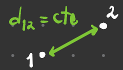
Dinâmica de Corpos Rígidos

→ $N = 2$:



↳ $3N = 6$ coordenadas, $m = 1$ vínculo $\Big\}$ $n = 5$ graus de liberdade

→ $N = 3$:

↳ $3N = 9$ coordenadas
$m = 3$ vínculos
$n = 9 - 3 = 6$ graus de liberdade

→ $N = 4$:

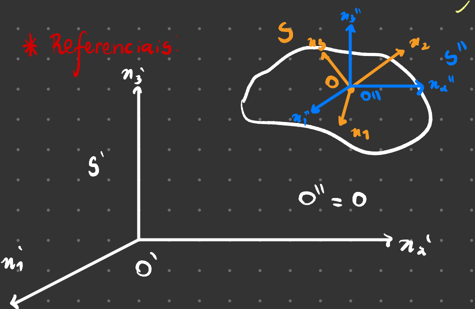
↳ $3N = 12$
$m = 6$
$n = 6$

→ $N' = N + 1$:

↳ $3N' = 3N + 3$
↳ $m' = m + 3$
↳ $n' = n = 6$

→ Graus de liberdade: $n = 6$ $\begin{cases} 3 \to \text{translação} \\ 3 \to \text{rotação} \end{cases}$

* Referenciais:



$O'' = O$

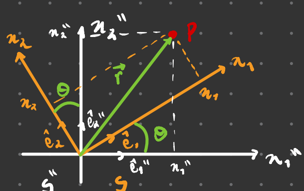
S': Referencial inercial do espaço fixo do laboratório

S: Referencial do corpo (não inercial)

S'': Referencial inercial, $O'' = O$

Rotações de eixos coordenados ($S \leftrightarrow S''$)

* Rotação em 2D ($x_3 = x_3''$):



Em S: $\vec{r}_S = \sum_{i=1}^{2} x_i \hat{e}_i$

Em S'': $\vec{r}_{S''} = \sum_{i=1}^{2} x_i'' \hat{e}_i''$

Mas: $\vec{r}_{S''} = \vec{r}_S = \vec{r} \Rightarrow \sum_{i=1}^{2} x_i'' \hat{e}_i'' = \sum_{i=1}^{2} x_i \hat{e}_i$

$$\begin{cases} \hat{e}_1 = \cos\theta\, \hat{e}_1'' + \operatorname{sen}\theta\, \hat{e}_2'' \\ \hat{e}_2 = -\operatorname{sen}\theta\, \hat{e}_1'' + \cos\theta\, \hat{e}_2'' \end{cases} \to \text{Notação matricial: } K = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix}$$

→ $\hat{e}_i = \sum_{j=1}^{2} K_{ji} \hat{e}_j''$

→ $\hat{e} = \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$, $\hat{e}'' = \begin{pmatrix} \hat{e}_1'' \\ \hat{e}_2'' \end{pmatrix}$

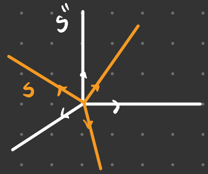
$\boxed{\hat{e} = K^T \hat{e}''}$ $\boxed{\hat{e}'' = (K^T)^{-1} \hat{e}}$

→ $r = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $r'' = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix}$ →

$\vec{r} = r^T \hat{e} = r''^T \hat{e}''$
$r''^T \hat{e}'' = r^T \hat{e} = r^T K^T \hat{e}''$
$(r''^T - r^T K^T)\, \hat{e}'' = 0$
$r''^T = r^T K^T$
$\therefore \boxed{\begin{matrix} r'' = Kr \\ r = K^{-1} r'' \end{matrix}}$

$$\begin{cases} x_1'' = x_1 \cos\theta\, \hat{e}_1 - x_2 \operatorname{sen}\theta\, \hat{e}_2 \\ x_2'' = x_2 \operatorname{sen}\theta\, \hat{e}_1 + x_2 \cos\theta\, \hat{e}_2 \end{cases}$$

* Rotação em 3D:



$$\hat{e} = K^T \hat{e}'' \longrightarrow \hat{e}_i = \sum_{K=1}^{3} K_{ki}\,\hat{e}_k'' \quad (i=1,2,3)$$

$$\left.\begin{array}{l} \hat{e}_i \cdot \hat{e}_j'' = \sum_{K=1}^{3} K_{ki}\, \underbrace{\hat{e}_k'' \cdot \hat{e}_j''}_{\delta_{kj}} = K_{ji} \\ \hat{e}_i'' \cdot \hat{e}_j = \cos\theta_{ij} \end{array}\right\} \boxed{K_{ij} = \cos\theta_{ij}}$$

cossenos diretores

$$r = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad r'' = \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix}, \quad r = K^{-1} r'' \longrightarrow$$

Matriz de rotação: $\boxed{R = K^{-1}}$

$$\left.\begin{array}{l} r = R r'' \\ r'' = R^{-1} r \end{array}\right\} \boxed{x_i = \sum_{j=1}^{3} R_{ij} X_j''}$$

* Condição de ortogonalidade: $\vec{r} = \vec{r}_S = \vec{r}_{S''}$

$$\|\vec{r}\| = \sqrt{\sum_{i=1}^{3} x_i^2} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\|\vec{r}_S\| = \|\vec{r}_{S''}\| \longrightarrow \sum_{i=1}^{3} x_i''^2 = \sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3} x_i x_i$$

$$\sum_i x_i''^2 = \sum_{i,j,k} \underbrace{R_{ij} x_j''}_{x_i} \underbrace{R_{ik} x_k''}_{x_k}$$

Condição: $\sum_{i=1}^{3} R_{ij} R_{ik} = \delta_{jk} \rightarrow \left.\begin{array}{l} R^T R = I_3 \\ R R^T = I_3 \end{array}\right\}$ $R^T R = R^T R = I_3 \Rightarrow \boxed{R^{-1} = R^T}$ matriz ortogonal

$R_{3\times3} \rightarrow$ 9 elementos
↳ Condição de ortogonalidade: 6 relações entre elementos } 3 cossenos diretores L.I.

$$RR^T = I_3 \Rightarrow \det(RR^T) = \det(I_3)$$
$$\det(R)\det(R^T) = 1$$
$$[\det(R)]^2 = 1 \Rightarrow \det(R) = \pm 1$$

- $1 \rightarrow$ rotações próprias
- $-1 \rightarrow$ rotações impróprias (inversão de eixos: dextrógiro ↔ levógiro)

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$$R = \begin{pmatrix} R_{11} & R_{12} & \cdots \\ R_{21} & R_{22} & \\ \vdots & & R_{33} \end{pmatrix} \rightarrow R = R(\theta_1, \theta_2, \theta_3)$$

* Taxa de variação temporal de vetores dinâmicos

→ Vetor $\vec{g} = \vec{g}(t)$

→ Em relação a S': $\vec{g} = \sum_{i=1}^{3} g_i' \hat{e}_i' \Rightarrow \left.\frac{d\vec{g}}{dt}\right|_{S'} = \left.\frac{d\vec{g}}{dt}\right|_{inercial} = \sum_{i=1}^{3} \frac{dg_i'}{dt}\hat{e}_i'$

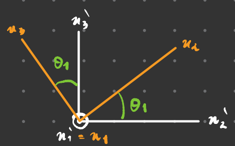
→ Em relação a S: $\vec{g} = \sum_{i=1}^{3} g_i \overset{\hat{e}_i(t)}{\hat{e}_i} \Rightarrow \left.\frac{d\vec{g}}{dt}\right|_{inercial} = \sum_{i=1}^{3} \frac{dg_i}{dt}\hat{e}_i + \sum_{i=1}^{3} g_i \frac{d\hat{e}_i}{dt}$

→ Mas: $\left.\frac{d\vec{g}}{dt}\right|_{corpo} = \sum_{i=1}^{3} \frac{dg_i}{dt}\hat{e}_i$

$$\therefore \boxed{\left.\frac{d\vec{g}}{dt}\right|_{inercial} = \left.\frac{d\vec{g}}{dt}\right|_{corpo} + \sum_{i=1}^{3} g_i \frac{d\hat{e}_i}{dt}} \quad (3.8)$$

* Matriz de rotações infinitesimais

→ Rotações em torno de x_1', x_2', x_3'



etc.

$$R^{(1)}(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \operatorname{sen}\theta_1 \\ 0 & -\operatorname{sen}\theta_1 & \cos\theta_1 \end{pmatrix}$$

$$R^{(2)}(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & -\operatorname{sen}\theta_2 \\ 0 & 1 & 0 \\ \operatorname{sen}\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$

$$R^{(3)}(\theta_3) = \begin{pmatrix} \cos\theta_3 & \operatorname{sen}\theta_3 & 0 \\ -\operatorname{sen}\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

* Rotações 2D: comutam → $R^{(3)}(\alpha_3)R^{(3)}(\theta_3) = R^{(3)}(\theta_3+\alpha_3) = R^{(3)}(\alpha_3)R^{(3)}(\theta_3)$

* Rotações 3D: não comutam → $R^{(j)}(\phi_j)R^{(i)}(\theta_i) \neq R^{(i)}(\theta_i)R^{(j)}(\theta_j)$ $(i,j = 1,2,3 \text{ e } i \neq j)$ → grupo $SO(3)$ (Não abeliano)

θ_i, ϕ_i finitos

→ $R^{(i)}(\theta_i) = \prod_{j=1}^{N} R^{(i)}(\Delta\theta_j)$, $\sum_{j=1}^{N}\Delta\theta_j = \theta_i$

→ $\Delta\theta_j = \delta\theta_j$ $(|\delta\theta_j| \ll 1)$, $N \gg 1$

Considerando só $\delta\theta_j$: $\operatorname{sen}(\delta\theta_j) \approx \delta\theta_j$, $\cos(\delta\theta_j) \approx 1$

$R^{(i)}(\delta\theta_i) \approx I_3 + \delta R^{(i)}$

$$\delta R^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta\theta_1 \\ 0 & -\delta\theta_1 & 0 \end{pmatrix}, \quad \delta R^{(2)} = \begin{pmatrix} 0 & 0 & -\delta\theta_2 \\ 0 & 0 & 0 \\ \delta\theta_2 & 0 & 0 \end{pmatrix}$$

+1: permutações pares de 1, 2, 3
-1: permutações ímpares de 1, 2, 3
0: $i=j$, $j=k$, $i=k$

$R_{ij}^{(k)} = \delta_{ij} + \varepsilon_{ijk}\delta\theta_k \rightarrow \left[R^{(k)}(\delta\theta_k)R^{(\ell)}(\delta\theta_\ell)\right]_{ij} = \delta_{ij} + \varepsilon_{ijk}\delta\theta_k + \varepsilon_{ij\ell}\delta\theta_\ell = \left[R^{(\ell)}(\delta\theta_\ell)R^{(k)}(\delta\theta_k)\right]_{ij}$

$$\delta\vec{\theta} = (\delta\theta_1, \delta\theta_2, \delta\theta_3) \longrightarrow \begin{cases} R(\delta\vec{\theta}) = R^{(1)}(\delta\theta_1)R^{(2)}(\delta\theta_2)R^{(3)}(\delta\theta_3) \\ R_{ij}(\delta\vec{\theta}) = \delta_{ij} + \sum_{k=1}^{3}\varepsilon_{ijk}\delta\theta_k \end{cases}$$

↳ $[dR]$

→ Retomando a (3.8): $\left.\frac{d\vec{g}}{dt}\right|_{S'} = \left.\frac{d\vec{g}}{dt}\right|_{S} + \sum_{i=1}^{3} g_i \frac{d\hat{e}_i}{dt}$

→ Em relação a S': $\hat{e}(t) \xrightarrow{t \to t+dt} \hat{e}(t+dt) = R(d\vec{\theta})\,\hat{e}(t) = \hat{e}(t) + dR(d\vec{\theta})\,\hat{e}(t)$

$$d\hat{e} = \hat{e}(t+dt) - \hat{e}(t) = dR(d\vec{\theta})\,\hat{e}(t)$$

$$d\hat{e}_i = \sum_{j=1}^{3} [dR(d\vec{\theta})]_{ij}\,\hat{e}_j = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\, d\theta_k\, \hat{e}_j$$

$$d\vec{g}\,\big|_{rot} = \sum_{i=1}^{3} g_i\, d\hat{e}_i = \sum_{i,j,k=1}^{3} g_i\,\varepsilon_{ijk}\, d\theta_k\, \hat{e}_j \quad i \to j,\ j \to k$$

$$= \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, d\theta_j\, g_k\, \hat{e}_i$$

$$\therefore \boxed{d\vec{g}\,\big|_{rot} = d\vec{\theta} \times \vec{g}}$$

→ $\left.\frac{d\vec{g}}{dt}\right|_{S'} = \left.\frac{d\vec{g}}{dt}\right|_{S} + \underbrace{\frac{d\vec{\theta}}{dt}}_{\vec{\omega}} \times \vec{g} \Rightarrow \boxed{\left.\frac{d\vec{g}}{dt}\right|_{S'} = \left.\frac{d\vec{g}}{dt}\right|_{S} + \vec{\omega} \times \vec{g}}$

→ Forma operacional: $\left.\frac{d}{dt}\right|_{inercial} = \left.\frac{d}{dt}\right|_{corpo} + \vec{\omega}\,\times$

* Leitura: 3.2.3.1, 3.2.3.2

※01/12/23

→ sistema de N partículas

→ Distâncias entre partículas fixas

→ n = 6 graus de liberdade (3 de translação, 3 de rotação)

→ Apenas rotação: existe um ponto fixo

1) Momento linear:

$$\frac{d\vec{P}}{dt} = \vec{F}^{(e)}, \quad \vec{P} = \sum_{a=1}^{N} m_a \vec{v}_a = \sum_a \vec{p}_a$$

2) Momento angular: genericamente:

P: ponto fixo

$$\vec{L}_p = \sum_{a=1}^{N} (\vec{r}_a - \vec{r}_p) \times (\vec{p}_a - m_a \cancel{\dot{\vec{r}}_p})$$

$$\dot{\vec{L}}_p = \vec{N}_p^{(e)} - M(\underbrace{\vec{R}}_{\text{pos. CM}} - \vec{r}_p) \times \ddot{\vec{r}}_p \rightarrow \text{se} \begin{cases} (i) \ \vec{r}_p = \vec{R} \\ (ii) \ \ddot{\vec{r}}_p = 0 \\ (iii) \ \ddot{\vec{r}}_p \parallel \vec{R} - \vec{r}_p \end{cases} \Rightarrow \boxed{\dot{\vec{L}}_p = \vec{N}_p^{(e)}}$$

* Corpo rígido: O (origem de S) ou O = P (ponto fixo) ou O = CM

Momentum angular total sobre O:

$$\rightarrow \vec{L} = \sum_{k=1}^{n} \vec{r}_k \times \vec{p}_k', \quad \vec{p}_k' = \vec{p}_k |_{S', \text{inercial}}$$

$$= \sum_{k=1}^{n} m_k (\vec{r}_k \times \vec{v}_k'), \quad \vec{v}_k' = \frac{d\vec{r}_k}{dt}\Big|_{S'}$$

$$\rightarrow \text{Neste caso, } \frac{d\vec{r}_k}{dt}\Big|_{S'} = \overbrace{\cancel{\frac{d\vec{r}_k}{dt}\Big|_{S}}}^{0} + \vec{\omega} \times \vec{r}_k = \vec{\omega} \times \vec{r}_k$$

$$\vec{L} = \sum_{k=1}^{N} m_k \vec{r}_k \times (\vec{\omega} \times \vec{r}_k)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$$\vec{L} = \sum_{k=1}^{N} m_k [r_k^2 \vec{\omega} - (\vec{r}_k \cdot \vec{\omega}) \vec{r}_k] = \sum_{i=1}^{3} L_i \hat{e}_i$$

$$\rightarrow L_i = \sum_{k=1}^{N} m_k \Big(r_k^2 \underbrace{\omega_i}_{\sum_j \delta_{ij} \omega_j} - \sum_{j=1}^{3} x_{k,j} x_{k,i} \omega_j \Big) = \sum_{j=1}^{3} \Big[\underbrace{\sum_{k=1}^{N} m_k (r_k^2 \delta_{ij} - x_{k,i} x_{k,j})}_{I} \Big] \omega_j$$

* Matriz de inércia:

↳ Discreto: $I_{ij} = \sum_{k=1}^{N} m_k (r_k^2 \delta_{ij} - x_{k,i} x_{k,j})$ $\left\{ \boxed{L_i = \sum_{j=1}^{3} I_{ij} \omega_j} \quad \boxed{L = I\omega} \right.$

↳ Contínuo:

$$\left.\begin{aligned} & x_{k,i} \rightarrow x_i \rightarrow \vec{r} = (x_1, x_2, x_3) \subset V \\ & m_k \rightarrow dm = \rho(\vec{r})\, d^3r \\ & N \rightarrow \infty: \sum_{k=1}^{N} m_k \rightarrow \int_V d^3r\, \rho(\vec{r}) \end{aligned}\right\} \boxed{I_{ij} = \int_V d^3r\, [r^2 \delta_{ij} - x_i x_j]\, \rho(\vec{r})}$$

$$\rho(\vec{r}) = \sum_{k=1}^{N} m_k \delta(\vec{r} - \vec{r}_k)$$

↳ Propriedades de I:

(i) $I_{ij} \in \mathbb{R}$ → I é real

(ii) $I_{ij} = I_{ji}$ → I é simétrica

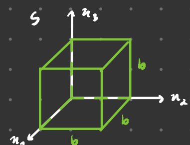
↳ Momentos de inércia: diagonal de I

$$I_{ii} = \int_V d^3r\,(\overbrace{r^2 - x_i^2}^{= x_i^2+x_j^2+x_k^2-x_i^2 \,=\, x_j^2+x_k^2})\,\rho(\vec r) = \int_V d^3r\,\underbrace{(x_j^2+x_k^2)}_{\geq 0}\overset{\geq 0}{\rho} \Rightarrow \boxed{I_{ii} \geq 0}$$

↳ Produtos de inércia: elementos fora da diagonal

$$I_{ij} = -\int_V d^3r\, x_i x_j\, \rho(\vec r)$$

* Exemplo: Matriz I de um cubo homogêneo



$$I_{ii} \overset{(i\neq k\neq i)}{=} \rho\int_V d^3r\,(x_j^2+x_k^2) = 2\rho\int_V d^3r\, x_j^2$$

$$= 2\rho\int_0^b dx_i \int_0^b dx_k \int_0^b dx_j\, x_j^2 = \frac{2}{3}\overset{M/b^3}{\rho}\, b^5 = \frac{2}{3}Mb^2$$

depende do referencial

Produtos: $I_{ij} \overset{i\neq j\neq k}{=} -\int d^3r\, x_i x_j \rho = -\rho\int_0^b dx_i\, x_i \int_0^b dx_j\, x_j \int_0^b dx_k \Rightarrow I_{ij} = -\frac{1}{4} Mb^2$

$$\mathrm{I} = Mb^2 \begin{pmatrix} 2/3 & -1/4 & -1/4\\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

→ Matriz de inércia → componentes do tensor de inércia

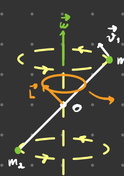
→ Rotação: $S \to \bar S$: $O = \bar O$ → $x_i' = \sum_{j=1}^{3} R_{ij} x_j$

I'_{ij}: componente ij de $\bar{\mathrm{I}}$ → $\boxed{I'_{ij} = \sum_{m,n=1}^{3} R_{im} R_{jm} I_{mn}}$ Tensor de posto 2 → $\boxed{I' = R I R^T}$

* Notação de diádica $\overleftrightarrow{I} = \sum_{i,j=1}^{3} I_{ij} \underbrace{\hat e_i \hat e_j}_{\text{diádica}} = I_{11}\hat e_1\hat e_1 + I_{12}\hat e_1 \hat e_2 + \dots + I_{33}\hat e_3\hat e_3$

$L_i = \sum_{j=1}^{3} I_{ij}\omega_j \to \vec L = \overleftrightarrow{I}\cdot\vec\omega = \left(\sum_{i,j} I_{ij}\hat e_i\hat e_j\right)\cdot\left(\sum_{k=1}^{3} \omega_k\hat e_k\right) = \sum_{i,j,k=1}^{3} I_{ij}\,\omega_k\, \hat e_i\, \overbrace{\hat e_j\cdot\hat e_k}^{\delta_{jk}} \Rightarrow \boxed{\vec L = \sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}\,\omega_j\, \hat e_i}$ → $\vec L$ não necessariamente paralelo a $\vec\omega$

↳ Haltere



$\vec\ell_1 = (\vec r_1 \times \vec v_1)\, m_1$

precessão do momento angular

Energia Cinética Rotacional

$\rightarrow T = \frac{1}{2}\sum_{a=1}^{N} m_a \dot{r}_a^2$

$\vec{v}_k' \cdot \vec{v}_k' = \vec{v}_k' \cdot \frac{d\vec{r}_k'}{dt} = \vec{v}_k' \cdot \vec{\omega} \times \vec{r}_k$

$\rightarrow T_{rot} = \frac{1}{2}\sum_{k=1}^{N} m_k \underbrace{v_k'^2} = \frac{1}{2}\sum_{k=1}^{N} m_k \left[\vec{v}_k' \cdot (\vec{\omega} \times \vec{r}_k)\right] = \frac{1}{2}\sum_{k} \overbrace{m_k \left[\vec{\omega} \cdot (\vec{r}_k \times \vec{v}_k')\right]}^{\vec{l}_k}$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

$\rightarrow \vec{L} = \sum_{k=1}^{N} \vec{l}_k \Rightarrow T_{rot} = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}\vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$

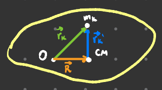
$\rightarrow$ Eixo instantâneo de rotação: $\hat{e}_n \Rightarrow \vec{\omega} = \omega \hat{e}_n \Rightarrow T = \frac{1}{2}\omega^2 \underbrace{(\hat{e}_n \cdot \overleftrightarrow{I} \cdot \hat{e}_n)} \Rightarrow \boxed{T_{rot} = \frac{1}{2}\omega^2 I}$

escalar: $I = \hat{e}_n \cdot \overleftrightarrow{I} \cdot \hat{e}_n$: Momento de inércia em relação ao eixo $\hat{e}_n$

↳ Eixo de simetria

*04/12/23

Teorema de Steiner



$$\begin{cases} M\vec{R} = \sum_{k=1}^{N} m_k \vec{r}_k \\ \vec{r}_k = \vec{R} + \vec{r}_k' \end{cases}$$

$\rightarrow \overleftrightarrow{I}_o = \sum_{k=1}^{N} m_k \left[r_k^2 \overleftrightarrow{1} - \vec{r}_k \vec{r}_k\right]$

↳ $\overleftrightarrow{1} = \sum_{i,j=1}^{3} \delta_{ij} \hat{e}_i \hat{e}_j$

↳ $\vec{r}_k \vec{r}_k = \sum_{i,j=1}^{3} x_{k,i} x_{k,j} \hat{e}_i \hat{e}_j$

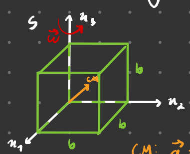
$\sum_k m_k \vec{r}_k' = 0$

$\rightarrow \overleftrightarrow{I}_o = \sum_k m_k \left[(\vec{R} + \vec{r}_k')^2 \overleftrightarrow{1} - (\vec{R} + \vec{r}_k')(\vec{R} + \vec{r}_k')\right] = \sum_k m_k \left[(R^2 + \cancel{2\vec{R}\vec{r}_k'}^{\,0} + r_k'^2)\overleftrightarrow{1} - \vec{R}\vec{R} - \cancel{\vec{R}\vec{r}_k'}^{\,0} - \cancel{\vec{r}_k'\vec{R}}^{\,0} - \vec{r}_k'\vec{r}_k'\right]$

$\overleftrightarrow{I}_o = \underbrace{\sum_k m_k}_{M} \left[R^2 \overleftrightarrow{1} - \vec{R}\vec{R}\right] + \underbrace{\sum_k m_k (r_k'^2 \overleftrightarrow{1} - \vec{r}_k' \vec{r}_k')}_{\overleftrightarrow{I}_{CM}} \Rightarrow \boxed{\overleftrightarrow{I}_o = \overleftrightarrow{I}_{CM} + M(R^2 \overleftrightarrow{1} - \vec{R}\vec{R})}$

↳ $$\boxed{\begin{aligned} (I_o)_{ij} &= (I_{CM})_{ij} + M(R^2 \delta_{ij} - R_i R_j) \\ (I_{CM})_{ij} &= \sum_{k=1}^{N} m_k (r_k'^2 \delta_{ij} - x_{k,i}' x_{k,j}') \end{aligned}}$$

* Exemplo: Matriz I de um cubo homogêneo



CM: $\vec{a} = \left(\frac{b}{2}, \frac{b}{2}, \frac{b}{2}\right)$

$\rho = cte = M/b^3$

$$I = Mb^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \Rightarrow L_i = \sum_{j=1}^{3} I_{ij}\omega_j \rightarrow \vec{L} \not\parallel \vec{\omega}$

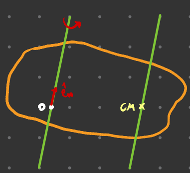
$T_{rot} = \frac{1}{2}\vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} = \frac{1}{2}\sum_{i,j=1}^{3} I_{ij}\omega_i\omega_j$

$\rightarrow (I_{CM})_{ij} = (I_o)_{ij} - M(a^2\delta_{ij} - a_i a_j) = (I_o)_{ij} - \frac{1}{4}Mb^2(3\delta_{ij} - 1)$

↳ $(I_{CM})_{ii} = (I_o)_{ii} - \frac{1}{2}Mb^2 = \frac{2}{3}Mb^2 - \frac{1}{2}Mb^2 = \frac{1}{6}Mb^2$

↳ $(I_{CM})_{ij} = (I_o)_{ij} + \frac{1}{4}Mb^2 = 0$

$$I_{CM} = \frac{1}{6}Mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$\overleftrightarrow{I} \longrightarrow \boxed{I = \hat{e}_n \cdot \overleftrightarrow{I} \cdot \hat{e}_n}$

Momento de inércia ao longo de $\hat{e}_n$

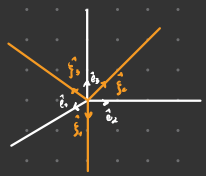
↳ Teorema dos eixos paralelos: $\boxed{I_o = I_{CM} + M(\hat{e}_n \times \vec{R})^2}$

Diagonalização do Tensor de Inércia

→ matriz de inércia: $I_{ij} = I_{ji}$ → autovalores reais e positivos

→ I diagonal: $I_{ij} = I_i \delta_{ij}$ → $L_i = I_i \omega_i$

$T_{rot} = \frac{1}{2} \sum_{i=1}^{3} I_i \omega_i^2$



$$\overleftrightarrow{I} = \sum_{i,j=1}^{3} I_{ij}\, \hat{e}_i \hat{e}_j = \sum_{i=1}^{3} I_i\, \hat{\xi}_i \hat{\xi}_i$$

(1) (2) (3)

$\{I_1, I_2, I_3\}$: Momentos principais de inércia

$\{\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3\}$: eixos principais de inércia

$$(1) = (3):\quad \overleftrightarrow{I} \cdot \hat{\xi}_k = \sum_{i=1}^{3} I_i \hat{\xi}_i \underbrace{\hat{\xi}_i \cdot \hat{\xi}_k}_{\delta_{ik}} = I_k \hat{\xi}_k = \sum_{i=1}^{3} I_k \xi_{ki}\, \hat{e}_i$$

$$(1) = (2):\quad \overleftrightarrow{I} \cdot \hat{\xi}_k = \sum_{i,j=1}^{3} I_{ij}\, \hat{e}_i \underbrace{\hat{e}_j \cdot \hat{\xi}_k}_{\xi_{kj}} = \sum_{i,j=1}^{3} I_{ij} \xi_{kj}\, \hat{e}_i$$

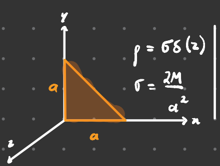
$$\sum_{i=1}^{3} \Big(\underbrace{\sum_{j=1}^{3} I_{ij} \xi_{kj} - I_k \xi_{ki}}_{=0} \Big) \hat{e}_i = 0 \;\Rightarrow\; \sum_{j=1}^{3} I_{ij} \xi_j = I \xi_i \quad (i = 1, 2, 3)$$

$$\underbrace{\begin{pmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{pmatrix}}_{\Delta} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0$$

→ $\det(\Delta) = 0$: polinômio de 3º grau para I → 3 raízes reais e positivas $\{I_1, I_2, I_3\}$

Normalização: $|\vec{\xi}| = 1$

* Ex.: Momentos e eixos principais de uma placa triangular



$\rho = \sigma \delta(z)$

$\sigma = \frac{2M}{a^2}$

$$I_{ij} = \int_V d^3r\, \rho (r^2 \delta_{ij} - x_i x_j)$$

↳ $I_{xx} = \int_V d^3r\, \rho (r^2 - x^2) = \int_V dx\, dy\, dz\, \sigma\delta(z) (y^2 + z^2) = \sigma \left[\int_0^a dy\, y^2 \int_0^{a-y} dx \int dz\, \delta(z) + \int_0^a dy\, y^2 \int_0^{a-y} dx \int dz\, \underbrace{\cancel{\delta(z)\, z^2}}_{0} \right]$

$$\therefore\; I_{xx} = I_{yy} = \frac{1}{12} \sigma a^4 = \frac{1}{6} M a^2$$

$$I_{zz} = 2 I_{xx} = \frac{1}{3} M a^2$$

↳ $I_{xy} = I_{yx} = -\int_V dx\, dy\, dz\, \sigma\delta(z)\, xy = -\sigma \int_0^a dy \int_0^{a-y} dx \int dz\, \delta(z)\, xy \;\Rightarrow\; I_{xy} = I_{yx} = -\frac{1}{12} M a^2$

↳ $I_{iz} = I_{zi} = \dots \int dz\, z\, \delta(z) = 0 \;\Rightarrow\; I_{iz} = I_{zi} = 0$

$$\therefore\; I = \frac{1}{12} M a^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\rightarrow \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \;\Rightarrow\; \begin{aligned} &(2-\lambda)^2(4-\lambda) - (4-\lambda) = 0 \\ &(4-\lambda)\left[(2-\lambda)^2 - 1\right] = 0 \\ &\lambda = 2 \pm 1,\; \lambda = 4 \end{aligned}$$

$\Rightarrow I_1 = \frac{1}{12} M a^2,\; I_2 = \frac{1}{4} M a^2,\; I_3 = \frac{1}{3} M a^2 \Rightarrow$ $\boxed{I_0 = \frac{Ma^2}{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}}$

$$\rightarrow \lambda_1 = 1:\; \begin{cases} (2-1)\xi_x - \xi_y = 0 \\ -\xi_x + (2-1)\xi_y = 0 \\ (4-1)\xi_z = 0 \end{cases} \Bigg|\; \begin{aligned} &\xi_{1,x} = \xi_{1,y} \\ &\xi_{1,z} = 0 \\ &|\vec{\xi}_1| = 1 \end{aligned} \;\Rightarrow\; \hat{\xi}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \rightarrow \boxed{\hat{\xi}_1 = \frac{1}{\sqrt{2}} (\hat{x} + \hat{y})}$$

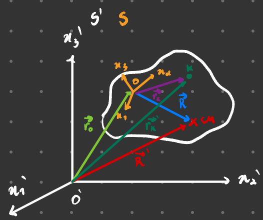
→ $\lambda_2 = 3$: $\boxed{\hat{\xi}_2 = \frac{1}{\sqrt{2}} (-\hat{x} + \hat{y})}$

↳ escolha para referencial dextrógiro

→ $\lambda_3 = 4$: $\boxed{\hat{\xi}_3 = (0, 0, 1) = \hat{z}}$

Leitura: Simetrias e eixos principais de inércia

Lagrangiana de um corpo rígido



$\rightarrow$ Em S': $\vec{r}_k, \vec{R}$ = ctes

$$\begin{cases} \vec{r}_k\,' = \vec{r}_0 + \vec{r}_k \\ \vec{R}\,' = \vec{r}_0 + \vec{R} \end{cases} \quad \begin{cases} M\vec{R}\,' = \sum_k m_k \vec{r}_k\,' \\ M\vec{R} = \sum_k m_k \vec{r}_k \end{cases} \quad \Bigg| \quad \begin{aligned} \dot{\vec{r}}_k\,' &= \vec{v}_0|_{S'} + \dot{\vec{r}}_k|_{S'} \\ \vec{V}\,' = \dot{\vec{R}}\,' &= \vec{v}_0|_{S'} + \dot{\vec{R}}|_{S'} \end{aligned}$$

* Energia cinética: $T = \frac{1}{2}\sum_k m_k v_k'^2 = \frac{1}{2}\sum_k m_k \dot{r}_k'^2$

(3.13): $\left.\frac{d}{dt}\right|_{S'} = \left.\frac{d}{dt}\right|_{S} + \vec{\omega}\times \quad \begin{cases} \dot{\vec{r}}_k|_{S'} = \cancelto{0}{\dot{\vec{r}}_k|_S} + \vec{\omega}\times\vec{r}_k \\ \dot{\vec{R}}|_{S'} = \vec{\omega}\times\vec{R} \end{cases}$

$$T = \frac{1}{2}\sum_k m_k(\vec{v}_0 + \dot{\vec{r}}_k)^2 = \frac{1}{2}\sum_k m_k(v_0^2 + \dot{r}_k^2 + 2\vec{v}_0\cdot\dot{\vec{r}}_k)$$

$$= \underbrace{\frac{1}{2}Mv_0^2}_{T_1} + \underbrace{\frac{1}{2}\sum_k m_k \dot{r}_k^2}_{T_2} + \underbrace{\sum_k m_k \vec{v}_0\cdot(\vec{\omega}\times\vec{r}_k)}_{T_3} \qquad \overbrace{\sum_k m_k \vec{r}_k}^{M\vec{R}}$$

$$T_2 = \sum_k m_k \dot{r}_k^2 = \sum_k m_k \dot{\vec{r}}_k\cdot\dot{\vec{r}}_k = \sum_k m_k \dot{\vec{r}}_k\cdot(\vec{\omega}\times\vec{r}_k) = \vec{\omega}\cdot\Big[\sum_k m_k(\overbrace{\vec{r}_k\times\dot{\vec{r}}_k}^{\vec{r}_k})\Big] = \vec{\omega}\cdot\vec{L}$$

$$\vec{L} = \overleftrightarrow{I}\cdot\vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}_0\cdot\vec{\omega} \Rightarrow T_2 = T_{rot} = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}_0\cdot\vec{\omega}$$

Portanto,

$$T = \underbrace{\frac{1}{2}Mv_0^2}_{\text{translacional}} + \underbrace{\frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}_0\cdot\vec{\omega}}_{\text{rotacional}} + M\vec{v}_0\cdot(\vec{\omega}\times\vec{R}) \quad \rightarrow \text{Ponto O qualquer}$$

$$\vec{r}_0 = \vec{R}\,' - \vec{R} \Rightarrow \vec{v}_0 = \vec{V}\,' - \dot{\vec{R}} = \vec{V}\,' - \vec{\omega}\times\vec{R} \Rightarrow \boxed{T = \frac{1}{2}MV'^2 + \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}_0\cdot\vec{\omega} - \frac{1}{2}M(\vec{\omega}\times\vec{R})^2}$$

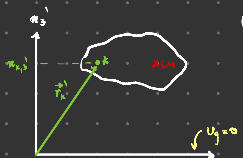
Se $O = CM$: $\vec{R} = 0 \Rightarrow \boxed{T = \frac{1}{2}MV'^2 + \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}_{cm}\cdot\vec{\omega}}$

* Energia potencial:

sistema de partículas: $U(\{\vec{r}_k\},\{\dot{\vec{r}}_k\}) = U^{(ext)}(\{\vec{r}_k\},\{\dot{\vec{r}}_k\}) + \underbrace{\sum_{k<\ell} U(|\vec{r}_{k\ell}|)}_{U^{(int)} = \text{cte}}$ $\rightarrow$ não será incluída na lagrangiana

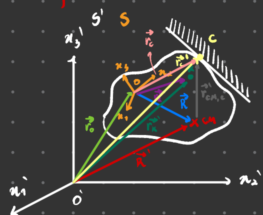
corpo rígido

$U^{(ext)} = U_{cm}(\vec{R}\,') + U_{rot}(\mathbb{R})$ matriz de rotação



$$U_g = \sum_k m_k g\, x_{k,3}' = g\sum_k m_k x_{k,3}' \Rightarrow U_g = Mg R_3'$$

* Condição de rolamento:



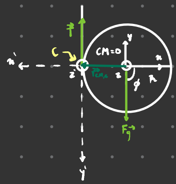
Ponto C: $\begin{cases} \text{contato instantâneo} \\ \text{em repouso em } S' \\ \text{em repouso instantâneo em } S \end{cases}$

Rolamento sem deslizamento:

$$\left.\begin{aligned} \vec{r}_c\,' &= \vec{r}_0 + \vec{r}_c = \vec{R}\,' + \vec{r}_{cm,c} \\ \dot{\vec{r}}_c\,' &= \vec{v}_0 + \dot{\vec{r}}_c = \vec{V} + \dot{\vec{r}}_{cm,c} \\ &= \vec{v}_0 + \vec{\omega}\times\vec{r}_c = \vec{V}\,' + \vec{\omega}\times\vec{r}_{cm,c} \end{aligned}\right\} \quad \begin{aligned} &\vec{v}_0 + \vec{\omega}\times\vec{r}_c = 0 \\ &\text{ou} \\ &\vec{V}\,' + \vec{\omega}\times\vec{r}_{cm,c} = 0 \end{aligned}$$

* Exemplo: Ioiô

$h \ll R$



$$\rho = \frac{M}{\pi R^2 h}$$

$O = CM: \quad T = T_{trans} + T_{rot}$

$$T_{rot} = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$$

$$\left.\begin{aligned} I_x = I_y &= \int_V d^3r\,(y^2+z^2)\,\rho = \frac{1}{4}M\Big(R^2 + \overbrace{\frac{h^2}{3}}^{\ll 1}\Big) \approx \frac{1}{4}MR^2 \\ I_z &= \int_V d^3r\,(x^2+y^2)\,\rho = \frac{1}{2}MR^2 \end{aligned}\right\} \quad I = \frac{1}{4}MR^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\vec{\omega} = -\dot{\phi}\hat{z}$$

$$T_{rot} = \frac{1}{2}(\dot{\phi}\hat{z})\cdot\overleftrightarrow{I}\cdot(\dot{\phi}\hat{z}) = \frac{1}{2}\sum_{i,j=1}^{3}\overbrace{\omega_i\omega_j}^{-\dot{\phi}\delta_{i3}}\overbrace{I_{ij}}^{I_i\delta_{ij}} \Rightarrow T_{rot} = \frac{1}{2}I_z\dot{\phi}^2$$

$$T_{trans} = \frac{1}{2}MV'^2 \Rightarrow T_{trans} = \frac{1}{2}M\dot{y}'^2$$

$$\boxed{T = \frac{1}{2}M\dot{y}'^2 + \frac{1}{2}I_z\dot{\phi}^2} \qquad \boxed{U_g = -Mgy'}$$

Rolamento sobre c: $\quad \vec{V}' + \vec{\omega}\times\vec{r}_{cm,c} = 0$

Vínculo

$$\dot{y}'\hat{y}' + (-\dot{\phi}\hat{z})\times(-R\hat{x}') = 0 \Rightarrow \dot{y}' - R\dot{\phi} = 0 \Rightarrow y' = R\phi \rightarrow \dot{\phi}^2 = \frac{\dot{y}'^2}{R^2}$$

$$\Rightarrow L\{y', \dot{y}'\} = \frac{1}{2}\left(M + \frac{I_z}{R^2}\right)\dot{y}'^2 + Mgy' \Rightarrow \boxed{L = \frac{3}{4}M\dot{y}'^2 + Mgy'}$$

$$\Rightarrow \frac{\partial L}{\partial y'} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}'} = 0 \Rightarrow Mg - \frac{3}{2}M\ddot{y}' = 0 \Rightarrow \boxed{\ddot{y}' = \frac{2}{3}g} \qquad \boxed{\ddot{\phi} = \frac{2}{3}\frac{g}{R}}$$

$$\Rightarrow \vec{L} = \overleftrightarrow{I}\cdot\vec{\omega} = -I_z\dot{\phi}\hat{z} \longrightarrow N^{(ext)} = \vec{r}_{cm,c}\times\vec{T} = \dot{\vec{L}}$$

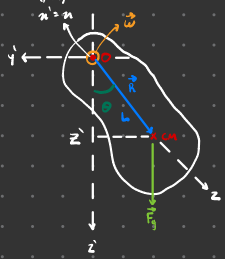
$$-RT\hat{z} = -I_z\ddot{\phi}\hat{z}$$

$$\boxed{T = \frac{1}{3}Mg}$$

※ 08/12/23

$[I] = ML^2$

∗ Exemplo: Pêndulo físico



velocidade de translação

$$T = \frac{1}{2} M \cancel{v_0^2} + \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}_0\cdot\vec{\omega} + M\cancel{\vec{v}_0}\cdot(\vec{\omega}\times\vec{R})$$

$$\vec{\omega} = \dot{\theta}\hat{n} \Rightarrow T = \frac{1}{2} I_{nn}\dot{\theta}^2$$

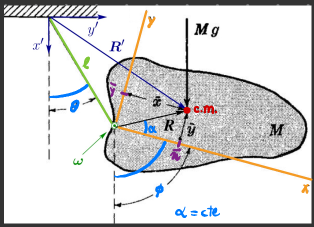
$$U_g = -mgz'$$

Vínculo: $z' = L\cos\theta \Rightarrow U_g = -MgL\cos\theta$

$$L\{\theta,\dot{\theta}\} = \frac{1}{2} I_{nn}\dot{\theta}^2 + MgL\cos\theta$$

$$\left.\begin{aligned} &\frac{\partial L}{\partial \theta} = -MgL\,\text{sen}\,\theta \\ &\frac{\partial L}{\partial \dot{\theta}} = I_{nn}\dot{\theta} \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = I_{nn}\ddot{\theta} \end{aligned}\right\} \quad \boxed{\ddot{\theta} + \Omega^2\theta = 0} \qquad \Omega^2 = \frac{MgL}{I_{nn}}$$

∗ Exemplo: Pêndulo duplo semifísico



$$T = \frac{1}{2}MV'^2 + \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}_0\cdot\vec{\omega} - \frac{1}{2}M(\vec{\omega}\times\vec{R})^2$$

$$\vec{V}' = (\dot{X}', \dot{Y}'), \quad \vec{\omega} = \dot{\phi}\hat{z}$$

$$\begin{cases} \hat{n}' = \cos\phi\,\hat{n} - \text{sen}\,\phi\,\hat{y} \\ \hat{y}' = \text{sen}\,\phi\,\hat{n} + \cos\phi\,\hat{y} \end{cases} \Rightarrow \begin{cases} \hat{n} = \cos\phi\,\hat{n}' + \text{sen}\,\phi\,\hat{y}' \\ \hat{y} = -\text{sen}\,\phi\,\hat{n}' + \cos\phi\,\hat{y}' \end{cases}$$

$$\vec{R} = \bar{x}\hat{n} + \bar{y}\hat{y} = R[\cos(\alpha+\phi)\,\hat{n}' + \text{sen}(\alpha+\phi)\,\hat{y}']$$

→ $\vec{r}_0 = \ell\cos\theta\,\hat{n}' + \ell\,\text{sen}\,\theta\,\hat{y}'$

não variam no tempo

$$\vec{R}' = \vec{r}_0 + \vec{R} = \underbrace{[\ell\cos\theta + R\cos(\alpha+\phi)]}_{X'}\hat{n}' + \underbrace{[\ell\,\text{sen}\,\theta + R\,\text{sen}(\alpha+\phi)]}_{Y'}\hat{y}'$$

$$\left.\begin{aligned} \dot{X}' &= -\ell\,\text{sen}\,\theta\dot{\theta} - R\,\text{sen}(\alpha+\phi)\dot{\phi} \\ \dot{Y}' &= \ell\cos\theta\dot{\theta} + R\cos(\alpha+\phi)\dot{\phi} \end{aligned}\right\} \quad V'^2 = \dot{X}'^2 + \dot{Y}'^2 = \ell^2\dot{\theta}^2 + R^2\dot{\phi}^2 + 2\ell R\cos(\theta-\phi-\alpha)\dot{\theta}\dot{\phi}$$

→ $\vec{\omega}\cdot\overleftrightarrow{I}_0\cdot\vec{\omega} = I_{zz}\dot{\phi}^2$

→ $\vec{\omega}\times\vec{R} = \dot{\phi}\hat{z}\times(\bar{x}\hat{n} + \bar{y}\hat{y}) = \dot{\phi}(\bar{x}\hat{y} - \bar{y}\hat{n}) \Rightarrow (\vec{\omega}\times\vec{R})^2 = R^2\dot{\phi}^2$

$$\left.\begin{aligned} T &= \frac{1}{2}M\ell^2\dot{\theta}^2 + \frac{1}{2}I_{zz}\dot{\phi}^2 + M\ell R\cos(\theta-\phi-\alpha)\dot{\theta}\dot{\phi} \\ U_g &= -MgX' = -Mg[\ell\cos\theta + R\cos(\alpha+\phi)] \end{aligned}\right\} \quad L = L\{\theta,\dot{\theta},\phi,\dot{\phi}\}$$

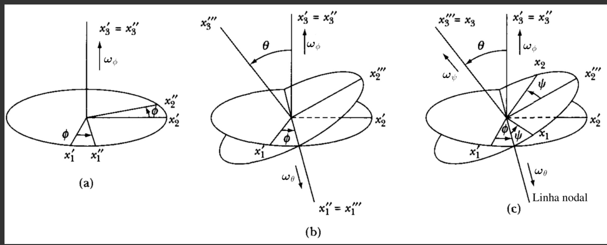
$$\begin{cases} \partial_\theta L - d_t\partial_{\dot{\theta}} L = 0 \\ \partial_\phi L - d_t\partial_{\dot{\phi}} L = 0 \end{cases}$$

→ solução de equilíbrio: $g\,\text{sen}\,\theta = 0 \Rightarrow \theta = 0$

$MgR\,\text{sen}(\phi+\alpha) = 0 \Rightarrow \phi = -\alpha$

Ângulos de Euler

→ Matriz R: (3×3) → $R^T R = I_3$ → 3 ângulos livres



(a) $\hat{e}'' = R_\phi \hat{e}'$, $r'' = R_\phi r'$

$$R_\phi = \begin{pmatrix} \cos\phi & \text{sen}\,\phi & 0 \\ -\text{sen}\,\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (0 \leq \phi \leq 2\pi)$$

Precessão

(b) $\hat{e}'' = R_\theta \hat{e}''$

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \text{sen}\,\theta \\ 0 & -\text{sen}\,\theta & \cos\theta \end{pmatrix} \quad (0 \leq \theta \leq \pi)$$

Nutação (?)

(c) $\hat{e} = R_\psi \hat{e}'''$

$$R_\psi = \begin{pmatrix} \cos\psi & \text{sen}\,\psi & 0 \\ -\text{sen}\,\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (0 \leq \psi \leq 2\pi)$$

Rotação

* Rotação completa: $S' \to S$

$\hat{e} = R_{Euler}\, \hat{e}'$

↳ $R_\psi \hat{e}''' \to R_\psi R_\theta \hat{e}'' \to R_\psi R_\theta R_\phi \hat{e}'$

$\boxed{R_{Euler} = R_\psi R_\theta R_\phi}$

Inversa: $\boxed{\hat{e}' = R^T_{Euler}\, \hat{e}}$, $R^T_{Euler} = R_\phi^T R_\theta^T R_\psi^T$

* Velocidade angular: $\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi$

→ $\vec{\omega}_\phi = \dot{\phi}\, \hat{n}_3'$

→ $\vec{\omega}_\theta = \dot{\theta}\, \hat{n}_1'' = \dot{\theta} \cos\phi\, \hat{e}_1' + \dot{\theta}\, \text{sen}\,\phi\, \hat{e}_2'$

→ $\vec{\omega}_\psi = \dot{\psi}\, \hat{n}_3 = \dot{\psi}\, \text{sen}\,\theta\, \text{sen}\,\phi\, \hat{e}_1' - \dot{\psi}\, \text{sen}\,\theta \cos\phi\, \hat{e}_2' + \dot{\psi} \cos\theta\, \hat{e}_3'$

∴ $\vec{\omega} = (\dot{\theta}\cos\phi + \dot{\psi}\,\text{sen}\,\theta\,\text{sen}\,\phi)\,\hat{e}_1' + (\dot{\theta}\,\text{sen}\,\phi - \dot{\psi}\,\text{sen}\,\theta\cos\phi)\,\hat{e}_2' + (\dot{\phi} + \dot{\psi}\cos\theta)\,\hat{e}_3'$

→ Leitura: 3.3.2. → $\vec{\omega} \neq \frac{d}{dt} \vec{\lambda}(\phi, \theta, \psi)$

※ 18/12/23

Equações de Euler

→ $\overleftrightarrow{I} = I_1 \hat{e}_1\hat{e}_1 + I_2 \hat{e}_2\hat{e}_2 + I_3 \hat{e}_3\hat{e}_3$

$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$

$$\begin{cases} I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 = N_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_1\omega_3 = N_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 = N_3 \end{cases}$$

* Rotação livre (sem torque):

$$\begin{cases} I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 = 0 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_1\omega_3 = 0 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 = 0 \end{cases}$$

* Pião simétrico ($I_2 = I_1$)

$$\begin{cases} I_1\dot{\omega}_1 - (I_1 - I_3)\,\omega_1\omega_3 = 0 \\ I_1\dot{\omega}_2 - (I_3 - I_1)\,\omega_1\omega_3 = 0 \\ I_3\dot{\omega}_3 = 0 \Rightarrow \boxed{\omega_3 = cte} \end{cases}$$

$\ddot{\omega}_1 + \Omega^2\omega_1 = 0$

$\Omega^2 = \frac{I_3 - I_1}{I_1}\,\omega_3$

$\Rightarrow$

$\omega_1(t) = \omega_\perp \cos(\Omega t)$

$\omega_2(t) = \omega_\perp \operatorname{sen}(\Omega t)$

$\omega_\perp = \sqrt{\omega_1^2 + \omega_2^2} = cte$

* Exercício 3.5

→ Ângulos de Euler: $\{\underset{\text{precessão}}{\phi}, \underset{\text{nutação}}{\theta}, \underset{\text{rotação}}{\psi}\}$

→ $\vec{\omega} = (\dot{\phi}\operatorname{sen}\theta\operatorname{sen}\psi + \dot{\theta}\cos\psi)\,\hat{e}_1 + (\dot{\phi}\operatorname{sen}\theta\cos\psi - \dot{\theta}\operatorname{sen}\psi)\,\hat{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$

→ Condição inicial: $x_1 = x_1' \Rightarrow \psi(t=0) = \phi(t=0) = 0$

↳ $\vec{\omega}(t=0)$: $\omega_1 = \dot{\theta}$, $\omega_2 = \dot{\phi}\operatorname{sen}\theta$, $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$

→ Rotação livre: $\vec{N} = 0 \Rightarrow \vec{L} = L\hat{e}_3' = 0$

→ Referencial S (do corpo): $\vec{L}|_S = (L_1, L_2, L_3)$

$$L_S = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix},\quad L_{S'} = \begin{pmatrix} 0 \\ 0 \\ L \end{pmatrix} \quad \rightarrow L_S = R_{Euler} L_{S'} = \begin{pmatrix} \cdots & \cdots & R_{13} \\ \cdots & \cdots & R_{23} \\ \cdots & \cdots & R_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ L \end{pmatrix}$$

↳ $t = 0$: $R_{13} = 0$, $R_{23} = \operatorname{sen}\theta$; $R_{33} = \cos\theta$

↳ S: $L_1 = 0$; $L_2 = L\operatorname{sen}\theta$; $L_{33} = L\cos\theta$

$L_1 = I_1\omega_1 = 0 \Rightarrow I_1\dot{\theta} = 0 \Rightarrow \boxed{\dot{\theta} = 0}$

$L_2 = I_1\omega_2 = L\operatorname{sen}\theta \Rightarrow I_1\dot{\phi}\operatorname{sen}\theta = L\operatorname{sen}\theta \Rightarrow \boxed{\dot{\phi} = \frac{\omega_2}{\operatorname{sen}\theta} = \frac{L}{I_1} = cte}$

$L_3 = I_3\omega_3 \Rightarrow \boxed{\omega_3 = \frac{L_3}{I_3} = \frac{L\cos\theta}{I_3}}$

↳ Quase horizontal: $\theta \ll 1$, $\cos\theta \approx 1$

$\omega_3 = \frac{L}{I_3} = \frac{L}{2I_1} \Rightarrow \boxed{\dot{\phi} \approx 2\omega_3}$

* Construção de Poinsot:

→ Constantes de movimento: $\vec{N} = 0 \Rightarrow \vec{L} = \text{cte}$

$$T_{rot} = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega} = \text{cte}$$

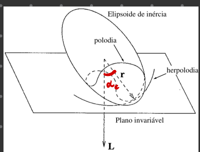
→ Vetor $\vec{r}$: $\vec{r} = \dfrac{\vec{\omega}}{\sqrt{2T}} \Rightarrow \vec{r}\cdot\overleftrightarrow{I}\cdot\vec{r} = 1 \Rightarrow \boxed{I_1 r_1^2 + I_2 r_2^2 + I_3 r_3^2 = 1}$ semieixos: $I_1^{-1}, I_2^{-1}, I_3^{-1}$

Elipsoide de inércia

→ $\nabla_{\vec{r}}(\vec{r}\cdot\overleftrightarrow{I}\cdot\vec{r})$: vetor ortogonal ao elipsoide → $\nabla_r(\vec{r}\cdot\overleftrightarrow{I}\cdot\vec{r}) = \sqrt{\frac{2}{L}}\,\vec{L}$

$$d = \frac{\vec{r}\cdot\vec{L}}{L} \rightarrow \vec{r}\cdot\vec{L} = rL\cos\gamma = dL \Rightarrow d = r\cos\gamma \rightarrow \boxed{d = \frac{\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}}{\sqrt{2T}\,L} = \frac{\sqrt{2T}}{L} = \text{cte}}$$

* Pião simétrico ($I_1 = I_2$)



$\operatorname{tg}\alpha_c = \dfrac{\omega_\perp}{\omega_3} = \text{cte}$

e

$\cos\alpha_c = \dfrac{\vec{\omega}\cdot\vec{L}}{\omega L} = \vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega} = \text{cte}$

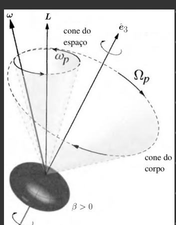
↳ cone do espaço

$$\overleftrightarrow{I} = \sum_{i=1}^{3} I_i \hat{e}_i\hat{e}_i = I_1\overleftrightarrow{1} + (I_3 - I_1)\hat{e}_3\hat{e}_3$$

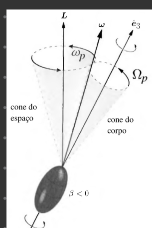
$$\overleftrightarrow{1} \rightsquigarrow I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{s} = \frac{\vec{L}}{\omega I_1} = \frac{\overleftrightarrow{I}\cdot\vec{\omega}}{\omega I_1} = \frac{\overleftrightarrow{I}\cdot\hat{n}}{I_1} \Rightarrow \boxed{\vec{s} = \hat{n} + \beta\cos\alpha_c\,\hat{e}_3}, \quad \beta = \frac{I_3 - I_1}{I_1}, \quad \cos\alpha_c = \hat{e}_3\cdot\hat{n}$$

a. $\beta > 0$ ($I_3 > I_1$: oblato)



b. $\beta < 0$ ($I_3 < I_1$: prolato)



$$\vec{s} = \hat{n} - |\beta|\cos\alpha_c\,\hat{e}_3$$

* Estabilidade das rotações livres

$$\begin{cases} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0 \end{cases}$$

solução "trivial": $\omega_3 = \text{cte}, \ \omega_2 = \omega_1 = 0$

$$\vec{\omega} = \delta\omega_1\hat{e}_1 + \delta\omega_2\hat{e}_2 + \omega_3\hat{e}_3, \quad |\delta\omega_1|, |\delta\omega_2| \ll \omega_3$$

$$\begin{cases} I_1\frac{d}{dt}(\delta\omega_1) - (I_2 - I_3)(\delta\omega_2)\omega_3 = 0 \\ I_2\frac{d}{dt}(\delta\omega_2) - (I_3 - I_1)(\delta\omega_1)\omega_3 = 0 \\ I_3\dot{\omega}_3 - (I_1 - I_2)(\delta\omega_1)(\delta\omega_2) = 0 \end{cases} \Rightarrow I_3\dot{\omega}_3 \approx 0 \Rightarrow \omega_3 \approx \text{cte}$$

$$\begin{cases} \frac{d}{dt}(\delta\omega_1) \approx \left(\frac{I_2 - I_3}{I_1}\omega_3\right)(\delta\omega_2) \\ \frac{d}{dt}(\delta\omega_2) \approx \left(\frac{I_3 - I_1}{I_2}\omega_3\right)(\delta\omega_1) \end{cases}$$

$$\frac{d^2}{dt^2}(\delta\omega_1) = -\frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}(\delta\omega_1)$$

→ solução geral: $\boxed{\delta\omega_1(t) = Ae^{i\Omega_1 t} + Be^{-i\Omega_1 t}}$

$$\Omega_1 = \sqrt{\frac{(I_3 - I_2)(I_3 - I_1)^{-1}}{I_1 I_2}}\,\omega_3$$

$$\boxed{\delta\omega_2(t) = -i\sqrt{\frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}}\,\delta\omega_1(t)}$$

i) ($I_3 > I_1$ e $I_3 > I_2$) ou ($I_3 < I_1$ e $I_3 < I_2$):

$\delta\omega_i = A_i\,\text{sen}(\Omega_1 t) + B_i\cos(\Omega_1 t) \quad (i = 1, 2)$

↳ Estável

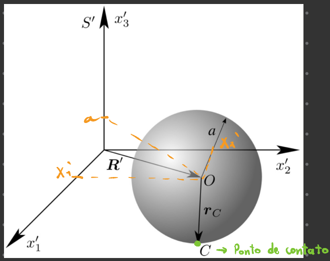
ii) ($I_3 > I_1$ e $I_3 < I_2$) ou ($I_3 < I_1$ e $I_3 > I_2$):

$\delta\omega_i = A_i\,\text{senh}(|\Omega_1| t) + B_i\cosh(|\Omega_1| t)$

↳ Instável

※ 20/12/23

Esferas rolando sobre superfícies



Posição instantânea do CM: $\vec{R}' = (X_1', X_2', a)$

Energia Cinética: $T = T_{rot} + T_{trans} = \frac{1}{2} M V'^2 + \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$

$$O = CM \Rightarrow \overleftrightarrow{I} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad I = \frac{2}{5} M a^2$$

$$T = \frac{1}{2} M (\dot{X}_1'^2 + \dot{X}_2'^2) + \frac{1}{2} I \omega^2$$

$$U_g = M g X_3' = M g a = cte$$

$$\therefore \boxed{L = T}$$

→ Ângulos de Euler: $\omega^2 = \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta$

$$L\{\dot{X}_1', \dot{X}_2', \phi, \theta, \dot{\theta}, \dot{\psi}\} = \frac{1}{2} M (\dot{X}_1'^2 + \dot{X}_2'^2) + \frac{1}{2} I (\dot{\theta} + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta)$$

→ Condição de rolamento: $\vec{V}' + \vec{\omega} \times \vec{r} = 0$; $\vec{r}_c = (0, 0, -a)$ → em relação ao CM

→ $f_1 = -\dot{\psi} \operatorname{sen}\theta \cos\phi + \theta \operatorname{sen}\phi - \frac{\dot{X}_1'}{a} = 0$ $\quad\Big|\quad \frac{\partial f_1}{\partial \dot{X}_1'} = -\frac{1}{a}$

$f_2 = \dot{\psi} \operatorname{sen}\theta \operatorname{sen}\phi + \dot{\theta}\cos\phi + \frac{\dot{X}_2'}{a} = 0$ $\quad\Big|\quad \frac{\partial f_2}{\partial \dot{X}_1'} = 0$

→ Eqs. Euler-Lagrange: (X_1'): $\frac{\partial L}{\partial X_1'} - \frac{d}{dt}\frac{\partial L}{\partial \dot{X}_1'} + \lambda_1 \frac{\partial f_1}{\partial \dot{X}_1'} + \lambda_2 \frac{\partial f_2}{\partial \dot{X}_1'} = 0$

(X_2'): ⋮

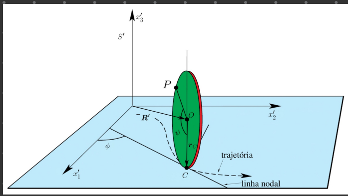
⋮

$$\begin{cases} M\ddot{X}_1' = -\frac{\lambda_1}{a} \\ M\ddot{X}_2' = \frac{\lambda_2}{a} \\ I\frac{d}{dt}(\dot{\phi} + \dot{\psi}\cos\theta) = I\dot{\omega}_3' = 0 \\ I(\ddot{\theta} + \dot{\phi}\dot{\psi}\operatorname{sen}\theta) = \lambda_1 \operatorname{sen}\phi + \lambda_2\cos\phi \\ I\frac{d}{dt}(\dot{\psi} + \dot{\phi}\cos\theta) = -(\lambda_1\cos\phi - \lambda_2\operatorname{sen}\phi)\operatorname{sen}\theta \end{cases}$$

→ $\left.\begin{aligned} \lambda_2 &= I\dot{\omega}_1' = M a \ddot{X}_2' \\ \lambda_2 &= -M a \dot{\omega}_1' \end{aligned}\right\}$ $I > 0,\ M > 0,\ a > 0 \Rightarrow \boxed{\lambda_1 = \lambda_2 = 0} \Rightarrow \boxed{\omega_1' = cte,\ \omega_2' = cte}$

✻ 08/01/24

Disco de Euler



$I_1 = I_2$

$I_3 = I_1 + I_2 = 2I_1$

$\vdots$

$$I = \frac{1}{4}Ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$T = T_{trans} + T_{rot} = \frac{1}{2}Mv'^2 + \frac{1}{2}\left[I_1(\omega_1^2 + \omega_2^2) + I_3\omega_3^2\right]$$

→ $\vec{R}' = (x_1', x_2', x_3') = (x_1', x_2', a)$

→ $\vec{v}' = (\dot{x}_1', \dot{x}_2')$

→ C.I.: $\dot{x}_{10}' = \dot{x}_{20}' = 0$

$\phi_0 = \pi, \; \theta_0 = \frac{\pi}{2}, \; \psi_0 = 0$

→ Matriz de rotação: $R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = -x_1' \\ x_2 = x_3' \\ x_3 = x_2' \end{matrix}$

→ Vínculo: $\theta = \theta_0 = \frac{\pi}{2}$ (não há tombamento)

→ Nos ângulos de Euler: $\theta = \theta_0 = \frac{\pi}{2}, \; \dot{\theta} = 0$

$\phi \to \phi + \pi$

$\Rightarrow$

$\omega_1 = \dot{\phi}\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\psi + \dot{\theta}\cos\psi$

$\omega_2 = \dot{\phi}\,\mathrm{sen}\,\theta\cos\psi - \dot{\theta}\,\mathrm{sen}\,\psi$

$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$

$\omega_1' = -\dot{\theta}\cos\phi - \dot{\psi}\,\mathrm{sen}\,\phi\,\mathrm{sen}\,\theta$

$\omega_2' = -\dot{\theta}\,\mathrm{sen}\,\phi + \dot{\psi}\cos\phi\,\mathrm{sen}\,\theta$

$\omega_3' = \dot{\phi} + \dot{\psi}\cos\theta$

$\omega_1 = \dot{\phi}\,\mathrm{sen}\,\psi$

$\omega_2 = \dot{\phi}\cos\psi$

$\omega_3 = \dot{\psi}$

$\omega_1' = -\dot{\psi}\,\mathrm{sen}\,\phi$

$\omega_2' = \dot{\psi}\cos\phi$

$\omega_3' = \dot{\phi}$

→ $U = U_{g,cm} = cte$ → L equivalente apenas com T

→ $L = T \begin{cases} T_{trans} = \frac{1}{2}M(\dot{x}_1'^2 + \dot{x}_2'^2) \\ T_{rot} = \frac{1}{2}I\left[\cancel{\dot{\theta}^2} + \dot{\phi}^2\,\mathrm{sen}^2\phi + 2(\dot{\psi} + \dot{\phi}\cancel{\cos\theta})^2\right] \end{cases}$

$$\Rightarrow \boxed{L\{\dot{x}_1', \dot{x}_2', \dot{\phi}, \dot{\psi}\} = \frac{1}{2}M\left(\dot{x}_1'^2 + \dot{x}_2'^2\right) + \frac{1}{2}I\left(\dot{\phi}^2 + 2\dot{\psi}^2\right)}$$

→ Demais vínculos: condição de rolamento sem deslizamento → $\vec{v}' + \vec{\omega} \times \vec{r}_c = 0$

↳ $\vec{r}_c = (0, 0, -a)$

$\vec{v}' = (\dot{x}_1, \dot{x}_2, 0)$

$$\vec{\omega} \times \vec{r}_c = \begin{vmatrix} \hat{e}_1' & \hat{e}_2' & \hat{e}_3' \\ \omega_1' & \omega_2' & \omega_3' \\ 0 & 0 & -a \end{vmatrix} = -a(\omega_2'\hat{e}_1 - \omega_1'\hat{e}_2)$$

↳ $\dot{x}_1' - a\omega_2' = 0$

$\dot{x}_2' + a\omega_1' = 0$

→ Vínculos não holônomos $\begin{cases} f_1 = \dot{x}_1' - a\dot{\psi}\cos\phi = 0 \\ f_2 = \dot{x}_2' - a\dot{\psi}\,\mathrm{sen}\,\phi = 0 \end{cases}$

→ Equações de Euler-Lagrange:

$$\left.\begin{matrix} \dfrac{\partial L}{\partial x_1'} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}_1'} + \lambda_1 \dfrac{\partial f_1}{\partial \dot{x}_1'} + \lambda_2 \cancel{\dfrac{\partial f_2}{\partial \dot{x}_1'}} = 0 \\ \vdots \\ \dfrac{\partial L}{\partial \psi} + \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\psi}} + \lambda_1 \dfrac{\partial f_1}{\partial \dot{\psi}} + \lambda_2 \dfrac{\partial f_2}{\partial \dot{\psi}} = 0 \end{matrix}\right\}$$

$-M\ddot{x}_1' + \lambda_1 = 0$

$-M\ddot{x}_2' + \lambda_2 = 0$

$I\ddot{\phi} = 0$

$2I\ddot{\psi} + \lambda_1 a\cos\phi + \lambda_2 a\,\mathrm{sen}\,\phi = 0$

→ $\ddot{\phi} = 0 \Rightarrow \phi(t) = \dot{\phi}_0 t$

→ $\ddot{x}_1' = a\ddot{\psi}\cos\phi - a\dot{\phi}\dot{\psi}\,\text{sen}\,\phi$
$\ddot{x}_2' = a\ddot{\psi}\,\text{sen}\,\phi + a\dot{\phi}\dot{\psi}\cos\phi$

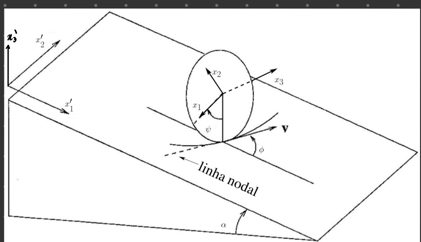
→ $\lambda_1 = M\ddot{x}_1' = Ma(\ddot{\psi}\cos\phi - \dot{\phi}_0\dot{\psi}\,\text{sen}\,\phi)$
$\lambda_2 = M\ddot{x}_2' = Ma(\ddot{\psi}\,\text{sen}\,\phi + \dot{\phi}_0\dot{\psi}\cos\phi)$

→ $2I\ddot{\psi} + Ma^2\ddot{\psi} = 0 \Rightarrow \ddot{\psi} = 0 \Rightarrow \psi(t) = \dot{\psi}_0 t$

→ $\dot{x}_1' = a\dot{\psi}\cos\phi = a\dot{\psi}_0\cos(\dot{\phi}_0 t)$
$\dot{x}_2' = a\dot{\psi}\,\text{sen}\,\phi = a\dot{\psi}_0\,\text{sen}(\dot{\phi}_0 t)$ $\overset{\dot{\phi}_0 \neq 0}{\Rightarrow}$ $\begin{cases} x_1'(t) = a\frac{\dot{\psi}_0}{\dot{\phi}_0}\,\text{sen}(\dot{\phi}_0 t) \\ x_2'(t) = a\frac{\dot{\psi}_0}{\dot{\phi}_0}[1 - \cos(\dot{\phi}_0 t)] \end{cases}$

$x_1'^2 + \left(x_2' - a\frac{\dot{\psi}_0}{\dot{\phi}_0}\right)^2 = \left(a\frac{\dot{\psi}_0}{\dot{\phi}_0}\right)^2$ →

CM faz circunferência



Tudo igual, exceto: $U = U_g \neq \text{cte}$
$U_g(x_1' = 0) = 0 \Rightarrow U_g = -Mg\,x_1'\,\text{sen}\,\alpha$

$\Rightarrow \boxed{L = \frac{1}{2}M(\dot{x}_1'^2 + \dot{x}_2'^2) + \frac{1}{2}I(\dot{\phi} + 2\dot{\psi}) + Mg\,x_1'\,\text{sen}\,\alpha}$

$\begin{cases} f_1 = \dot{x}_1' - a\dot{\psi}\cos\phi = 0 \\ f_2 = \dot{x}_2' - a\dot{\psi}\,\text{sen}\,\phi = 0 \end{cases}$

→ Equações E-L: $\begin{cases} Mg\,\text{sen}\,\alpha - M\ddot{x}_1' + \lambda_1 = 0 \\ -M\ddot{x}_2' + \lambda_2 = 0 \\ -I\ddot{\phi} = 0 \rightarrow \boxed{\phi(t) = \dot{\phi}_0 t} \\ 2I\ddot{\psi} + \lambda_1 a\cos\phi + \lambda_2 a\,\text{sen}\,\phi = 0 \end{cases}$

→ $\lambda_1 = Ma(\ddot{\psi}\cos\phi - \dot{\phi}_0\dot{\psi}\,\text{sen}\,\phi) - Mg\,\text{sen}\,\alpha$
→ $\lambda_2 = Ma(\ddot{\psi}\,\text{sen}\,\phi + \dot{\phi}_0\dot{\psi}\cos\phi)$

→ $\ddot{\psi} = \frac{2g}{3a}\,\text{sen}\,\alpha\cos(\dot{\phi}_0 t) \Rightarrow \psi(t) = \dot{\psi}_0 t + \frac{2g\,\text{sen}\,\alpha}{3a\dot{\phi}_0}[1 - \cos(\dot{\phi}_0 t)]$

→ $\dot{x}_1' = a\dot{\psi}\cos\psi = a\left[\dot{\psi}_0 + \frac{2g\,\text{sen}\,\alpha}{3a\dot{\phi}_0}\,\text{sen}(\dot{\phi}_0 t)\right]\cos(\dot{\phi}_0 t)$

→ $\dot{x}_2' = a\dot{\psi}\,\text{sen}\,\phi$

$\Rightarrow$ $x_1'(t) = [\beta + \gamma\,\text{sen}(\dot{\phi}_0 t)]\,\text{sen}(\dot{\phi}_0 t)$
$x_2'(t) = \beta[1 - \cos(\dot{\phi}_0 t)] + \gamma[\dot{\phi}_0 t - \text{sen}(2\dot{\phi}_0 t)]$
$\beta = a\frac{\dot{\psi}_0}{\dot{\phi}_0}$, $\gamma = \frac{g\,\text{sen}\,\alpha}{3\dot{\phi}_0^2}$

→ Se $\ddot{\phi}_0 \neq 0$ e $\dot{\psi}_0 = 0$ (disco solto do repouso) → $\beta = 0$

↳ $\left(x_1' - \frac{g\,\text{sen}\,\alpha}{6\dot{\phi}_0^2}\right)^2 + \left(x_2' - \frac{g\,\text{sen}\,\alpha}{3\dot{\phi}_0}t\right)^2 = \left(\frac{g\,\text{sen}\,\alpha}{6\dot{\phi}_0^2}\right)^2$

Centro: $\vec{r}_{centro} = \frac{g\,\text{sen}\,\alpha}{6\dot{\phi}_0^2}(\hat{e}_1' + \overset{\text{deriva}}{2\dot{\phi}_0 t}\,\hat{e}_2')$