Tarefa 1) expandir explicitamente o geopotencial U até o termo $n = 3$, separando os termos zonais dos termos tesserais e setoriais

• Substituições :

$\rightarrow C_{n0} = -J_n$ (1)

$\rightarrow \eta_m = \cos m\lambda \cos^m \varphi$ (2) artigo (2.57)

$\rightarrow i_m = \operatorname{sen} m\lambda \cos^m \varphi$ (3) artigo (2.58)

$\rightarrow A_{nm}(\operatorname{sen}\varphi) = \dfrac{P_{nm}(\operatorname{sen}\varphi)}{\cos^m \varphi}$ (4) artigo (2.55)

$\rightarrow C_{00} = 1$ (5) artigo (pag 16)

$\rightarrow C_{10} = C_{11} = S_{11} = S_{00} = S_{n0} = 0$ (6) artigo (pag 16)

. Potencial da Terra:

$$\rightarrow U = \frac{\mu}{r} - \frac{\mu}{r}\sum_{n=2}^{N}\left(\frac{a_e}{r}\right)^n J_n A_n(\operatorname{sen}\varphi) + \frac{\mu}{r}\sum_{n=2}^{N}\sum_{m=1}^{n}\left(\frac{a_e}{r}\right)^n (C_{nm} r_m + S_{nm} i_m) A_{nm}(\operatorname{sen}\varphi)$$

. Para $n=0$ e $m=0$:

$$\rightarrow U_{00} = \frac{\mu}{r} - \frac{\mu}{r}\left[\left(\frac{a_e}{r}\right)^0 J_0 A_0(\operatorname{sen}\varphi)\right] + \frac{\mu}{r}\left[\left(\frac{a_e}{r}\right)^0 (C_{00} r_0 + S_{00} i_0) A_{00}(\operatorname{sen}\varphi)\right]$$

$$U_{00} = \frac{\mu}{r} - \frac{\mu}{r}(J_0 A_0 \operatorname{sen}\varphi) + \frac{\mu}{r}\left[(C_{00} r_0 + S_{00} i_0) A_0 \operatorname{sen}\varphi\right]$$

. Substituindo (2) e (3), temos:

$$\rightarrow U_{00} = \frac{\mu}{r} - \frac{\mu}{r}(J_0 A_0 \operatorname{sen}\varphi) + \frac{\mu}{r}\left[(C_{00}\cos 0 \cos^0\varphi + \cancel{S_{00}\operatorname{sen} 0 \cos^0\varphi}) A_0 \operatorname{sen}\varphi\right]$$

$$U_{00} = \frac{\mu}{r} - \frac{\mu}{r}(J_0 A_0 \operatorname{sen}\varphi) + \frac{\mu}{r}(C_{00} A_0 \operatorname{sen}\varphi)$$

. Usando (1), temos:

$$\rightarrow U_{00} = \frac{\mu}{r} - 2\frac{\mu}{r}(J_0 A_0 \operatorname{sen}\varphi)$$

. Com (4), (1) e (5), obtemos $J_0 = 1$ e então:

$$\rightarrow U_{00} = \frac{\mu}{r} - 2\frac{\mu}{r}\,\frac{P_0 \operatorname{sen}\varphi}{\cos^0\varphi}$$

$$U_{00} = \frac{\mu}{r} - 2\frac{\mu}{r} P_0 \operatorname{sen}\varphi$$

Pela tabela 2.2 da pag 9 do artigo, temos:

$$\rightarrow \boxed{\cancel{U_{00} = -\frac{\mu}{r}}} \qquad (\mathrm{I})$$

Para $n=1$ e $m=0$:

$$\rightarrow U_{10} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r} J_1 A_1 \operatorname{sen}\varphi\right) + \frac{\mu}{r}\left[\frac{a_e}{r}\left(C_{10} r_0 + S_{10} i_0\right) A_{10} \operatorname{sen}\varphi\right]$$

Como já vimos de (2) e (3); $r_0 = 1$ e $i_0 = 0$, portanto:

$$\rightarrow U_{10} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r} J_1 A_1 \operatorname{sen}\varphi\right) + \frac{\mu}{r}\left(\frac{a_e}{r} C_{10} A_1 \operatorname{sen}\varphi\right)$$

Usando (1) e (6), temos que $C_{10} = J_1 = 0$, então

$$\rightarrow \boxed{\cancel{U_{10} = \frac{\mu}{r}}} \qquad (\mathrm{II})$$

Para $n=2$ e $m=0$

$$\rightarrow U_{20} = \cancel{\frac{\mu}{r}} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 A_2 \operatorname{sen}\varphi + \frac{\mu}{r}\left[\left(\frac{a_e}{r}\right)^2 \left(C_{20} r_0 + S_{20} i_0\right) A_{20} \operatorname{sen}\varphi\right]$$

Como $r_0 = 1$ e $i_0 = 0$:

$$\rightarrow U_{20} = \cancel{\frac{\mu}{r}} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 A_2 \operatorname{sen}\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 C_{20} A_2 \operatorname{sen}\varphi$$

De (1), obtemos:

$$\rightarrow U_{20} = \cancel{\frac{\mu}{r}} - 2\,\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 A_2 \operatorname{sen}\varphi$$

• Substituindo (4), temos que

$$\rightarrow U_{20} = \cancel{\frac{\mu}{r}} - 2\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \frac{P_2 \operatorname{sen}\varphi}{\cos^0\varphi}$$

$$U_{20} = \cancel{\frac{\mu}{r}} - 2\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 P_2 \operatorname{sen}\varphi$$

• Pela tabela 2.2 da pag 9 do artigo:

$$\rightarrow U_{20} = \cancel{\frac{\mu}{r}} - \cancel{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \frac{1}{\cancel{4}_{2}}\left(1 - 3\cos(2\varphi)\right)$$

$$U_{20} = \cancel{\frac{\mu}{r}} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \left(1 - 3\cos(2\varphi)\right)$$

• Temos a relação $\cos(2a) = 2\cos^2 a - 1$, portanto:

$$\cos^2\varphi = 1 - \operatorname{sen}^2\varphi$$

$$\rightarrow U_{20} = \cancel{\frac{\mu}{r}} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \left(1 - 3(2\cos^2\varphi - 1)\right)$$

$$U_{20} = \cancel{\frac{\mu}{r}} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \left(1 - 3(2 - 2\operatorname{sen}^2\varphi - 1)\right)$$

$$U_{20} = \cancel{\frac{\mu}{r}} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \left(1 - (3 - 6\operatorname{sen}^2\varphi)\right)$$

$$U_{20} = \cancel{\frac{\mu}{r}} - \frac{1}{\cancel{2}}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \left(\cancel{6}^{3}\operatorname{sen}^2\varphi - \cancel{2}\right)$$

$$\boxed{U_{20} = \cancel{\frac{\mu}{r}} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \left(3\operatorname{sen}^2\varphi - 1\right)} \qquad (III)$$

• Para $n=3$ e $m=0$

$$\rightarrow U_{30} = \cancel{\frac{\mu}{r}} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 A_3 \,\mathrm{sen}\,\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 \left(C_{30}\, r_0 + S_{30}\, i_0\right) A_{30} \,\mathrm{sen}\,\varphi$$

• Como $r_0 = 1$ e $i_0 = 0$, temos:

$$\rightarrow U_{30} = \cancel{\frac{\mu}{r}} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 A_3 \,\mathrm{sen}\,\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 C_{30} A_3 \,\mathrm{sen}\,\varphi$$

• De (1), obtemos:

$$\rightarrow U_{30} = \cancel{\frac{\mu}{r}} - 2\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 A_3 \,\mathrm{sen}\,\varphi$$

• Substituindo (4), temos que:

$$\rightarrow U_{30} = \cancel{\frac{\mu}{r}} - 2\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \frac{P_3 \,\mathrm{sen}\,\varphi}{\cos^0 \varphi}$$

$$U_{30} = \cancel{\frac{\mu}{r}} - 2\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 P_3 \,\mathrm{sen}\,\varphi$$

• Pela tabela 2.2 da pag 9 do artigo:

$$\rightarrow U_{30} = \cancel{\frac{\mu}{r}} - \cancel{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \frac{1}{\cancel{8}_{4}} \left(3 \,\mathrm{sen}\,\varphi - 5 \,\mathrm{sen}(3\varphi)\right)$$

$$U_{30} = \cancel{\frac{\mu}{r}} - \frac{1}{4}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \left(3 \,\mathrm{sen}\,\varphi - 5 \,\mathrm{sen}(3\varphi)\right)$$

• Temos a relação $\mathrm{sen}(3a) = (4\cos^2 a - 1)\,\mathrm{sen}\, a$, portanto:

$\cos^2\varphi = 1 - \mathrm{sen}^2\varphi$

$$\rightarrow U_{30} = \cancel{\frac{\mu}{r}} - \frac{1}{4}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \left[3 \,\mathrm{sen}\,\varphi - 5\left(4\cos^2\varphi - 1\right) \mathrm{sen}\,\varphi\right]$$

$$U_{30} = \cancel{\frac{\mu}{r}} - \frac{1}{4}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \left[3\,\text{sen}\,\varphi - 5\,(4 - 4\,\text{sen}^2\varphi - 1)\,\text{sen}\,\varphi\right]$$

$$U_{30} = \cancel{\frac{\mu}{r}} - \frac{1}{4}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \left(3\,\text{sen}\,\varphi - 15\,\text{sen}\,\varphi + 20\,\text{sen}^3\varphi\right)$$

$$U_{30} = \cancel{\frac{\mu}{r}} - \frac{1}{\cancel{4}}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \left(\cancel{20}^{\,5}\,\text{sen}^3\varphi - 1\cancel{2}^{\,3}\,\text{sen}\,\varphi\right)$$

$$\boxed{U_{30} = \cancel{\frac{\mu}{r}} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \left(5\,\text{sen}^3\varphi - 3\,\text{sen}\,\varphi\right)} \qquad (IV)$$

- <u>Soma dos zonais até $n=3$</u> :

$\rightarrow$ $U_z = U_{00} + U_{10} + U_{20} + U_{30}$ $\rightarrow$ zonais

$$U_z = \cancel{-\frac{\mu}{r} + \frac{\mu}{r}}\ \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2\left(3\,\text{sen}^2\varphi - 1\right) + \cancel{\frac{\mu}{r}} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3\left(5\,\text{sen}^3\varphi - 3\,\text{sen}\,\varphi\right)$$

$$\boxed{U_z = \frac{\mu}{r}\left[1 - \left(\frac{a_e}{r}\right)^2 \frac{J_2}{2}\left(3\,\text{sen}^2\varphi - 1\right) - \left(\frac{a_e}{r}\right)^3 \frac{J_3}{2}\left(5\,\text{sen}^3\varphi - 3\,\text{sen}\,\varphi\right)\right]}$$

. Para $n = 1$ e $m = 1$;

$$\rightarrow U_{11} = \frac{\mu}{r} - \frac{\mu}{r}\frac{a_e}{r} J_1 A_1 \operatorname{sen}\varphi + \frac{\mu}{r}\frac{a_e}{r}\left(C_{11} r_1 + S_{11} i_1\right) A_{11} \operatorname{sen}\varphi$$

. De (1) e (6), temos que $J_1 = C_{11} = S_{11} = 0$, portanto:

$$\rightarrow \boxed{U_{11} = \frac{\mu}{r}} \qquad (V)$$

. Para $n=2$ e $m=1$;

$$\rightarrow U_{21} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 A_2 \operatorname{sen}\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left(C_{21} r_1 + S_{21} i_1\right) A_{21} \operatorname{sen}\varphi$$

. De (2) e (3), temos:

$$\rightarrow U_{21} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 A_2 \operatorname{sen}\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left(C_{21} \cos\lambda \cos\varphi + S_{21} \operatorname{sen}\lambda \cos\varphi\right) A_{21} \operatorname{sen}\varphi$$

. Substituindo (4), temos que:

$$\rightarrow U_{21} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \frac{P_2 \operatorname{sen}\varphi}{\cos^0\varphi} + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left(C_{21} \cos\lambda \cancel{\cos\varphi} + S_{21} \operatorname{sen}\lambda \cancel{\cos\varphi}\right) \frac{P_{21} \operatorname{sen}\varphi}{\cancel{\cos\varphi}}$$

$$U_{21} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 P_2 \operatorname{sen}\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left(C_{21} \cos\lambda + S_{21} \operatorname{sen}\lambda\right) P_{21} \operatorname{sen}\varphi$$

→ Pela tabela 2.2 da pag. 9 do artigo:

$$\rightarrow U_{21} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \frac{1}{4}(1 - 3\cos(2\varphi)) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left(C_{21} \cos\lambda + S_{21} \operatorname{sen}\lambda\right) \frac{3}{2} \operatorname{sen}(2\varphi)$$

• Como já vimos, $\frac{1}{4}(1-3\cos(2\varphi)) = \frac{1}{2}(3\,\text{sen}^2\varphi-1)$, portanto

$$\rightarrow U_{21} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2\left(3\,\text{sen}^2\varphi-1\right) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2\left(C_{21}\cos\lambda + S_{21}\,\text{sen}\,\lambda\right)\frac{3}{2}\,\text{sen}(2\varphi)$$

• Temos a relação $\text{sen}(2a) = 2\cos a\,\text{sen}\,a$, portanto:

$$\rightarrow U_{21} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2\left(3\,\text{sen}^2\varphi-1\right) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2\left(C_{21}\cos\lambda + S_{21}\,\text{sen}\,\lambda\right)\frac{3}{\cancel{2}}\cdot\cancel{2}\cos\varphi\,\text{sen}\,\varphi$$

$$\boxed{U_{21} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2\left(3\,\text{sen}^2\varphi-1\right) + 3\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2\left(C_{21}\cos\lambda + S_{21}\,\text{sen}\,\lambda\right)\cos\varphi\,\text{sen}\,\varphi} \quad (VI)$$

• Para $n=3$ e $m=1$:

$$\rightarrow U_{31} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 A_3\,\text{sen}\,\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3\left(C_{31} r_1 + S_{31} r_1\right) A_{31}\,\text{sen}\,\varphi$$

• De (2), (3) e (4), temos

$$\rightarrow U_{31} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \frac{P_3\,\text{sen}\,\varphi}{\cos^0\varphi} + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3\left(C_{31}\cos\lambda\cancel{\cos\varphi} + S_{31}\,\text{sen}\,\lambda\cancel{\cos\varphi}\right)\frac{P_{31}\,\text{sen}\,\varphi}{\cancel{\cos\varphi}}$$

$$U_{31} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 P_3\,\text{sen}\,\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3\left(C_{31}\cos\lambda + S_{31}\,\text{sen}\,\lambda\right) P_{31}\,\text{sen}\,\varphi$$

• Pela tabela 2.2 da pag. 9 do artigo:

$$\rightarrow U_{31} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3\,\frac{1}{8}\left(3\,\text{sen}\,\varphi - 5\,\text{sen}(3\varphi)\right) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3\left(C_{31}\cos\lambda + S_{31}\,\text{sen}\,\lambda\right)\frac{3}{8}\left(\cos\varphi - 5\cos(3\varphi)\right)$$

• Como já vimos, $\frac{1}{8}(3\,\text{sen}\,\varphi - 5\,\text{sen}(3\varphi)) = \frac{1}{2}(5\,\text{sen}^3\varphi - 3\,\text{sen}\,\varphi)$, portanto:

$$\rightarrow U_{31} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3\left(5\,\text{sen}^3\varphi - 3\,\text{sen}\,\varphi\right) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3\left(C_{31}\cos\lambda + S_{31}\,\text{sen}\,\lambda\right)\frac{3}{8}\left(\cos\varphi - 5\cos(3\varphi)\right)$$

. Temos a relação $\cos(3\varphi) = 4\cos^3\varphi - 3\cos\varphi$, portanto:

$$\rightarrow U_{31} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 (5\,sen^3\varphi - 3\,sen\varphi) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (C_{31}\cos\lambda + S_{31}\,sen\,\lambda)\,\frac{3}{8}\left(\cos\varphi - 5\left(4\cos^3\varphi - 3\cos\varphi\right)\right)$$

$$\boxed{U_{31} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 (5\,sen^3\varphi - 3\,sen\varphi) + \frac{3}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (C_{31}\cos\lambda + S_{31}\,sen\lambda)\left(4\cos\varphi - 5\cos^3\varphi\right)} \quad (VII)$$

. Para $n=2$ e $m=2$:

$$\rightarrow U_{22} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 A_2\,sen\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 (C_{22}\,r_2 + S_{22}\,i_2)\,A_{22}\,sen\varphi$$

. De (2), (3) e (4), temos:

$$\rightarrow U_{22} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \frac{P_2\,sen\varphi}{\cos^0\varphi} + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left(C_{22}\cos(2\lambda)\cancel{\cos^2\varphi} + S_{22}\,sen(2\lambda)\cancel{\cos^2\varphi}\right)\frac{P_{22}\,sen\varphi}{\cancel{\cos^2\varphi}}$$

$$U_{22} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 P_2\,sen\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left(C_{22}\cos(2\lambda) + S_{22}\,sen(2\lambda)\right) P_{22}\,sen\varphi$$

. Temos a relação $\cos(2a) = 2\cos^2 a - 1$ e $sen(2a) = 2\cos a\,sen\,a$, portanto:

$$\rightarrow U_{22} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 P_2\,sen\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left[C_{22}(2\cos^2\lambda - 1) + 2S_{22}\cos\lambda\,sen\lambda\right] P_{22}\,sen\varphi$$

$$U_{22} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 P_2\,sen\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left(2C_{22}\cos^2\lambda - C_{22} + 2S_{22}\cos\lambda\,sen\lambda\right) P_{22}\,sen\varphi$$

. Pela tabela 2.2 da pag 9 do artigo temos:

$$\rightarrow U_{22} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 \frac{1}{4}(1 - 3\cos 2\varphi) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 \left(2C_{22}\cos^2\lambda - C_{22} + 2S_{22}\cos\lambda\,sen\lambda\right)\frac{3}{2}(1 + \cos 2\varphi)$$

• Como já vimos, $\frac{1}{4}(1-3\cos(2\varphi)) = \frac{1}{2}(3\,\text{sen}^2\varphi - 1)$, portanto

$$\to U_{22} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2(3\,\text{sen}^2\varphi - 1) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 (2C_{22}\cos^2\lambda - C_{22} + 2S_{22}\cos\lambda\,\text{sen}\,\lambda)\frac{3}{2}(1+\cos(2\varphi))$$

• Temos a relação $\cos(2a) = 2\cos^2 a - 1$, portanto:

$$\to U_{22} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2(3\,\text{sen}^2\varphi - 1) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 (2C_{22}\cos^2\lambda - C_{22} + 2S_{22}\cos\lambda\,\text{sen}\,\lambda)\frac{3}{2}(1+(2\cos^2\varphi - 1))$$

$$\boxed{U_{22} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2(3\,\text{sen}^2\varphi - 1) + 3\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 (2C_{22}\cos^2\lambda - C_{22} + 2S_{22}\cos\lambda\,\text{sen}\,\lambda)\cos^2\varphi} \quad (VIII)$$

• Para $n = 3$ e $m = 2$:

$$\to U_{32} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 A_3\,\text{sen}\,\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (C_{32}R_2 + S_{32}I_2)A_{32}\,\text{sen}\,\varphi$$

• De (2), (3) e (4), temos:

$$\to U_{32} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3\frac{P_3\,\text{sen}\,\varphi}{\cos^0\varphi} + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (C_{22}\cos(2\lambda)\cancel{\cos^2\varphi} + S_{22}\,\text{sen}(2\lambda)\cancel{\cos^2\varphi})\frac{P_{32}\,\text{sen}\,\varphi}{\cancel{\cos^2\varphi}}$$

$$U_{32} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 P_3\,\text{sen}\,\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (2C_{32}\cos^2\lambda - C_{32} + 2S_{32}\cos\lambda\,\text{sen}\,\lambda)P_{32}\,\text{sen}\,\varphi$$

• Pela tabela 2.2 da pag 9 do artigo, temos:

$$\to U_{32} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3\frac{1}{8}(3\,\text{sen}\,\varphi - 5\,\text{sen}(3\varphi)) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (2C_{32}\cos^2\lambda - C_{32} + 2S_{32}\cos\lambda\,\text{sen}\,\lambda)\frac{15}{4}(\text{sen}\,\varphi + \text{sen}(3\varphi))$$

• Como já vimos, $\frac{1}{8}(3\,\text{sen}\,\varphi - 5\,\text{sen}(3\varphi)) = \frac{1}{2}(5\,\text{sen}^3\varphi - 3\,\text{sen}\,\varphi)$, portanto:

$$\to U_{32} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3(5\,\text{sen}^3\varphi - 3\,\text{sen}\,\varphi) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (2C_{32}\cos^2\lambda - C_{32} + 2S_{32}\cos\lambda\,\text{sen}\,\lambda)\frac{15}{4}(\text{sen}\,\varphi + \text{sen}(3\varphi))$$

. Temos a relação $\operatorname{sen} 3a = (4\cos^2 a - 1)\operatorname{sen} a$, portanto:

$$\rightarrow U_{32} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{2}\left(\frac{a_e}{r}\right)^3 J_3(5\operatorname{sen}^3\varphi - 3\operatorname{sen}\varphi) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (2C_{32}\cos^2\lambda - C_{32} + 2S_{32}\cos\lambda\operatorname{sen}\lambda)\frac{15}{4}\left(\operatorname{sen}\varphi + \left[(4\cos^2\varphi - 1)\operatorname{sen}\varphi\right]\right)$$

. Calculando, $\frac{15}{4}\left(\operatorname{sen}\varphi + \left[(4\cos^2\varphi - 1)\operatorname{sen}\varphi\right]\right) = 15(\operatorname{sen}\varphi - \operatorname{sen}^3\varphi)$, portanto

$$\rightarrow \boxed{U_{32} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3(5\operatorname{sen}^3\varphi - 3\operatorname{sen}\varphi) + 15\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3(2C_{32}\cos^2\lambda - C_{32} + 2S_{32}\cos\lambda\operatorname{sen}\lambda)(\operatorname{sen}\varphi - \operatorname{sen}^3\varphi)} \quad (IX)$$

. Para $n=3$ e $m=3$:

$$\rightarrow U_{33} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 A_3 \operatorname{sen}\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (C_{33} r_3 + S_{33} i_{33}) A_{33} \operatorname{sen}\varphi$$

. De (2), (3) e (4), temos:

$$\rightarrow U_{33} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \frac{P_3 \operatorname{sen}\varphi}{\cos^0\varphi} + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (C_{33}\cos(3\lambda)\cancel{\cos^3\varphi} + S_{33}\operatorname{sen}(3\lambda)\cancel{\cos^3\varphi}) \frac{P_{33}\operatorname{sen}\varphi}{\cancel{\cos^3\varphi}}$$

. Temos a relação $\cos(3a) = 4\cos^3 a - 3\cos a$ e $\operatorname{sen}(3a) = (4\cos^2 a - 1)\operatorname{sen} a$, portanto:

$$\rightarrow U_{33} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 P_3 \operatorname{sen}\varphi + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (4C_{33}\cos^3\lambda - 3C_{33}\cos\lambda + 3S_{33}\operatorname{sen}\lambda - 4S_{33}\operatorname{sen}^3\lambda) P_{33}\operatorname{sen}\varphi$$

. Pela tabela 2.2 da pag 9 do artigo, temos:

$$\rightarrow U_{33} = \frac{\mu}{r} - \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 \frac{1}{8}(3\operatorname{sen}\varphi - 5\operatorname{sen}(3\varphi)) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (4C_{33}\cos^3\lambda - 3C_{33}\cos\lambda + 3S_{33}\operatorname{sen}\lambda - 4S_{33}\operatorname{sen}^3\lambda)\frac{15}{4}(3\cos\varphi + \cos(3\varphi))$$

. Como já vimos $\frac{1}{8}(3\operatorname{sen}\varphi - 5\operatorname{sen}(3\varphi)) = \frac{1}{2}(5\operatorname{sen}^3\varphi - 3\operatorname{sen}\varphi)$ portanto:

$$\rightarrow U_{33} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3(5\operatorname{sen}^3\varphi - 3\operatorname{sen}\varphi) + \frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (4C_{33}\cos^3\lambda - 3C_{33}\cos\lambda + 3S_{33}\operatorname{sen}\lambda - 4S_{33}\operatorname{sen}^3\lambda)\frac{15}{4}(3\cos\varphi + \cos(3\varphi))$$

Temos a relação $\cos(3a) = 4\cos^3 a - 3\cos a$, substituindo em $\frac{15}{4}(3\cos\varphi + \cos(3\varphi))$ e calculando, temos que é igual a $15\cos^3\varphi$, portanto:

$$\rightarrow \boxed{U_{33} = \frac{\mu}{r} - \frac{1}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 J_3 (5\,\mathrm{sen}^3\varphi - 3\,\mathrm{sen}\,\varphi) + 15\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (4C_{33}\cos^3\lambda - 3C_{33}\cos\lambda + 3S_{33}\,\mathrm{sen}\,\lambda - 4S_{33}\,\mathrm{sen}^3\lambda)\cos^3\varphi} \quad (X)$$

. Soma dos tesserais até $n = 3$:

$$\rightarrow U_T = U_{11} + U_{21} + U_{31} + U_{22} + U_{32} + U_{33} \quad \rightarrow \text{tesserais}$$

$$U_T = \frac{\mu}{r} + \cancel{\frac{\mu}{r}} - \frac{1}{2}\frac{\mu}{r}\cancel{\left(\frac{a_e}{r}\right)^2 J_2}(3\,\mathrm{sen}^2\varphi - 1) + 3\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 (C_{21}\cos\lambda + S_{21}\,\mathrm{sen}\,\lambda)\cos\varphi\,\mathrm{sen}\,\varphi + \ldots$$

$$\ldots + \cancel{\frac{\mu}{r}} - \frac{1}{2}\frac{\mu}{r}\cancel{\left(\frac{a_e}{r}\right)^3 J_3}(5\,\mathrm{sen}^3\varphi - 3\,\mathrm{sen}\,\varphi) + \frac{3}{2}\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (C_{31}\cos\lambda + S_{31}\,\mathrm{sen}\,\lambda)(4\cos\varphi - 5\cos^3\varphi) + \ldots$$

$$\ldots + \cancel{\frac{\mu}{r}} - \frac{1}{2}\frac{\mu}{r}\cancel{\left(\frac{a_e}{r}\right)^2 J_2}(3\,\mathrm{sen}^2\varphi - 1) + 3\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 (2C_{22}\cos^2\lambda - C_{22} + 2S_{22}\cos\lambda\,\mathrm{sen}\,\lambda)\cos^2\varphi + \ldots$$

$$\ldots + \cancel{\frac{\mu}{r}} - \frac{1}{2}\frac{\mu}{r}\cancel{\left(\frac{a_e}{r}\right)^3 J_3}(5\,\mathrm{sen}^3\varphi - 3\,\mathrm{sen}\,\varphi) + 15\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (2C_{32}\cos^2\lambda - C_{32} + 2S_{32}\cos\lambda\,\mathrm{sen}\,\lambda)(\mathrm{sen}\,\varphi - \mathrm{sen}^3\varphi) + \ldots$$

$$\ldots + \cancel{\frac{\mu}{r}} - \frac{1}{2}\frac{\mu}{r}\cancel{\left(\frac{a_e}{r}\right)^3 J_3}(5\,\mathrm{sen}^3\varphi - 3\,\mathrm{sen}\,\varphi) + 15\frac{\mu}{r}\left(\frac{a_e}{r}\right)^3 (4C_{33}\cos^3\lambda - 3C_{33}\cos\lambda + 3S_{33}\,\mathrm{sen}\,\lambda - 4S_{33}\,\mathrm{sen}^3\lambda)\cos^3\varphi$$

$$U_T = \frac{\mu}{r}\left[1 - \left(\frac{a_e}{r}\right)^2 \cancel{J_2}(3\,\mathrm{sen}^2\varphi - 1) - \frac{3}{2}\left(\frac{a_e}{r}\right)^3 \cancel{J_3}(5\,\mathrm{sen}^3\varphi - 3\,\mathrm{sen}\,\varphi) + \dots\right.$$

$$\dots + 3\left(\frac{a_e}{r}\right)^2 (C_{21}\cos\lambda + S_{21}\,\mathrm{sen}\,\lambda)\cos\varphi\,\mathrm{sen}\,\varphi + \frac{3}{2}\left(\frac{a_e}{r}\right)^3 (C_{31}\cos\lambda + S_{31}\,\mathrm{sen}\,\lambda)(4\cos\varphi - 5\cos^3\varphi) + \dots$$

$$\dots + 3\left(\frac{a_e}{r}\right)^2 (2C_{22}\cos^2\lambda - C_{22} + 2S_{22}\cos\lambda\,\mathrm{sen}\,\lambda)\cos^2\varphi + 15\left(\frac{a_e}{r}\right)^3 (2C_{32}\cos^2\lambda - C_{32} + 2S_{32}\cos\lambda\,\mathrm{sen}\,\lambda)(\mathrm{sen}\,\varphi - \mathrm{sen}^3\varphi) + \dots$$

$$\dots + 15\left(\frac{a_e}{r}\right)^3 (4C_{33}\cos^3\lambda - 3C_{33}\cos\lambda + 3S_{33}\,\mathrm{sen}\,\lambda - 4S_{33}\,\mathrm{sen}^3\lambda)\cos^3\varphi$$

2) Calcular o valor do potencial para $X = Y = Z = a_e$

. a) Para $X = a_e$, $Y = 0$ e $Z = 0$, observando a fig 2.10 do artigo, temos:

$\rightarrow r = a_e$	$\rightarrow \mathrm{sen}\,\lambda = 0$	$\rightarrow \mathrm{sen}\,\varphi = 0$	(a)
	$\rightarrow \cos\lambda = 1$	$\rightarrow \cos\varphi = 1$	

. b) Para $X = 0$, $Y = a_e$ e $Z = 0$, temos:

$\rightarrow r = a_e$	$\rightarrow \mathrm{sen}\,\lambda = 1$	$\rightarrow \mathrm{sen}\,\varphi = 0$	(b)
	$\rightarrow \cos\lambda = 0$	$\rightarrow \cos\varphi = 1$	

. c) Para $X = 0$, $Y = 0$ e $Z = a_e$, temos

$\rightarrow r = a_e$	$\rightarrow \mathrm{sen}\,\lambda = 0$	$\rightarrow \mathrm{sen}\,\varphi = 1$	(c)
	$\rightarrow \cos\lambda = 0$	$\rightarrow \cos\varphi = 0$	

• d) Para $X = a_e$, $Y = a_e$ e $Z = 0$, temos

$$\rightarrow r = \sqrt{2}\,a_e \quad \left|\begin{array}{l} \rightarrow \operatorname{sen}\lambda = \dfrac{\sqrt{2}}{2} \\ \rightarrow \cos\lambda = \dfrac{\sqrt{2}}{2} \end{array}\right| \begin{array}{l} \rightarrow \operatorname{sen}\psi = 0 \\ \rightarrow \cos\psi = 1 \end{array} \qquad (d)$$

• e) Para $X = a_e$, $Y = 0$ e $Z = a_e$, temos:

$$\rightarrow r = \sqrt{2}\,a_e \quad \left|\begin{array}{l} \rightarrow \operatorname{sen}\lambda = 0 \\ \rightarrow \cos\lambda = 1 \end{array}\right| \begin{array}{l} \rightarrow \operatorname{sen}\psi = \dfrac{\sqrt{2}}{2} \\ \rightarrow \cos\psi = \dfrac{\sqrt{2}}{2} \end{array} \qquad (e)$$

• f) Para $X = 0$, $Y = a_e$ e $Z = a_e$, temos:

$$\rightarrow r = \sqrt{2}\,a_e \quad \left|\begin{array}{l} \rightarrow \operatorname{sen}\lambda = 1 \\ \rightarrow \cos\lambda = 0 \end{array}\right| \begin{array}{l} \rightarrow \operatorname{sen}\psi = \dfrac{\sqrt{2}}{2} \\ \rightarrow \cos\psi = \dfrac{\sqrt{2}}{2} \end{array} \qquad (f)$$

• g) Para $X = a_e$, $Y = a_e$, $Z = a_e$, temos:

$$\rightarrow r = \sqrt{3}\,a_e \quad \left|\begin{array}{l} \rightarrow \operatorname{sen}\lambda = \dfrac{\sqrt{2}}{2} \\ \rightarrow \cos\lambda = \dfrac{\sqrt{2}}{2} \end{array}\right| \begin{array}{l} \rightarrow \operatorname{sen}\psi = \dfrac{\sqrt{3}}{3} \\ \rightarrow \cos\psi = \dfrac{\sqrt{6}}{3} \end{array} \qquad (g)$$

• I) Zonais:

↓

a) Potencial zonal para $X = a_e$ $(Y = Z = 0)$

. Usando as relações de (a) na expressão de U_z, temos:

$$\rightarrow U_z = \frac{\mu}{a_e}\left[1 - \left(\frac{a_e}{a_e}\right)^2 J_2(3.0^2 - 1) - \left(\frac{a_e}{a_e}\right)^3 J_3(5.0^3 - 3.0)\right]$$

$$\boxed{U_{z_a} = \frac{\mu}{a_e}(1 + J_2)} \rightarrow X = a_e$$

b) Potencial zonal para $Y = a_e$ $(X = Z = 0)$

. Usando as relações de (b) na expressão de U_z, temos:

$$\rightarrow U_{z_b} = \frac{\mu}{a_e}\left[1 - \left(\frac{a_e}{a_e}\right)^2 J_2(3.0^2 - 1) - \left(\frac{a_e}{a_e}\right)^3 J_3(5.0^3 - 3.0)\right]$$

$$\boxed{U_{z_b} = \frac{\mu}{a_e}(1 + J_2)} \rightarrow Y = a_e$$

c) Potencial zonal para $Z = a_e$ $(X = Y = 0)$

. Usando as relações de (c) na expressão de U_z, temos:

$$\rightarrow U_{z_c} = \frac{\mu}{a_e}\left[1 - \left(\frac{a_e}{a_e}\right)^2 J_2(3.1^2 - 1) - \left(\frac{a_e}{a_e}\right)^3 J_3(5.1^3 - 3.1)\right]$$

$$\boxed{U_{z_c} = \frac{\mu}{a_e}(1 - 2J_2 - 2J_3)} \rightarrow Z = a_e$$

d) Potencial zonal para $X = a_e$ e $Y = a_e$ $(Z=0)$

. Usando as relações de (d) na expressão de U_z, temos:

$$\rightarrow U_{zd} = \frac{\mu}{\sqrt{2}\,a_e}\left[1 - \left(\frac{\cancel{a_e}}{\sqrt{2}\,\cancel{a_e}}\right)^2 J_2\,(3.0^2 - 1) - \left(\frac{\cancel{a_e}}{\sqrt{2}\,\cancel{a_e}}\right)^3 J_3\,(5.0^3 - 3.0)\right]$$

$$\boxed{U_{zd} = \frac{\sqrt{2}\,\mu}{2a_e}\left(1 + \frac{J_2}{2}\right)} \rightarrow X = a_e \text{ e } Y = a_e$$

e) Potencial zonal para $X = a_e$ e $Z = a_e$ $(Y = 0)$

. Usando as relações de (e) na expressão de U_z, temos:

$$\rightarrow U_{ze} = \frac{\mu}{\sqrt{2}\,a_e}\left[1 - \left(\frac{\cancel{a_e}}{\sqrt{2}\,\cancel{a_e}}\right)^2 J_2\left(3.\left(\frac{\sqrt{2}}{2}\right)^2 - 1\right) - \left(\frac{\cancel{a_e}}{\sqrt{2}\,\cancel{a_e}}\right)^3 J_3\left(5.\left(\frac{\sqrt{2}}{2}\right)^3 - 3.\left(\frac{\sqrt{2}}{2}\right)\right)\right]$$

$$U_{ze} = \frac{\sqrt{2}\,\mu}{2a_e}\left[1 - \left(\frac{1}{2}\right) J_2\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{4}\right) J_3\left(5\,\frac{\sqrt{2}}{4} - 3\,\frac{\sqrt{2}}{2}\right)\right]$$

$$\boxed{U_{ze} = \frac{\sqrt{2}\,\mu}{2a_e}\left(1 - \frac{1}{4}J_2 + \frac{1}{8}J_3\right)} \rightarrow X = a_e \text{ e } Z = a_e$$

f) Potencial zonal para $Y = a_e$ e $Z = a_e$ $(X = 0)$

. Usando as relações de (f) na expressão de U_z, temos:

$$\rightarrow U_{zf} = \frac{\mu}{\sqrt{2}\,a_e}\left[1 - \left(\frac{\cancel{a_e}}{\sqrt{2}\,\cancel{a_e}}\right)^2 J_2\left(3\left(\frac{\sqrt{2}}{2}\right)^2 - 1\right) - \left(\frac{a_e}{\sqrt{2}\,a_e}\right)^3 J_3\left(5\left(\frac{\sqrt{2}}{2}\right)^3 - 3.\frac{\sqrt{2}}{2}\right)\right]$$

$$\boxed{U_{zf} = \frac{\sqrt{2}\,\mu}{2a_e}\left(1 - \frac{1}{4}J_2 + \frac{1}{8}J_3\right)} \rightarrow Y = a_e \text{ e } Z = a_e$$

g) Potencial zonal para $X = a_e$, $Y = a_e$ e $Z = a_e$

. Usando as relações de (g) na expressão de U_Z, temos:

$$\rightarrow U_{Z_g} = \frac{\mu}{\sqrt{3}\,a_e}\left[1 - \left(\frac{\cancel{a_e}}{\sqrt{3}\,\cancel{a_e}}\right)^2 J_2 \left(3 \cdot \left(\frac{\sqrt{3}}{3}\right)^2 - 1\right) - \left(\frac{\cancel{a_e}}{\sqrt{3}\,\cancel{a_e}}\right)^3 J_3 \left(5\left(\frac{\sqrt{3}}{3}\right)^3 - 3\left(\frac{\sqrt{3}}{3}\right)\right)\right]$$

$$U_{Z_g} = \frac{\sqrt{3}\,\mu}{3\,a_e}\left[1 - \left(\frac{1}{3}\right)^2 J_2 \,\cancel{(1-1)} - \left(\frac{\sqrt{3}}{9}\right) J_3 \left(5\frac{\sqrt{3}}{9} - \sqrt{3}\right)\right.$$

$$\boxed{U_{Z_g} = \frac{\sqrt{3}\,\mu}{3\,a_e}\left(1 - \frac{4}{27} J_3\right)} \rightarrow X = a_e,\ Y = a_e \text{ e } Z = a_e$$

II) Tesserais:

a) Potencial tesseral para $X = a_e$ $(Y = Z = 0)$

. Usando as relações (a) na expressão de U_T, temos:

$$\rightarrow \boxed{U_{T_a} = \frac{\mu}{a_e}\left(6 + J_2 - \frac{3}{2} C_{31} + 3 C_{22} + 15 C_{33}\right)} \rightarrow X = a_e$$

b) Potencial tesseral para $Y = a_e$ $(X = Z = 0)$

. Usando as relações de (b) na expressão de U_T, temos:

$$\rightarrow \boxed{U_{T_b} = \frac{\mu}{a_e}\left(6 + J_2 - \frac{3}{2} S_{31} - 3 C_{22} - 15 S_{33}\right)} \rightarrow Y = a_e$$

c) Potencial tesseral para $Z = a_e$ $(x = y = 0)$

· Usando as relações de (c) na expressão de U_T, temos:

$$\rightarrow \boxed{U_{T_c} = \frac{\mu}{a_e}\left(6 - 2J_2 - 3J_3\right)} \rightarrow Z = a_e$$

d) Potencial tesseral para $X = a_e$ e $Y = a_e$ $(z=0)$

· Usando as relações de (d) na expressão de U_T, temos:

$$\rightarrow U_{T_d} = \frac{\sqrt{2}\,\mu}{2\,a_e}\left[6 + \frac{J_2}{2} - \frac{3}{2}\frac{\sqrt{2}}{4}\left(C_{31}\frac{\sqrt{2}}{2} + S_{31}\frac{\sqrt{2}}{2}\right) + 3\,\frac{1}{2}\left(\cancel{2}C_{22}\frac{1}{\cancel{2}} - C_{22} + \cancel{2}S_{22}\frac{1}{\cancel{2}}\right) + \ldots\right.$$

$$\left.\ldots + 15\frac{\sqrt{2}}{4}\left(\cancel{4}C_{33}\frac{\sqrt{2}}{\cancel{4}} - 3C_{33}\frac{\sqrt{2}}{2} + 3S_{33}\frac{\sqrt{2}}{2} - \cancel{4}S_{33}\frac{\sqrt{2}}{\cancel{4}}\right)\right]$$

$$\boxed{U_{T_d} = \frac{\sqrt{2}\,\mu}{2\,a_e}\left(6 + \frac{J_2}{2} - \frac{3}{8}C_{31} - \frac{3}{8}S_{31} + \frac{3}{2}S_{22} - \frac{15}{4}C_{33} + \frac{15}{4}S_{33}\right)} \rightarrow X = a_e \text{ e } Y = a_e$$

e) Potencial tesseral para $X = a_e$ e $Z = a_e$ $(Y=0)$

· Usando as relações de (e) na expressão de U_T, temos:

$$\rightarrow U_{T_e} = \frac{\sqrt{2}\,\mu}{2\,a_e}\left[6 - \frac{1}{4}J_2 - \frac{3}{2}\frac{\sqrt{2}}{4}J_3\left(5\frac{\sqrt{2}}{4} - 3\frac{\sqrt{2}}{2}\right) + 3\,\frac{1}{2}C_{21}\frac{1}{2} + \ldots\right.$$

$$\ldots + \frac{3}{2}\frac{\sqrt{2}}{4}C_{31}\left(\cancel{4}^{2}\frac{\sqrt{2}}{\cancel{2}} - 5\frac{\sqrt{2}}{4}\right) + 3\cdot\frac{1}{2}C_{22}\frac{1}{2} + 15\frac{\sqrt{2}}{4}C_{32}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right) + \ldots$$

$$\left.\ldots + 15\frac{\sqrt{2}}{4}C_{33}\frac{\sqrt{2}}{4}\right]$$

$$\boxed{U_{T_e} = \frac{\sqrt{2}\,\mu}{2\,a_e}\left(6 - \frac{J_2}{4} + \frac{3}{16}J_3 + \frac{3}{4}C_{21} + \frac{9}{16}C_{31} + \frac{3}{4}C_{22} + \frac{15}{8}C_{32} + \frac{15}{8}C_{33}\right)}$$

$\hookrightarrow X = a_e$ e $Z = a_e$

f) Potencial tesseral para $Y = a_e$ e $Z = a_e$ $(X = 0)$

· Usando as relações de (f) na expressão de U_T, temos:

$$\rightarrow U_{T_f} = \frac{\sqrt{2}\,\mu}{2\,a_e}\left[6 - \frac{1}{4}J_2 - \frac{3}{2}\frac{\sqrt{2}}{4}J_3\left(5\frac{\sqrt{2}}{4} - 3\frac{\sqrt{2}}{2}\right) + 3\,\frac{1}{2}S_{21}\frac{1}{2} + \dots\right.$$

$$\dots + \frac{3}{2}\frac{\sqrt{2}}{4}S_{31}\left(4\frac{\sqrt{2}}{2} - 5\frac{\sqrt{2}}{4}\right) - 3\,\frac{1}{2}C_{22}\frac{1}{2} - 15\frac{\sqrt{2}}{4}C_{32}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right) + \dots$$

$$\left.\dots - 15\frac{\sqrt{2}}{4}S_{33}\frac{\sqrt{2}}{4}\right]$$

$$\boxed{U_{T_f} = \frac{\sqrt{2}\,\mu}{2\,a_e}\left(6 - \frac{J_2}{4} + \frac{3}{16}J_3 + \frac{3}{4}S_{21} + \frac{9}{16}S_{31} - \frac{3}{4}C_{22} - \frac{15}{8}C_{32} - \frac{15}{8}S_{33}\right)}$$

$\hookrightarrow Y = a_e$ e $Z = a_e$

g) Potencial tesseral para $X = a_e$, $Y = a_e$ e $Z = a_e$

· Usando as relações de (g) na expressão de U_T, temos:

$$\rightarrow U_{T_g} = \frac{\sqrt{3}\,\mu}{3\,a_e}\left[6 - 0 - \frac{3}{2}\frac{\sqrt{3}}{9}J_3\left(5\frac{\sqrt{3}}{9} - 3\frac{\sqrt{3}}{3}\right) + \cancel{3}\frac{1}{\cancel{3}}\left(C_{21}\frac{\sqrt{2}}{2} + S_{21}\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{3} + \dots\right.$$

$$\dots + \frac{\cancel{3}}{2}\frac{\sqrt{3}}{\cancelto{3}{9}}\left(C_{31}\frac{\sqrt{2}}{2} + S_{31}\frac{\sqrt{2}}{2}\right)\left(4\frac{\sqrt{6}}{3} - 5\cdot\frac{2\sqrt{6}}{9}\right) + \cancel{3}\frac{1}{\cancel{3}}\left(\cancel{2}\cancel{C_{22}}\frac{1}{\cancel{2}} - \cancel{C_{22}} + \cancel{2}S_{22}\frac{1}{\cancel{2}}\right)\frac{2}{3} + \dots$$

$$\left.\dots + \cancelto{5}{15}\frac{\sqrt{3}}{\cancelto{3}{9}}\left(\cancel{2}\cancel{C_{32}}\frac{1}{\cancel{2}} - \cancel{C_{32}} + \cancel{2}S_{32}\frac{1}{\cancel{2}}\right)\left(\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{9}\right) + \cancelto{5}{15}\frac{\sqrt{3}}{\cancelto{3}{9}}\left(\cancel{4}C_{33}\frac{\sqrt{2}}{\cancel{4}} - 3C_{33}\frac{\sqrt{2}}{2} + 3S_{33}\frac{\sqrt{2}}{2} - \cancel{4}S_{33}\frac{\sqrt{2}}{\cancel{4}}\right)\frac{2\sqrt{6}}{9}\right.$$

$$\boxed{U_{T_g} = \frac{\sqrt{3}\,\mu}{3a_e}\left(6 + \frac{2}{9}J_3 + \frac{1}{3}C_{21} + \frac{1}{3}S_{21} + \frac{1}{9}C_{31} + \frac{1}{9}S_{31} + \frac{2}{3}S_{22} + \frac{10}{9}S_{32} - \frac{10}{9}C_{33} + \frac{10}{9}S_{33}\right)}$$

$\hookrightarrow X = a_e, Y = a_e$ e $Z = a_e$