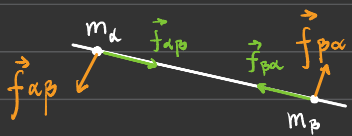
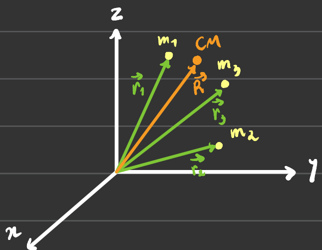
Sistemas de partículas



$\vec{f}_{\alpha\beta}$: força que β faz em α

* 3ª Lei de Newton: $\begin{cases} \text{Forma "fraca": } \vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha} \quad (*) \\ \text{Forma "forte": } (*) \text{ e força central} \end{cases}$

Centro de massa



$m_\alpha \quad (\alpha = 1, 2, \dots, n)$

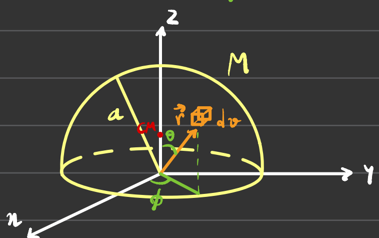
$\vec{r}_\alpha \quad (\alpha = 1, 2, \dots, n)$

$M = \sum_{\alpha=1}^{n} m_\alpha$

Posição do CM:

	Discreto	Contínuo
	$\boxed{\vec{R} = \sum_{\alpha=1}^{n} \frac{m_\alpha}{M} \vec{r}_\alpha}$	$\boxed{\vec{R} = \frac{1}{M} \int_V dm\, \vec{r}}$

* Exemplo: Hemisfério sólido uniforme



$\rightarrow V = \frac{1}{2}\frac{4}{3}\pi a^3 = \frac{2}{3}\pi a^3$

$\rightarrow \rho(\vec{r}) = \frac{dm}{dv}$

$\rightarrow \vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$Z = \frac{\rho}{M} \int_V dv\, z$

↳ $dv = r^2 \operatorname{sen}\theta \, d\theta \, d\phi \, dr$

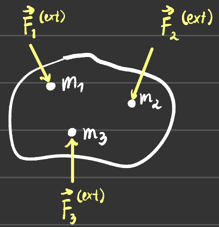
↳ $z = r\cos\theta$

$$Z = \frac{\rho}{M} \int_0^a dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi \; r^3 \operatorname{sen}\theta \cos\theta = \frac{2\pi\rho}{M} \left(\frac{r^4}{4}\right)_0^a \left(-\frac{1}{2}\cos^2\theta\right)_0^{\frac{\pi}{2}} = \frac{\pi\rho}{M}\frac{a^4}{4}$$

$$Z = \frac{\pi a^4}{4M} \cdot M \cdot \frac{3}{2\pi a^3} \Rightarrow \boxed{Z = \frac{3}{8}a}$$

* Momento linear:

n partículas



$\vec{F}_\alpha^{(ext)}$: força resultante externa sobre α

$\vec{f}_{\alpha\beta}$: força que β faz em A

→ Força resultante sobre α: $\vec{f}_\alpha = \vec{F}_\alpha^{(ext)} + \sum_{\substack{\beta=1 \\ (\neq\alpha)}}^{n} \vec{f}_{\alpha\beta} = m_\alpha \ddot{\vec{r}}_\alpha$

$$\sum_{\alpha=1}^{n} m_\alpha \ddot{\vec{r}}_\alpha = \underbrace{\sum_{\alpha=1}^{n} \vec{F}_\alpha^{(ext)}}_{\vec{F}_{ext}} + \sum_{\substack{\alpha,\beta=1 \\ (\alpha\neq\beta)}}^{n} \vec{f}_{\alpha\beta}$$

↳ $\sum_{\alpha=1}^{n} m_\alpha \ddot{\vec{r}}_\alpha = \frac{d^2}{dt^2}\left(\overbrace{\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha}^{M\vec{R}}\right) = M\ddot{\vec{R}}$

↳ $\sum_{\substack{\alpha\beta=1 \\ (\alpha\neq\beta)}}^{n} \vec{f}_{ab} = \vec{f}_{12} + \vec{f}_{13} + \cdots + \vec{f}_{21} + \vec{f}_{23} + \cdots + \vec{f}_{31} + \vec{f}_{32} + \cdots$

$$\sum_{\alpha\beta=1}^{n} \vec{f}_{ab} = \sum_{\substack{\alpha\beta=1 \\ (\alpha<\beta)}}^{n} \left(\vec{f}_{\alpha\beta} + \vec{f}_{\beta\alpha}\right)$$

↳ $M\ddot{\vec{R}} = \vec{F}_{ext} + \sum_{\substack{\alpha\beta=1 \\ (\alpha<\beta)}}^{n} \left(\vec{f}_{\alpha\beta} + \vec{f}_{\beta\alpha}\right)$

↳ 3ª Lei: $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$

$\boxed{\vec{F}_{ext} = M\ddot{\vec{R}}}$ → independe da natureza das forças internas

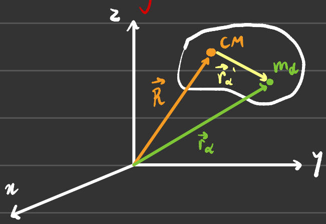
→ Momento linear do sistema: $\vec{P} = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n$

$$\vec{P} = \sum_{\alpha=1}^{n} m_\alpha \vec{v}_\alpha = \frac{d}{dt}\left(\overbrace{\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha}^{M\vec{R}}\right) \Rightarrow \boxed{\vec{P} = M\dot{\vec{R}}}$$

→ $\vec{F}_{ext} = \frac{d}{dt}(M\dot{\vec{R}}) \Rightarrow \boxed{\vec{F}_{ext} = \dot{\vec{P}}}$

↳ Conservação do momento linear do sistema: $\boxed{\vec{F}_{ext} = 0 \Rightarrow \vec{P} = cte}$

*** Momento angular:**



$$\vec L_\alpha = \vec r_\alpha \times \vec p$$

$$\vec L = \sum_{\alpha=1}^{n} \vec L_\alpha = \sum_{\alpha=1}^{n} m_\alpha (\vec r_\alpha \times \dot{\vec r}_\alpha)$$

$$\vec R + \vec r_\alpha' = \vec r_\alpha$$

$$\vec L = \sum_{\alpha=1}^{n} m_\alpha (\vec R + \vec r_\alpha') \times (\dot{\vec R} + \dot{\vec r}_\alpha')$$

$$= \sum_{\alpha=1}^{n} m_\alpha (\vec R \times \dot{\vec R} + \vec R \times \dot{\vec r}_\alpha' + \vec r_\alpha' \times \dot{\vec R} + \vec r_\alpha' \cdot \dot{\vec r}_\alpha')$$

$$= (\vec R \times \dot{\vec R}) \underbrace{\sum_{\alpha=1}^{n} m_\alpha}_{M} + \vec R \times \underbrace{\left(\sum_{\alpha=1}^{n} m_\alpha \dot{\vec r}_\alpha'\right)}_{0} + \underbrace{\left(\sum_{\alpha=1}^{n} \vec r_\alpha' m_\alpha\right)}_{0} \times \dot{\vec R} + \sum_{\alpha=1}^{n} m_\alpha (\vec r_\alpha' \times \dot{\vec r}_\alpha') \qquad \vec p_\alpha'$$

$$\therefore \boxed{\vec L = \underbrace{M(\vec R \times \dot{\vec R})}_{\text{rotação do sistema}} + \underbrace{\sum_{\alpha=1}^{n} \vec r_\alpha' \times \vec p_\alpha'}_{\text{rotação das partículas em torno do CM}}}$$

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→ $\dot{\vec{L}} = \underbrace{\sum_{\alpha=1}^{n} \dot{\vec{r}}_\alpha \times \overset{m_\alpha \dot{\vec{r}}_\alpha}{\vec{p}_\alpha}}_{0} + \sum_{\alpha=1}^{n} \vec{r}_\alpha \times \dot{\vec{p}}_\alpha$

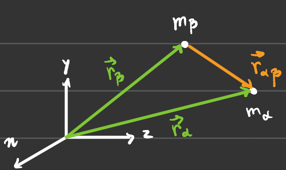
→ 2ª Lei: $\dot{\vec{p}}_\alpha = \vec{F}_\alpha^{(ext)} + \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{n} \vec{f}_{\alpha\beta}$

→ $\dot{\vec{L}} = \sum_{\alpha=1}^{n} \vec{r}_\alpha \times \vec{F}_\alpha^{(ext)} + \underbrace{\sum_{\substack{\alpha,\beta=1 \\ (\alpha\neq\beta)}}^{n} \vec{r}_\alpha \times \vec{f}_{\alpha\beta}}_{\text{Torque interno total}}$

→ Torque interno sobre α: $\vec{N}_\alpha = \vec{r}_\alpha \times \sum_{\substack{\beta=1 \\ (\alpha\neq\beta)}}^{n} \vec{f}_{\alpha\beta}$

→ Torque interno total: $\sum_{\substack{\alpha,\beta=1 \\ (\alpha\neq\beta)}}^{n} \vec{r}_\alpha \times \vec{f}_{\alpha\beta} = \sum_{\substack{\alpha,\beta=1 \\ (\alpha<\beta)}}^{n} \left(\vec{r}_\alpha \times \vec{f}_{\alpha\beta} + \vec{r}_\beta \times \vec{f}_{\beta\alpha} \right)$

→ 3ª Lei: $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$



$\vec{r}_\beta + \vec{r}_{\alpha\beta} = \vec{r}_\alpha$

$\vec{r}_{\alpha\beta} = \vec{r}_\alpha - \vec{r}_\beta$

→ $\dot{\vec{L}} = \sum_{\alpha=1}^{n} \vec{r}_\alpha \times \vec{F}_\alpha^{(ext)} + \sum_{\substack{\alpha,\beta=1 \\ \alpha<\beta}}^{n} \left[\overbrace{(\vec{r}_\alpha - \vec{r}_\beta)}^{\vec{r}_{\alpha\beta}} \times \vec{f}_{\alpha\beta} \right]$

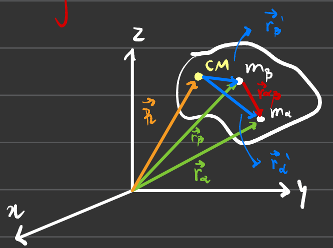
escala da Mecânica Clássica: coulombiana e gravitacional

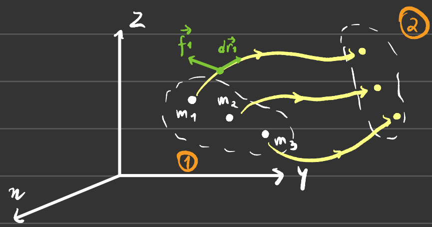
→ Assumimos que as forças $\vec{f}_{\alpha\beta}$ são centrais ⇒ $\vec{r}_{\alpha\beta} \times \vec{f}_{\alpha\beta} = 0$

$\therefore \boxed{\dot{\vec{L}} = \sum_{\alpha=1}^{n} \vec{r}_\alpha \times \vec{F}_\alpha^{(ext)} = \vec{N}_{ext}}$

→ Conservação do momento angular: $\boxed{\vec{N}_{ext} = 0 \Rightarrow \vec{L} = cte}$

* Energia:





Configuração:

$$\begin{cases} \vec{r}_1, \vec{r}_2, \dots, \vec{r}_n \\ \dot{\vec{r}}_1, \dot{\vec{r}}_2, \dots, \dot{\vec{r}}_n \end{cases}$$

→ Trabalho total sobre o sistema: $W = \sum_{\alpha=1}^{n} \int_1^2 \vec{f}_\alpha \cdot d\vec{r}_\alpha$

→ $\vec{f}_\alpha \cdot d\vec{r}_\alpha = (\vec{f}_\alpha \cdot \vec{v}_\alpha)\,dt$, $\quad \vec{v}_\alpha = \dfrac{d\vec{r}_\alpha}{dt}$

$\vec{f}_\alpha = m_\alpha \dfrac{d\vec{v}_\alpha}{dt}$

$$\vec{f}_\alpha \cdot d\vec{r}_\alpha = m_\alpha \left(\frac{d\vec{v}_\alpha}{dt} \cdot \vec{v}_\alpha\right) dt = \frac{m_\alpha}{2}\frac{d}{dt}\left(\vec{v}_\alpha \cdot \vec{v}_\alpha\right) dt = \frac{m_\alpha}{2}\frac{d}{dt}\left(v_\alpha^2\right) dt$$

→ Energia cinética: $T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2 \Rightarrow \vec{f}_\alpha \cdot d\vec{r}_\alpha = \dfrac{d}{dt} T_\alpha\, dt = dT_\alpha$

→ $W = \sum_{\alpha=1}^{n} \int_1^2 dT_\alpha = \sum_{\alpha=1}^{n} \left(T_\alpha^{(2)} - T_\alpha^{(1)}\right)$

$\therefore \boxed{W = \Delta T}$, $\quad T = \sum_{\alpha=1}^{n} T_\alpha$

Teorema trabalho-energia cinética

$$\rightarrow T_\alpha = \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha$$
$$= \frac{1}{2} m_\alpha \left(\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}' \right) \cdot \left(\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}' \right)$$
$$= \frac{1}{2} m_\alpha \left(\dot{\vec{R}} \cdot \dot{\vec{R}} + \dot{\vec{r}}_\alpha{}' \cdot \dot{\vec{r}}_\alpha{}' + 2 \dot{\vec{R}} \cdot \dot{\vec{r}}_\alpha{}' \right)$$

$$\rightarrow \dot{\vec{r}}_\alpha{}' = \vec{v}_\alpha{}', \quad \dot{\vec{R}} = \vec{V}$$

$$\frac{d}{dt}\left(\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha{}' \right)$$

$$\rightarrow T = \sum_{\alpha=1}^{n} T_\alpha = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{\alpha=1}^{n} m_\alpha v_\alpha{}'^2 + \dot{\vec{R}} \underbrace{\sum_{\alpha=1}^{n} m_\alpha \dot{\vec{r}}_\alpha{}'}$$

$$\therefore \boxed{T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{\alpha=1}^{n} m_\alpha v_\alpha{}'^2}$$

$$\Rightarrow W = \sum_{\alpha=1}^{n} \int_1^2 \vec{F}_\alpha^{(ext)} \cdot d\vec{r}_\alpha + \underbrace{\sum_{\substack{\alpha,\beta=1 \\ (\alpha\neq\beta)}}^{n} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha}_{W_{int}}$$

→ supondo que as forças são conservativas:

$$\hookrightarrow \vec{F}_\alpha^{(ext)} = -\vec{\nabla}_\alpha U_\alpha^{(ext)}, \quad U_\alpha^{(ext)} = U_\alpha^{(ext)}(\vec{r}) \quad \Big| \quad \vec{\nabla}_\alpha = \frac{\partial}{\partial x_\alpha}\hat{i} + \frac{\partial}{\partial y_\alpha}\hat{j} + \frac{\partial}{\partial z_\alpha}\hat{k}$$

$$\hookrightarrow \vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha U_{\alpha\beta}, \quad U_{\alpha\beta} = U_{\alpha\beta}(\vec{r}_\alpha, \vec{r}_\beta)$$

$$\hookrightarrow \int_1^2 \vec{F}_\alpha^{(ext)} \cdot d\vec{r}_\alpha = -\int_1^2 \left(\vec{\nabla}_\alpha U_\alpha^{(ext)} \right) \cdot d\vec{r}_\alpha, \quad d\vec{r}_\alpha = dx_\alpha\hat{i} + dy_\alpha\hat{j} + dz_\alpha\hat{k}$$
$$\left(\vec{\nabla}_\alpha U_\alpha^{(ext)} \right) \cdot d\vec{r}_\alpha = \frac{\partial U_\alpha^{(ext)}}{\partial x_\alpha} dx_\alpha + \frac{\partial U_\alpha^{(ext)}}{\partial y_\alpha} dy_\alpha + \frac{\partial U_\alpha^{(ext)}}{\partial z_\alpha} dz_\alpha = dU_\alpha^{(ext)}$$

$$\therefore \int_1^2 \vec{F}_\alpha^{(ext)} \cdot d\vec{r}_\alpha = -\int_1^2 dU_\alpha^{(ext)} = -\Delta U_\alpha^{(ext)}$$

$$\hookrightarrow W_{int} = \sum_{\substack{\alpha,\beta=1 \\ (\alpha<\beta)}}^{n} \left[\int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha + \int_1^2 \vec{f}_{\beta\alpha} \cdot d\vec{r}_\beta \right]$$

$$= - \sum_{\substack{\alpha\beta=1 \\ (\alpha<\beta)}}^{n} \left[\int_1^2 (\vec{\nabla}_\alpha U_{\alpha\beta}) \cdot d\vec{r}_\alpha + \int_1^2 (\vec{\nabla}_\beta U_{\beta\alpha}) \cdot d\vec{r}_\beta \right]$$

$$\hookrightarrow U_{\alpha\beta}(\vec{r}_\alpha, \vec{r}_\beta) \longrightarrow dU_{\alpha\beta} = (\vec{\nabla}_\alpha U_{\alpha\beta}) \cdot d\vec{r}_\alpha + (\vec{\nabla}_\beta U_{\alpha\beta}) \cdot d\vec{r}_\beta$$

$$\Rightarrow W_{int} = - \sum_{\substack{\alpha,\beta=1 \\ (\alpha<\beta)}}^{n} \left\{ \int_1^2 \left[dU_{\alpha\beta} - (\vec{\nabla}_\beta U_{\alpha\beta}) \cdot d\vec{r}_\beta \right] + \int_1^2 (\vec{\nabla}_\beta U_{\beta\alpha}) \cdot d\vec{r}_\beta \right\}$$

$$= - \sum_{\substack{\alpha,\beta=1 \\ (\alpha<\beta)}}^{n} \left[\Delta U_{\alpha\beta} + \int_1^2 \underbrace{(\vec{\nabla}_\beta U_{\beta\alpha} - \vec{\nabla}_\beta U_{\alpha\beta})}_{=0} \cdot d\vec{r}_\beta \right]$$

$\hookrightarrow$ Hipótese: forças internas centrais $\Rightarrow U_{\alpha\beta} = U(|\vec{r}_\alpha - \vec{r}_\beta|) \Rightarrow U_{\alpha\beta} = U_{\beta\alpha}$

$$\therefore \boxed{W_{int} = -\Delta U_{int}}, \quad U_{int} = \sum_{\substack{\alpha,\beta=1 \\ (\alpha<\beta)}}^{n} U_{\alpha\beta}$$

$$\rightarrow W_T = W_{ext} + W_{int} \Rightarrow \boxed{W = -\Delta U}, \quad U = \sum_{\alpha=1}^{n} U_\alpha^{(ext)} + \sum_{\substack{\alpha,\beta=1 \\ (\alpha<\beta)}}^{n} U_{\alpha\beta}$$

$\hookrightarrow$ apenas para forças centrais

$$W_{ext} = \sum_{\alpha=1}^{n} \int_1^2 \vec{F}_\alpha^{(ext)} \cdot d\vec{r}_\alpha, \quad \vec{F}_\alpha^{(ext)} = -\vec{\nabla}_\alpha U_\alpha^{(ext)} \quad \boxed{W_{ext} = -\Delta U_{ext}}, \quad U_{ext} = \sum_{\alpha=1}^{n} U_\alpha^{(ext)}$$

$$W_{int} = \sum_{\substack{\alpha\beta=1 \\ (\alpha<\beta)}}^{n} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha, \quad \vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha U_{\alpha\beta} \quad \boxed{W_{int} = -\Delta U_{int}}, \quad U_{int} = \sum_{\substack{\alpha\beta=1 \\ (\alpha<\beta)}}^{n} U_{\alpha\beta}$$

+ centrais

Centrais: $U = U_{int} + U_{ext}$

$$\left.\begin{matrix} W = \Delta T \\ W = -\Delta U \end{matrix}\right\} \quad \left.\begin{matrix} \Delta T = -\Delta U \\ E = T + U \end{matrix}\right\} \quad \begin{matrix} \Delta T + \Delta U = 0 \\ \boxed{\Delta E = 0} \end{matrix}$$

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9-15. A smooth rope is placed above a hole in a table (Figure 9-D). One end of the rope falls through the hole at $t = 0$, pulling steadily on the remainder of the rope. Find the velocity and acceleration of the rope as a function of the distance to the end of the rope x. Ignore all friction. The total length of the rope is L.

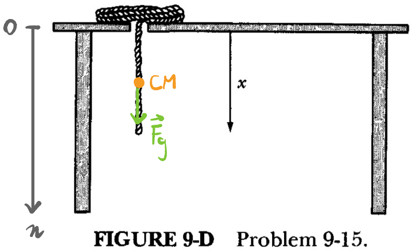


FIGURE 9-D Problem 9-15.

→ Corda uniforme: $\rho = M/L = m/x$

→ Velocidade do CM em função de x?

→ 2ª lei: $\vec{F}_g = \dfrac{d\vec{p}}{dt}$, $p_x = m V_x$ ↳ massa do pedaço da corda $m = m(t)$

$m = \rho x$, $\dot{m} = \rho \dot{x}$

$$mg = \frac{d}{dt}\left[m V_x\right]$$

$$mg = \dot{m} V_x + m \dot{V}_x$$

$$x\cancel{\rho} g = \cancel{\rho} \dot{x} V_x + \rho x \dot{V}_x$$

→ Posição do CM: $X = \dfrac{x}{2} \Rightarrow V_x = \dot{X} = \dfrac{1}{2}\dot{x}$

→ $xg = \dot{x} V_x + x \dot{V}_x \Rightarrow 2Xg = 2\dot{X} V_x + 2X \dot{V}_x \Rightarrow Xg = \underbrace{\dot{X}}_{V_x} V_x + X \dot{V}_x$

→ $\dfrac{dV_x}{dt} = \dfrac{dV_x}{dX}\dfrac{dX}{dt} = \dot{V}_x \dfrac{dV_x}{dX} \Rightarrow \boxed{Xg = V_x^2 + X V_x \dfrac{dV_x}{dX}}$

→ Chute: $V_x(X) = AX^n$, $\frac{dV_x}{dt} = AnX^{n-1}$

$$Xg = A^2X^{2n} + XAX^n Anx^{n-1}$$

$$Xg = A^2(1+n)X^{2n} \Rightarrow n = \frac{1}{2}, \quad g = A^2(1+n)$$

$$\therefore \boxed{V_x = \sqrt{\frac{2g}{3}}\, X^{\frac{1}{2}}}$$

→ corda → extenso

07/08/23

EXAMPLE 9.4

A projectile of mass M explodes while in flight into three fragments (Figure 9-8). One mass ($m_1 = M/2$) travels in the original direction of the projectile, mass m_2 ($= M/6$) travels in the opposite direction, and mass m_3 ($= M/3$) comes to rest. The energy E released in the explosion is equal to five times the projectile's kinetic energy at explosion. What are the velocities?

$E_f = 5T_i$

$k_1 = ?\ k_2 = ?$

energia interna associada à explosão é transformada em energia cinética

Solution. Let the velocity of the projectile of mass M be $\mathbf{v}$. The three fragments have the following masses and velocities:

$m_1 = \frac{M}{2}$, $\mathbf{v}_1 = k_1\mathbf{v}$ Forward direction, $k_1 > 0$

$m_2 = \frac{M}{6}$, $\mathbf{v}_2 = -k_2\mathbf{v}$ Opposite direction, $k_2 > 0$

$m_3 = \frac{M}{3}$, $\mathbf{v}_3 = 0$ At rest

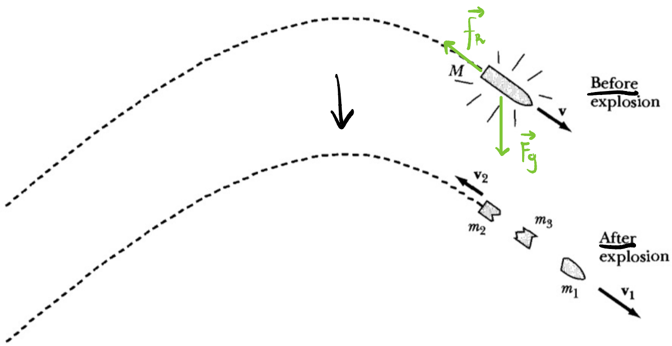


FIGURE 9-8 Example 9.4. A projectile of mass M explodes in flight into three fragments of masses m_1, m_2, and m_3.

* Aproximando a conservação do momento: $\Delta\vec{P} = \int_{t_1}^{t_2} \vec{F}_{ext}\, dt$

$\Delta t \ll 1$: $\Delta\vec{P} \approx \vec{F}_{ext}\Delta t$

$\rightarrow M\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 \Rightarrow M\vec{v} = \frac{M}{2}k_1\vec{v} - \frac{M}{6}k_2\vec{v}$

$\therefore 3k_1 - k_2 = 6$

$E = 5\left(\frac{1}{2}Mv^2\right)$

* Conservação de energia: $E + \frac{1}{2}Mv^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

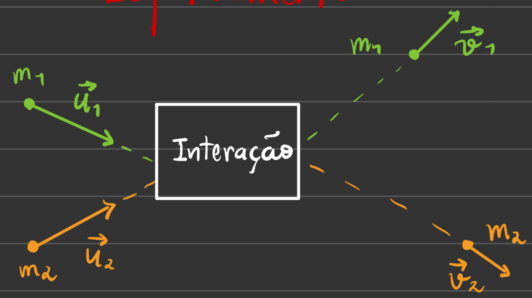
$\frac{6}{2}Mv^2 = \frac{1}{2}\cdot\frac{M}{2}k_1^2v^2 + \frac{1}{2}\frac{M}{6}k_2^2v^2$

$\therefore \frac{1}{4}k_1^2 + \frac{1}{12}k_2^2 = 3$

$\rightarrow 3k_1^2 + (3k_1 - 6)^2 = 36 \Rightarrow 3k_1^2 + 9k_1^2 - 36k_1 + \cancel{36} = \cancel{36} \Rightarrow k_1^2 - 3k_1 = 0 \Rightarrow \boxed{k_1 = 3}$

$\rightarrow 3k_1 - k_2 = 6 \Rightarrow k_2 = 3\cdot3 - 6 \Rightarrow \boxed{k_2 = k_1 = 3}$

Espalhamento



→ $\vec{u}_i$: velocidades iniciais
→ $\vec{v}_i$: velocidades finais
} medidas no referencial do laboratório

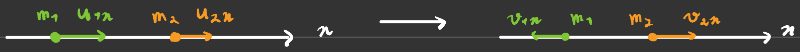
* Leis de conservação: $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$

$$Q + \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$Q + T_i = T_f \, , \quad [Q] = J$$

→ $Q = \Delta T = T_f - T_i$
- $Q = 0$: espalhamento elástico
- $Q > 0$: espalhamento exoérgico
- $Q < 0$: espalhamento endoérgico (inelástico) → ex.: colisões macroscópicas, interação relativística

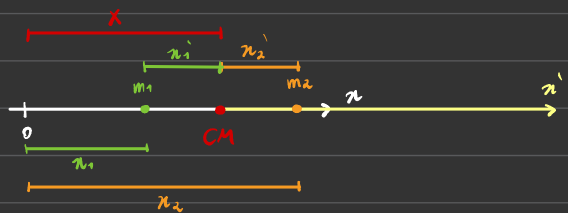
Colisão frontal – Duas partículas



→ $m_1 u_{1x} + m_2 u_{2x} = m_1 v_{1x} + m_2 v_{2x}$

→ $\frac{1}{2} m_1 u_{1x}^2 + \frac{1}{2} m_2 u_{2x}^2 + Q = m_1 v_{1x}^2 + m_2 v_{2x}^2$

→ Coeficiente de restituição: $\boxed{\varepsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|} = \frac{|v_2' - v_1'|}{|u_2' - u_1'|}}$, $v_i \neq |v_i|$
↳ componente

→ 

Referencial do CM

$$\begin{cases} x_i = X + x_i' \\ \dot{x}_i = \dot{X} + \dot{x}_i' \end{cases}$$

→ $\dot{x}_2 - \dot{x}_1 = \dot{X} + \dot{x}_2' - \dot{X} - \dot{x}_1' \Rightarrow \dot{x}_2 - \dot{x}_1 = \dot{x}_2' - \dot{x}_1' \Rightarrow \varepsilon$ independe do referencial

→ $$\begin{aligned} Q = T_F - T_I &= \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 - \tfrac{1}{2} m_1 u_1^2 - \tfrac{1}{2} m_2 u_2^2 \\ &= \tfrac{1}{2} m_1 (V + v_1')^2 + \tfrac{1}{2} m_2 (V + v_2')^2 - \tfrac{1}{2} m_1 (V + u_1')^2 - \tfrac{1}{2} m_2 (V + u_2')^2 \\ &= \tfrac{1}{2} m_1 \left[\cancel{V^2} + 2V v_1' + v_1'^2\right] + \tfrac{1}{2} m_2 \left[\cancel{V^2} + 2V v_2' + v_2'^2\right] - \tfrac{1}{2} m_1 \left[\cancel{V^2} + 2V u_1' + u_1'^2\right] - \tfrac{1}{2} m_2 \left[\cancel{V^2} + 2V u_2' + u_2'^2\right] \\ &= m_1 V v_1' + \tfrac{1}{2} m_1 v_1'^2 + m_2 V v_2' + \tfrac{1}{2} m_2 v_2'^2 - m_1 V u_1' - \tfrac{1}{2} m_1 u_1'^2 - m_2 V u_2' - \tfrac{1}{2} m_2 u_2'^2 \\ &= \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2 + V \underbrace{(m_1 v_1' + m_2 v_2')}_{0} - \tfrac{1}{2} m_1 u_1' - \tfrac{1}{2} m_2 u_2' - V \underbrace{(m_1 u_1' + m_2 u_2')}_{0} \end{aligned}$$

→ $V = \frac{m_1 \dot{x}_1 + m_2 \dot{x}_2}{m_1 + m_2} \Rightarrow \cancel{(m_1 + m_2) V} = m_1(\cancel{V} + \dot{x}_1') + m_2(\cancel{V} + \dot{x}_2') \Rightarrow m_1 \dot{x}_1' + m_2 \dot{x}_2' = 0$
↳ $\frac{\dot{x}_1'}{\dot{x}_2'} = -\frac{m_2}{m_1}$

$\therefore \boxed{Q = T_F - T_I = T_F' - T_I'}$

$$\rightarrow \varepsilon^2 = \frac{v_2'^2 + v_1'^2 - 2v_2'v_1'}{u_2'^2 + u_1'^2 - 2u_2'u_1'} = \frac{v_2'(v_2'-v_1') + v_1'(v_1'-v_2')}{u_2'(u_2'-u_1') + u_1'(u_1'-u_2')}$$

$$= \frac{v_2'^2\left(1-\frac{v_1'}{v_2'}\right) + v_1'^2\left(1-\frac{v_2'}{v_1'}\right)}{u_2'^2\left(1-\frac{u_1'}{u_2'}\right) + u_1'^2\left(1-\frac{u_2'}{u_1'}\right)}$$

$$= \frac{v_2'^2\left(1-\frac{m_2}{m_1}\right) + v_1'^2\left(1-\frac{m_1}{m_2}\right)}{u_2'^2\left(1+\frac{m_2}{m_1}\right) + u_2^2\left(1+\frac{m_2}{m_1}\right)} \begin{matrix} \times\, m_2 m_1 \\ \times\, m_2 m_1 \end{matrix}$$

$$= \frac{m_2 v_2'^2 + m_1 v_1'^2}{m_2 u_2'^2 + m_1 u_1'^2} = \frac{2T_F'}{2T_I'}$$

$$\therefore \boxed{\varepsilon^2 = \frac{T_F'}{T_I'}} \rightarrow \varepsilon^2 = 1 + \frac{Q}{T_I'} \rightarrow \boxed{Q = T_I'(\varepsilon^2 - 1) = \Delta T}$$

$$* \begin{cases} \varepsilon = 1 \Rightarrow Q = 0 \Rightarrow \text{elástico} \\ \varepsilon = 0 \Rightarrow Q = -T_I' \Rightarrow \text{perfeitamente inelástico} \\ 0 < \varepsilon < 1 \Rightarrow \text{inelástico} \\ \varepsilon > 1 \Rightarrow Q > 0 \Rightarrow \text{“explosão”} \end{cases}$$

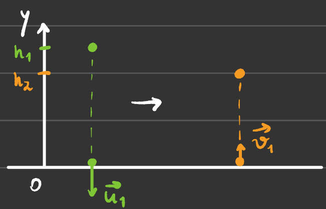
$$\rightarrow \boxed{v_{1n} = \frac{(m_1 - \varepsilon m_2)}{(m_1 + m_2)} u_{1n} + \frac{m_2(1+\varepsilon)}{(m_1+m_2)} u_{2n}} \qquad \boxed{v_{2n} = \frac{(m_2 - \varepsilon m_1)}{(m_1 + m_2)} u_{2n} + \frac{m_1(1+\varepsilon)}{(m_1+m_2)} u_{1n}}$$

$*\ \varepsilon = 0 \Rightarrow v_{1n} = v_{2n}$

$*\ \varepsilon = 1$ e $m_1 = m_2 \Rightarrow v_{1n} = u_{2n}$, $v_{2n} = u_{1n}$ } colisão elástica entre massas iguais: troca de velocidades

09/08/23

9-41. A rubber ball is dropped from rest onto a linoleum floor a distance h_1 away. The rubber ball bounces up to a height h_2. What is the coefficient of restitution? What fraction of the original kinetic energy is lost in terms of ε?



$$u_{1y}(h_1) = 0$$
$$v_{1y}(h_2) = 0$$
$$\vec{g} = cte$$

$$\varepsilon = \frac{|v_{1y} - \cancelto{0}{v_{2y}}|}{|u_{1y} - \cancelto{0}{u_{2y}}|} \rightarrow \text{piso}$$

$$\rightarrow u_{1y} = \cancelto{0}{u_{1y}(t=0)} - g t_1$$

$$\rightarrow \overset{0}{y} - \overset{h_1}{y_0} = u_{1y}(t=0)\, t_1 - \frac{1}{2} g t_1^2 \Rightarrow h_1 = \frac{1}{2} g t_1^2 \Rightarrow t_1 = \sqrt{\frac{2h_1}{g}}$$

$$\rightarrow u_{1y} = -g t_1 = -g\sqrt{\frac{2h_1}{g}} \Rightarrow u_{1y} = -\sqrt{2 g h_1}$$

$$\rightarrow \cancelto{0}{v_{1y}(h_2)} = v_{1y} - g t_2 \Rightarrow t_2 = \frac{v_{1y}}{g}$$

$$\rightarrow \overset{h_2}{y} - \overset{0}{y_0} = v_{1y} t_2 - \frac{1}{2} g t_2^2 \Rightarrow h_2 = v_{1y} t_2 - \frac{1}{2} g t_2^2 = v_{1y}^2 \cdot \frac{1}{g} - \frac{1}{2} g \cdot \frac{1}{g^2} v_{1y}^2 = \frac{1}{2g} v_{1y}^2$$

$$\rightarrow v_{1y}^2 = 2 g h_2 \Rightarrow v_{1y} = \sqrt{2 g h_2}$$

$$\rightarrow \varepsilon = \frac{\sqrt{2 g h_2}}{\sqrt{2 g h_1}} \Rightarrow \boxed{\varepsilon = \sqrt{\frac{h_2}{h_1}}}$$

$$\rightarrow \left.\begin{aligned} T_{perd} &= T_I - T_F = -\Delta T \\ T_{perd} &= -Q \end{aligned}\right\} \quad \frac{T_{perd}}{T_I} = -\frac{Q}{T_I} = -\overset{1}{\frac{T_I'}{T_I}}\,(\varepsilon^2 - 1)$$

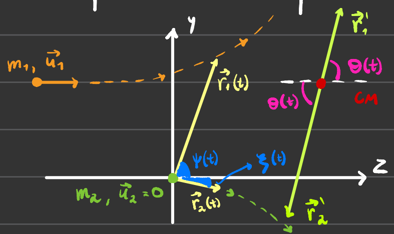
$$\rightarrow T_I = \frac{1}{2} m_1 u_1^2 \rightarrow \text{laboratório} \quad \Big| \quad T_I' = \frac{1}{2} m_1 u_1'^2 \rightarrow \text{centro de massa}$$

Piso: $M \gg m_1 \Rightarrow u_1' \approx u_1$

$$\Rightarrow \frac{T_{perd}}{T_I} = -(\varepsilon^2 - 1) \Rightarrow \boxed{\frac{T_{perd}}{T_I} = (1 - \varepsilon^2)}$$

Espalhamento elástico — Duas partículas

→ Uma partícula em repouso (inicialmente)



$m_2 \to$ alvo | $\vec{r}_1(t), \vec{r}_2(t) \to$ lab

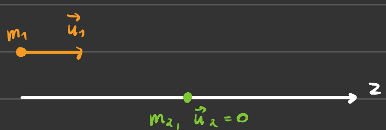
$m_1 \to$ incidente

→ Quando partículas deixam de interagir → trajetórias retilíneas

Lab

$$\begin{cases} \psi = \lim_{t\to\infty} \psi(t) \\ \xi = \lim_{t\to\infty} \xi(t) \end{cases}$$

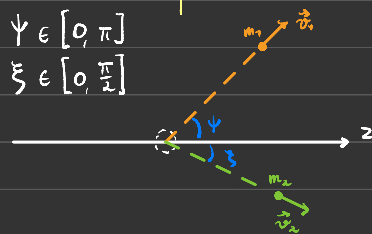
Antes



Depois

$\psi \in [0, \pi]$

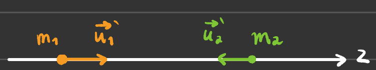
$\xi \in [0, \frac{\pi}{2}]$



→ No referencial do CM: $\theta = \lim_{t\to\infty} \theta(t)$

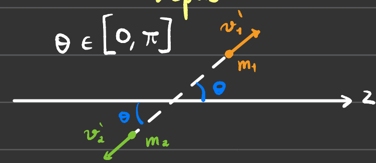
$\hookrightarrow m_1 \dot{\vec{r}}_1' + m_2 \dot{\vec{r}}_2' = 0 \Rightarrow \dot{\vec{r}}_1' = -\dfrac{m_2}{m_1} \dot{\vec{r}}_2'$

Antes



Depois

$\theta \in [0, \pi]$



$$\rightarrow \vec{R} + \vec{r}_i' = \vec{r}_i, \quad i = 1, 2 \Rightarrow \begin{cases} \dot{\vec{r}}_i = \dot{\vec{R}} + \dot{\vec{r}}_i' \\ \vec{u}_i = \vec{V} + \vec{u}_i' \\ \vec{v}_i = \vec{V} + \vec{v}_i' \end{cases}$$

$$\rightarrow \vec{V} = \frac{1}{M}(m_1 \vec{u}_1 + m_2 \cancel{\vec{u}_2}) \Rightarrow \vec{V} = \frac{m_1}{m_1 + m_2} \vec{u}_1$$

$$\rightarrow \text{Ângulo } \psi: \quad \vec{v}_1' = \vec{v}_1 - \vec{V} \Rightarrow \begin{cases} z: \ v_1' \cos\theta = v_1 \cos\psi - V \\ y: \ v_1' \operatorname{sen}\theta = v_1 \operatorname{sen}\psi \end{cases}$$

$$\rightarrow \frac{\operatorname{sen}\psi}{\cos\psi} = \frac{v_1' \operatorname{sen}\theta}{v_1' \cos\theta + V} \Rightarrow \tan\psi = \frac{\operatorname{sen}\theta}{\cos\theta + \frac{V}{v_1'}}$$

$$\rightarrow \text{CM}: \quad m_1 u_1' + m_2 u_2' = m_1 \vec{v}_1' + m_2 \vec{v}_2' = 0$$

$$\vec{u}_2' = -\frac{m_1}{m_2} \vec{u}_1' \Rightarrow u_2' = \frac{m_1}{m_2} u_1'$$

$$\vec{v}_2' = -\frac{m_1}{m_2} \vec{v}_1' \Rightarrow v_2' = \frac{m_1}{m_2} v_1'$$

$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

↳ Ex.: Mostre que $u_i' = v_i'$ (módulos)

$$\left.\begin{aligned} &\rightarrow \frac{V}{v_1'} = \frac{V}{u_1'} = \frac{m_1}{m_1 + m_2} \cdot \frac{u_1}{u_1'} \\ &\rightarrow \vec{u}_1' = \vec{u}_1 - \vec{V} = \frac{m_2}{m_1 + m_2} \vec{u}_1 \end{aligned}\right\} \quad \boxed{\frac{V}{v_1'} = \frac{m_1}{m_2}}$$

LAB CM

$$\therefore \boxed{\tan\psi = \frac{\operatorname{sen}\theta}{\cos\theta + \frac{m_1}{m_2}}}$$

$\rightarrow$ Ângulo ξ: $\vec{v}_2' = \vec{v}_2 - \vec{V} \Rightarrow \begin{cases} x: & -v_2' \cos\theta = v_2 \cos\xi - V \\ y: & -v_2' \operatorname{sen}\theta = -v_2 \operatorname{sen}\xi \end{cases}$

$\rightarrow \tan\xi = \dfrac{\operatorname{sen}\theta}{\frac{V}{v_2'} - \cos\theta}$

$\rightarrow v_2' = \dfrac{m_1 v_1'}{m_2} \Rightarrow \dfrac{V}{v_2'} = \dfrac{m_2}{m_1}\dfrac{V}{v_1'} = \dfrac{m_2}{m_1} \cdot \dfrac{m_1}{m_2} = 1$

$\rightarrow \tan\xi = \dfrac{\operatorname{sen}\theta}{1 - \cos\theta} = \cot\left(\dfrac{\theta}{2}\right)$

$\rightarrow \cot x = \tan\left(\dfrac{\pi}{2} - x\right)$

$$\tan\xi = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\xi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\therefore \boxed{2\xi = \pi - \theta} \rightarrow \text{CM}$$

LAB

11/08/23

$\rightarrow$ $\vec{v}_1 = \vec{v}_1' + \vec{V}$

$\vec{v}_1'$ $\rightarrow$ velocidade da partícula no referencial do CM

$\vec{V}$ $\rightarrow$ velocidade do CM no referencial do lab

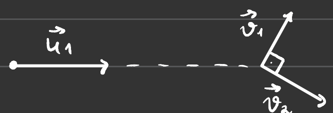
$\vec{v}_1$ $\rightarrow$ velocidade da partícula no referencial do lab

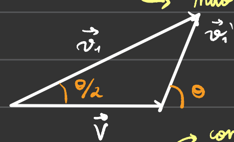
$\rightarrow$ $\dfrac{V}{v_1'} = \dfrac{m_1}{m_2}$

* $m_2 \gg m_1$: $\tan\psi \approx \tan\theta \Rightarrow \begin{cases} \psi \approx \theta \\ 2\xi \approx \pi - \psi \end{cases}$ | $\dfrac{V}{v_1'} \approx 0$

* $m_2 = m_1$: $\tan\psi = \dfrac{\operatorname{sen}\theta}{1+\cos\theta} = \tan\left(\dfrac{\theta}{2}\right) \Rightarrow \begin{cases} \psi = \frac{\theta}{2} \\ \xi + \psi = \frac{\pi}{2} \end{cases}$ | $\dfrac{V}{v_1'} = 1$

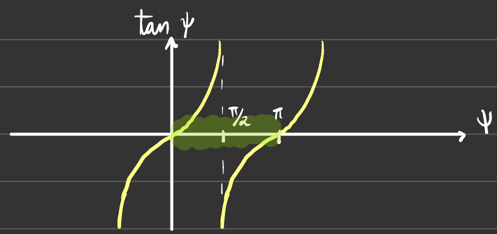
$\hookrightarrow$ não há espalhamento para trás





* $m_2 \ll m_1$: $\tan\psi \approx 0 \Rightarrow \psi \approx 0 < \begin{matrix} \theta = 0 \Rightarrow \cancel{\xi = \frac{\pi}{2}} \\ \theta = \pi \Rightarrow \xi = 0 \end{matrix}$ | $\dfrac{V}{v_1'} \gg 1$

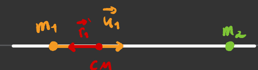
$\rightarrow$ conserva momento apenas trivialmente

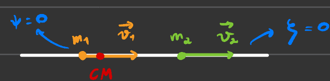


Solução correta: $\psi \in [0, \pi]$

$\hookrightarrow$ $V = \dfrac{m_1 u_1}{m_1 + m_2} \approx \dfrac{u_1}{1 + \frac{m_2}{m_1}} \approx u_1 - \dfrac{m_2}{m_1} u_1 < u_1$

$\hookrightarrow$ $m_1 \vec{r}_1' + m_2 \vec{r}_2' = 0 \Rightarrow \vec{r}_1' = -\dfrac{m_2}{m_1} \vec{r}_2'$





$\hookrightarrow$ $V = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \Rightarrow v_1 \approx u_1 - \dfrac{m_2}{m_1} v_2 \Rightarrow \dfrac{V}{v_1} \approx 1 + \dfrac{m_2}{m_1} \rightarrow V > v_1$

* Energia cinética em termos dos ângulos de deflexão:

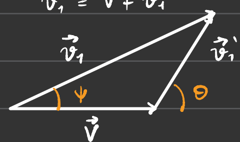
→ T_{1F}, T_{2F}: energias cinéticas "finais" de m_1 e m_2 medidas no lab

→ $T_I = T_{1I} + T_{2I}$: energia cinética inicial total no lab

→ $\dfrac{T_{1F}}{T_I} = ?$ | $\dfrac{T_{2F}}{T_I} = ?$ | $\dfrac{T_{1F}}{T_I} + \dfrac{T_{2F}}{T_I} = 1$

→ $$\frac{T_{1F}}{T_I} = \frac{\frac{1}{2} m_1 v_1^2}{\frac{1}{2} m_1 u_1^2} = \frac{v_1^2}{u_1^2}$$

$\vec{v}_1 = \vec{V} + \vec{v}_1'$



$v_1'^2 = v_1^2 + V^2 - 2 v_1 V \cos\psi$

$v_1' \operatorname{sen}\theta = v_1 \operatorname{sen}\psi \Rightarrow v_1 = \dfrac{\operatorname{sen}\theta}{\operatorname{sen}\psi} v_1'$

→ $$\frac{v_1^2}{u_1^2} = \frac{v_1'^2}{u_1^2} - \frac{V^2}{u_1^2} + \frac{2 v_1 V}{u_1^2}\cos\psi$$

↳ $\vec{V} = \dfrac{m_1}{(m_1+m_2)} \vec{u}_1 \Rightarrow \dfrac{V}{u_1} = \dfrac{m_1}{m_1+m_2}$

m_2/m_1 → aula passada

↳ $\dfrac{v_1'}{u_1} = \boxed{\dfrac{v_1'}{V}} \dfrac{m_1}{(m_1+m_2)} \Rightarrow \dfrac{v_1'}{u_1} = \dfrac{m_2}{m_1+m_2}$

↳ $$\frac{v_1}{u_1} = \frac{\operatorname{sen}\theta}{\operatorname{sen}\psi}\frac{v_1'}{u_1} = \frac{m_2}{(m_1+m_2)}\frac{\operatorname{sen}\theta}{\operatorname{sen}\psi}$$

$$\rightarrow \frac{T_{1F}}{T_I} = \frac{v_1^2}{u_1^2} = \frac{m_2^2}{(m_1+m_2)^2} - \frac{m_1^2}{(m_1+m_2)^2} + 2\cos\psi \frac{m_1}{(m_1+m_2)} \frac{\operatorname{sen}\theta}{\operatorname{sen}\psi} \frac{m_2}{(m_1+m_2)}$$

$$\rightarrow \frac{T_{1F}}{T_I} = \frac{(m_2^2 - m_1^2)}{(m_1+m_2)^2} + \frac{2m_1m_2}{(m_1+m_2)^2} \frac{\operatorname{sen}\theta}{\tan\psi}$$

$$\rightarrow \tan\psi = \frac{\operatorname{sen}\theta}{\cos\theta + \frac{m_1}{m_2}}$$

$$\frac{T_{1F}}{T_I} = \frac{(m_1+m_2)^2 - 2m_1m_2 + 2m_1m_2\cos\theta}{(m_1+m_2)^2}$$

$$\therefore \boxed{\frac{T_{1F}}{T_I} = 1 + \frac{2m_1m_2}{(m_1+m_2)^2}(\cos\theta - 1)}$$

$$\rightarrow \frac{T_{2F}}{T_I} = 1 - \frac{T_{1F}}{T_I} \Rightarrow \boxed{\frac{T_{2F}}{T_I} = \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta)}$$

$$* \ \psi = \frac{\pi}{2}: \quad \tan\psi = \frac{\operatorname{sen}\theta}{\cos\theta + \frac{m_1}{m_2}} \rightarrow \infty \Rightarrow \cos\theta + \frac{m_1}{m_2} = 0$$

$$\cos\theta = -\frac{m_1}{m_2}$$

$$\rightarrow \frac{T_{1F}}{T_I} = 1 + \frac{2m_1m_2}{(m_1+m_2)^2}\left(-\frac{m_1}{m_2} - 1\right) = 1 - \frac{2m_1\cancel{m_2}}{(m_1+m_2)^{\cancel{2}}} \frac{\cancel{(m_1+m_2)}}{\cancel{m_2}} = 1 - \frac{2m_1}{m_1+m_2}$$

$$\rightarrow \text{Se } \frac{T_{1F}}{T_I} = \frac{1}{2}: \quad \frac{1}{2} = 1 - \frac{2m_1}{m_1+m_2} \Rightarrow \frac{m_2}{m_1} = 3 \ //$$

14/08/23

EXAMPLE 9.8

Particles of mass m_1 elastically scatter from particles of mass m_2 at rest. (a) At what LAB angle should a magnetic spectrometer be set to detect particles that lose one-third of their momentum? (b) Over what range m_1/m_2 is this possible? (c) Calculate the scattering angle for $m_1/m_2 = 1$.

a) $\psi = ?$ | m_1 perde $\frac{1}{3}$ do momento

$$\tan\psi = \frac{\operatorname{sen}\theta}{\cos\theta + \frac{m_1}{m_2}} \qquad \Bigg| \qquad \frac{T_{1F}}{T_I} = 1 - \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta)$$

$$\rightarrow \frac{T_{1F}}{T_I} = \frac{\frac{1}{2}m_1v_1^2}{\frac{1}{2}m_1u_1^2} = \frac{v_1^2}{u_1^2}$$

$$\rightarrow m_1v_1 = \frac{2m_1u_1}{3} \Rightarrow \frac{v_1}{u_1} = \frac{2}{3}$$

$$\frac{T_{1F}}{T_I} = \frac{4}{9}$$

$$\rightarrow \frac{4}{9} = 1 - \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta) \Rightarrow \frac{5}{9} = \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta) \Rightarrow \cos\theta = 1 - \underbrace{\frac{5}{18}\frac{(m_1+m_2)^2}{m_1m_2}}_{y} = 1-y$$

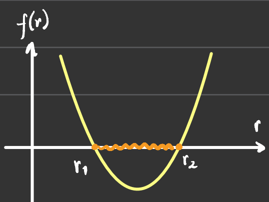
$$\rightarrow \operatorname{sen}^2\theta = 1-\cos^2\theta = 1-(1-y)^2 \Rightarrow \operatorname{sen}\theta = \sqrt{1-(1+y^2-2y)} = \sqrt{2y-y^2}$$

$$\therefore \boxed{\tan\psi = \frac{\sqrt{2y-y^2}}{1-y+\frac{m_1}{m_2}}}$$

c) $m_1 = m_2 \Rightarrow y = \frac{20}{18} \approx 1{,}1 \Rightarrow \tan\psi \approx \frac{1}{0{,}89} \approx 1{,}12 \Rightarrow \boxed{\psi \approx 48°}$

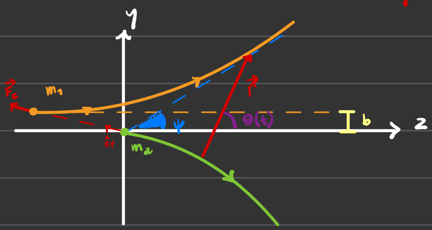
b) solução real: $2y - y^2 > 0 \Rightarrow y(2-y) > 0 \Rightarrow y < 2$ | ($y \to 0$ apenas se $\frac{m_2}{m_1} \to \infty$)

$$r = \frac{m_1}{m_2} \Rightarrow y = \frac{5}{18}\frac{m_2^2\left(1+\frac{m_1}{m_2}\right)^2}{m_1m_2} = \frac{5}{18}\frac{(1+r)^2}{r} < 2 \Rightarrow 1+r^2+2r < \frac{36}{5}r \rightarrow f(r) = 5r^2 - 26r + 5 < 0$$



$$r_1 = \frac{1}{5},\ r_2 = 5 \Rightarrow \boxed{r \in \left[\frac{1}{5}, 5\right]}$$

Espalhamento — Forças centrais



b: parâmetro de impacto

$\vec{r} = \vec{r}_1 - \vec{r}_2$

$\vec{F_c} = F(r)\hat{e}_r$

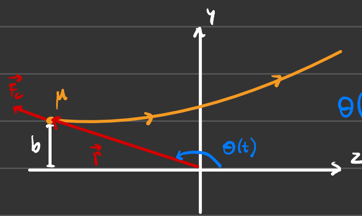
→ Condições iniciais: $\vec{r}_2(0) = 0$, $\dot{\vec{r}}_2(0) = 0$

$z_1(0) \to -\infty$, $y_1(0) = b$, $\dot{\vec{r}}_1(0) = u_1\hat{k}$

→ Problema equivalente: $\mu\ddot{\vec{r}} = \vec{F_c}$, $\mu = \dfrac{m_1 m_2}{m_1+m_2}$

↳ $t=0$: $r(0) \to \infty$ | $\vec{r}(0) = \vec{r}_1(0) - \vec{r}_2(0)$

$\theta(0) \to \pi$ | $\dot{\vec{r}}(0) = u_1\hat{k}$



$\theta(t\to\infty) = \Theta$

* Energia: $E = \frac{1}{2}\mu|\dot{\vec{r}}|^2 \Rightarrow E = \frac{1}{2}\mu u_1^2 = cte$

* Momento angular: $\vec{L} = \mu(\vec{r}\times\dot{\vec{r}}) = \mu\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & y & z\\ 0 & \dot{y} & \dot{z}\end{vmatrix} = \mu(y\dot{z}-\dot{y}z)\hat{i}$

$t=0 \to y=b$, $\dot{y}=0$, $\dot{z}=u_1$ $\Big\}$ $\vec{L} = \mu b u_1\hat{i}$

→ $\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$, $\vec{r} = r\hat{e}_r$ $\Big\}$ $\vec{L} = \mu(\vec{r}\times\dot{\vec{r}}) = \mu r^2\dot{\theta}\,\overbrace{(\hat{e}_r\times\hat{e}_\theta)}^{-\hat{i}} \Rightarrow \vec{L} = -\mu r^2\dot{\theta}\hat{i}$

→ $-\mu r^2\dot{\theta} = \mu b u_1 \Rightarrow \boxed{\dot{\theta} = -\dfrac{bu_1}{r^2} < 0}$ $\quad t\to\infty \Rightarrow r\to\infty \Rightarrow \dot{\theta}\to 0$

$$\rightarrow E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + U(r)$$

$$\rightarrow \dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{L^2}{\mu^2 r^2}}$$

Início: $\dot{r} < 0 \Rightarrow r$ diminui

Fim: $\dot{r} > 0 \Rightarrow r$ aumenta

depende de $\dot{r}$

$$\rightarrow \frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = -\frac{L^2}{\mu r^2}\frac{dr}{d\theta} \Rightarrow d\theta = \pm \frac{(-L)}{\mu r^2}\frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{L^2}{\mu^2 r^2}}}$$

$$\rightarrow \int_{\pi}^{\theta} d\theta' = -\frac{1}{\mu}\int_{-\infty}^{r_{min}} \frac{(-L)\,dr}{r^2\sqrt{\frac{2}{\mu}\left[E - U(r)\right] - \frac{L^2}{\mu^2 r^2}}} + \frac{1}{\mu}\int_{r_{min}}^{\infty} \frac{(-L)\,dr}{r^2\sqrt{\frac{2}{\mu}\left[E - U(r)\right] - \frac{L^2}{\mu^2 r^2}}}$$

$$\therefore\ \theta = \pi - \frac{2L'}{\mu}\int_{r_{min}}^{\infty} \frac{dr}{r^2\left[\frac{2}{\mu}\left(E' - U(r)\right) - \frac{L'^2}{\mu^2 r^2}\right]^{\frac{1}{2}}}$$

$$\rightarrow L' = \mu b u_1 \Rightarrow \boxed{\theta = \pi - 2bu_1\int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{u_1^2 - \frac{2}{\mu}U(r) - \frac{b^2 u_1^2}{r^2}}}}$$

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* Problema original:

↳ LAB: $E = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 + U(|\vec{r}_1 - \vec{r}_2|) = \frac{1}{2} m_1 u_1^2$

↳ CM: $E' = \frac{1}{2} m_1 |\dot{\vec{r}}_1'|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2'|^2 + U(|\vec{r}_1' - \vec{r}_2'|)$

↳ $\vec{r}_1 = \vec{r}_1' + \vec{R}$, $\vec{r}_2 = \vec{r}_2' + \vec{R}$ | $m_1 \vec{r}_1' + m_2 \vec{r}_2' = 0 \Rightarrow \vec{r}_1' = -\frac{m_2}{m_1} \vec{r}_2'$, $\vec{r}_2' = -\frac{m_1}{m_2} \vec{r}_1'$

$\Rightarrow E = E' + \frac{1}{2}(m_1 + m_2) V^2$, $\vec{V} = \dot{\vec{R}}$

↳ $\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{r}_1' - \vec{r}_2' \Rightarrow \vec{r}_1' = \frac{m_2}{m_1 + m_2} \vec{r}$, $\vec{r}_2' = -\frac{m_1}{m_1 + m_2} \vec{r}$

↳ $\dot{\vec{r}}_1(0) = u_1 \hat{k}$, $\dot{\vec{r}}_2(0) = 0$, $U = 0$

$\Rightarrow \boxed{E' = \frac{1}{2} \mu |\dot{\vec{r}}|^2 + U(r) = \frac{1}{2} \mu u_1^2}$

↳ LAB: $\vec{L} = m_1 (\vec{r}_1 \times \dot{\vec{r}}_1) + m_2 (\vec{r}_2 \times \dot{\vec{r}}_2)$

↳ CM: $\vec{L}' = m_1 (\vec{r}_1' \times \dot{\vec{r}}_1') + m_2 (\vec{r}_2' \times \dot{\vec{r}}_2') \Rightarrow \boxed{\vec{L}' = \mu (\vec{r} \times \dot{\vec{r}}) = \mu b u_1 \hat{i}}$

* Potencial de Newton - Coulomb

$$\rightarrow U(r) = \frac{k}{r} \begin{cases} k>0 \rightarrow \text{repulsiva} \\ k<0 \rightarrow \text{atrativa} \end{cases}$$

$$\rightarrow E = \frac{1}{2}\mu \dot{r}^2 + V_{eff}(r)$$

$$\rightarrow V_{eff}(r) = \frac{L^2}{2\mu r^2} + \frac{k}{r} = \frac{\mu^2 b^2 u_1^2}{2\mu r^2} + \frac{k}{r} = \frac{1}{2}\mu \frac{b^2 u_1^2}{r^2} + \frac{k}{r} \Rightarrow V_{eff}(r) = \frac{Eb^2}{r^2} + \frac{k}{r}$$

$\rightarrow$ Ponto de retorno: $\dot{r} = 0$

$$E = V_{eff}(r_{min}) = \frac{Eb^2}{r_{min}^2} + \frac{k}{r_{min}} \Rightarrow r_{min}^2 E - k r_{min} - Eb^2 = 0$$

$$\boxed{r_{min} = \frac{k + \sqrt{k^2 + 4E^2b^2}}{2E}}$$

$$\rightarrow u = \frac{b}{r}, \quad du = -\frac{b}{r^2}dr \Rightarrow \theta = \pi + 2\int_{\frac{b}{r_{min}}}^{0} \frac{du}{\sqrt{1 - \frac{ku}{Eb} - u^2}} = \pi + 2\int_{\frac{b}{r_{min}}}^{0} \frac{du}{\left[1 - \left(u + \frac{k}{2bE}\right)^2 + \frac{k^2}{4b^2E^2}\right]^{\frac{1}{2}}}$$

$$\rightarrow v = u + \frac{k}{2Eb}, \quad dv = du \Rightarrow \theta = \pi + 2\int_{\frac{b}{r_{min}} + \frac{k}{2Eb}}^{\frac{k}{2Eb}} \frac{dv}{\sqrt{1 + \frac{k^2}{4E^2b^2} - v^2}} \quad \Bigg| \quad \frac{d}{dx} = \cos^{-1}\left(\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}}$$

$$\rightarrow \theta = \pi - 2\cos^{-1}\left[\frac{v}{1 + \sqrt{\frac{k^2}{4E^2b^2}}}\right]_{\frac{b}{r_{min}} + \frac{k}{2Eb}}^{\frac{k}{2Eb}} = \pi - 2\cos^{-1}\left[\frac{\frac{k}{2bE}}{\sqrt{1 + \frac{k^2}{4E^2b^2}}}\right] + 2\cos^{-1}\left[\frac{\frac{b}{r_{min}} + \frac{k}{2bE}}{1 + \sqrt{\frac{k^2}{4b^2E^2}}}\right]$$

$$\rightarrow K \equiv \frac{|k|}{2E} > 0$$

$$\rightarrow r_{min} = \frac{k + 2Eb\sqrt{1+\frac{k^2}{4E^2b^2}}}{2E} \Rightarrow r_{min} = sgn(k)\,K + b\sqrt{1+\frac{K^2}{b^2}}$$

$$sgn(k) = \begin{cases} 1, & k>0 \\ -1, & k<0 \end{cases}$$

$$\Rightarrow \theta = \pi - 2\left[\cos^{-1}\left(\frac{sgn(k)\,K}{\sqrt{b^2+K^2}}\right) - \cos^{-1}\left(\frac{\frac{b^2}{r_{min}} + sgn(k)\,K}{\sqrt{b^2+K^2}}\right)\right]$$

$$\rightarrow \cos^{-1}(x) + \cos^{-1}(y) = \cos^{-1}\left[xy + \sqrt{(1-x^2)(1-y^2)}\right]$$

$$\rightarrow \Phi = \cos^{-1}\left[\frac{K\,sgn(k)}{(b^2+K^2)}\left(\frac{b^2}{r_{min}} + K\,sgn(k)\right) + \sqrt{\left(1-\frac{K^2}{(b^2+K^2)}\right)\left(1-\frac{\left(\frac{b^2}{r_{min}}+K\,sgn(k)\right)^2}{(b^2+K^2)}\right)}\right]$$

$\vdots$

$$\Phi = \cos^{-1}\left[\frac{1}{(b^2+K^2)}\left(\frac{b^2K\,sgn(k)}{r_{min}} + K^2 + \frac{b^2}{r_{min}}\sqrt{r_{min}^2 - b^2 - 2K\,sgn(k)\,r_{min}}\right)\right]$$

$$\rightarrow r_{min}^2 - b^2 - 2K\,sgn(k)\,r_{min} = \cancel{K^2} + \cancel{b^2}\left(\cancel{1} + \frac{K^2}{\cancel{b^2}}\right) + \cancel{2Kb\,sgn(k)\sqrt{1+\frac{K^2}{b^2}}} - \cancel{b^2} - \cancel{2K^2} - \cancel{2K\,sgn(k)\,b\sqrt{1+\frac{K^2}{b^2}}} = 0 \quad //$$

$$\rightarrow \cos\Phi = \frac{1}{b^2+K^2}\left(K^2 + \frac{b^2\overbrace{K\,sgn(k)}^{=r_{min}-\sqrt{b^2+K^2}}}{r_{min}}\right) = 1 - \frac{b^2}{r_{min}\sqrt{b^2+K^2}}$$

$$\rightarrow \theta = \pi - 2\Phi \Rightarrow \Phi = \frac{\pi-\theta}{2} \Rightarrow \cos\Phi = \cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = sen\left(\frac{\theta}{2}\right)$$

$$\rightarrow \overbrace{\text{sen}^2}^{\cancel{1}-\cos^2\left(\frac{\theta}{2}\right)}\left(\frac{\theta}{2}\right) = \cancel{1} + \frac{b^4}{r_{min}^2\left(b^2+k^2\right)} - \frac{2b^2}{r_{min}\sqrt{b^2+k^2}}$$

$$\rightarrow \cos^2\left(\frac{\theta}{2}\right) = b^2\left[\frac{2}{r_{min}\sqrt{b^2+k^2}} - \frac{b^2}{r_{min}^2\left(b^2+k^2\right)}\right] = b^2\,\frac{\left(\overbrace{2r_{min}\sqrt{b^2+k^2}-b^2}^{r_{min}^2}\right)}{r_{min}^2\left(b^2+k^2\right)}$$

→ substituir r_{min}: $\cos^2\left(\frac{\theta}{2}\right) = \frac{b^2}{b^2+k^2} = \frac{1}{1+\frac{k^2}{b^2}}$

$$\cos\left(\frac{\theta}{2}\right) = \pm\frac{1}{\sqrt{1+\frac{k^2}{b^2}}}$$

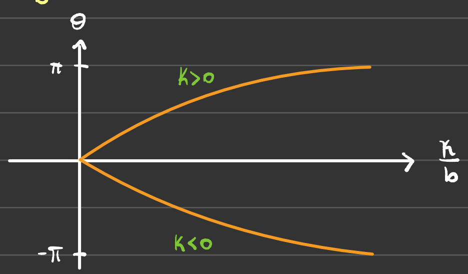
$\theta > 0 \rightarrow K > 0$ (repulsivo)

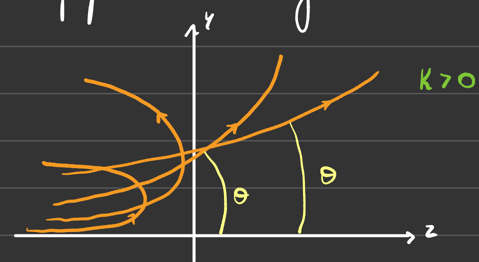
$$\therefore \boxed{\theta = \pm 2\arccos\left(\frac{1}{\sqrt{1+\frac{k^2}{b^2}}}\right)}, \quad k \equiv \frac{|K|}{2E}$$

$\theta < 0 \rightarrow K < 0$ (atrativo)

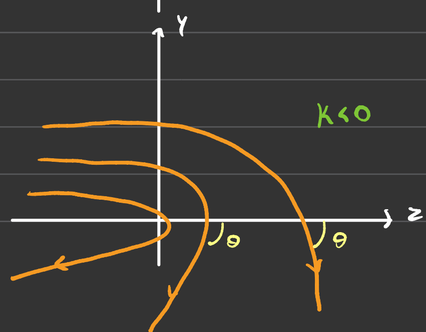
↳ $\frac{k}{b} \rightarrow 0 \Rightarrow \theta = 0$ (parâmetro de impacto grande)

↳ $\frac{k}{b} \rightarrow \infty \Rightarrow \theta = \pm\pi$ (parâmetro de impacto pequeno ou energia baixa)





$$\boxed{\varepsilon = \sqrt{1+\frac{2EL^2}{\mu K^2}} = \sqrt{1+\frac{b^2}{k^2}}}$$



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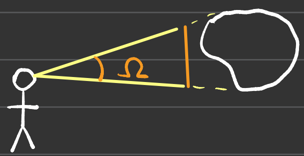
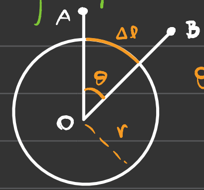
Ângulo sólido

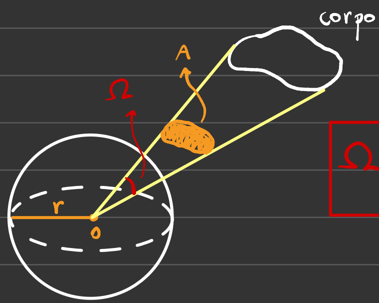
Ângulo sólido



Ângulo planar



$$\theta = \frac{\Delta l}{r}$$



$$\boxed{\Omega = \frac{A}{r^2}}$$

A → área transversal da projeção na esfera

$[\Omega] = sr$

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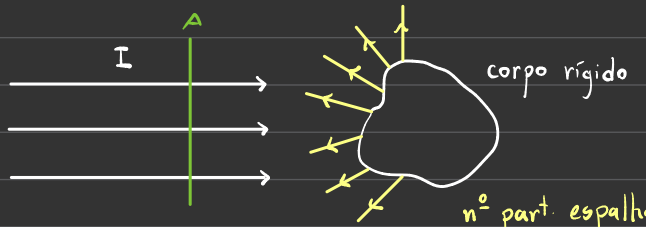
Seção de choque / espalhamento



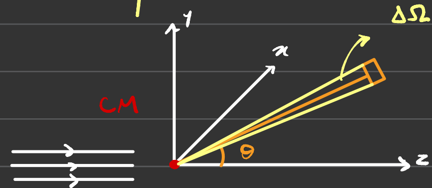
$[I] = \frac{1}{m^2 s}$

$N \propto I$

$\boxed{N = \sigma I}$ → intensidade do feixe

↳ Seção de choque $[\sigma] = m^2$

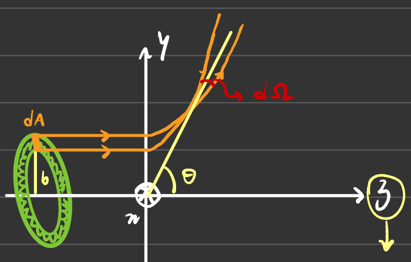
→ A seção de choque é a "área efetiva" do corpo



→ $\Delta N \propto I \Delta\Omega$ → número de partículas espalhadas ao longo de um ângulo sólido

$\Delta N = \sigma(\theta, \phi) I \Delta\Omega$

→ $\Delta\Omega$ → $d\Omega$: $\boxed{dN = \sigma(\theta, \phi) I d\Omega}$



↗ anel é círculo no plano xy

→ $da = b\, db\, d\phi$

$dN = I da = I b db d\phi$

$\cancel{I} b\, db\, d\phi = \sigma(\theta, \phi) \cancel{I} d\Omega$

→ Força central: simetria axial

↳ $\sigma(\theta, \phi) = \sigma(\theta)$

→ Coord. esféricas: $dA = r^2 \operatorname{sen}\theta\, d\theta\, d\phi$

$d\Omega = \frac{dA}{r^2} = \operatorname{sen}\theta\, d\theta\, d\phi$

→ $b\, db\, \cancel{d\phi} = \sigma(\theta) \operatorname{sen}\theta\, d\theta\, \cancel{d\phi}$

$\boxed{\sigma(\theta) = \frac{b}{\operatorname{sen}\theta} \left|\frac{db}{d\theta}\right|}$ → no referencial do CM

no referencial do LAB: p. 326

Espalhamento de Rutherford

$\rightarrow U(r) = \frac{K}{r}$ ($K > 0$) $\rightarrow \theta = \pm 2\cos^{-1}\left[\frac{1}{\sqrt{1+\frac{\kappa^2}{b^2}}}\right], \quad \kappa = \frac{|K|}{2E}$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{1+\frac{\kappa^2}{b^2}}$$

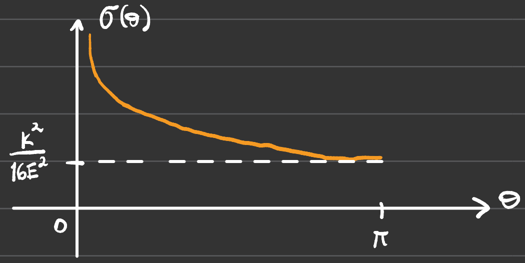
$$\frac{\kappa^2}{b^2}\cos^2\left(\frac{\theta}{2}\right) = 1-\cos^2\left(\frac{\theta}{2}\right) = \operatorname{sen}^2\left(\frac{\theta}{2}\right)$$

$$\boxed{b(\theta) = \kappa \cot\left(\frac{\theta}{2}\right)}$$

$\rightarrow \frac{d}{dx}\cot x = -\frac{1}{\operatorname{sen}^2 x} \Rightarrow \frac{db}{d\theta} = -\frac{\kappa}{2\operatorname{sen}^2\left(\frac{\theta}{2}\right)} \Rightarrow \sigma(\theta) = \frac{\kappa^2\cot\left(\frac{\theta}{2}\right)}{2\operatorname{sen}\theta\,\operatorname{sen}^2\left(\frac{\theta}{2}\right)}$

$\rightarrow \operatorname{sen}\theta = 2\operatorname{sen}\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \Rightarrow \sigma(\theta) = \frac{\kappa^2\cot\left(\frac{\theta}{2}\right)}{4\operatorname{sen}^3\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = \frac{\kappa^2}{4\operatorname{sen}^4\left(\frac{\theta}{2}\right)}$

$\rightarrow \kappa = \frac{|K|}{2E} \Rightarrow \boxed{\sigma(\theta) = \frac{K^2}{16E^2\operatorname{sen}^4\left(\frac{\theta}{2}\right)}}$ $\rightarrow$ Seção de choque de Rutherford

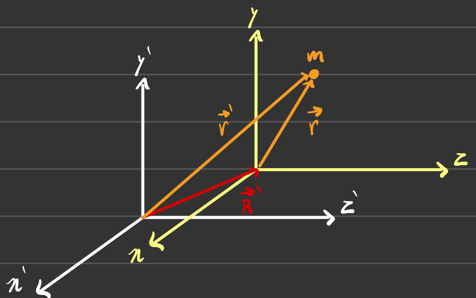


21/08/23

Referenciais não-inerciais

→ Acelerados em relação a um referencial inercial

→ Força centrífuga, força de Coriolis



$S': (x', y', z') \rightarrow$ inercial

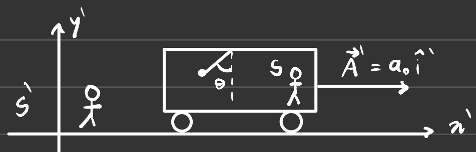
$S: (x, y, z) \rightarrow$ não-inercial

$\ddot{\vec{R}}' \neq 0$

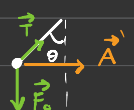
$$\begin{cases} \vec{r}\,' = \vec{R}\,' + \vec{r} \\ \vec{v}\,' = \vec{V}\,' + \vec{v} \\ \vec{a}\,' = \vec{A}\,' + \vec{a} \end{cases}$$

→ 2ª lei em S': $\vec{F}_{res} = m\vec{a}\,' \Rightarrow \vec{F}_{res} = m(\vec{A}\,' + \vec{a}) \Rightarrow \boxed{m\vec{a} = \vec{F}_{res} - m\vec{A}\,'}$

* Exemplo:



↳ Ref. S':

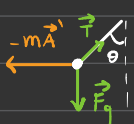


$\vec{F}_g + \vec{T} = m\vec{A}\,'$

$$\left.\begin{aligned} y' &: mg = T\cos\theta \\ x' &: T\sin\theta = ma_0 \end{aligned}\right\} \tan\theta = \frac{a_0}{g}$$

} em S', a causa da inclinação do pêndulo é a aceleração do trem

↳ Ref. S:

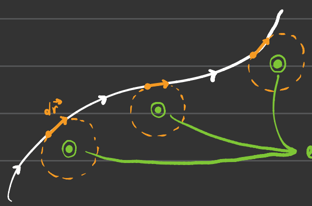


$$\left.\begin{aligned} m\vec{a} &= \vec{F}_{res} - m\vec{A}\,' \\ \vec{a} &= 0 \\ \vec{F}_{res} &= \vec{F}_g + \vec{T} \end{aligned}\right\} m\vec{A}\,' = \vec{F}_g + \vec{T} \Big\}$$

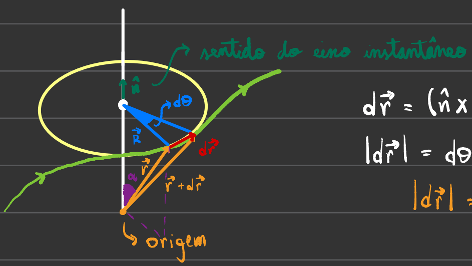
força "fictícia"

em S, surge uma força que contrapõe a força resultante para deixar o pêndulo em repouso

Velocidade angular



* $d\vec{r}$ pode ser interpretado como um arco de circunferência



$$d\vec{r} = (\hat{n} \times \vec{r})\, d\theta$$

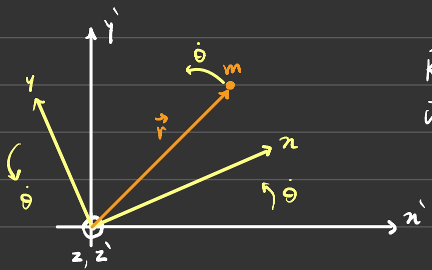
$$|d\vec{r}| = d\theta\, |\hat{n} \times \vec{r}| = d\theta\, \overbrace{r \operatorname{sen}\alpha}^{R}$$

$$|d\vec{r}| = R\, d\theta$$

$$\frac{d\vec{r}}{dt} = \frac{d\theta}{dt}(n \times \vec{r}) \Rightarrow \vec{v} = \dot{\theta}(\hat{n} \times \vec{r}) \Rightarrow \boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

$$\vec{\omega} = \dot{\theta}\,\hat{n}$$

Transformação de variáveis cinemáticas



$\hat{k} = \hat{k}'$

$\vec{\omega} = \dot\theta \hat{k}$

S: $\vec{r} = x\hat{i} + y\hat{j}$ → $\hat{i}, \hat{j}$ dependem do tempo

$\vec{v} \equiv \dot{x}\hat{i} + \dot{y}\hat{j} = 0$ — x, y não dep. do tempo

S': $\vec{r}' = x'\hat{i}' + y'\hat{j}'$ → $\hat{i}', \hat{j}'$ não dep. do tempo

$\vec{v}' = \dot{x}'\hat{i}' + \dot{y}'\hat{j}'$

→ Se a origem é a mesma: $\vec{r} = \vec{r}'$ → mesmo vetor, representação diferente

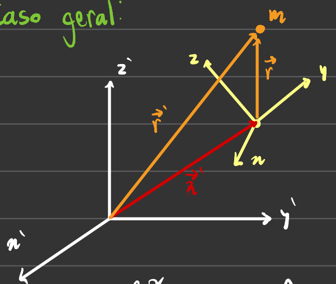
↳ $x'\hat{i}' + y'\hat{j}' = x\hat{i} + y\hat{j} \Rightarrow \dot{x}'\hat{i}' + \dot{y}'\hat{j}' = \dot{x}\hat{i} + x\dot{\hat{i}} + \dot{y}\hat{j} + y\dot{\hat{j}}$

$$\vec{v}' = \cancel{\vec{v}} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt}$$

↳ $\dfrac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \Rightarrow \dfrac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i}, \quad \dfrac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j}$

$$\Rightarrow \vec{v}' = x(\vec{\omega} \times \hat{i}) + y(\vec{\omega} \times \hat{j}) = \vec{\omega} \times \overbrace{(x\hat{i} + y\hat{j})}^{\vec{r}} \Rightarrow \boxed{\vec{v}' = \vec{\omega} \times \vec{r}}$$

* Caso geral:



$$\vec{\omega} = \dot{\theta}\,\hat{n}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$\vec{r}' = x'\hat{i}' + y'\hat{j}' + z'\hat{k}' \qquad \vec{v}' = \dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}'$$

$$\vec{R}' = X'\hat{i}' + Y'\hat{j}' + Z'\hat{k}' \qquad \vec{V}' = \dot{X}'\hat{i}' + \dot{Y}'\hat{j}' + \dot{Z}'\hat{k}'$$

→ Não-inercial: $(x, y, z) \rightarrow \hat{i}, \hat{j}, \hat{k}$ dependem do tempo

→ Inercial: $(x', y', z') \rightarrow \hat{i}', \hat{j}', \hat{k}'$ fixos

$$\rightarrow \frac{d\vec{r}'}{dt} = \frac{d\vec{R}'}{dt} + \frac{d\vec{r}}{dt} \Rightarrow \vec{v}' = \vec{V}' + \frac{d\vec{r}}{dt}$$

$$\rightarrow \vec{v}' = \vec{V}' + \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} + x\dot{\hat{i}} + y\dot{\hat{j}} + z\dot{\hat{k}} \quad \Big| \quad \dot{\hat{i}} = \vec{\omega}\times\hat{i},\ \dot{\hat{j}} = \vec{\omega}\times\hat{j},\ \dot{\hat{k}} = \vec{\omega}\times\hat{k}$$

$$\therefore \boxed{\vec{v}' = \vec{V}' + \vec{v} + \vec{\omega}\times\vec{r}}$$

$$\rightarrow \vec{a}' = \ddot{x}'\hat{i}' + \ddot{y}'\hat{j}' + \ddot{z}'\hat{k}'$$

$$\vec{A}' = \ddot{X}'\hat{i}' + \ddot{Y}'\hat{j}' + \ddot{Z}'\hat{k}'$$

$$\rightarrow \overbrace{\frac{d\vec{v}'}{dt}}^{\vec{a}'} = \overbrace{\frac{d\vec{V}'}{dt}}^{\vec{A}'} + \frac{d\vec{v}}{dt} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\overbrace{\dot{\vec{r}}}^{\vec{v}+\vec{\omega}\times\vec{r}}$$

$$\rightarrow \frac{d\vec{v}}{dt} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} + \dot{x}\dot{\hat{i}} + \dot{y}\dot{\hat{j}} + \dot{z}\dot{\hat{k}} = \vec{a} + \vec{\omega}\times\vec{v}$$

$$\Rightarrow \vec{a}' = \vec{A}' + \vec{a} + \vec{\omega}\times\vec{v} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\vec{v} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\boxed{\vec{a}' = \vec{A}' + \vec{a} + \underbrace{2\vec{\omega}\times\vec{v}}_{\text{Coriolis}} + \dot{\vec{\omega}}\times\vec{r} + \underbrace{\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{Centrífuga}}}$$

translação da origem

aceleração no referencial não inercial

Dinâmica de uma partícula — Ref. não inercial

→ 2ª lei: $\vec{F}_{res} = m\vec{a}' = m\left(\vec{A}' + \vec{a} + 2\vec{\omega}\times\vec{v} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right)$

→ Inercial (S'): $(\vec{r}', \vec{a}', \vec{v}')$

Não-inercial (S): $(\vec{r}, \vec{a}, \vec{v})$

"forças" não-inerciais

→ $m\vec{a} = \vec{F}_{res} - m\vec{A}' - 2m(\vec{\omega}\times\vec{v}) - m(\dot{\vec{\omega}}\times\vec{r}) - m(\vec{\omega}\times(\vec{\omega}\times\vec{r}))$

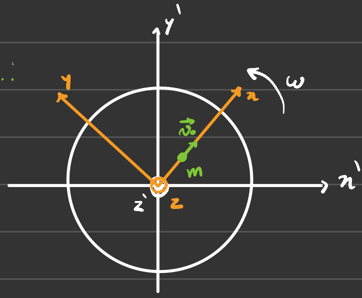
↳ interação

* Força de Coriolis: $\vec{F}_{cor} \equiv -2m(\vec{\omega}\times\vec{v})$

* Força centrífuga: $\vec{F}_{cent} \equiv -m\left[\vec{\omega}\times(\vec{\omega}\times\vec{r})\right]$ → para fora

* Força transversal: $\vec{F}_{Trans} \equiv -m(\dot{\vec{\omega}}\times\vec{r})$ → aceleração angular

* Ex.:



$\vec{F}_{res} = ?$

$\omega = cte \Rightarrow \dot{\omega} = 0 \Rightarrow \vec{F}_{trans} = 0$

$v_0 = cte$ em S (não-inercial) $\Rightarrow \vec{a} = 0$

origens coincidem

$$\cancel{m\vec{a}}^{\,0} = -\cancel{m\vec{A}}^{\,0} + \vec{F}_{res} + \vec{F}_{cor} + \cancel{\vec{F}_{trans}}^{\,0} + \vec{F}_{cent}$$

$$\vec{F}_{res} = -\vec{F}_{cor} - \vec{F}_{cent}$$

* Representar os vetores no sistema de coordenadas de S: $\begin{cases} \vec{\omega} = \omega\hat{k} \\ \vec{v} = v_0\hat{i} \\ \vec{r} = x\hat{i} \end{cases}$

$\rightarrow \vec{F}_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -m\omega\hat{k} \times (\omega\hat{k} \times x\hat{i}) \; -m\omega\hat{k} \times \omega x\hat{j} = m\omega^2 x\hat{i}$

$\rightarrow \vec{F}_{cor} = -2m(\vec{\omega} \times \vec{v}) = -2m(\omega\hat{k} \times v_0\hat{i}) = -2m\omega v_0\hat{j}$

$\boxed{\vec{F}_{res} = -mx\omega^2\hat{i} + 2mv_0\omega\hat{j}}$ $\rightarrow$ força resultante medida em S' e representado em S

23/08/23

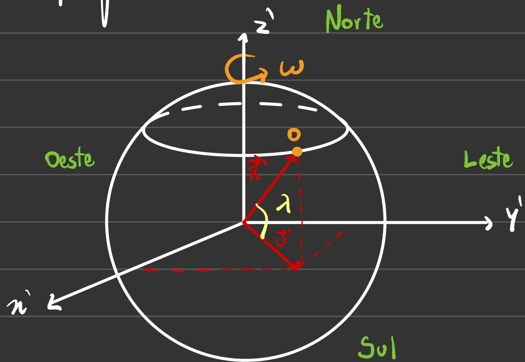
Referencial na superfície da Terra

→ $\omega_{Rot} \approx 7 \cdot 10^{-5}\ 1/s$ → rotação em torno do eixo polar

→ $\omega_{orb} \approx 2 \cdot 10^{-7}\ 1/s$ → rotação em torno do sol

$\Big\}\ \omega_{Rot} \gg \omega_{orb}$

→ Centro da Terra é aproximadamente um referencial inercial em relação à superfície da Terra

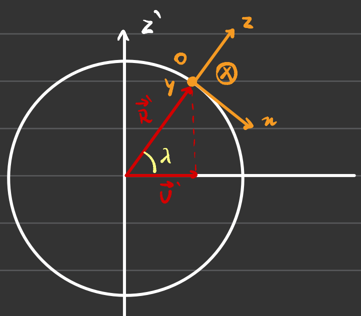


O → origem de S

eixo z' → polar

plano x'y' → equatorial

$\lambda \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ → latitude



$$\vec{A}' = \frac{d\vec{V}'}{dt} = \frac{d}{dt}(\vec{\omega}' \times \vec{R}') = \vec{\omega}' \times \vec{R}' + \vec{\omega}' \times \vec{V}'$$

$$\vec{A}' = \vec{\omega}' \times (\vec{\omega}' \times \vec{R}')$$

$$\boxed{\vec{g} = \vec{g}' - \vec{\omega}' \times (\vec{\omega}' \times \vec{R}')}$$

$\vec{r} = 0,\ \vec{v} = 0,\ \vec{a} = 0$

$$\cancel{m\vec{a}} = \vec{F}_{res} - m\vec{A}' + \cancel{\vec{F}_{cor}} + \cancel{\vec{F}_{cent}}$$

$$0 = \vec{F}_{res} - m\vec{A}'$$

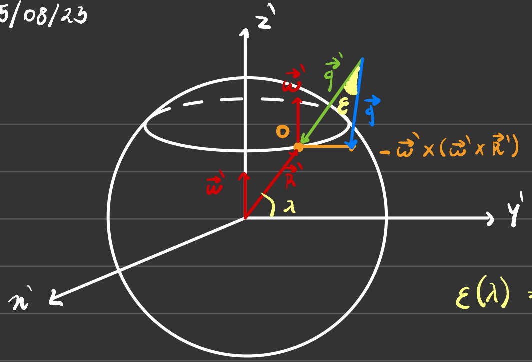
$$\vec{F}_{res} = \vec{F}_N + \vec{F}_g = \vec{F}_N \underbrace{- \frac{mGM_T}{R_T^2}\hat{k}}_{g'}$$

$$\vec{F}_{res} = \vec{F}_N + m\vec{g}'$$

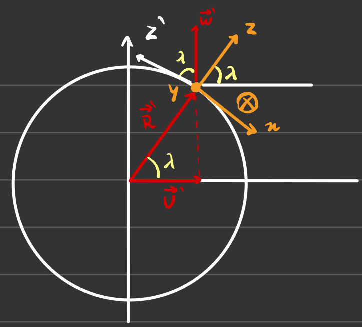
$$0 = \vec{F}_N + m\vec{g}' - m\vec{A}' = \vec{F}_N + m(\vec{g}' - \vec{A}') = 0$$

$$\boxed{\vec{F}_N + m\vec{g} = 0} \qquad \vec{g} = \vec{g}' - \vec{A}'$$

25/08/23



$\varepsilon(\lambda) = ?$



→ Representar vetores no sistema de coordenadas não inercial

→ $\omega_y' = 0$, $\omega_x' = -\omega\cos\lambda$, $\omega_z' = \omega\,\text{sen}\,\lambda \Rightarrow \vec{\omega}' = -\omega\cos\lambda\,\hat{i} + \omega\,\text{sen}\,\lambda\,\hat{k}$

→ $\vec{R}' = R_T\,\hat{k}$

→ $\vec{\omega}' \times \vec{R}' = -\omega\cos\lambda\,R_T(\hat{i}\times\hat{k}) + \omega\,\text{sen}\,\lambda\,R_T(\hat{k}\times\hat{k}) \Rightarrow \vec{\omega}'\times\vec{R}' = R_T\,\omega\cos\lambda\,\hat{j}$

→ $\vec{\omega}' \times (\vec{\omega}'\times\vec{R}') = -\omega\cos\lambda \cdot R_T\,\omega\cos\lambda\,(\hat{i}\times\hat{j}) + \omega\,\text{sen}\,\lambda\,R_T\,\omega\cos\lambda\,(\hat{k}\times\hat{j})$

$\Rightarrow \vec{\omega}' \times (\vec{\omega}'\times\vec{R}') = -R_T\omega^2\cos^2\lambda\,\hat{k} - R_T\omega^2\cos\lambda\,\text{sen}\,\lambda\,\hat{i}$

$$\therefore \boxed{\vec{g} = \underbrace{R_T\omega^2\cos\lambda\,\text{sen}\,\lambda}_{g_x}\,\hat{i} + \underbrace{(R_T\omega^2\cos^2\lambda - \vec{g}')}_{g_z}\,\hat{k}}$$

→ $\tan\varepsilon = \dfrac{g_x}{g_z}$, $\tan\varepsilon \approx \varepsilon \Rightarrow \boxed{\varepsilon = \dfrac{R_T\omega^2\cos\lambda\,\text{sen}\,\lambda}{R_T\omega^2\cos^2\lambda - g'}}$

→ Módulo de g: $\boxed{g = \sqrt{R_T^2\omega^4\cos^2\lambda + g'^2 - 2g'R_T\omega^2\cos^2\lambda}}$

→ $\lambda_{max} \approx 45° \Rightarrow \boxed{\varepsilon(\lambda_{max}) = 1{,}7\cdot 10^{-3} \approx 0{,}1°}$

* Equador: $\lambda = 0$

↳ $\varepsilon = 0$

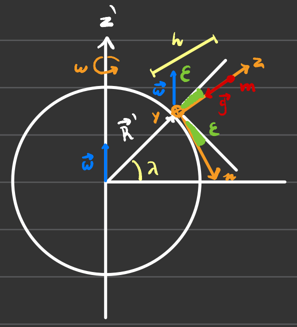
↳ $g = g' - \underbrace{R_T\omega^2}_{0{,}034\ m/s^2}$

* Polo Norte: $\lambda = \frac{\pi}{2}$

↳ $\varepsilon = 0$

↳ $g = g'$

Queda Livre



$t=0: \quad \vec{r}(0) = h\hat{k}$

$\vec{v}(0) = 0$

$\vec{F}_{res} = m\vec{g}', \quad g' = 9{,}8\, m/s^2$

rotação dos eixos de S

Ref. não - inercial: $m\vec{a} = m\vec{g}' - m\vec{A}' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m(\vec{\omega} \times \vec{v})$

$\vec{A}' = \vec{\omega} \times (\vec{\omega}' \times \vec{R}')$

rotação da origem de S

$\rightarrow \vec{\omega} = \vec{\omega}'$

$\rightarrow \vec{a} = \vec{g}' - \vec{\omega} \times (\vec{\omega} \times \vec{R}') - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2(\vec{\omega} \times \vec{v})$

$\vec{a} = \vec{g}' - \vec{\omega} \times (\vec{\omega} \times \underbrace{(\vec{R}' + \vec{r}')}_{\approx \vec{R}'}) - 2(\vec{\omega} \times \vec{v})$

$\vec{a} \approx \underbrace{\vec{g}' - \vec{\omega} \times (\vec{\omega} \times \vec{R}')}_{\vec{g}} - 2(\vec{\omega} \times \vec{v})$

$\therefore \vec{a} = \vec{g} - \overbrace{2(\vec{\omega} \times \vec{v})}^{\text{Coriolis}}$

$\rightarrow \vec{\omega} = \omega \hat{k}' \Rightarrow \vec{\omega} \approx -\omega \cos\lambda\, \hat{i} + \omega\, \text{sen}\,\lambda\, \hat{k}$

$\rightarrow \vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$

$\rightarrow \vec{\omega} \times \vec{v} = -\omega\cos\lambda\, \dot{y}\, (\hat{i} \times \hat{j}) - \omega\cos\lambda\, \dot{z}\, (\hat{i} \times \hat{k}) + \omega\,\text{sen}\,\lambda\, \dot{x}\, (\hat{k} \times \hat{i}) + \omega\,\text{sen}\,\lambda\, \dot{y}\, (\hat{k} \times \hat{j})$

$= -\omega\cos\lambda\, \dot{y}\, \hat{k} + \omega\cos\lambda\, \dot{z}\, \hat{j} + \omega\,\text{sen}\,\lambda\, \dot{x}\, \hat{j} - \omega\,\text{sen}\,\lambda\, \dot{y}\, \hat{i}$

$\therefore \vec{\omega} \times \vec{v} = -\omega \dot{y}\, \text{sen}\,\lambda\, \hat{i} + \omega(\dot{z}\cos\lambda + \dot{x}\,\text{sen}\,\lambda)\hat{j} - \omega \dot{y} \cos\lambda\, \hat{k}$

$$\therefore \boxed{\begin{aligned} \ddot{x} &= 2\omega \dot{y}\, \text{sen}\,\lambda \\ \ddot{y} &= -2\omega(\dot{x}\,\text{sen}\,\lambda + \dot{z}\cos\lambda) \\ \ddot{z} &= -g + 2\omega\dot{y}\cos\lambda \end{aligned}}$$

→ Integrando:

$$\boxed{\begin{aligned} v_x &= 2\omega(\operatorname{sen}\lambda)y + A_x \\ v_y &= -2\omega x \operatorname{sen}\lambda - 2\omega z\cos\lambda + A_y \\ v_z &= -gt + 2\omega y\cos\lambda + A_z \end{aligned}}$$

→ $\ddot{y} = -2\omega \operatorname{sen}\lambda\,(2\omega y \operatorname{sen}\lambda + A_x) - 2\omega\cos\lambda\,(-gt + 2\omega y\cos\lambda + A_z)$

→ Desprezando os termos quadráticos: $\ddot{y} = 2\omega g t\cos\lambda - 2\omega(A_x \operatorname{sen}\lambda + A_z\cos\lambda)$

$$\therefore y(t) \approx \frac{1}{3}\omega g t^3\cos\lambda - \omega t^2(A_x\operatorname{sen}\lambda + A_z\cos\lambda) + D_y t + B_y$$

→ Usando em $\ddot{x}$ e $\ddot{z}$:

$$x(t) \approx \omega\operatorname{sen}\lambda\, D_y t^2 + 2\omega t\operatorname{sen}\lambda\, B_y + A_x t + B_x$$

$$z(t) \approx -\frac{1}{2}gt^2 + \omega\cos\lambda\, D_y t^2 + 2\omega\cos\lambda\, B_y t + A_y t + B_z$$

$x(0) \equiv x_0 = B_x$	$\dot{x}(0) \equiv v_{0x} = 2\omega\operatorname{sen}\lambda\, B_y + A_x$
→ $y(0) \equiv y_0 = B_y$	$\dot{y}(0) \equiv v_{0y} = D_y$
$z(0) \equiv z_0 = B_z$	$\dot{z}(0) \equiv v_{0z} = 2\omega\cos\lambda\, B_y + A_z$

→ Substituindo:

$$\boxed{\begin{aligned} x(t) &= \omega\operatorname{sen}\lambda\, v_{0y} t^2 + v_{0x} t + x_0 \\ y(t) &= \frac{1}{3}\omega\cos\lambda\, g t^3 - \omega t^2\left[v_{0x}\operatorname{sen}\lambda + v_{0z}\cos\lambda - 2\omega y_0\right] + v_{0y} t + y_0 \\ z(t) &= -\frac{1}{2}gt^2 + \omega\cos\lambda\, v_{0y} t^2 + v_{0z} t + z_0 \end{aligned}}$$

→ Especificando as condições iniciais: $\begin{cases} x_0 = y_0 = 0, \quad z_0 = h \\ v_{0x} = v_{0y} = v_{0z} = 0 \end{cases}$

$$\boxed{\begin{aligned} x(t) &= 0 \\ z(t) &= -\frac{1}{2}gt^2 + h \\ y(t) &= \frac{1}{3}\omega\cos\lambda\, g t^3 \end{aligned}}$$

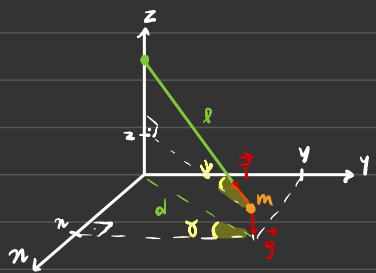
→ $t = t_*$: atinge a superfície

$$z(t_*) = 0 \Rightarrow -\frac{1}{2}gt_*^2 + h = 0 \Rightarrow t_* = \sqrt{\frac{2h}{g}} \Rightarrow y(t_*) = \frac{1}{3}\omega\cos\lambda\sqrt{\frac{2h^3}{g}}$$

→ $\left.\begin{aligned} h &= 100\,m \\ \lambda &= 45^\circ \\ g &= 9{,}8\ m/s^2 \\ \omega &= 7\cdot 10^{-5}\ 1/s \end{aligned}\right\}$ $y(t_*) \approx 1{,}55\ cm$

28/08/23

Pêndulo de Foucault



Equilíbrio: $x = y = z = 0$

$(x, y, z) \rightarrow$ não-inercial

$$\begin{cases} T_z = T \operatorname{sen} \psi \\ T_y = -T \cos\psi \cos\gamma \\ T_x = -T\cos\psi \operatorname{sen}\gamma \end{cases} \qquad \ell^2 = d^2 + (\ell - z)^2 \qquad \begin{matrix} \operatorname{sen}\psi = (\ell - z)/\ell, \; \cos\psi = d/\ell \\ \operatorname{sen}\gamma = |x|/d, \; \cos\gamma = |y|/d \end{matrix}$$

$$\therefore \; T_z = T\left(1 - \frac{z}{\ell}\right), \quad T_y = -T\frac{y}{\ell}, \quad T_x = -T\frac{x}{\ell}$$

$\rightarrow$ Eq. dinâmica: $\vec{a} = \vec{g} + \dfrac{\vec{T}}{m} - 2(\vec{\omega} \times \vec{v}), \quad \vec{g} = -g\hat{k}$

$$\vec{\omega} = -\omega\cos\lambda\,\hat{i} + \omega\operatorname{sen}\lambda\,\hat{k}$$

$$\Rightarrow \begin{cases} \ddot{x} = -\dfrac{T}{m}\dfrac{x}{\ell} + 2\omega\dot{y}\operatorname{sen}\lambda \\ \ddot{y} = -\dfrac{T}{m}\dfrac{y}{\ell} - 2\omega(\dot{x}\operatorname{sen}\lambda + \dot{z}\cos\lambda) \\ \ddot{z} = -g + \dfrac{T}{m}\left(1 - \dfrac{z}{\ell}\right) + 2\omega\dot{y}\cos\lambda \end{cases}$$

$\rightarrow \frac{|x|}{\ell} \ll 1, \quad \frac{|y|}{\ell} \ll 1$

$$\rightarrow \ell^2 = x^2 + y^2 + (\ell - z)^2 \Rightarrow \ell - z = \sqrt{\ell^2 - x^2 - y^2}$$

$$z = \ell - \ell\sqrt{1 - \frac{x^2}{\ell^2} - \frac{y^2}{\ell^2}} \approx \ell - \ell\left[1 - \frac{x^2}{2\ell^2} - \frac{y^2}{2\ell^2}\right]$$

$$\frac{z}{\ell} \approx \frac{1}{2}\left(\frac{x^2}{\ell^2} + \frac{y^2}{\ell^2}\right) \Rightarrow z \approx \frac{1}{2}\left(\frac{x^2}{\ell} + \frac{y^2}{\ell}\right) \Rightarrow z \approx 0, \ \dot{z} \approx 0, \ \ddot{z} \approx 0$$

$$\Rightarrow 0 = -g + \frac{T}{m} + 2\omega\dot{y}\cos\lambda \Rightarrow T = mg - 2\omega\dot{y}m\cos\lambda$$

$$\rightarrow \ddot{x} = -\frac{g}{\ell}x + 2\dot{y}\cancel{\omega\frac{x}{\ell}}^{\to 0}\cos\lambda + 2\omega\dot{y}\,\mathrm{sen}\,\lambda$$

$$\ddot{y} = -\frac{g}{\ell}y + 2\dot{y}\cancel{\omega\frac{y}{\ell}}^{\to 0}\cos\lambda - 2\omega\dot{x}\,\mathrm{sen}\,\lambda$$

$$\Rightarrow \boxed{\begin{aligned}\ddot{x} &= -\frac{g}{\ell}x + 2\omega\dot{y}\,\mathrm{sen}\,\lambda \\ \ddot{y} &= -\frac{g}{\ell}y - 2\omega\dot{x}\,\mathrm{sen}\,\lambda\end{aligned}}$$

$\rightarrow$ Seja $q = x + iy$, $q \in \mathbb{C}$:

$$\ddot{q} = \overbrace{-\frac{g}{\ell}x - i\frac{g}{\ell}y}^{-g/\ell\, q} + 2\omega\,\mathrm{sen}\,\lambda\,\dot{y} - 2i\omega\,\mathrm{sen}\,\lambda\,\dot{x} = -\frac{g}{\ell}q - 2i\omega\,\mathrm{sen}\,\lambda\overbrace{(\dot{x} + i\dot{y})}^{\dot{q}}$$

$$\ddot{q} = -\frac{g}{\ell}q - 2i\omega\,\mathrm{sen}\,\lambda\,\dot{q}$$

$$\rightarrow \omega_0^2 = g/\ell, \quad \omega_z = \omega\,\mathrm{sen}\,\lambda \Rightarrow \ddot{q} + \omega_0^2 q + 2i\omega_z\dot{q} = 0$$

$$q(t) = e^{-i\omega_z t}\left[Ae^{i\sqrt{\omega_0^2+\omega_z^2}\,t} + Be^{-i\sqrt{\omega_0^2+\omega_z^2}\,t}\right]$$

→ Referencial inercial: $\omega_z = 0 \Rightarrow q(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \equiv q_0(t)$

→ $\sqrt{\omega_0^2 + \omega_z^2} \approx \omega_0 \Rightarrow q(t) \approx e^{-i\omega_z t}\left[Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right]$

$q(t) \approx e^{-i\omega_z t} q_0$

→ $\left.\begin{array}{l} q(t) = x(t) + iy(t) \\ q_0(t) = x_0(t) + iy_0(t) \end{array}\right\} \quad x + iy = \left[\cos(\omega_z t) - i\,\text{sen}(\omega_z t)\right](x_0 + iy)$

$\vdots$

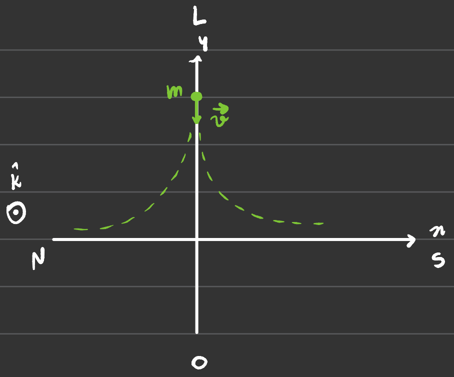
$$x(t) = x_0(t)\cos(\omega_z t) + y_0(t)\,\text{sen}(\omega_z t)$$
$$y(t) = y_0(t)\cos(\omega_z t) - x_0(t)\,\text{sen}(\omega_z t)$$

→ Matricialmente:

$$\boxed{\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{ref. não-inercial}} = \overbrace{\begin{pmatrix} \cos(\omega_z t) & \text{sen}(\omega_z t) \\ -\text{sen}(\omega_z t) & \cos(\omega_z t) \end{pmatrix}}^{\text{matriz de rotação}} \underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{\text{ref. inercial}}}$$

→ Período: $\tau_F = \dfrac{2\pi}{\omega_z} = \dfrac{2\pi}{\omega\,\text{sen}\lambda}$

$\left.\begin{array}{l} \text{Paris: } \lambda = 48° \\ \omega = 7{,}3 \cdot 10^{-5}\ \frac{1}{s} \end{array}\right\} \quad \tau_F \approx 31{,}7\,h$



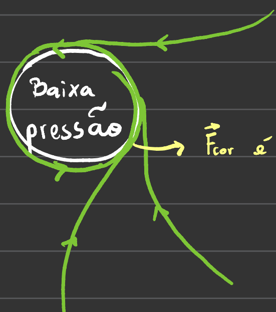
$$\vec{v} = -v\hat{j} \Rightarrow \vec{\omega}\times\vec{v} = \omega v\cos\lambda\,\hat{k} + \overbrace{\omega v\,\mathrm{sen}\,\lambda\,\hat{i}}^{\text{deflexão N-S}}$$

$$\vec{F}_{cor} = -2m(\vec{\omega}\times\vec{v})$$

$$\hookrightarrow \vec{F}_{cor,x} = -2m\omega v\,\mathrm{sen}\,\lambda$$

Norte $(\lambda>0) \Rightarrow F_{cor,x} < 0$

Sul $(\lambda<0) \Rightarrow F_{cor,x} > 0$



$\vec{F}_{cor}$ é "cancelada" pelo gradiente de pressão