Segundo Trabalho de Cálculo II

Vinícius Menezes Monte

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Questões

I. Se $f : [a, b] \longrightarrow \mathbb{R}$ **possui derivada integrável, ponha** $m = \dfrac{a+b}{2}$ **e prove que:**

$$f(a) + f(b) = \frac{2}{b-a} \int_a^b [f(x) + (x-m) \cdot f'(x)]\, dx$$

II. Sejam m **e** n **números naturais diferentes de zero. Verifique que:**

$$\int_0^1 x^n \cdot (1-x)^m\, dx = \frac{m}{n+1} \cdot \int_0^1 x^{n+1} \cdot (1-x)^{m-1}\, dx$$

III. Verifique que, $\forall n \in \mathbb{N}$ **e** $s > 0$**, vale:**

$$\int t^n \cdot e^{-st}\, dt = -\frac{1}{s} \cdot t^n \cdot e^{-st} + \frac{n}{s} \cdot \int t^{n-1} \cdot e^{-st}\, dt + k$$

IV. Seja $n \geq 2$**, mostre que:**

$$(a) \quad \int \sin^n x\, dx = -\frac{1}{n} \cdot \sin^{n-1}(x) \cdot \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x)\, dx$$

$$(b) \quad \int \cos^n x\, dx = \frac{1}{n} \cdot \cos^{n-1}(x) \cdot \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x)\, dx$$

V. Mostre que $\tanh^{-1}(x) = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$

Soluções

I. Se $f : [a,b] \longrightarrow \mathbb{R}$ possui derivada integrável, ponha $m = \dfrac{a+b}{2}$ e prove que:

$$f(a) + f(b) = \frac{2}{b-a} \int_a^b [f(x) + (x-m) \cdot f'(x)]\, dx$$

O operador $\int_a^b$ é linear, então:

$$\begin{aligned}\int_a^b [f(x) + (x-m) \cdot f'(x)]\, dx &= \int_a^b [f(x) + x \cdot f'(x) - m \cdot f'(x)]\, dx \\ &= \int_a^b f(x)\, dx + \int_a^b x \cdot f'(x)\, dx - m \int_a^b f'(x)\, dx\end{aligned}$$

Pelo Teorema Fundamental do Cálculo, temos:

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

Portanto:

$$\int_a^b [f(x) + (x-m) \cdot f'(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b x \cdot f'(x)\, dx - m(f(b) - f(a)) \tag{1}$$

Aplicando Integração por Partes, temos:

$$\int x \cdot f'(x)\, dx.$$

Faça $u = x$.

$$\frac{du}{dx} = \frac{dx}{dx} \Rightarrow du = dx.$$

E ainda

$$dv = f'(x)dx \Rightarrow \int dv = \int f'(x)dx \Rightarrow v = f(x)$$

Portanto

$$\int x \cdot f'(x)\, dx = \int udv = uv - \int vdu = x \cdot f(x) - \int f(x)dx$$

Pelo Teorema Fundamental do Cálculo, temos:

$$\int_a^b x \cdot f'(x)\, dx = b \cdot f(b) - a \cdot f(a) - \int_a^b f(x)\, dx$$

Voltando para a equação (1), temos:

$$\begin{aligned}
\int_a^b [f(x) + (x-m) \cdot f'(x)]\, dx &= \cancel{\int_a^b f(x)\, dx} + b \cdot f(b) - a \cdot f(a) - \cancel{\int_a^b f(x)\, dx} - m(f(b) - f(a)) \\
&= b \cdot f(b) - a \cdot f(a) - \left(\frac{a+b}{2}\right)(f(b) - f(a)) \\
&= \frac{2bf(b) - 2af(a) - af(b) + af(a) - bf(b) + bf(a)}{2} \\
&= \frac{(b-a)}{2}(f(b) + f(a))
\end{aligned}$$

Ou seja:

$$f(b) + f(a) = \frac{2}{b-a} \int_a^b [f(x) + (x-m) \cdot f'(x)]\, dx$$

II. Sejam m e n números naturais diferentes de zero. Verifique que:

$$\int_0^1 x^n \cdot (1-x)^m\, dx = \frac{m}{n+1} \cdot \int_0^1 x^{n+1} \cdot (1-x)^{m-1}\, dx$$

Façamos Integração por Partes:

$$u = (1-x)^m \Rightarrow \frac{du}{dx} = \frac{d(1-x)^m}{dx} = m(1-x)^{m-1} \cdot (-1) = -m(1-x)^{m-1}$$

$$du = -m(1-x)^{m-1}\, dx$$

$$dv = x^n\, dx \Rightarrow \int dv = \int x^n\, dx = \frac{x^{n+1}}{n+1}$$

$$v = \frac{x^{n+1}}{n+1}$$

$$\begin{aligned}
\int udv = uv - \int vdu &= (1-x)^m \cdot \frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1} \cdot (-m(1-x)^{m-1})\, dx \\
&= (1-x)^m \cdot \frac{x^{n+1}}{n+1} + \frac{m}{n+1} \int x^{n+1} \cdot (1-x)^{m-1}\, dx
\end{aligned}$$

Pelo Teorema Fundamental do Cálculo, temos:

$$\int_0^1 x^n \cdot (1-x)^m \, dx = (1-1)^m \cdot \frac{1^{n+1}}{n+1} - (1-0)^m \cdot \frac{0^{n+1}}{n+1} + \frac{m}{n+1} \int_0^1 x^{n+1} \cdot (1-x)^{m-1} \, dx$$
$$= 0 - 0 + \frac{m}{n+1} \int_0^1 x^{n+1} \cdot (1-x)^{m-1} \, dx$$
$$= \frac{m}{n+1} \int_0^1 x^{n+1} \cdot (1-x)^{m-1} \, dx$$

Finalmente:

$$\int_0^1 x^n \cdot (1-x)^m \, dx = \frac{m}{n+1} \int_0^1 x^{n+1} \cdot (1-x)^{m-1} \, dx$$

III. Verifique que, $\forall n \in \mathbb{N}$ e $s > 0$, vale:

$$\int t^n \cdot e^{-st} \, dt = -\frac{1}{s} \cdot t^n \cdot e^{-st} + \frac{n}{s} \cdot \int t^{n-1} \cdot e^{-st} \, dt + k$$

Façamos integração por partes.
Faça

$$u = t^n \Rightarrow \frac{du}{dt} = n \cdot t^{n-1} \Rightarrow du = n \cdot t^{n-1} \, dt$$

$$du = n \cdot t^{n-1} \, dt$$

E ainda

$$dv = e^{-st} \, dt \Rightarrow \int dv = \int e^{-st} \, dt$$

$$v = \int e^{-st} \, dt$$

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Faça

$$w = -st \Rightarrow \frac{dw}{dt} = -\frac{st}{dt} \Rightarrow dw = -s \, dt \Rightarrow dt = -\frac{1}{s} dw$$

$$\int e^w (-\frac{1}{s}) dw = -\frac{1}{s} \int e^w dw = -\frac{1}{s} \cdot e^w = -\frac{1}{s} \cdot e^{-st}$$

Portanto

$$v = -\frac{1}{s} \cdot e^{-st}$$

Prossigamos com a integração por partes:

$$\int udv = uv - \int vdu \rightarrow \int t^n \cdot e^{-st}\, dt = -\frac{t^n \cdot e^{-st}}{s} - \int -\frac{1}{s} \cdot e^{-st} \cdot n \cdot t^{n-1}\, dt$$

Finalmente:

$$\int t^n \cdot e^{-st}\, dt = -\frac{1}{s} \cdot t^n \cdot e^{-st} + \frac{n}{s} \int t^{n-1} e^{-st}\, dt + k$$

IV. Seja $n \geq 2$, mostre que:

$$(a) \qquad \int \sin^n x\, dx = -\frac{1}{n} \cdot \sin^{n-1}(x) \cdot \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x)\, dx$$

$$\sin^n(x)dx = \sin^{n-1} \sin(x)(x)dx$$

Faça

$$u = \sin^{n-1}(x) => \frac{du}{dx} = \frac{\sin^{n-1}(x)}{dx} = (n-1)(\sin^{n-2}(x))\cos(x)$$

$$du = (n-1)(\sin^{n-2}(x))\cos(x)\, dx$$

E ainda

$$dv = \sin(x)\, dx \Rightarrow \int dv = \int \sin(x)\, dx$$

$$v = -\cos(x)$$

$$\int udv = uv - \int vdu$$

$$\begin{aligned}
\int \sin^{n-1}(x)\sin(x)\, dx &= -\cos(x)\sin^{n-1}(x) - \int -\cos(x)(n-1)(\sin^{n-2}(x))\cos(x)\, dx \\
&= -\cos(x)\sin^{n-1}(x) + \int (n-1)(\sin^{n-2}(x))\cos(x)\cos(x)\, dx \\
&= -\cos(x)\sin^{n-1}(x) + (n-1)\int (\sin^{n-2}(x))\cos^2(x)\, dx \\
&= -\cos(x)\sin^{n-1}(x) + (n-1)\int (\sin^{n-2}(x))(1 - sin^2(x))\, dx \\
&= -\cos(x)\sin^{n-1}(x) + (n-1)\int sin^{n-2}(x)\, dx - (n-1)\int sin^n(x)\, dx
\end{aligned}$$

Ou seja:

$$\int \sin^{n-1}(x)\sin(x)\,dx+(n-1)\int sin^n(x)\,dx = -\cos(x)\sin^{n-1}(x)+(n-1)\int sin^{n-2}(x)\,dx$$

$$n\int \sin^n(x)\,dx = -\cos(x)\sin^{n-1}(x)+(n-1)\int sin^{n-2}(x)\,dx$$

Finalmente temos:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x)+\frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

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$$(b) \qquad \int \cos^n x\,dx = \frac{1}{n}\cdot\cos^{n-1}(x)\cdot\sin(x)+\frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$\cos^n(x)dx = \cos^{n-1}\cos(x)(x)dx$$

Faça

$$u = \cos^{n-1}(x) => \frac{du}{dx} = \frac{\cos^{n-1}(x)}{dx} = -(n-1)(\cos^{n-2}(x))\sin(x)$$

$$du = -(n-1)(\cos^{n-2}(x))\sin(x)dx$$

E ainda

$$dv = \cos(x)\,dx \Rightarrow \int dv = \int \cos(x)\,dx$$

$$v = \sin(x)$$

Então:

$$\int udv = uv - \int vdu$$

$$\begin{aligned}
\int \cos^n(x)\,dx &= \sin(x)\cos^{n-1}(x) - \int \sin(x)(-(n-1)\cos^{n-2}(x)\sin(x))\,dx \\
&= \sin(x)\cos^{n-1}(x) + \int (n-1)\cos^{n-2}(x)\sin^2(x)\,dx \\
&= \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)(1-\cos^2(x))\,dx \\
&= \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\int \cos^n(x)\,dx
\end{aligned}$$

Ou seja:

$$\int \cos^n(x)\,dx + (n-1)\int \cos^n(x)\,dx = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx$$

$$n\int \cos^n(x)\,dx = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx$$

Finalmente:

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

V. Mostre que $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$$y = \tanh^{-1}(x) \Rightarrow \tanh(y) = x \Rightarrow \frac{\sinh(y)}{\cosh(y)} = x \Rightarrow \frac{\frac{e^y - e^{-y}}{2}}{\frac{e^y + e^{-y}}{2}} = x \Rightarrow$$

$$\Rightarrow x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \Rightarrow \frac{e^y - \frac{1}{e^y}}{e^y + \frac{1}{e^y}} = x \Rightarrow \frac{\frac{e^{2y}-1}{e^y}}{\frac{e^{2y}+1}{e^y}} = x \Rightarrow \frac{e^{2y}-1}{e^{2y}+1} = x \Rightarrow$$

$$\Rightarrow x \cdot e^{2y} + x = e^{2y} - 1 \Rightarrow x \cdot e^{2y} - e^{2y} = -x - 1 \Rightarrow e^{2y} - x \cdot e^{2y} = x + 1 \Rightarrow$$

$$\Rightarrow e^{2y} = \frac{x+1}{1-x} \Rightarrow \ln(e^{2y}) = \ln\left(\frac{x+1}{1-x}\right) \Rightarrow 2y = \ln\left(\frac{x+1}{1-x}\right) \Rightarrow y = \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$$

com $|x| < 1$