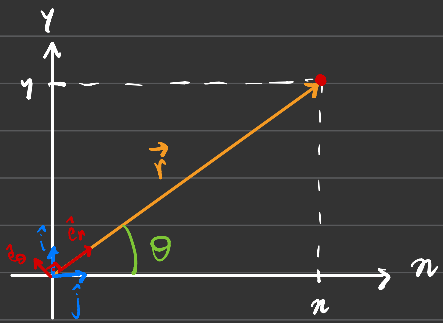
Coordenadas polares



Coordenada radial: $r \in [0, \infty)$

Coordenada angular: $\theta \in [0, 2\pi)$

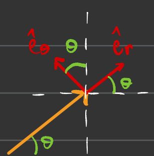
* $(r,\theta) \to (x,y)$: $x = r\cos\theta$
  $y = r\,\text{sen}\,\theta$

* $(x,y) \to (r,\theta)$: $r = \sqrt{x^2+y^2}$
  $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

direção do aumento do θ

* Vetores unitários: $\hat e_r \perp \hat e_\theta$ → dependem do tempo

* Posição: $\boxed{\vec r = r\hat e_r}$ → para derivar: $\hat e_r = \hat e_r(t)$ → $\vec v = (\dots)\hat e_r + (\dots)\hat e_\theta$



$|\hat e_r| = |\hat e_\theta| = 1$

$\hat e_r = \cos\theta\,\hat i + \text{sen}\,\theta\,\hat j$

$\hat e_\theta = -\text{sen}\,\theta\,\hat i + \cos\theta\,\hat j$

$$\left\}\begin{aligned} \dot{\hat e}_r &= -\text{sen}\,\theta\,\dot\theta\,\hat i + \cos\theta\,\dot\theta\,\hat j = \dot\theta\,\hat e_\theta \\ \dot{\hat e}_\theta &= -\cos\theta\,\dot\theta\,\hat i - \text{sen}\,\theta\,\dot\theta\,\hat j = -\dot\theta\,\hat e_r \end{aligned}\right.$$

* Velocidade: $\dot{\vec r} = \dot r\,\hat e_r + r\,\dot{\hat e}_r \Rightarrow \boxed{\dot{\vec r} = \dot r\,\hat e_r + r\dot\theta\,\hat e_\theta}$

* Aceleração: $\ddot{\vec r} = \ddot r\,\hat e_r + \dot r\,\dot{\hat e}_r + (\dot r\dot\theta + r\ddot\theta)\,\hat e_\theta + r\dot\theta\,\dot{\hat e}_\theta$

$= \ddot r\,\hat e_r + \dot r\dot\theta\,\hat e_\theta + \dot r\dot\theta\,\hat e_\theta + r\ddot\theta\,\hat e_\theta - r\dot\theta^2\,\hat e_r$

$$\boxed{\ddot{\vec r} = (\ddot r - r\dot\theta^2)\,\hat e_r + (r\ddot\theta + 2\dot r\dot\theta)\,\hat e_\theta}$$

* Exemplo: MCU
  → $r = \text{cte}$, $\dot\theta = \text{cte}$
  → $\vec a = (\cancel{\ddot r} - r\dot\theta^2)\,\hat e_r +$
  $+ (\cancel{r\ddot\theta} + \cancel{\dot r\dot\theta})\,\hat e_\theta$
  ⇒ $\boxed{\vec a = -r\dot\theta^2\,\hat e_r}$

Gradiente

→ $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ } taxa de variação da função ao longo das direções do espaço

→ $\vec{\nabla} f = \underbrace{F_r \hat{e}_r}_{\text{componente radial}} + \overbrace{F_\theta \hat{e}_\theta}^{\text{componente angular}}$

→ $\begin{cases} \hat{e}_r = \cos\theta\,\hat{i} + \operatorname{sen}\theta\,\hat{j} \\ \hat{e}_\theta = -\operatorname{sen}\theta\,\hat{i} + \cos\theta\,\hat{j} \end{cases}$ → $\begin{cases} \operatorname{sen}\theta\,\hat{e}_r = \operatorname{sen}\theta\cos\theta\,\hat{i} + \operatorname{sen}^2\theta\,\hat{j} \\ \cos\theta\,\hat{e}_\theta = -\operatorname{sen}\theta\cos\theta\,\hat{i} + \cos^2\theta\,\hat{j} \end{cases}$

$$\boxed{\operatorname{sen}\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = \hat{j}}$$

→ analogamente, $\boxed{\hat{i} = \cos\theta\,\hat{e}_r - \operatorname{sen}\theta\,\hat{e}_\theta}$

→ $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$, $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$

* $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

↳ $\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{r\cos\theta}{r} \Rightarrow \frac{\partial r}{\partial x} = \cos\theta$

↳ $\frac{\partial \theta}{\partial x} = \left(1 + \frac{y^2}{x^2}\right)^{-1}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\operatorname{sen}\theta}{r^2} \Rightarrow \frac{\partial \theta}{\partial x} = -\frac{\operatorname{sen}\theta}{r}$

$$\therefore \boxed{\frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\operatorname{sen}\theta}{r}\frac{\partial f}{\partial \theta}}$$

↳ analogamente, $\boxed{\frac{\partial f}{\partial y} = \operatorname{sen}\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}}$

$$\rightarrow \vec{\nabla} f = \left(\cos\theta \frac{\partial f}{\partial r} - \frac{\operatorname{sen}\theta}{r}\frac{\partial f}{\partial \theta}\right)\hat{i} + \left(\operatorname{sen}\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right)\hat{j}$$

$$= \left(\cos\theta \frac{\partial f}{\partial r} - \frac{\operatorname{sen}\theta}{r}\frac{\partial f}{\partial \theta}\right)\left(\cos\theta\,\hat{e}_r - \operatorname{sen}\theta\,\hat{e}_\theta\right) +$$

$$+ \left(\operatorname{sen}\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right)\left(\operatorname{sen}\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta\right)$$

$$= \left[\cos^2\theta \frac{\partial f}{\partial r} - \cancel{\frac{\operatorname{sen}\theta\cos\theta}{r}\frac{\partial f}{\partial \theta}} + \operatorname{sen}^2\theta \frac{\partial f}{\partial r} + \cancel{\frac{\cos\theta\operatorname{sen}\theta}{r}\frac{\partial f}{\partial \theta}}\right]\hat{e}_r +$$

$$+ \left[-\cancel{\cos\theta\operatorname{sen}\theta\frac{\partial f}{\partial r}} + \frac{\operatorname{sen}^2\theta}{r}\frac{\partial f}{\partial \theta} + \cancel{\operatorname{sen}\theta\cos\theta\frac{\partial f}{\partial r}} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial \theta}\right]\hat{e}_\theta$$

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta}$$

Elemento de área

→ $dxdy$: área infinitesimal em coordenadas polares

↳ $dxdy = |f(r,\theta)|\,drd\theta$

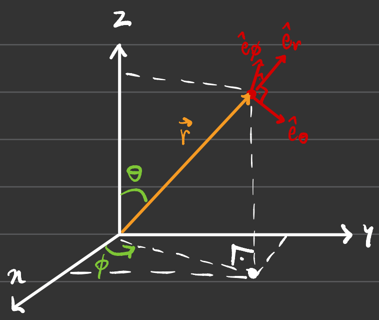
↳ fator decorrente da mudança de coordenadas

$$f(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \rightarrow \text{Jacobiano}$$

$$\rightarrow \left.\begin{array}{l} x = r\cos\theta \\ y = r\operatorname{sen}\theta \end{array}\right\} f(r,\theta) = \begin{vmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\operatorname{sen}^2\theta = r$$

$$\therefore \boxed{dxdy = r\,dr\,d\theta}$$

Coordenadas esféricas



$$\begin{cases} r \in [0, \infty) \\ \phi \in [0, 2\pi) \\ \theta \in [0, \pi] \end{cases}$$

$* \ (r, \theta, \phi) \rightarrow (x, y, z)$

$$\begin{cases} x = r \operatorname{sen}\theta \cos\phi \\ y = r \operatorname{sen}\theta \operatorname{sen}\phi \\ z = r\cos\theta \end{cases}$$

$* \ (x, y, z) \rightarrow (r, \theta, \phi)$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \varphi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

26/06/23

* Vetores unitários:

$$\rightarrow \hat{e}_r \begin{cases} (\hat{e}_r)_z = \cos\theta \\ (\hat{e}_r)_x = \text{sen}\,\theta \cos\phi \\ (\hat{e}_r)_y = \text{sen}\,\theta\, \text{sen}\,\phi \end{cases} \rightarrow \boxed{\hat{e}_r = \text{sen}\,\theta \cos\phi\, \hat{i} + \text{sen}\,\theta\, \text{sen}\,\phi\, \hat{j} + \cos\theta\, \hat{k}}$$

$$\rightarrow \hat{e}_\theta \begin{cases} (\hat{e}_\theta)_z = -\text{sen}\,\theta \\ (\hat{e}_\theta)_x = \cos\theta \cos\phi \\ (\hat{e}_\theta)_y = \cos\theta\, \text{sen}\,\phi \end{cases} \rightarrow \boxed{\hat{e}_\theta = \cos\theta \cos\phi\, \hat{i} + \cos\theta\, \text{sen}\,\phi\, \hat{j} - \text{sen}\,\theta\, \hat{k}}$$

$$\rightarrow \boxed{\hat{e}_\phi = -\text{sen}\,\phi\, \hat{i} + \cos\phi\, \hat{j}}$$

* Velocidade: $\boxed{\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\phi}\, \text{sen}\,\theta\, \hat{e}_\phi}$

* Aceleração: pág. 34 — Fowles

Gradiente

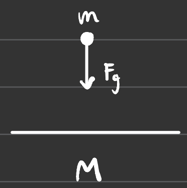
$$\rightarrow \boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\,\text{sen}\,\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi}$$

Elemento de volume

$$\rightarrow dx\,dy\,dz = |f(r,\theta,\phi)|\, dr\,d\theta\,d\phi \Rightarrow \boxed{dx\,dy\,dz = r^2\, \text{sen}\,\theta\, dr\,d\theta\,d\phi}$$

Gravitação

* Kepler: lei "harmônica" → $|F_g| \propto 1/r^2$



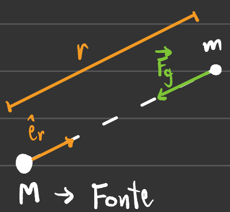
deve depender da massa

equivalência das massas gravitacional e inercial

→ $\frac{c}{r^2} = ma$ → $\frac{Km}{r^2} = ma$

→ Para satisfazer à 3ª lei: $\boxed{F_g \propto \frac{KmM}{r^2}}$

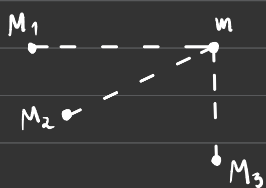
Lei da Gravitação



$$\boxed{\vec{F}_g = -\frac{GMm}{r^2}\hat{e}_r}$$

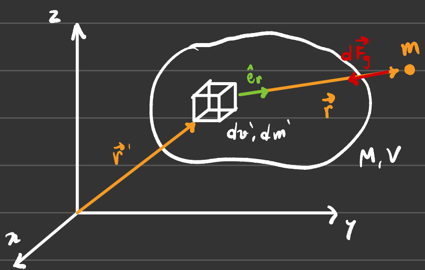
$$G = 6{,}67 \cdot 10^{-11} \frac{Nm^2}{Kg^2}$$

* Princípio de superposição:



$$\boxed{\vec{F}_g = \sum_{i=1}^{n} \vec{F}_{g,i}}$$

* Força de uma distribuição contínua:



Densidade: $\rho(\vec{r}\,') = \dfrac{dm'}{dv'}$

Força: $d\vec{F}_g = -\dfrac{Gm\,dm'}{r^2}\,\hat{e}_r$

→ Pelo princípio da superposição: $\vec{F}_g = \int_V d\vec{F}_g = -Gm\int_V \dfrac{dm'}{r^2}\,\hat{e}_r$

$$\boxed{\vec{F}_g = -Gm\int_V \frac{\rho(\vec{r})\,dv'}{r^2}\,\hat{e}_r}$$

Campo gravitacional

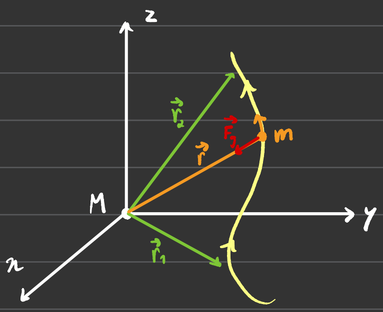
m → partícula teste

* Campo gravitacional:

$\vec{g} = \lim_{m\to 0} \dfrac{\vec{F}_g}{m}$ $\qquad [g] = \dfrac{m}{s^2}$

$$\boxed{\vec{g} = -G\int_V \frac{\rho(\vec{r})\,dv'}{r^2}\,\hat{e}_r}$$

Energia potencial gravitacional



$$W_g = \int_1^2 \vec{F}_g \cdot d\vec{r}$$

$$d\vec{r} = dr\,\hat{e}_r + dr_\theta\,\hat{e}_\theta + dr_\phi\,\hat{e}_\phi$$

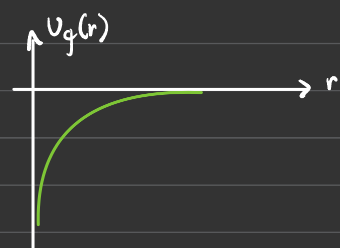
$$\Rightarrow W_g = -GMm \int_1^2 \frac{\hat{e}_r \cdot d\vec{r}}{r^2}$$

$$\rightarrow \hat{e}_r \cdot d\vec{r} = dr\,\hat{e}_r \cdot \hat{e}_r + \cancel{dr_\theta\,\hat{e}_r \cdot \hat{e}_\theta} + \cancel{dr_\phi\,\hat{e}_r \cdot \hat{e}_\phi} \Rightarrow \hat{e}_r \cdot d\vec{r} = dr$$

$$\rightarrow W_g = -GMm \int_1^2 \frac{dr}{r^2} = -GMm\left(-\frac{1}{r}\right)\Bigg|_{r_1}^{r_2} \Rightarrow \boxed{W_g = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

força conservativa

$$\rightarrow \vec{F}_g \text{ conservativa} \Leftrightarrow \exists\, U_g \text{ t.q. } \vec{F}_g = -\vec{\nabla} U_g$$



$$\hookrightarrow \vec{\nabla} U_g = \frac{\partial U_g}{\partial r}\hat{e}_r + \underbrace{\frac{1}{r}\frac{\partial U_g}{\partial \theta}\hat{e}_\theta + \frac{1}{r\,\text{sen}\,\theta}\frac{\partial U_g}{\partial \phi}\hat{e}_\phi}_{=0}$$

$$\hookrightarrow \vec{F}_g = -\frac{GMm}{r^2}\hat{e}_r \qquad \Rightarrow U_g = U_g(r)$$

escolha $A = 0$

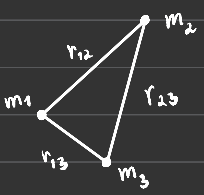
$$\rightarrow -\frac{GMm}{r^2}\hat{e}_r = -\frac{dU_g}{dr}\hat{e}_r \Rightarrow \boxed{U_g(r) = -\frac{GMm}{r} + A} \Rightarrow \boxed{W_g = -\Delta U_g}$$

$$\rightarrow \begin{cases} \Delta U_g > 0 \quad (r_2 > r_1) \rightarrow \text{m se afasta de M} \quad (W_g < 0) \\ \Delta U_g < 0 \quad (r_2 < r_1) \rightarrow \text{m se aproxima de M} \quad (W_g > 0) \end{cases}$$

Energia potencial gravitacional

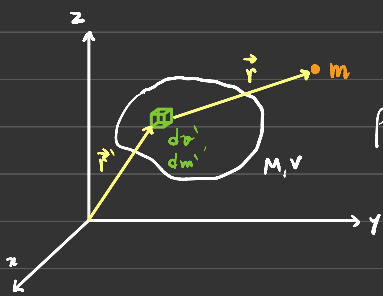
$\rightarrow \; U_g(r) = -\dfrac{GMm}{r}$

$\rightarrow \; W_g = -\Delta U_g$



$$U_{g,sist} = U_g(r_{12}) + U_g(r_{13}) + U_g(r_{23})$$

n partículas: $\boxed{U_{g,sist} = \dfrac{1}{2} \displaystyle\sum_{\substack{\alpha,\beta=1 \\ (\alpha\neq\beta)}}^{n} U_g(r_{\alpha\beta})}$

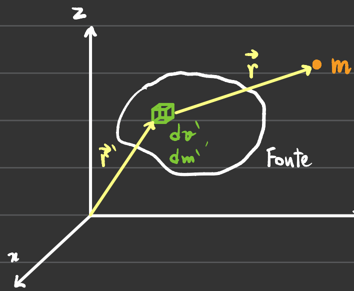


$$\rho(\vec{r}') = \frac{dm'}{dv'}$$

$$dU_g = -\frac{Gmm'}{r}$$

$$\boxed{U_g = -Gm\int_V \frac{\rho(\vec{r}')\,dv'}{r}}$$

Potencial gravitacional



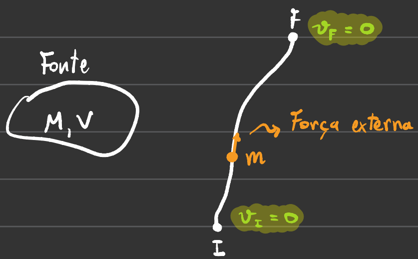
m → partícula teste
(x, y, z)

$$\boxed{\Phi = \lim_{m \to 0} \frac{U_g}{m}}$$ Campo escalar

$[\Phi] = J/kg$

$\Phi = \Phi(x, y, z)$

$$\boxed{\Phi = -G \int_V \frac{\rho(\vec{r}')\,dv'}{r}}$$



$W_{Total} = \Delta K = 0$

$W_{Ext} + W_g = 0$

$W_{Ext} = -W_g = \Delta U_g$

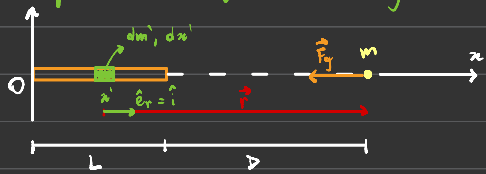
$$\boxed{\frac{W_{Ext}}{m} = \frac{\Delta U_g}{m} = \Delta \Phi}$$

$\Delta\Phi > 0$: m se afasta da fonte → $W_{Ext} > 0$

$\Delta\Phi < 0$: m se aproxima da fonte → $W_{Ext} < 0$

$\to \vec{F}_g = -\vec{\nabla} U_g \Rightarrow \frac{\vec{F}_g}{m} = \vec{\nabla}\left(\frac{U_g}{m}\right) \Rightarrow \boxed{m \to 0:\ \vec{g} = -\vec{\nabla}\Phi}$

* Exemplo: Barra fina e homogênea



Densidade: $\rho(x') = \dfrac{dm'}{dx'} = \rho = cte$

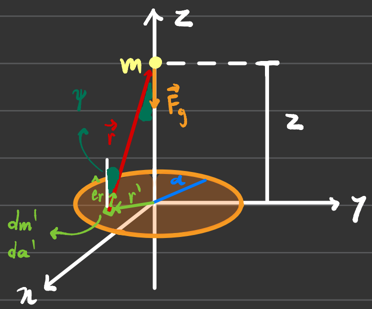
$x' \in [0, L], \quad r = L + D - x'$

$$\rightarrow \vec{F}_g = -Gm \int_V \frac{\rho(\vec{r}')\,\hat{e}_r}{r^2}\,dv' = -Gm\rho \int_0^L \frac{dx'}{r^2}\,\hat{i}$$

$$\vec{F}_g = -Gm\rho\,\hat{i} \int_0^L \frac{dx'}{(L+D-x')^2} = Gm\rho\left(\frac{1}{L+D} - \frac{1}{D}\right)\hat{i}$$

$$\therefore \boxed{\vec{F}_g = -\frac{GmM}{D(L+D)}\,\hat{i}} \qquad \vec{g} = \frac{\vec{F}_g}{m} = -\frac{GM}{D(L+D)}\,\hat{i}$$

* Exemplo: Disco fino e homogêneo



$$\vec{F}_g = -Gm \int_A \frac{\rho(\vec{r}')\hat{e}_r}{r^2}\, da' = -Gm\rho \int_A \frac{r'\hat{e}_r}{(r'^2 + z^2)}\, dr'd\theta$$

$$da' = r'd\theta dr', \quad r' \in [0, a], \quad \theta \in [0, 2\pi)$$

$$r^2 = r'^2 + z^2$$

$$\hat{e}_r = (\hat{e}_r)_x \hat{i} + (\hat{e}_r)_y \hat{j} + (\hat{e}_r)_z \hat{k}$$

$$\Rightarrow \vec{F}_g = -Gm\rho \int_0^a dr' \int_0^{2\pi} d\theta \frac{r'(\hat{e}_r)_z \hat{k}}{(\vec{r}'^2 + z^2)}$$

$$\rightarrow (\hat{e}_r)_z = \underbrace{|\hat{e}_r|}_{=1} \cos\psi = \frac{z}{\sqrt{r'^2 + z^2}}$$

$$\rightarrow \vec{F}_g = -Gm\rho z \int_0^a dr' \int_0^{2\pi} \frac{d\theta\, r'}{(r'^2 + z^2)^{\frac{3}{2}}}\, \hat{k}$$

$$= -\hat{k}\, 2\pi G m\rho z \int_0^a \frac{r'dr'}{(r'^2 + z^2)^{\frac{3}{2}}}$$

$$= -\hat{k}\, 2\pi G m \rho z \left(-\frac{1}{\sqrt{r'^2 + z^2}} \right)\Bigg|_0^a$$

$$\therefore \boxed{\vec{F}_g = -2\pi\rho G m \left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)\hat{k}}$$

30/06/23

↳ Potencial gravitacional:

$$\rightarrow \Phi = -G\int_A \frac{\rho(\vec{r}')da'}{r} = -G\rho\int_A \frac{da'}{r}$$

$$\rightarrow da' = r'd\theta dr', \quad z^2 + r'^2 = r^2$$

$$\Phi = -G\rho\int_0^a dr' \int_0^{2\pi} \frac{r'}{\sqrt{z^2+r'^2}} d\theta$$

$$= -2\pi G \int_0^a \frac{dr' r'}{\sqrt{z^2+r'^2}}$$

$$= -2\pi G\rho \sqrt{z^2+r'^2}\Big|_0^a$$

$$\therefore \Phi = -2\pi G\rho\left(\sqrt{z^2+a^2} - z\right)$$

$$\rightarrow \vec{g} = -\vec{\nabla}\Phi = -\frac{\partial\Phi}{\partial z}\hat{k}$$

$$\frac{d\Phi}{dz} = -2\pi G\rho\left[\frac{1}{2}(z^2+a^2)^{-\frac{1}{2}} \cdot 2z - 1\right]$$

$$\vec{g} = -2\pi G\rho\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]\hat{k}$$

$$\therefore \boxed{\vec{F}_g = -2\pi G\rho m\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right]\hat{k}}$$

$$\rightarrow \text{Expandindo: } \vec{g} = -2\pi G\rho\left[1 - \frac{1}{\sqrt{1+a^2/z^2}}\right]\hat{k}$$

$$\rightarrow z \gg a: \quad \frac{1}{\sqrt{1+\frac{a^2}{z^2}}} \approx 1 - \frac{1}{2}\frac{a^2}{z^2}$$

$$\vec{g} \approx -\cancel{2}\pi G\rho\frac{1}{\cancel{2}}\frac{a^2}{z^2}\hat{k}$$

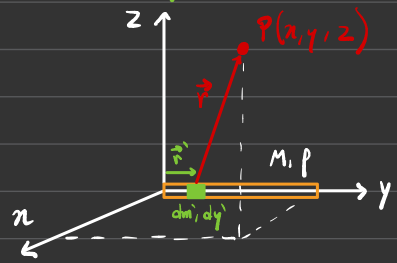
$\rho \cdot A = m$

$$\approx -\frac{G\rho\pi a^2}{z^2}\hat{k}$$

$$\therefore \boxed{\vec{g} \approx -\frac{GM}{z^2}\hat{k}}$$

limite geral para qualquer geometria

* Barra fina e homogênea



$\rightarrow \Phi = -G\rho \int_0^L \frac{dy'}{r}$

$\rightarrow$ Coordenadas de dm': $(x', y', z') = (0, y', 0)$

$$r = \sqrt{x^2 + (y-y')^2 + z^2}$$

$$\Rightarrow \Phi = -G\rho \int_0^L \frac{dy'}{\sqrt{x^2 + (y-y')^2 + z^2}}$$

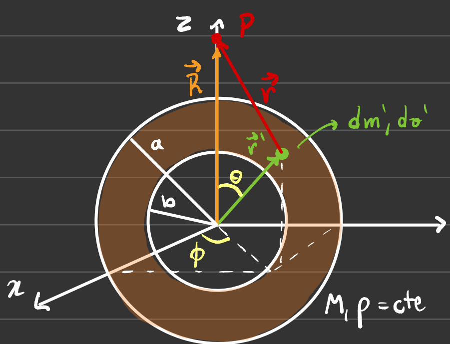
$$\rightarrow \begin{cases} u = y - y' \\ du = -dy' \end{cases} \Rightarrow \Phi = G\rho \int_y^{y-L} \frac{du}{\sqrt{x^2 + u^2 + z^2}} = G\rho \frac{1}{2} \ln\left[\frac{u + \sqrt{x^2 + u^2 + z^2}}{\sqrt{x^2 + u^2 + z^2} - u}\right]_y^{y-L}$$

$$\boxed{\Phi(x,y,z) = \frac{G\rho}{2} \ln\left[\frac{y - L + \sqrt{x^2 + (y-L)^2 + z^2}}{\sqrt{x^2 + (y-L)^2 + z^2} - y + L}\right] - \frac{G\rho}{2} \ln\left[\frac{y + \sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + y^2 + z^2} - y}\right]}$$

* Camada esférica

→ Esfera "oca": "esfera com um buraco esférico no meio"

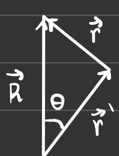
Ao longo de cada "anel" (plano) as forças paralelas a esse plano se cancelam e a força resultante aponta através da esfera, para baixo



Potencial em P: $\Phi = -G\rho \int_V \frac{dv'}{r}$

$dv' = r'^2 \operatorname{sen}\theta \, dr' \, d\theta \, d\phi$

$\Phi = -G\rho \int_b^a dr' \, r'^2 \int_0^\pi d\theta \operatorname{sen}\theta \int_0^{2\pi} \frac{d\phi}{r}$



Lei dos cossenos:

$r^2 = R^2 + r'^2 - 2r'R\cos\theta$

$$\Rightarrow \Phi = -G\rho \int_b^a dr' \, r'^2 \int_0^\pi d\theta \operatorname{sen}\theta \int_0^{2\pi} \frac{d\phi}{\sqrt{R^2 + r'^2 - 2r'\cos\theta}}$$

$$\Rightarrow \Phi = -G\rho \int_b^a dr' r'^2 \int_0^{\pi} d\theta \, sen\theta \int_0^{2\pi} \frac{d\phi}{\sqrt{R^2 + r'^2 - 2r' \cos\theta}}$$

$$\rightarrow r = \sqrt{R^2 + r'^2 - 2r' R\cos\theta}$$

$$dr = \frac{1}{2}\left(R^2 + r'^2 - 2r' R\cos\theta\right)^{-\frac{1}{2}} \cdot 2r' R \, sen\theta \, d\theta = \frac{r' R \, sen\theta \, d\theta}{r}$$

$$\rightarrow sen\theta \, d\theta = \frac{r\,dr}{r' R}$$

$$\Rightarrow \Phi = -2\pi G\rho \int_b^a dr' r'^2 \int_{r(\theta=0)}^{r(\theta=\pi)} \frac{r\,dr}{r' R r}$$

$$= -\frac{2\pi G\rho}{R} \int_b^a dr' r' \int_{r(\theta=0)}^{r(\theta=\pi)} dr$$

$$\rightarrow \text{Limites: } \begin{cases} \theta = \pi \rightarrow r = \sqrt{r'^2 + R^2 + 2r' R} \\ \quad r = \sqrt{(r' + R)^2} \\ \quad r = r' + R \\ \theta = 0 \rightarrow r = \sqrt{r'^2 + R^2 - 2r' R} \\ \quad r = \sqrt{(R - r')^2} \\ \quad r = \pm(R - r') \rightarrow \text{depende de onde o ponto estiver} \\ \quad r = |R - r'| \end{cases}$$

$$\rightarrow \Phi = -\frac{2\pi G\rho}{R} \int_b^a dr' r' \int_{|R-r'|}^{|r'+R|} dr$$

$$\therefore \Phi = -\frac{2\pi G\rho}{R} \int_b^a dr' r' \left[r' + R - |R - r'| \right]$$

03/07/23

→ Fora da camada: $R > a$

$$\Phi = -\frac{2\pi G\rho}{R}\int_b^a dr' r' \left[R + r' - (R - r')\right]$$

$$= -\frac{4\pi G\rho}{R}\int_b^a r'^2 dr'$$

$$= -\frac{4\pi G\rho}{R}\frac{r'^3}{3}\Big|_b^a$$

equivalente ao potencial de uma partícula de massa M localizada no centro

$$\therefore \Phi = -\frac{G\rho}{R}\underbrace{\frac{4\pi}{3}(a^3 - b^3)}_{\text{volume da camada}} \longrightarrow \boxed{\Phi = -\frac{GM}{R}}, \quad R > a$$

→ Dentro da camada: $R < b$

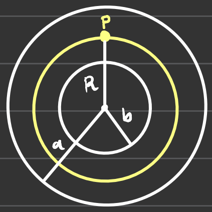
$$\Phi = -\frac{2\pi G\rho}{R}\int_b^a dr' r' \left[R + r' - (r' - R)\right]$$

$$= -4\pi G\rho\int_b^a r' dr'$$

$$= -4\pi G\rho \frac{r'^2}{2}\Big|_b^a$$

$$\therefore \boxed{\Phi = -2\pi G\rho(a^2 - b^2)}, \quad R < b \longrightarrow \text{independe de } R$$

→ Na camada: $b < R < a$



$$\Phi = \Phi_{int} + \Phi_{ext}$$

↱ potencial devido à massa externa ao raio R

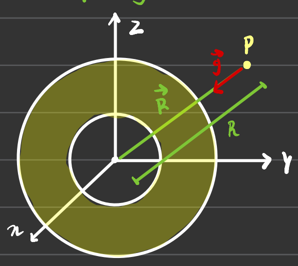
↳ potencial devido à massa interna ao raio R

$$\rightarrow \Phi_{int} = -\frac{2\pi G\rho}{R}\int_b^R dr'\,r'\left[R+r'-(R-r')\right] \Rightarrow \boxed{\Phi_{int} = -\frac{G\rho}{R}\frac{4\pi}{3}\left(R^3-b^3\right)}$$

$$\rightarrow \Phi_{ext} = -\frac{2\pi G\rho}{R}\int_R^a dr'\,r'\left[R+r'-(r'-R)\right] \Rightarrow \boxed{\Phi_{ext} = -2\pi G\rho\left(a^2-R^2\right)}$$

$$\therefore \boxed{\Phi = -\frac{G\rho}{R}\frac{4}{3}\pi\left(R^3-b^3\right) - 2\pi G\rho\left(a^2-R^2\right)}$$

→ Campo gravitacional: $\vec{g} = -\vec{\nabla}\Phi$



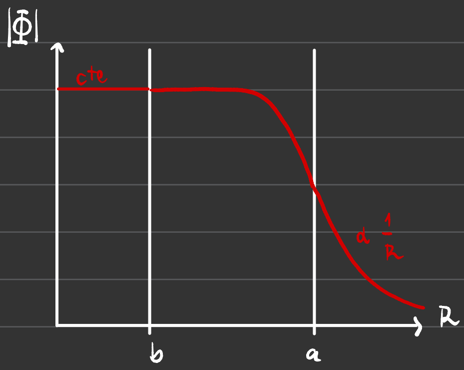
$\Phi = \Phi(R)$ → coordenada radial → $\vec{g} = -\frac{d\Phi}{dR}\hat{e}_R$

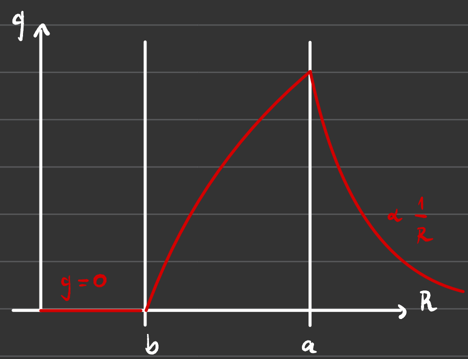
$$R > a: \quad \frac{d\Phi}{dR} = \frac{GM}{R^2} \Rightarrow \boxed{\vec{g} = -\frac{GM}{R^2}\hat{e}_R}$$

$$R < b: \quad \frac{d\Phi}{dR} = 0 \Rightarrow \boxed{\vec{g} = 0}$$

$$b < R < a: \quad \frac{d\Phi}{dR} = 4\pi G\rho R - G\rho\frac{8}{3}\pi R - G\rho\frac{4}{3}\pi\frac{b^3}{R^2} = -\frac{4}{3}\pi G\rho\left(\frac{b^3}{R^2} - R\right)$$

$$\boxed{\vec{g} = \frac{4}{3}\pi G\rho\left(\frac{b^3}{R^2} - R\right)\hat{e}_R}$$



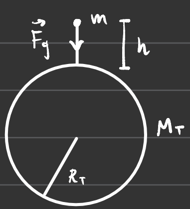


* Resultado geral: válido para qualquer distribuição de massa que dependa apenas da direção radial

* Esfera compacta e uniforme (raio a):

$b \to 0$: $R > a \to \boxed{\vec{g} = -\frac{GM}{R^2}\hat{e}_r}$ $\qquad 0 < R < a \to \boxed{\vec{g} = -\frac{4}{3}\pi G \rho R \hat{e}_r}$

* Para a Terra:



$\to F_g = \frac{GM_T m}{(R_T + h)^2}$, $R_T \gg h \to F_g \approx \underbrace{\frac{GM_T}{R_T^2}}_{g} m$

$\to U_g = -\frac{GM_T m}{(R_T + h)}$

$R_T = 6371 \text{ km}$

$M_T = 5{,}97 \cdot 10^{24} \text{ kg}$

Se eleva a uma altura h: $\Delta U_g = U_g(R_T + h) - U_g(R_T)$

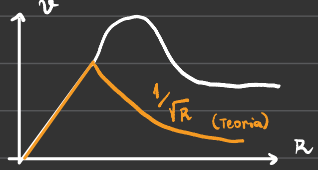
$$\Delta U_g = -GM_T m\left[\frac{1}{R_T + h} - \frac{1}{R_T}\right] = -\frac{GM_T m}{R_T}\left[\frac{1}{1 + \frac{h}{R_T}} - 1\right]$$

$$\left(1 + \frac{h}{R_T}\right)^{-1} \approx 1 - \frac{h}{R_T} \Rightarrow \Delta U_g = -\frac{GM_T m}{R_T}\left[1 - \frac{h}{R_T} - 1\right]$$

$$\therefore \boxed{\Delta U_g \approx \frac{GM_T}{R_T^2} m h = mgh}$$

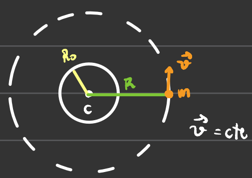
* Galáxia espiral:

Experimento:



matéria escura

Modelo:



$R < R_0$: $\frac{4}{3}\pi G \rho R m = \frac{m v^2}{R}$

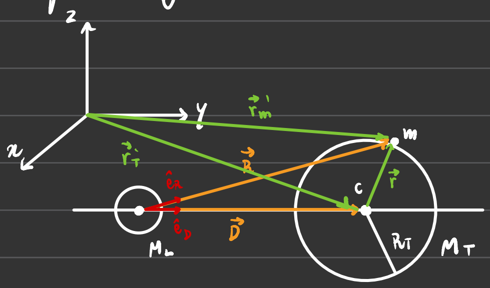
$$v = R\sqrt{\frac{4}{3}\pi G \rho}$$

$R > R_0$: $\frac{GMm}{R^2} = \frac{m v^2}{R}$

$$v = \sqrt{\frac{GM}{R}}$$

Forças de maré

→ Variação da força gravitacional ao longo de uma superfície



$r = R_T$

$\vec{r}_T' + \vec{r} = \vec{r}_m'$

2ª Lei sobre m: $-\dfrac{GM_T m}{R_T^2}\hat{e}_r - \dfrac{GM_L m}{R^2}\hat{e}_R = m\ddot{\vec{r}}_m'$

2ª Lei sobre o centro da Terra: $-\dfrac{GM_T M_L}{D^2}\hat{e}_D = M_T\ddot{\vec{r}}_T'$

Em relação à Terra: força sobre m $= m\ddot{\vec{r}}$

$\ddot{\vec{r}}_T' + \ddot{\vec{r}} = \ddot{\vec{r}}_m'$

↱ campo gravitacional da Lua medido no ponto C

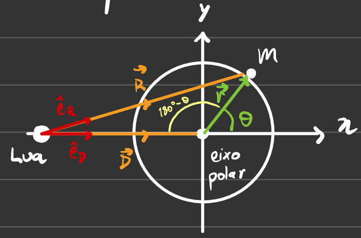
$\ddot{\vec{r}} = -\dfrac{GM_T}{R_T^2}\hat{e}_r - \dfrac{GM_L}{R^2}\hat{e}_R + \dfrac{GM_L}{D^2}\hat{e}_D$

↳ campo gravitacional gerado pela Terra

→ Força de maré: $\boxed{\vec{F}_R = -GmM_L\left(\dfrac{\hat{e}_R}{R^2} - \dfrac{\hat{e}_D}{D^2}\right)}$

→ Visão azimutal (Equador):

$r = R_T$
$\vec{D} = D\hat{i}$, $\hat{e}_D = \hat{i}$
$\vec{R} = R\hat{e}_R$



$$\rightarrow \vec{R} = \vec{D} + \vec{r} = D\hat{i} + r\cos\theta\,\hat{i} + r\,\text{sen}\,\theta\,\hat{j} \Rightarrow \hat{e}_R = \frac{(D + R_T\cos\theta)\hat{i} + R_T\,\text{sen}\,\theta\,\hat{j}}{R}$$

→ Pela lei dos cossenos: $R^2 = r^2 + D^2 - 2rD\cos(180° - \theta) = r^2 + D^2 + 2rD\cos\theta$

$$\therefore \hat{e}_R = \frac{(D + R_T\cos\theta)\hat{i} + R_T\,\text{sen}\,\theta\,\hat{j}}{\sqrt{R_T^2 + D^2 + 2R_T D\cos\theta}}$$

$$\rightarrow \hat{e}_R = \frac{\left(1 + \frac{R_T}{D}\cos\theta\right)\hat{i} + \frac{R_T}{D}\,\text{sen}\,\theta\,\hat{j}}{\sqrt{1 + \cancel{R_T^2/D^2} + 2\,R_T/D\,\cos\theta}}, \quad \frac{R_T}{D} \approx 0{,}02$$

$$\rightarrow \frac{1}{\sqrt{1 + \frac{2R_T}{D}\cos\theta}} = 1 - \frac{R_T}{D}\cos\theta + \cancel{\mathcal{O}\left(\frac{R_T^2}{D^2}\right)}$$

$$\Rightarrow \hat{e}_R \approx \left[\left(1 + \frac{R_T}{D}\cos\theta\right)\hat{i} + \frac{R_T}{D}\,\text{sen}\,\theta\,\hat{j}\right]\left(1 - \frac{R_T}{D}\cos\theta\right)$$
$$\approx \left(1 + \cancel{\frac{R_T}{D}\cos\theta} - \cancel{\frac{R_T}{D}\cos\theta} - \cancel{\frac{R_T^2}{D^2}\cos^2\theta}\right)\hat{i} + \left(\frac{R_T}{D}\,\text{sen}\,\theta - \cancel{\frac{R_T^2}{D^2}\,\text{sen}\,\theta\cos\theta}\right)\hat{j}$$

$$\therefore \hat{e}_R \approx \hat{i} + \frac{R_T}{D}\,\text{sen}\,\theta\,\hat{j}$$

$$\rightarrow \vec{F}_M = -GmM_L\left[\frac{\hat{i}+\frac{R_T}{D}\,\text{sen}\,\theta\,\hat{j}}{D^2+R_T^2+2R_TD\cos\theta}-\frac{\hat{i}}{D^2}\right]$$

$$= -\frac{GmM_L}{D^2}\left[\frac{\hat{i}+\frac{R_T}{D}\,\text{sen}\,\theta\,\hat{j}}{1+\cancel{\frac{R_T^2}{D^2}}+2\frac{R_T}{D}\cos\theta}-\hat{i}\right]$$

$$\vec{F}_M \approx -\frac{GmM_L}{D^2}\left[\left(\hat{i}+\frac{R_T}{D}\,\text{sen}\,\theta\,\hat{j}\right)\left(1-2\frac{R_T}{D}\cos\theta\right)-\hat{i}\right]$$

$$\approx -\frac{GmM_L}{D^2}\left[\cancel{\hat{i}}-\frac{2R_T}{D}\cos\theta\,\hat{i}+\frac{R_T}{D}\,\text{sen}\,\theta\,\hat{j}-\cancel{\hat{i}}\right]$$

$$f=(1+x)^{-1}$$

$$\hookrightarrow f' = -(1+x)^{-2}$$

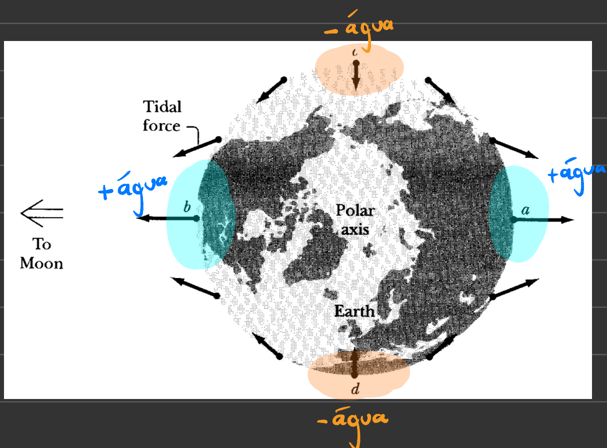
$$(1+x)^{-1}\approx 1-x$$

$$\therefore \boxed{\vec{F}_M = \frac{2GmM_L}{D^3}R_T\cos\theta\,\hat{i}-\frac{GmM_L}{D^3}R_T\,\text{sen}\,\theta\,\hat{j}}$$

→ gradiente sobre a superfície da Terra

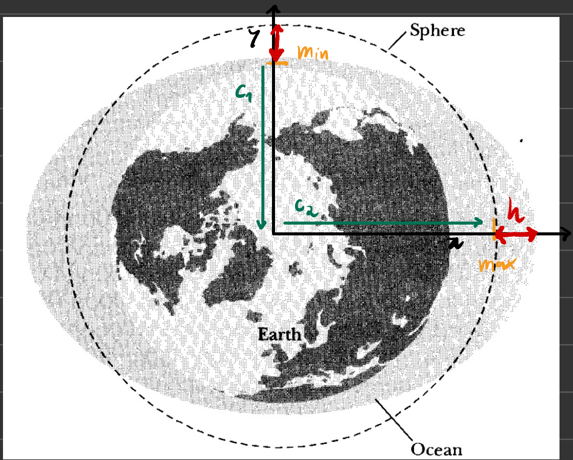
↳ cai muito rápido com a distância entre os corpos

↳ Ou, fazendo $F_0 \equiv \frac{GmM_LR_T}{D^3}$: $\boxed{\vec{F}_M = 2F_0\cos\theta\,\hat{i}-F_0\,\text{sen}\,\theta\,\hat{j}}$



$$\begin{cases}\theta=\frac{\pi}{2} \Rightarrow \vec{F}_M = -F_0\hat{j}\\ \theta=-\frac{\pi}{2} \Rightarrow \vec{F}_M = F_0\hat{j}\end{cases}$$

$$\begin{cases}\theta=0 \Rightarrow \vec{F}_M = 2F_0\hat{i}\\ \theta=\pi \Rightarrow \vec{F}_M = -2F_0\hat{i}\end{cases}$$



$h = ?$

→ Trabalho de $\vec{F}_M$: $W_M = mgh$

→ $W_M = \int_{min}^{max} \vec{F}_M \cdot d\vec{r}$

$= \int_{C_1} \vec{F}_M \cdot d\vec{r} + \int_{C_2} \vec{F}_M \cdot d\vec{r}$

conservativa

→ $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

→ Para um ponto no interior do círculo:

$$\vec{F}_M = \frac{2GmM_L}{D^3} r\cos\theta\,\hat{i} - \frac{GmM_L}{D} r\,\text{sen}\theta\,\hat{j}$$

→ C_1: $\theta = \frac{\pi}{2}$, $x = 0$
C_2: $\theta = 0$, $y = 0$

$R_T + h \approx R_T$

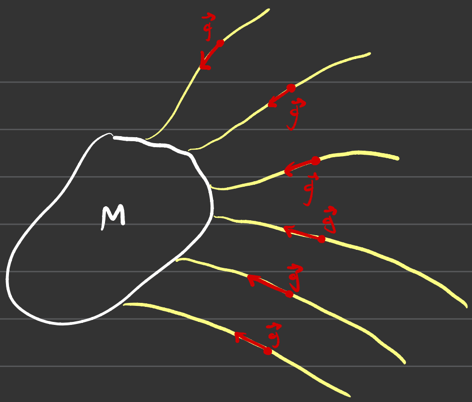
$$F_M = \frac{GmM_L}{D^3}\left[\int_{y=R_T}^{y=0} -y\,dy + \int_{x=0}^{x=R_T} 2x\,dx\right]$$

$$= \frac{GmM_L}{D^3}\left[\frac{y^2}{2}\Big|_0^{R_T} + x^2\Big|_0^{R_T}\right]$$

$$\therefore F_M = \frac{3}{2}\frac{GmM_L R_T^2}{D^3}$$

→ $\frac{3}{2}\frac{G\cancel{m}M_L R_T^2}{D^3} = \cancel{m}gh \Rightarrow \boxed{h = \frac{3}{2}\frac{GM_L R_T^2}{D^3 g}}$ ⇝ $h \approx 0{,}54\,\text{m}$ //

Linhas de campo

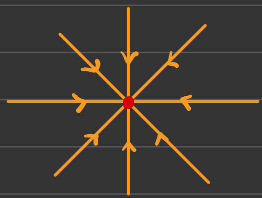


→ Se estendem da fonte ao infinito

→ $\vec{g}$: tangente às linhas
↳ $\vec{F}_g$ é atrativa ⇒ linhas "apontam" para o corpo

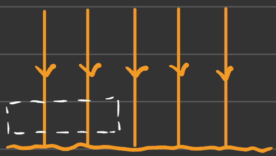
→ Densidade de linhas de campo ∝ intensidade do campo (g)

* Partícula pontual:



$$\vec{g} = -\frac{Gm}{r^2}\hat{e}_r$$

* Próximo à superfície da Terra:

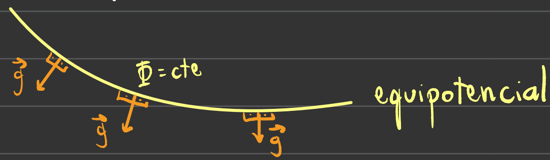


$$\vec{g} = g\hat{j}$$

→ Conceito útil para visualizar o campo no espaço (sem massa)

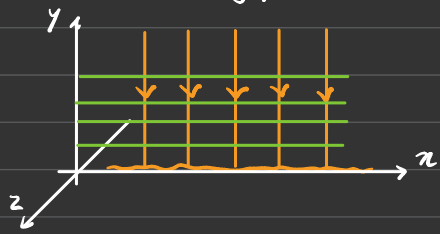
* Superfície equipotencial: $\Phi(x, y, z) = cte$ (arbitrária)

$\vec{g} = -\vec{\nabla}\Phi$, $\Phi = cte$ na superfície $\Rightarrow$ $\vec{g} \perp$ superfície



↳ Partícula pontual: $\Phi(r) = -\dfrac{Gm}{r} = cte \Rightarrow r = cte \Rightarrow$ esfera

↳ Próximo à sup. da Terra: $\Phi(y) = gy = cte \Rightarrow y = cte \Rightarrow$ plano



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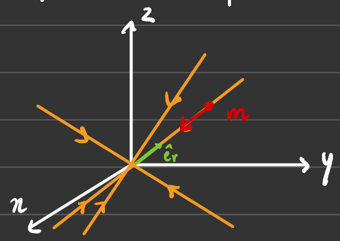
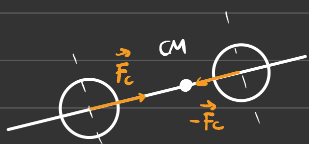
Força central

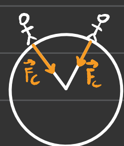
→ Linha de ação da força passa por um único ponto, independentemente da posição da partícula



* $\vec{F_c} = f(r, \theta, \phi)\,\hat{e}_r$

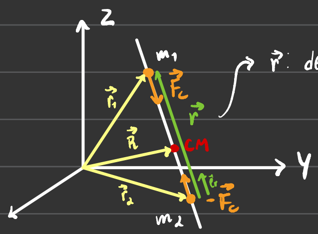
↳ Isotrópica: $\vec{F_c} = F(r)\,\hat{e}_r$ → depende apenas da coordenada radial





Leis de Kepler

1. Lei das elipses (1609) → órbitas elípticas
2. Lei das áreas (1609) → áreas iguais em tempos iguais
3. Lei harmônica (1618) → $T^2 \propto a^3$



$\vec{r}$: de 2 para 1

Central e isotrópica: $\vec{F}_c = \overbrace{F(r)}^{<0}\,\hat{e}_r$

$\rightarrow m_1\ddot{\vec{r}}_1 = F(r)\hat{e}_r, \quad m_2\ddot{\vec{r}}_2 = -F(r)\hat{e}_r$

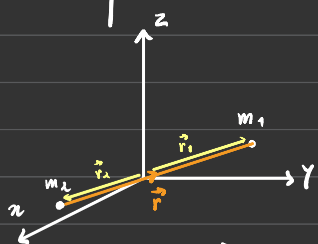
$\rightarrow \vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \Rightarrow (m_1+m_2)\ddot{\vec{R}} = \overbrace{m_1\ddot{\vec{r}}_1}^{\vec{F}_c} + \overbrace{m_2\ddot{\vec{r}}_2}^{-\vec{F}_c} = 0$

CM

$\therefore \boxed{\dot{\vec{R}} = cte}$ → Conservação do momento linear

* CM define um referencial inercial

→ No referencial do CM:



$\rightarrow \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = 0 \Rightarrow \vec{r}_1 = -\dfrac{m_2}{m_1}\vec{r}_2$

$\rightarrow \vec{r}_2 + \vec{r} = \vec{r}_1 \Rightarrow \vec{r} = \vec{r}_1 - \vec{r}_2$

$\hookrightarrow \vec{r} = -\dfrac{m_2}{m_1}\vec{r}_2 - \vec{r}_2 = -\left(\dfrac{m_2}{m_1} + 1\right)\vec{r}_2$

$\hookrightarrow \vec{r} = \vec{r}_1 + \dfrac{m_1}{m_2}\vec{r}_1 = \left(1 + \dfrac{m_1}{m_2}\right)\vec{r}_1$

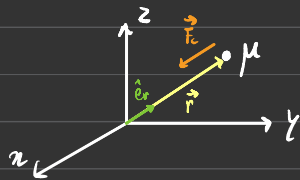
$$\left\{ \begin{aligned} \vec{r}_1 &= \frac{m_2}{m_1+m_2}\vec{r} \\ \vec{r}_2 &= \frac{-m_1}{m_1+m_2}\vec{r} \end{aligned} \right.$$

→ $\ddot{\vec{r}} = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \frac{\vec{F}_c}{m_1} + \frac{\vec{F}_c}{m_2} = \frac{m_1 + m_2}{m_1 m_2} \vec{F}_c$

→ Massa reduzida: $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$$\ddot{\vec{r}} = \frac{1}{\mu} \vec{F}_c \Rightarrow \boxed{\mu \ddot{\vec{r}} = F(r)\, \hat{e}_r}$$

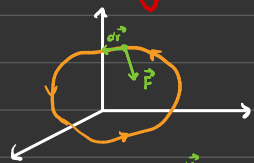
* Problema equivalente:



→ Lei de conservação → reduzir graus de liberdade

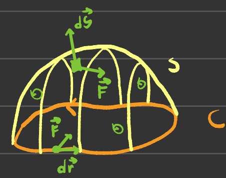
Conservação de energia

→ Força conservativa $\vec{F}$: $\oint_c \vec{F}\cdot d\vec{r} = 0$



→ Teorema de Stokes: $\oint_c \vec{F}\cdot d\vec{r} = \int_s \overbrace{(\vec{\nabla}\times\vec{F})}^{\text{rotacional}}\cdot d\vec{S}$

$\vec{F} = \vec{F}(x, y, z)$



→ Conservativa: $\int_s (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = 0 \rightarrow \boxed{\vec{\nabla}\times\vec{F} = 0}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \Rightarrow \vec{\nabla}\times\vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

→ Suponhamos: $\vec{F} = \vec{\nabla} f$, $f = f(x, y, z) \Rightarrow \boxed{\vec{\nabla}\times\vec{\nabla} f = 0}$

→ Força central: $\vec{F}_c = F(r)\,\hat{e}_r$, $r = \sqrt{x^2+y^2+z^2}$, $\hat{e}_r = \frac{\vec{r}}{r} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$

$$\vec{F}_c = \frac{x F(r)}{r}\hat{i} + \frac{y F(r)}{r}\hat{j} + \frac{z F(r)}{r}\hat{k}$$

↳ $$\frac{\partial F_{c,z}}{\partial y} = \frac{z}{r}\frac{\partial F}{\partial y} - \frac{F(r) z}{r^2}\frac{\partial r}{\partial y} = \frac{z}{r}\frac{\partial F}{\partial r}\frac{\partial r}{\partial y} - \frac{F(r) z}{r^2}\frac{\partial r}{\partial y} = \frac{yz}{r^2}\frac{\partial F}{\partial r} - \frac{yz}{r^3}F(r)$$

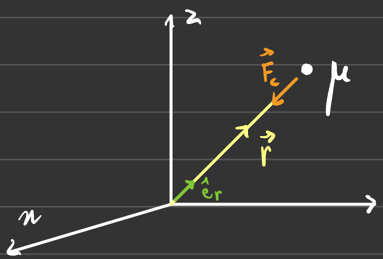
$$\frac{\partial F_{c,y}}{\partial z} = \frac{y}{r}\frac{\partial F}{\partial r}\frac{\partial r}{\partial z} - \frac{F(r) y}{r^2}\frac{\partial r}{\partial z} = \frac{yz}{r^2}\frac{\partial F}{\partial r} - \frac{yz}{r^3}F(r)$$

$$\therefore \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0$$

↳ $$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2y = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

12/07/23



$$\mu \ddot{\vec{r}} = \vec{F}_c$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \begin{cases} \vec{r}_1 = \dfrac{m_2}{m_1+m_2}\,\vec{r} \\ \vec{r}_2 = -\dfrac{m_1}{m_1+m_2}\,\vec{r} \end{cases}$$

convenção da física: $E = T + U$

→ $\vec{\nabla} \times \vec{F}_c = 0$ → $\vec{F}_c$ é conservativa ⇒ $\vec{F}_c = -\vec{\nabla} U$, $U = U(r, \theta, \phi)$

→ Em coordenadas esféricas: $\vec{F}_c = -\dfrac{\partial U}{\partial r}\hat{e}_r - \dfrac{1}{r}\dfrac{\partial U}{\partial \theta}\hat{e}_\theta - \dfrac{1}{r \operatorname{sen}\theta}\dfrac{\partial}{\partial \phi}\hat{e}_\phi = F(r)\,\hat{e}_r$

$$\Rightarrow \frac{\partial U}{\partial \phi} = \frac{\partial U}{\partial \theta} = 0 \Rightarrow U = U(r)$$

$$\therefore \boxed{F(r) = -\frac{dU}{dr}}$$

→ $\boxed{E = \dfrac{1}{2}\mu |\dot{\vec{r}}|^2 + U(r) = cte}$

→ Problema original: $E_* = \dfrac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \dfrac{1}{2} m_2 |\dot{\vec{r}}_2|^2 + U(r)$

↳ Exercício: demonstrar

$$\left. \begin{array}{l} \dot{\vec{r}} = \dfrac{m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2}{m_1 + m_2}, \quad \mu = \dfrac{m_1 m_2}{m_1 + m_2} \\[2ex] \vec{r}_1 = \dfrac{m_2}{m_1+m_2}\vec{r}, \quad \vec{r}_2 = \dfrac{-m_1}{m_1+m_2}\vec{r} \end{array} \right\}$$

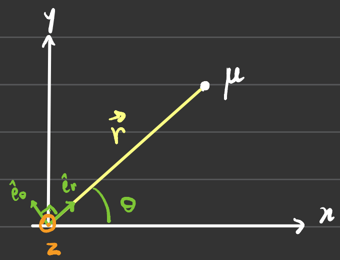
→ Momento angular: $\vec{L} = \mu(\vec{r} \times \dot{\vec{r}})$

→ Torque: $\vec{N} = \vec{r} \times \vec{F}_c = 0 \Rightarrow \vec{L} = cte$ → orientação do plano não muda

$\therefore$ $\vec{r}$ e $\dot{\vec{r}}$ estão no mesmo plano → reduzimos p/ coord. polares

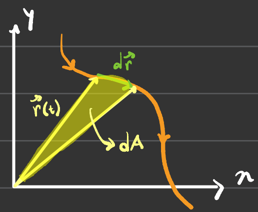
→ Problema original: $\vec{L}_{sist} = \vec{\ell}_1 + \vec{\ell}_2 = m_1(\vec{r}_1 \times \dot{\vec{r}}_1) + m_2(\vec{r}_2 \times \dot{\vec{r}}_2)$

↳ Exercício: demonstrar



$$\left.\begin{array}{l} \vec{r} = r\hat{e}_r \\ \dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \end{array}\right\} \vec{L} = \mu\left[r\dot{r}\,\cancel{(\hat{e}_r \times \hat{e}_r)}^{\,0} + r^2\dot{\theta}\,\overbrace{(\hat{e}_r \times \hat{e}_\theta)}^{\hat{k}}\right]$$

$$\boxed{\vec{L} = \mu r^2 \dot{\theta}\,\hat{k}}$$



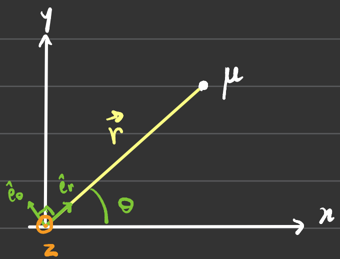
$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}|, \quad d\vec{r} = \vec{v}\,dt$$

$$= \frac{1}{2}|\vec{r} \times \vec{v}\,dt|$$

$$= \frac{1}{2}dt\,|\vec{r} \times \vec{v}|$$

$$= \frac{1}{2}dt\,\frac{|\vec{L}|}{\mu}$$

$$\Rightarrow \frac{dA}{dt} = \frac{|\vec{L}|}{2\mu} = cte \Rightarrow \left.\boxed{\Delta A = \frac{|\vec{L}|}{2\mu}\Delta t}\right\} \text{2ª Lei de Kepler}$$



$\vec{L} = \ell \hat{k}$

$\ell = \mu r^2 \dot{\theta} = cte$

$\vec{r} = \vec{r}(t), \ \theta = \theta(t)$

Para $\ell = cte$, $\dot{\theta}$ não pode trocar de sinal

$\therefore \theta(t)$ é função monotônica de t

→ 2ª lei: $\mu \ddot{\vec{r}} = F(r)\hat{e}_r$

$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

$\mu(\ddot{r} - r\dot{\theta}^2) = F(r)$

$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$

$\frac{\mu}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$ → conservação de $\vec{L}$

$\dot{\theta} = \frac{\ell}{\mu r^2}$

→ $\mu\ddot{r} - \mu r\dot{\theta}^2 = F(r)$

$\mu\ddot{r} - \mu r \frac{\ell^2}{\mu^2 r^4} = F(r)$

$$\boxed{\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}}$$

"força" centrífuga sempre "repulsiva"

→ $r = r(\theta) = ?$

$$-\frac{\ell^2 u^2}{\mu}\frac{d^2u}{d\theta^2} = F\left(\frac{1}{u}\right) + \frac{u^3\ell^2}{\mu}$$

→ $r = \frac{1}{u} \Rightarrow \dot{r} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$

→ $r^2 = \frac{\ell}{\mu\dot{\theta}} = \frac{1}{u^2}$

$\dot{r} = -\frac{\ell}{\mu\dot{\theta}}\frac{du}{d\theta}\dot{\theta} \Rightarrow \dot{r} = -\frac{\ell}{\mu}\frac{du}{d\theta}$

→ $\ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = -\frac{\ell}{\mu}\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -\frac{\ell}{\mu}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\dot{\theta} \Rightarrow \ddot{r} = -\frac{\ell^2u^2}{\mu^2}\frac{d^2u}{d\theta^2}$

$$\rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2u^2}F\left(\frac{1}{u}\right)}$$

Equação diferencial para a órbita

↳ Solução: $u = u(\theta)$

↳ Útil para determinar a força que origina dada órbita (problema inverso)

$\rightarrow E = \frac{1}{2}\mu|\dot{\vec{r}}|^2 + U(r)$

$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$\Big\} \; E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$

$\rightarrow \ell = \mu r^2\dot{\theta} \Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = cte$

$\rightarrow \dot{r}^2 = \frac{2}{\mu}[E - U(r)] - \frac{\ell^2}{\mu^2 r^2} \Rightarrow \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}[E-U(r)] - \frac{\ell^2}{\mu^2 r^2}}$

$\rightarrow \frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{\ell}{\mu r^2}\frac{dr}{d\theta} \Rightarrow d\theta = \pm\frac{\ell}{\mu}\frac{dr}{r^2\sqrt{\frac{2}{\mu}[E-U(r)] - \frac{\ell^2}{\mu^2 r^2}}}$

$$\therefore \boxed{\theta(r) = \pm\frac{\ell}{\mu}\int\frac{dr}{r^2\sqrt{\frac{2}{\mu}[E-U(r)] - \frac{\ell^2}{\mu^2 r^2}}} + \theta_0}$$

↳ Escolha de sinal: ditada pela conservação de momento angular e cond. inicial

→ Casos solúveis: $F(r) \propto r^n$, $U(r) \propto r^{n+1}$ $\rightarrow$

$n = 1 \rightarrow$ força restauradora

$n = -2 \rightarrow$ força eletrostática, gravitacional

$n = -3$

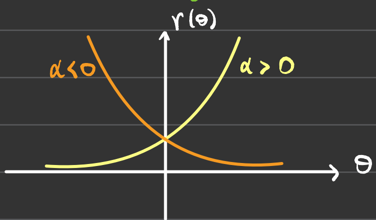
14/07/23

→ Da aula anterior: $\begin{cases} \ell = \mu r^2 \dot{\theta} \\ E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \end{cases}$

EXAMPLE 8.1

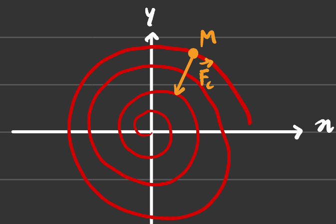
Find the force law for a central-force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where k and α are constants.

→ Espiral logarítmica: $r(\theta) = ke^{\alpha\theta}$, $k > 0$



$\Rightarrow \quad x = ke^{\alpha\theta}\cos\theta$

$\quad\;\; y = ke^{\alpha\theta}\operatorname{sen}\theta$



→ Lei da força? $r(t)$? $\theta(t)$?

→ Da aula anterior: $\frac{d^2u}{d\theta^2} + u = \frac{-M}{\ell^2 u^2} F\left(\frac{1}{u}\right)$, $u = \frac{1}{r}$

→ $u = \frac{1}{r} = \frac{1}{k}e^{-\alpha\theta} \Rightarrow \frac{du}{d\theta} = -\frac{\alpha}{k}e^{-\alpha\theta} = -\alpha u \Rightarrow \frac{du^2}{d\theta^2} = \frac{\alpha^2}{k}e^{-\alpha\theta} = \alpha^2 u$

→ $\alpha^2 u + u = \frac{-M}{\ell^2 u^2} F\left(\frac{1}{u}\right) \Rightarrow F\left(\frac{1}{u}\right) = -u(\alpha^2+1)\cdot\frac{u^2\ell^2}{M} = -\frac{u^3\ell^2(\alpha^2+1)}{M}$

→ $F(r) = -\left(\frac{1}{r}\right)^3 \frac{\ell^2(\alpha^2+1)}{M} \Rightarrow \boxed{F(r) = -\frac{\ell^2(\alpha^2+1)}{M}\frac{1}{r^3}}$

$$\rightarrow \frac{d\theta}{dt} = \frac{\ell}{Mr^2} = \frac{\ell}{Mk^2 e^{2\alpha\theta}}$$

$$\int d\theta\, e^{2\alpha\theta} = \frac{\ell}{Mk^2}\int dt \Rightarrow \frac{e^{2\alpha\theta}}{2\alpha} = \frac{\ell}{Mk^2} t + C \Rightarrow 2\alpha\theta = \ln\left[\frac{2\alpha\ell}{Mk^2} t + C_0\right]$$

$$\therefore \boxed{\theta(t) = \frac{1}{2\alpha}\ln\left[\frac{2\alpha\ell}{Mk^2} t + C_0\right]}$$

$$\rightarrow r(\theta) = k e^{\alpha\theta} = k \exp\left[\alpha \cdot \frac{1}{\alpha}\ln\left(\frac{2\alpha\ell}{Mk^2} t + C_0\right)^{\frac{1}{2}}\right]$$

$$\therefore \boxed{r(\theta) = k\sqrt{\frac{2\alpha\ell}{Mk^2} t + C_0}}$$

→ Condições iniciais: $\theta(0) = 0$, $\dot{\theta}(0) = \omega$

$$\hookrightarrow \theta(0) = \frac{1}{2\alpha}\ln C_0 = 0 \Rightarrow C_0 = 1$$

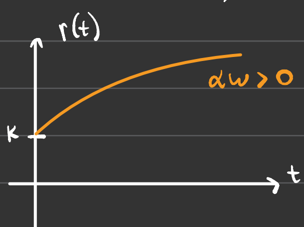
$$\hookrightarrow \ell = M\underbrace{r^2(0)}_{k^2}\dot{\theta}(0) = Mk^2\omega$$

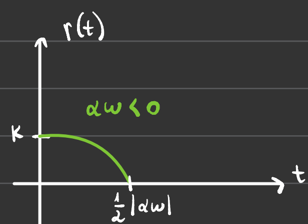
$$r(t) = k\sqrt{\frac{2\alpha \cancel{Mk^2}\omega t}{\cancel{Mk^2}} + 1} \Rightarrow$$

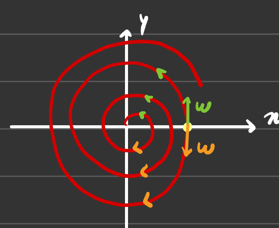
Para o caso específico

$$\boxed{r(t) = k\sqrt{2\alpha\omega t + 1}}$$

Sinal de $\alpha\omega$ determina a dinâmica







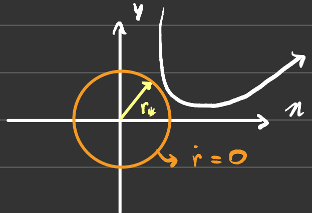
* A força é atrativa: por que espirala para fora?

→ $E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$

→ Velocidade radial: $\dot{r} = \pm\sqrt{\frac{2\left(E-U(r)\right)}{\mu} - \frac{\ell^2}{\mu^2 r^2}}$

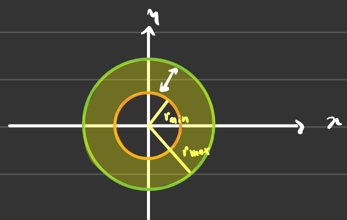
→ Pontos de retorno: $r = r_*$ tal que $\dot{r} = 0 \Rightarrow E - U(r_*) - \frac{\ell^2}{2\mu r_*^2} = 0$

Um ponto de retorno:



Estado espalhado

Dois pontos de retorno:



Estados ligados / confinados

→ Equação dinâmica para $r(t)$: $\mu\ddot{r} = F(r) + \boxed{\frac{\ell^2}{\mu r^3}}$ → "força" centrífuga repulsiva

↳ $\frac{\ell^2}{\mu r^3} = -\frac{dU_{cf}}{dr} \Rightarrow U_{cf}(r) = -\frac{\ell^2}{\mu}\int\frac{dr}{r^3} = \frac{\ell^2}{2\mu r^2} + \overbrace{A}^{\equiv 0}$

↳ $E = \underbrace{\frac{1}{2}\mu\dot{r}^2}_{\text{cinética}} + \underbrace{\frac{\ell^2}{2\mu r^2}}_{\text{centrífuga}} + \underline{U(r)}$

"real" → decorrente de uma interação

∴ $\boxed{E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r)}$

Potencial efetivo: $\boxed{V_{eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r)}$

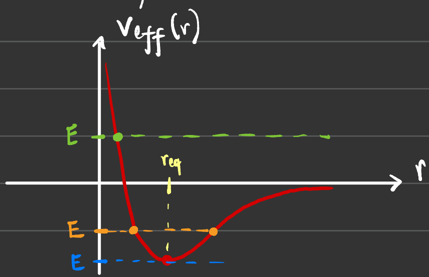
Potencial de Newton - Coulomb

* Potencial atrativo: $U(r) = -\frac{|k|}{r}$, $k < 0$

$\rightarrow V_{eff}(r) = \frac{\ell^2}{2\mu r^2} - \frac{|k|}{r}$ $\begin{cases} r \to 0: & V_{eff} \to \infty \\ r \to \infty: & V_{eff} \to 0 \end{cases}$

$\rightarrow \left.\frac{dV_{eff}}{dr}\right|_{r=r_{eq}} = 0 = \frac{|k|}{r_{eq}^2} - \frac{\ell^2}{\mu r_{eq}^3} \Rightarrow r_{eq} = \frac{\ell^2}{\mu|k|}$

$\hookrightarrow V_{eff}(r_{eq}) = \frac{\ell^2}{2\mu}\frac{\mu^2 k^2}{\ell^4} - \frac{|k|\mu|k|}{\ell^2} = -\frac{1}{2}\frac{k^2\mu}{\ell^2} < 0$



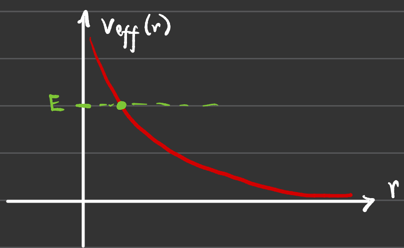
Regiões permitidas: $E \geq V_{eff}(r)$

$\begin{cases} E \geq 0: \text{ espalhado} \rightarrow \text{repulsivo} \\ E < 0: \text{ ligado} \rightarrow \text{atrativo} \\ E = V_{eff}(r_{eq}) \Rightarrow r(t) = r_{eq}: \text{ circular} \end{cases}$ $\rightsquigarrow$ ex.: estrelas binárias

* Potencial repulsivo: $U(r) = \frac{k}{r}$, $k > 0$

→ $V_{eff}(r) = \frac{\ell^2}{2\mu r^2} + \frac{k}{r}$

→ $\left.\frac{dV_{eff}}{dr}\right|_{r=r_{eq}} = 0 = -\frac{\ell^2}{\mu r_{eq}^3} - \frac{k}{r_{eq}^2}$ ⇒ não tem solução



* Em potencial repulsivo, não há estados ligados

17/07/23

* Potencial efetivo:

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}(r)$$

$$V_{eff} = \frac{\ell^2}{2\mu r^2} + U(r), \quad U(r) = \pm \frac{K}{r}$$

* Equação para a trajetória:

$$\theta(r) = \pm \frac{\ell}{\mu}\int \frac{dr}{r^2\left[\frac{2}{\mu}\left(E - U(r)\right) - \frac{\ell^2}{\mu^2 r^2}\right]^{\frac{1}{2}}} + \theta_0$$

* Força gravitacional:

$\rightarrow$ $U(r) = -\frac{K}{r}$, $k = Gm_1 m_2 > 0$

$$\rightarrow \theta(r) = \pm \frac{\ell}{\mu}\int \frac{dr}{r^2\left[\frac{2}{\mu}\left(E + \frac{K}{r}\right) - \frac{\ell^2}{\mu^2 r^2}\right]^{\frac{1}{2}}} + \theta_0$$

$$u \equiv \frac{\ell}{r}, \quad du = -\frac{\ell}{r^2}\,dr$$

$$\theta(r) = \pm\left[-\int \frac{du}{\left[2\mu E + \frac{2\mu K u}{\ell} - u^2\right]^{\frac{1}{2}}}\right] + \theta_0$$

$$\theta = \pm I + \theta_0$$

$$\rightarrow \left(u - \frac{\mu K}{\ell}\right)^2 = u^2 + \frac{\mu^2 K^2}{\ell^2} - \frac{2u\mu K}{\ell}$$

$$I = -\int \frac{du}{\sqrt{2\mu E + \frac{\mu^2 K^2}{\ell^2} - \left(u - \frac{\mu K}{\ell}\right)^2}} \quad \left\} \begin{array}{l} v = u - \frac{\mu K}{\ell} \\ dv = du \end{array} \right. \Rightarrow I = -\int \frac{dv}{\sqrt{A^2 - v^2}}$$

$$A^2 \equiv 2\mu E + \frac{\mu^2 K^2}{\ell^2}$$

$$\rightarrow \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\rightarrow I = \cos^{-1}\left(\frac{v}{A}\right) = \cos^{-1}\left[\frac{\frac{\ell}{r} - \frac{\mu K}{\ell}}{\sqrt{2\mu E + \frac{\mu^2 K^2}{\ell^2}}}\right]$$

$$\therefore \theta = \pm \arccos\left[\frac{\frac{\ell}{r} - \frac{\mu K}{\ell}}{\sqrt{2\mu E + \frac{\mu^2 K^2}{\ell^2}}}\right] + \theta_0$$

$$\rightarrow \cos(\theta - \theta_0) = \cos I = \frac{\frac{\ell}{r} - \frac{\mu K}{\ell}}{\sqrt{2\mu E + \frac{\mu^2 K^2}{\ell^2}}}$$

$$\hookrightarrow \alpha \equiv \frac{\ell^2}{\mu K} > 0$$

$$\hookrightarrow \varepsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu K^2}} \geq 0$$

Excentricidade

$$\cos(\theta - \theta_0) = \frac{\frac{\ell^2}{\mu K r} - 1}{\sqrt{\frac{2\mu E \ell^2}{\mu^2 K^2} + 1}}$$

$$\cos(\theta - \theta_0) = \frac{\frac{\alpha}{r} - 1}{\varepsilon}$$

$$\therefore r(\theta) = \frac{\alpha}{1 + \varepsilon\cos(\theta - \theta_0)}$$

$$\rightarrow r(\theta = \theta_0) = \frac{\alpha}{1+\varepsilon}$$

menor valor possível (mais próxima do centro de força)

$\rightarrow$ Tomamos $\theta_0 = 0$: $\boxed{r(\theta) = \frac{\alpha}{1 + \varepsilon\cos\theta}}$

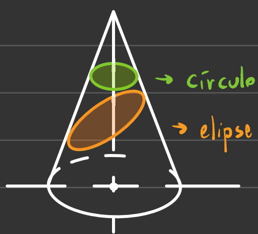
→ Em coordenadas retangulares:

↳ $x = r\cos\theta$

↳ $r = \dfrac{\alpha}{1+\varepsilon\frac{x}{r}} = \dfrac{r\alpha}{r+\varepsilon x} \Rightarrow r + \varepsilon x = \alpha$

↳ $r^2 = \alpha^2 + \varepsilon^2 x^2 - 2\alpha\varepsilon x = x^2 + y^2$

$\therefore (1-\varepsilon^2)x^2 + y^2 + 2\alpha\varepsilon x - \alpha^2 = 0$ } Seção cônica



$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$

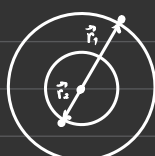
→ Fowles

* $E = V_{eff}(r_{eq}) = -\dfrac{1}{2}\dfrac{k^2\mu}{\ell^2}$ → energia mínima

↳ $\varepsilon = 0 \Rightarrow x^2 + y^2 = \alpha^2$ → círculo centrado em $(0,0)$ e raio α } MCU

↳ $\dot{\theta} = \dfrac{\ell}{\mu r^2} \Rightarrow \dot{\theta} = \text{cte}$

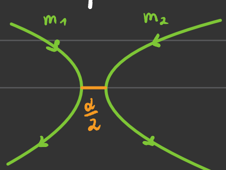
↳ $\vec{r}_1 = \dfrac{m_2\vec{r}}{(m_1+m_2)}$, $\vec{r}_2 = \dfrac{-m_1\vec{r}}{(m_1+m_2)}$

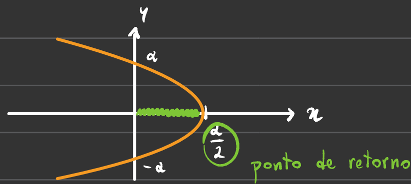


sistema binário

* $E = 0$

↳ $\varepsilon = 1 \Rightarrow y^2 = \alpha^2 - 2\alpha x$ → parábola





órbita aberta

$*\ \varepsilon > 1$ e $0 < \varepsilon < 1$

$$\frac{x^2(1-\varepsilon^2)}{\alpha^2} + \frac{2\varepsilon x}{\alpha} + \frac{y^2}{\alpha^2} = 1$$

$$\frac{x^2}{a^2} + \frac{2\varepsilon x}{\alpha(1-\varepsilon^2)} + \frac{y^2}{\alpha^2(1-\varepsilon^2)} = \frac{1}{1-\varepsilon^2}$$

$$\hookrightarrow \left(\frac{x}{\alpha} + \frac{\varepsilon}{(1-\varepsilon)^2}\right)^2 = \frac{x^2}{\alpha^2} + \frac{\varepsilon^2}{(1-\varepsilon^2)^2} + \frac{2\varepsilon x}{\alpha(1-\varepsilon^2)}$$

$$\hookrightarrow \left(\frac{x}{\alpha} + \frac{\varepsilon}{(1-\varepsilon^2)}\right)^2 - \frac{\varepsilon^2}{(1-\varepsilon^2)^2} + \frac{y^2}{\alpha^2(1-\varepsilon^2)} = \frac{1}{1-\varepsilon^2}$$

$$\hookrightarrow \left(\frac{x(1-\varepsilon^2)}{\alpha} + \varepsilon\right)^2 - \cancel{\varepsilon^2} + \frac{(1-\varepsilon^2)}{\alpha^2}y^2 = 1 - \cancel{\varepsilon^2}$$

$$\boxed{\frac{(1-\varepsilon^2)^2}{\alpha^2}\left(x + \frac{\varepsilon\alpha}{(1-\varepsilon^2)}\right)^2 + \frac{(1-\varepsilon^2)}{\alpha^2}y^2 = 1}$$

19/07/23

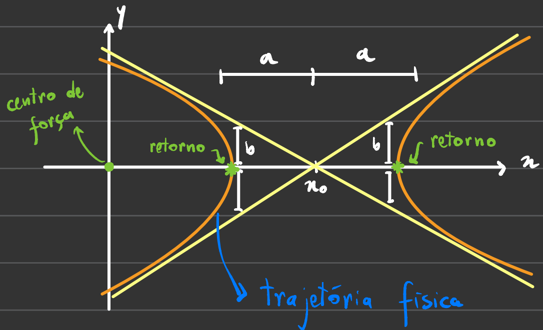
* $E > 0$

↳ $\varepsilon > 1$

↳ $a \equiv \dfrac{\alpha}{\varepsilon^2-1}$, $b \equiv \dfrac{1}{\sqrt{\varepsilon^2-1}}$, $x_0 \equiv \dfrac{\varepsilon\alpha}{\varepsilon^2-1}$

↳ $\dfrac{(1-\varepsilon^2)^2}{\alpha^2}\left[x + \dfrac{\varepsilon\alpha}{(1-\varepsilon^2)}\right]^2 + \dfrac{(1-\varepsilon^2)}{\alpha^2}y^2 = 1 \Rightarrow \dfrac{(x-x_0)^2}{a^2} - \dfrac{y^2}{b^2} = 1 \rightarrow$ hipérbole centrada em $(x_0, 0)$

↳ $\begin{cases} y = 0 \Rightarrow x - x_0 = \pm a \Rightarrow x_+ = x_0 + a,\ x_- = x_0 - a \\ x = x_0 \end{cases}$



↳ $\dfrac{b^2}{a^2} = \dfrac{\alpha^2}{\varepsilon^2-1}\dfrac{(\varepsilon^2-1)^2}{\alpha^2} = \varepsilon^2 - 1 \Rightarrow \varepsilon = \sqrt{1 + \dfrac{b^2}{a^2}}$

↳ Menor ponto de retorno: $x_0 - a = \dfrac{\alpha}{1+\varepsilon} = r(\theta_0)$ — tomamos $\theta_0 = 0$

* $V_{eff}(r_{eq}) < E < 0$

↳ $0 < \varepsilon < 1$

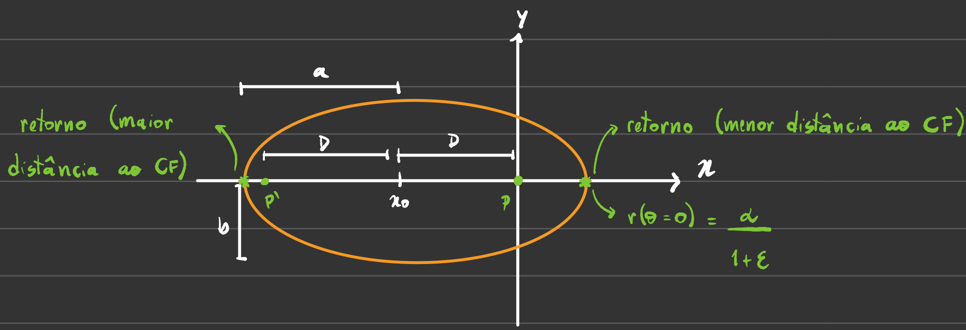
↳ $a \equiv \dfrac{\alpha}{1-\varepsilon^2}$, $b \equiv \dfrac{\alpha}{\sqrt{1-\varepsilon^2}}$, $x_0 \equiv \dfrac{-\varepsilon\alpha}{(1-\varepsilon^2)}$

↳ $\dfrac{(x-x_0)^2}{a^2} + \dfrac{y^2}{b^2} = 1$ → elipse centrada em $(x_0, 0)$

↳ $a = \dfrac{b}{\sqrt{1-\varepsilon^2}} > b$ → a: semieixo maior

b: semieixo menor

↳ $\varepsilon = \sqrt{1 - \dfrac{b^2}{a^2}}$



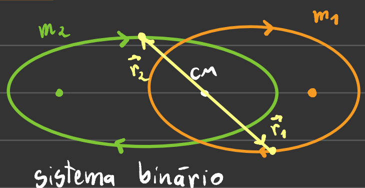
↳ Distância do centro a um ponto focal: $D = a\varepsilon = \dfrac{\alpha\varepsilon}{1-\varepsilon^2} = |x_0|$

↳ $\alpha = \dfrac{\ell^2}{\mu|k|}$, $\varepsilon = \sqrt{1 + \dfrac{2E\ell^2}{\mu k^2}}$

$$\Rightarrow a = \frac{\ell^2}{\mu|k|\left[1-\left(1-\frac{2|E|\ell^2}{\mu k^2}\right)\right]} \Rightarrow \boxed{a = \frac{k}{2|E|}} \quad \boxed{b = \frac{|\ell|}{\sqrt{2\mu|E|}}}$$

($E<0$)

↳ Duas partículas: $\vec{r}_1 = \dfrac{m_2}{m_1+m_2}\vec{r}$, $\vec{r}_2 = \dfrac{-m_1}{m_1+m_2}\vec{r}$



↳ vetores posição antiparalelos (CM em repouso) ⇒ período igual

$$\rightarrow \frac{dA}{dt} = \frac{|\vec{L}|}{2\mu} = \frac{|\ell|}{2\mu} \Rightarrow dt = \frac{2\mu}{|\ell|} dA$$

$$\rightarrow \tau = \int_0^\tau dt = \frac{2\mu A}{|\ell|}$$ → um período equivale a varrer a área total da elipse

$$\rightarrow A = \pi a b \Rightarrow \tau = \frac{2\pi a b \mu}{|\ell|} = \frac{\cancel{2}\pi\mu}{\cancel{|\ell|}} \cdot \frac{\overbrace{K}^{a}}{\cancel{2}|E|} \cdot \frac{\cancel{|\ell|}}{\sqrt{2\mu|E|}} = \pi\sqrt{\frac{\mu}{2}}\,\frac{K}{|E|^{\frac{3}{2}}}$$

$$\tau^2 = \pi^2 \frac{\mu}{2} \cdot K^2 \cdot \frac{8a^3}{K^3}$$

$$\therefore \boxed{\tau^2 = \frac{4\pi^2\mu}{K} a^3}$$

$U(r) = -\frac{K}{r}$

* Energia mecânica: $E = \frac{1}{2}\mu|\dot{\vec{r}}|^2 - \frac{K}{r}$

$$\begin{cases} E < 0 \Rightarrow T < |U| \rightarrow \text{órbita fechada} \\ E \geq 0 \Rightarrow T > |U| \rightarrow \text{órbita aberta} \end{cases}$$

↳ Condições iniciais:
$$\begin{cases} \text{Distância entre partículas: } R \\ \dot{\vec{r}}(0) = \dot{\vec{r}}_1(0) - \dot{\vec{r}}_2(0) = \vec{v}_0 \\ E = \frac{1}{2}\mu\vec{v}_0 - \frac{K}{r} \end{cases}$$

↳ Velocidade de escape: $E \geq 0 \Rightarrow \frac{1}{2}\mu v_0^2 \geq \frac{K}{R} \Rightarrow \boxed{v_0 \geq \sqrt{\frac{2G(m_1+m_2)}{R}}}$

* Recuperando as leis de Kepler:

→ M: massa do sol
m: massa de um planeta

$\left.\begin{array}{l} M: \text{massa do sol} \\ m: \text{massa de um planeta} \end{array}\right\} M \gg m$

→ Planeta: $\vec{r}_1 = \dfrac{M}{m+M}\vec{r} \approx \vec{r}$

Sol: $\vec{r}_2 = \dfrac{-m}{m+M}\vec{r} \approx 0$

1ª lei:

sol vai para a origem do sistema e é um dos focos da elipse da trajetória do planeta

→ $K = GmM$

$\mu = \dfrac{Mm}{m+M}$

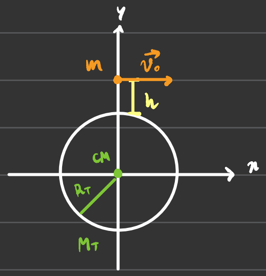
$$\tau^2 = \frac{4\pi^2}{G\cancel{mM}} \cdot \frac{\cancel{mM}}{m+M} \cdot a^3 = \frac{4\pi^2 a^3}{G(m+M)}$$

$$\therefore \boxed{\tau^2 \approx \frac{4\pi^2}{GM} a^3}$$

3ª lei

21/07/23

Lançamento de projéteis



$M_T \gg m \Rightarrow \mu \approx m$

Condições iniciais $(t=0)$:

$$\begin{cases} \vec{v}(0) = v_0\hat{i}, \quad \vec{r}(0) = (R_T + h)\hat{j} \approx R_T\hat{j} \\ v_0 = -R_T\underbrace{\dot{\theta}(0)}_{<0} \Rightarrow \dot{\theta}(0) = -\dfrac{v_0}{R_T} \end{cases}$$

$\rightarrow E = \dfrac{1}{2}mv_0^2 - \dfrac{GmM_T}{R_T}, \quad \ell = mr^2(0)\dot{\theta}(0) = -mR_Tv_0$

$\rightarrow V_{eff}(r_{eq}) = -\dfrac{1}{2}\dfrac{k^2\mu}{\ell^2} = -\dfrac{1}{2}\dfrac{G^2\cancel{m^2}M_T^2 m}{\cancel{m^2}R_T^2v_0^2} = -\dfrac{1}{2}\dfrac{mG^2M_T^2}{R_T^2v_0^2}$

$$\left.\begin{array}{l} \hookrightarrow M_T = 6\cdot 10^{24}\,\text{kg} \\ \quad R_T = 6{,}4\cdot 10^6\,\text{m} \\ \quad \dfrac{GM_T}{R_T} \approx 6{,}3\cdot 10^7\,\dfrac{\text{J}}{\text{kg}} \\ \hookrightarrow m = 1\,\text{kg} \\ \quad v_0 = 10^3\,\text{m/s} \end{array}\right\} \quad \begin{array}{l} E \approx -6{,}3\cdot 10^7\,\text{J} < 0 \\ |\ell| \approx 6{,}4\cdot 10^9\,\text{J}\cdot\text{s} \\ V_{eff}(r_{eq}) \approx -2\cdot 10^9\,\text{J} \end{array}$$

localmente parece uma parábola

$\hookrightarrow$ Parâmetros da órbita:

$$\left.\begin{array}{l} a = \dfrac{k}{2|E|} = \dfrac{GmM_T}{2|E|} \Rightarrow a \approx 3{,}2\cdot 10^6\,\text{m} \\ b = \dfrac{|\ell|}{\sqrt{2m|E|}} \Rightarrow b \approx 0{,}6\cdot 10^6\,\text{m} \end{array}\right\} \quad \varepsilon = \sqrt{1 - \dfrac{b^2}{a^2}} \Rightarrow \varepsilon \approx 0{,}98$$

→ Condição para órbita circular: $E = V_{eff}(r_{eq})$

$$\frac{1}{2} v_0^2 m - \frac{G m M_T}{R_T} = -\frac{1}{2} \frac{m G^2 M_T^2}{R_T^2 v_0^2}$$

$$R_T^2 v_0^4 - 2 R_T G M_T v_0^2 + G^2 M_T^2 = 0 \Rightarrow \boxed{v_0 = \sqrt{\frac{G M_T}{R_T}}} \Rightarrow v_0 \approx 8000 \text{ m/s}$$

Estabilidade de órbitas circulares

→ $V_{eff}(r) = \dfrac{\ell^2}{2\mu r^2} + U(r)$

$$\begin{cases} F(r) = -\dfrac{dU}{dr} \rightarrow \text{interação} \\ F_{CF}(r) = \dfrac{\ell^2}{\mu r^3} \rightarrow \text{centrífuga} \end{cases}$$

→ $\mu \ddot{r} = F_{res}(r)$, $F_{res}(r) = F(r) + \dfrac{\ell^2}{\mu r^3} = -\dfrac{dV_{eff}}{dr}$

→ Órbita circular: $\left.\begin{array}{l} F_{res}(r_{eq}) = 0 \\ F(r_{eq}) + \dfrac{\ell^2}{\mu r_{eq}^3} = 0 \end{array}\right\}$ $F(r_{eq}) = -\dfrac{\ell^2}{\mu r_{eq}^3}$

→ Perturbação na equação dinâmica:

↳ $\mu \ddot{r} = F(r) + \dfrac{\ell^2}{\mu r^3}$

↳ Perturbação: $r(t) = r_{eq} + x(t)$, $|x(t)| \ll 1$

$$\mu \ddot{x} = F(r_{eq} + x(t)) + \frac{\ell^2}{\mu (r_{eq} + x(t))^3}$$

↳ $F(r_{eq} + x(t)) = F(r_{eq}) + \left.\dfrac{dF}{dx}\right|_{x=0} (x(t) - 0)$

$$= F(r_{eq}) + \left.\frac{dF}{dr}\right|_{r_{eq}} \underbrace{\frac{dr}{dx}}_{1} x(t)$$

$$\hookrightarrow \frac{1}{(r_{eq} + x(t))^3} \approx \frac{1}{r_{eq}^3} - \frac{3x(t)}{r_{eq}^4}$$

$$\rightarrow F(r_{eq}) = -\frac{\ell^2}{\mu r_{eq}^3}$$

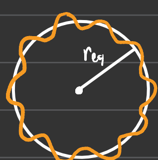
$$\hookrightarrow \mu \ddot{x} = \cancel{F(r_{eq})} + \left.\frac{dF}{dr}\right|_{r_{eq}} x(t) + \cancel{\frac{\ell^2}{\mu r_{eq}^3}} - \frac{3\ell^2 x(t)}{\mu r_{eq}^4}$$

$$\mu \ddot{x} + \left[\frac{3\ell^2}{\mu r_{eq}^4} - \left.\frac{dF}{dr}\right|_{r_{eq}} \right] x(t) = 0$$

$$\ddot{x} + \left[-\frac{3F(r_{eq})}{\mu r_{eq}} - \frac{1}{\mu} \left.\frac{dF}{dr}\right|_{r_{eq}} \right] x(t) = 0$$

* Caso 1: $-\frac{3F(r_{eq})}{r_{eq}} - \left.\frac{dF}{dr}\right|_{r_{eq}} > 0$

$$\hookrightarrow \ddot{x} + \omega_0^2 x = 0, \quad \omega_0 = \sqrt{-\frac{3F(r_{eq})}{\mu r_{eq}} - \left.\frac{dF}{\mu dr}\right|_{r_{eq}}} \rightarrow \text{MHS}$$



$\rightarrow$ Estável $\}$ $r(t) \in [\underbrace{r_{min}, r_{max}}_{\text{distâncias apsidais}}]$

* Caso 2: $-\frac{3F(r_{eq})}{r_{eq}} - \left.\frac{dF}{dr}\right|_{r_{eq}} < 0 \rightarrow x(t)$ diverge $\rightarrow$ instável

* Lei de potência: $F(r) = -\frac{k}{r^n}$, $k > 0$

$$\to \left.\frac{dF}{dr}\right|_{r_{eq}} = \frac{kn}{r_{eq}^{n+1}}$$

$$\to \frac{3k}{r_{eq}^{n+1}} - \frac{kn}{r_{eq}^{n+1}} = \frac{k}{r_{eq}^{n+1}}(3-n) \quad \begin{cases} n < 3 \to \text{estável} \\ n > 3 \to \text{instável} \end{cases} \quad \Big| \quad n = 3 \to ?$$

* Pontos apsidais da órbita elíptica: $\begin{cases} r_{min} \to \text{periélio} \\ r_{max} \to \text{afélio} \end{cases}$

* Ângulo apsidal ψ: ângulo entre pontos apsidais

↳ Ex.: elipse $\to \psi = \pi$

$\to \psi$ na órbita circular estável: $\ell = \mu r^2 \dot{\theta} \to \frac{d\theta}{dt} = \frac{\ell}{\mu(r_{eq} + x(t))^2} \approx \frac{\ell}{\mu r_{eq}^2}$ $(|x(t)| \ll 1)$

↳ $\psi = \int_{\theta(r_{min})}^{\theta(r_{max})} d\theta = \frac{\ell}{\mu r_{eq}^2} \int dt \Rightarrow \psi = \frac{\ell \Delta t}{\mu r_{eq}^2}$, $\Delta t = \underbrace{t(r_{max}) - t(r_{min})}_{\tau/2} \Rightarrow \psi = \frac{\ell \tau}{2 \mu r_{eq}^2}$

↳ $\frac{2\pi}{\tau} = \omega_0 = \sqrt{-\frac{3F(r_{eq})}{\mu r_{eq}} - \frac{1}{\mu}\left.\frac{dF}{dr}\right|_{r_{eq}}}$

↳ $\ell = \pm\sqrt{-\mu r_{eq}^3 F(r_{eq})}$

$$\boxed{\psi = \pm\pi \left[\frac{F(r_{eq})}{3F(r_{eq}) + r_{eq}\left.\frac{dF}{dr}\right|_{r_{eq}}}\right]^{\frac{1}{2}}}$$

$$\psi = \frac{\pm\sqrt{-\mu r_{eq}^3 F(r_{eq})} \cdot \cancel{2}\pi}{\cancel{2}\mu r_{eq}^2 \sqrt{-\frac{3F(r_{eq})}{\mu r_{eq}} - \frac{1}{\mu}\left.\frac{dF}{dr}\right|_{r_{eq}}}} = \pm\pi \sqrt{\frac{\cancel{\mu} r_{eq}^3 F(r_{eq})}{3F(r_{eq})\cancel{\mu} r_{eq}^3 + \cancel{\mu} r_{eq}^4 \left.\frac{dF}{dr}\right|_{r_{eq}}}}$$

* Para leis de potência: $F(r) = -\frac{k}{r^n}$, $\psi = \frac{\pm\pi}{\sqrt{3-n}}$, $n < 3$

$m \in \mathbb{Z}$

* Órbita fechada: $\boxed{m\psi = 2\pi}$

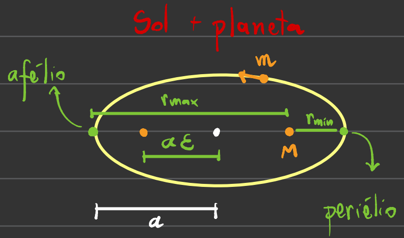
↳ $n = 2 \Rightarrow \psi = \pm\pi$

↳ $n = -1 \Rightarrow \psi = \pm\frac{\pi}{2}$

* Órbita aberta:

↳ $n = 1 \Rightarrow \psi = \pm\frac{\pi}{\sqrt{2}}$ } avanço/regressão do periélio

24/07/23



$M \gg m$

$r_{min} = a(1-\varepsilon)$

$r_{max} = a(1+\varepsilon)$

$F(r) = -\dfrac{k}{r^2}$

$K = GmM$

Órbita de Mercúrio

→ Deslocamento do periélio: $\begin{cases} \text{Experimental: 100 anos} \rightarrow \tilde{\Delta} = 574'' = 0{,}1594° \\ \text{Teoria newtoniana: 100 anos} \rightarrow \tilde{\Delta} = 531'' \end{cases} > \underline{43''}$

→ Equação para a Trajetória: $\dfrac{d^2u}{d\theta^2} + u = -\dfrac{\mu}{\ell^2 u^2} F\left(\dfrac{1}{u}\right)$, $u = \dfrac{1}{r}$

$\left.\begin{array}{l} \hookrightarrow F\left(\frac{1}{u}\right) = -ku^2 \\ \hookrightarrow \mu = \dfrac{mM}{m+M} \approx m \end{array}\right\}$ $\dfrac{d^2u}{d\theta^2} + u = \dfrac{mk}{\ell^2} = \dfrac{GMm^2}{\ell^2} = \dfrac{1}{\alpha^2}$

Solução: $u(\theta) = \dfrac{1}{\alpha}(1+\varepsilon\cos\theta)$

→ Periélio: $\dfrac{1}{r_{min}} = u(\theta = 0) = \dfrac{1}{\alpha}(1+\varepsilon)$

→ Pela teoria da relatividade geral: $\dfrac{d^2u}{d\theta^2} + u = \dfrac{1}{\alpha} + \delta u^2$, $\delta = \dfrac{3GM}{c^2}$

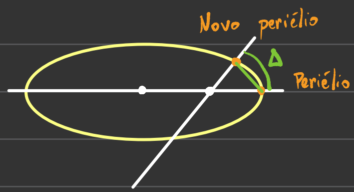
Solução aproximada: $u(\theta) = \dfrac{1}{\alpha}\left[1 + \varepsilon\cos\theta + \dfrac{\delta\varepsilon}{\alpha}\theta\,\mathrm{sen}\,\theta\right]$

$$\rightarrow u(\theta) = \frac{1}{\alpha}\left[1 + \varepsilon\left(\cos\theta + \frac{\delta}{\alpha}\theta \operatorname{sen}\theta\right)\right]$$

$$\rightarrow \cos\left(\theta - \frac{\delta}{\alpha}\theta\right) = \cos\theta \overbrace{\cos\left(\frac{\delta}{\alpha}\theta\right)}^{\simeq 1} + \operatorname{sen}\theta \overbrace{\operatorname{sen}\left(\frac{\delta}{\alpha}\theta\right)}^{\approx \frac{\delta}{\alpha}\theta}$$

$$\therefore u(\theta) \approx \frac{1}{\alpha}\left[1 + \varepsilon\cos\left(\theta - \frac{\delta}{\alpha}\theta\right)\right]$$

→ Próximo periélio: $\cos\left(\theta - \frac{\delta}{\alpha}\theta\right) = 1 \Rightarrow \theta - \frac{\delta}{\alpha}\theta = 2\pi \Rightarrow \theta = \frac{2\pi}{1 - \frac{\delta}{\alpha}} \approx 2\pi + 2\pi\frac{\delta}{\alpha}$



$$\Delta = 2\pi\frac{\delta}{\alpha}$$

$$\rightarrow \frac{\delta}{\alpha} = \frac{3GM}{c^2}\frac{GMm^2}{\ell^2} = \frac{3G^2M^2m^2}{c^2\ell^2}$$

$$\rightarrow \varepsilon = \sqrt{1 - \frac{2|E|\ell^2}{\mu k^2}} = \sqrt{1 - \frac{\ell^2}{a\mu k}} \Rightarrow \ell^2 = Gm^2Ma(1-\varepsilon^2)$$

$$\rightarrow a = \frac{k}{2|E|}$$

$$\frac{\delta}{\alpha} = \frac{3GM}{c^2a(1-\varepsilon^2)}$$

$$\therefore \boxed{\Delta = \frac{6\pi GM}{c^2a(1-\varepsilon^2)}}$$

depois de 1 volta

$\rightarrow$ Mercúrio: $\begin{cases} \varepsilon = 0{,}2056 \\ a = 5{,}791 \cdot 10^{10}\ m \\ M = 1{,}989 \cdot 10^{30}\ kg \end{cases} \Rightarrow \Delta \approx \pi \cdot 1{,}5957 \cdot 10^{-7}\ rad = 0{,}1034''$

$\tau = 0{,}2408$ anos $\Rightarrow \dfrac{100}{0{,}2408} = 415{,}28$ rev.

$\rightarrow \Delta_{100} = 415{,}28 \cdot \Delta \Rightarrow \boxed{\Delta_{100} \approx 42{,}94''}$