega_k^\sigma = \sigma k c \quad (\sigma = \pm 1) \Rightarrow v_{\phi,\sigma} = \sigma c$

$$\eta(x,t) = \frac{1}{2\pi}\left[\underbrace{\int dk\, u_+(k) e^{ik(x-ct)}}_{f(x-ct)} + \underbrace{\int dk\, u_- e^{ik(x+ct)}}_{g(x+ct)}\right]$$

Meios dispersivos

* Corda oscilando em meio viscoso: $\frac{\partial^2 \eta}{\partial t^2} + 2D\frac{\partial \eta}{\partial t} - c^2 \frac{\partial^2 \eta}{\partial x^2} = 0$

→ aplicando TF: $\frac{\partial}{\partial t} \to -i\omega$, $\frac{\partial}{\partial x} \to ik \longrightarrow (\omega^2 + 2iD\omega - k^2c^2)\,\eta(k,\omega) = 0$

→ Relações de dispersão: $\boxed{\omega_k^\sigma = \sigma\sqrt{k^2c^2 - D^2} - iD}$ $(\sigma = \pm 1)$

Região 1: $|k| < D/c \to \omega_k^\sigma = i\left(\sigma\sqrt{D^2 - k^2c^2} - D\right)$ $\begin{cases} \text{Re}\,\omega_k^\sigma = 0 \to \text{sem propagação} \\ \text{Im}\,\omega_k^\sigma < 0\ \forall \sigma \to \text{abrorção} \end{cases}$

Região 2: $|k| \geq D/c \to \omega_k^\sigma = \sigma\sqrt{k^2c^2 - D^2} - iD$ $\begin{cases} \text{Re}\,\omega_k^\sigma = \sigma\sqrt{k^2c^2 - D^2} \to \text{propagação} \\ \text{Im}\,\omega_k^\sigma = -D < 0 \to \text{abrorção} \end{cases}$

↳ $v_{\phi,\sigma}(k) = \frac{\sqrt{k^2c^2/D^2 - 1}}{kc/D}$