digit recognition

Y = Output

$= \{0, 1, 2, \ldots, 9\}$

= label of the number

X = Input space

= Vectorized discretized Image of the digit

$r \longrightarrow \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}_{rc} = X = (X_1, X_2, \ldots, X_{rc})$

c

$f : (X_1, X_2, \ldots, X_{rc}) \longmapsto \{0, 1, 2, \ldots, 9\}$

$P\{Y = j \mid X\}$

(I) Shape of the data (n, p). $\frac{n}{p} < 1 \Rightarrow$ Ultra

(II) Content of data (Type) $\frac{n}{p} > 1$ Thin

Consider the i^{th} obs. $X_i = (X_{i1}, X_{i2}, \ldots, X_{ip})^T$ p-dim input

What is X_{ij}?

(i) $X_{ij} \in \{0, 1\}$ Binary input

$X_i \in \{0, 1\}^p = \mathcal{X}$

(ii) $X_{ij} \in [0, 1]$ Unitary Input

Neural Networks

$(X_i, Y_i) \overset{iid}{\sim} P_{XY}(x, y)$

$f : \mathcal{X} \longrightarrow \mathcal{Y}$

$x \longmapsto f(x)$.

$\mathcal{X}$: Input space.

$\mathcal{Y}$: Output space

Def. f is a function ref to as a learning machine

f encodes the pattern of relationship (association) b/w X and Y.

$f \in \mathcal{H}$

eg: $\mathcal{Y}^{\mathcal{X}}$ = All possible mappings from $\mathcal{X}$ to $\mathcal{Y}$

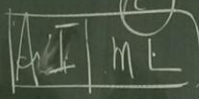
$\mathcal{H} \overset{def}{=}$ function space. Hypothesis space.

Note: (a) $\mathcal{H}$ maybe finite and countable

$\mathcal{H} = \{f_1, f_2, \ldots, f_m\}$

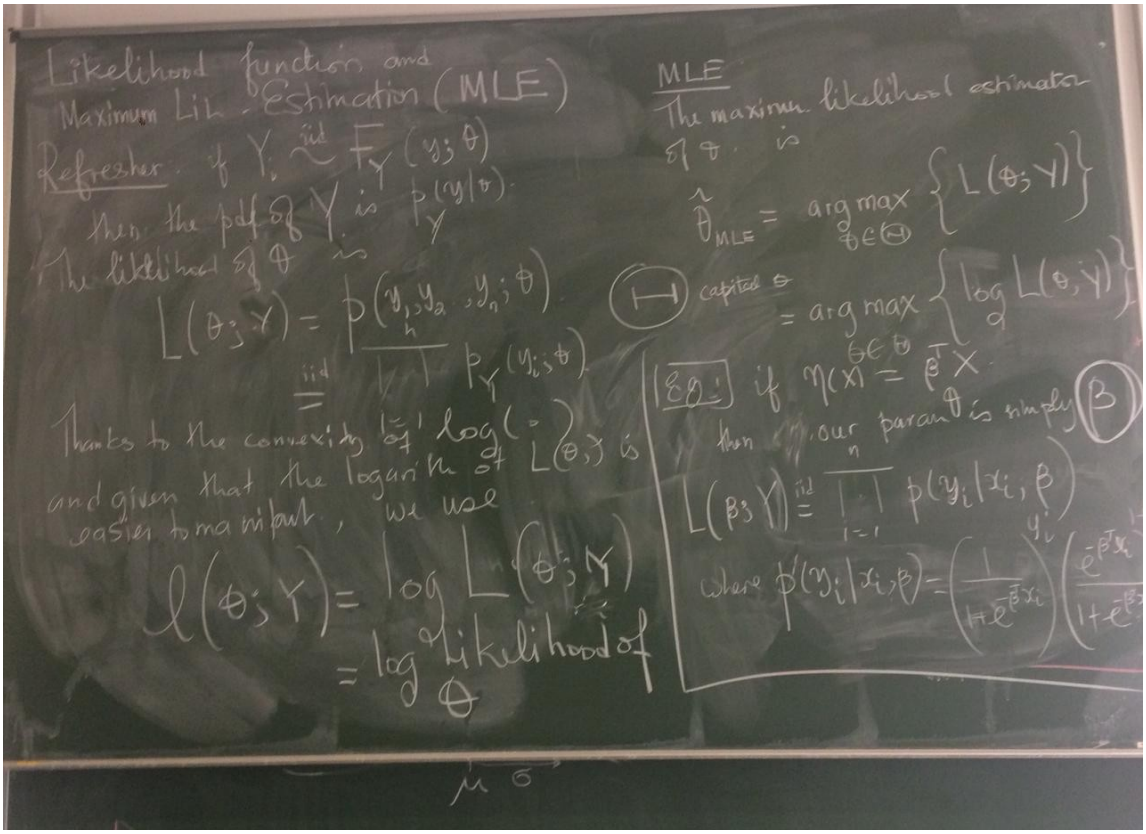
(b) $\mathcal{H} \equiv$ countably infinite

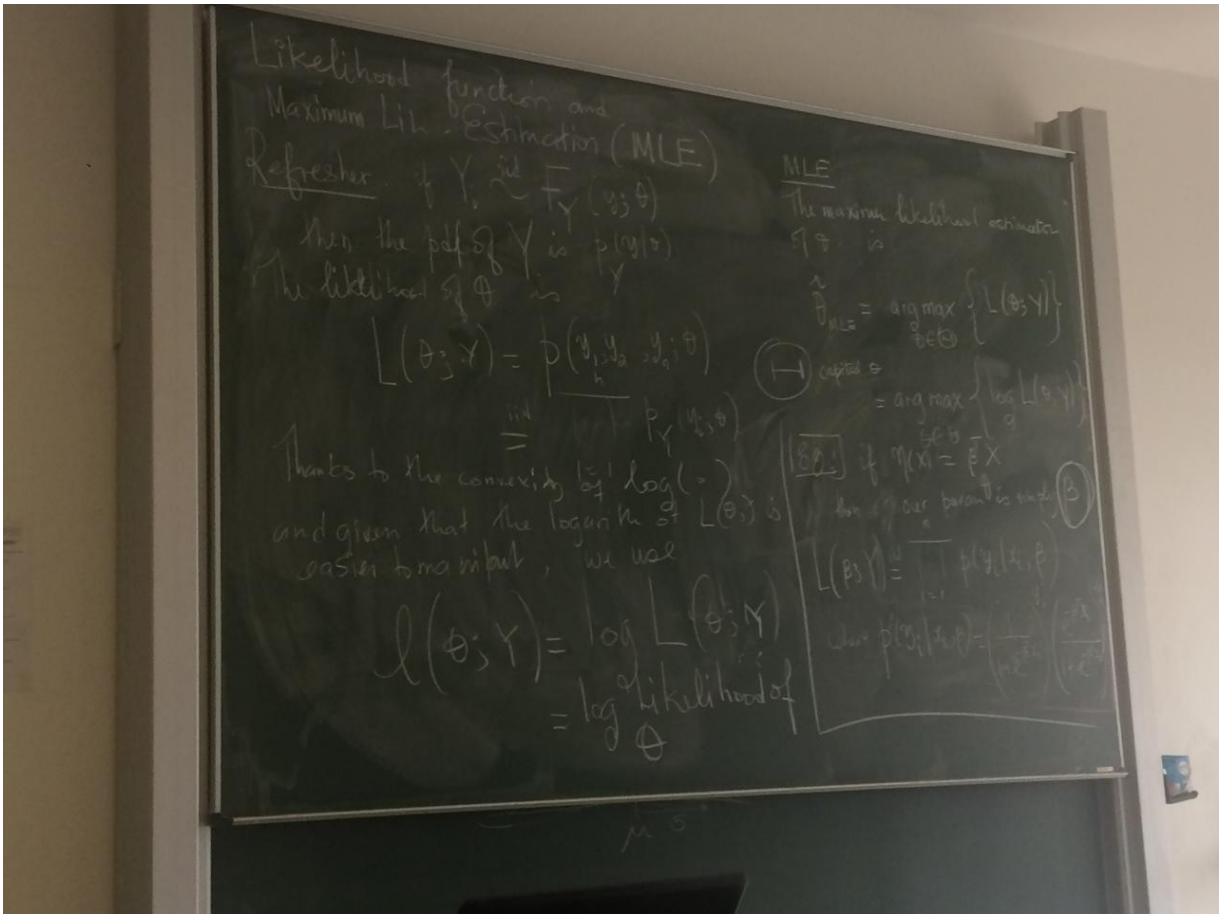
(c) $\mathcal{H}$ = Infinite



Choosing $\mathcal{H}$ is an act of approximation

eg $\mathcal{H}$ = Polynomials of degree m


Likelihood function and
Maximum Lik. Estimation (MLE)
Refresher
MLE
The maximum likelihood estimator
Thanks to the convexity of log(·)
and given that the logarithm of L(θ;Y) is
easier to manipulate, we use
log likelihood of
capital Θ
if
then our parameter is simply
where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Focus on $p_{Y|x}(y|x)$ posterior density of Y given $X=x$

Central to AI and ML b/c we are typically given $X=x$ and we seek to predict Y given $X=x$

AI = Artificial Intelligence

ML = Machine Learning

Eg: Digit Reg

- Take a picture of a given digit
- Extract the grayscale $r \times c$ matrix
- Vectorize $[\,] \rightarrow [\,]$

$$\begin{cases} \mathbb{P}(Y=0|x) \\ \mathbb{P}(Y=1|x) \\ \vdots \\ \mathbb{P}(Y=9|x) \end{cases}$$

$\hat{f}(x)$ = label of x

Majority rule

$$= \begin{cases} 0 & \text{if } \mathbb{P}(Y=0|x) \text{ is largest} \\ 1 & \text{if } \mathbb{P}(Y=1|x) \text{ is largest} \\ \vdots \\ 9 & \text{if } \mathbb{P}(Y=9|x) \text{ is largest} \end{cases}$$

$$\hat{f}(x) = \underset{j=0,1,\dots,9}{\arg\max}\left(\mathbb{P}(Y=j|x)\right)$$

$$\tilde{x}_{ij} \rightarrow \frac{x_{ij} - x^j_{(1)}}{x^j_{(n)} - x_{(1)}}$$

$$x_{ij} = \tilde{x}_{ij}\left(x^j_{(n)} - x^j_{(1)}\right) + x^{(j)}_{(1)}$$

(III) Distribution

$$p_{XY}(x, y)$$

(IV) Missing Ness

Pattern and incidence of missing values

(V) Outlier and Novelty Detection

Understanding $p_{XY}(x,y)$

- Focus on $p_X(x)$ ref to density estimation in $\mathcal{X}$ key to Cluster analysis
- Focus on $p_{X|Y}(x|y)$ ref estimation of the so called class conditional density. Pillar of generative pattern recog via discriminant Analysis
- Focus of $p_Y(y)$ is usually the easiest in PR thus is simply discrete prob estim

Eg:

Binary Classification

$Y = \{0, 1\}$

$X \subset \mathbb{R}^p$ $\quad x = (x_1, \ldots, x_p)^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$

$x_j \in \mathbb{R}$

Hypothesized:

Logistic Regr

Assume $P\{Y=1|x\} = \frac{1}{1+e^{-\eta(x)}}$

where $\eta : X \to \mathbb{R}$ ✓

$\eta(x) = \beta^T x$

$\eta(x)$ = nonlinear

$P(Y=0|x) = 1 - P(Y=1|x)$

$= 1 - \pi(x)$

$Y=0$ or $Y=1$

$P(A^c) = 1 - P(A)$

$p(y|x) = \pi(x)^y (1-\pi(x))^{1-y}$

$= \left(\frac{1}{1+e^{-\eta(x)}}\right)^y \left(\frac{e^{-\eta(x)}}{1+e^{-\eta(x)}}\right)^{1-y}$

Goal: Estimate $\eta(x)$

(ii) $X_{ij} \in \{a_1, a_2, \ldots, a_m\}$

(iv) $X_{ij} \in [a, b] \subset \mathbb{R}$

$X_i \in X \subset \mathbb{R}^p$

(v) $X_{ij} \equiv$ Frequency of word j in document i

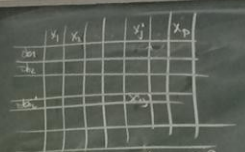
$\in \mathbb{R}^{p \times 1}$

$\equiv$ Text Mining with Bag of Words assumptions.

(vi) $X_{ij} \in \{$strongly disagree, disagree, Neutral, agree, strongly agree$\}$ 5 values

$\{0, 1\}$ 2 values

$\Rightarrow$ Ordinal Input



$x_{ij} \Rightarrow \frac{x_{ij} - \min}{\text{Range}(x)}$

ultra Thin

$\frac{n}{f}$

saturated desp

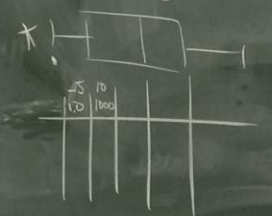
Cubitize

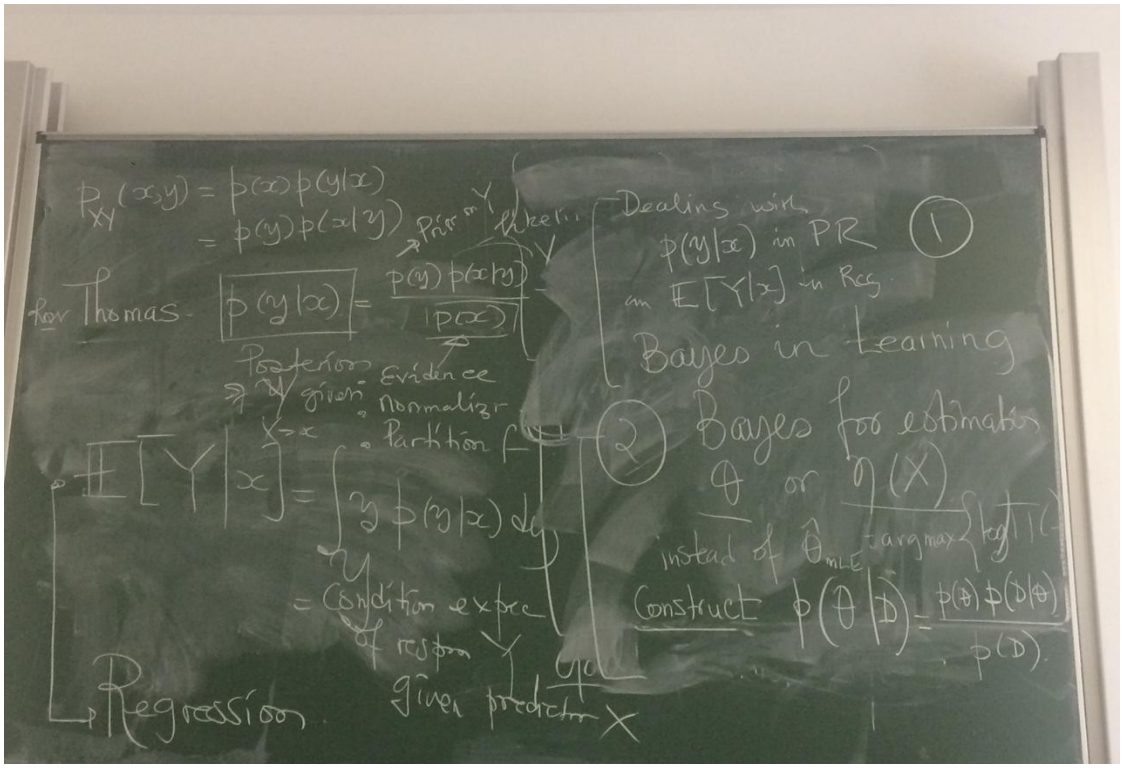
$X_i \in [a, b]^p$

$X_i \in [0, 1]^p$

$x_{ij} \to \frac{x_{ij} - \min_j x_{ij}}{\max_j x_{ij} - \min_j x_{ij}}$

Outlier: $x^* > Q_3 + 3 IQR$




$P_{XY}(x,y) = p(x)p(y|x)$
$= p(y)p(x|y)$
Prior on Y
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$\mathbb{E}[Y|x] = \int_y y\, p(y|x)\, dy$
= Condition expec of respon Y given predictor X
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Dealing with $p(y|x)$ in PR
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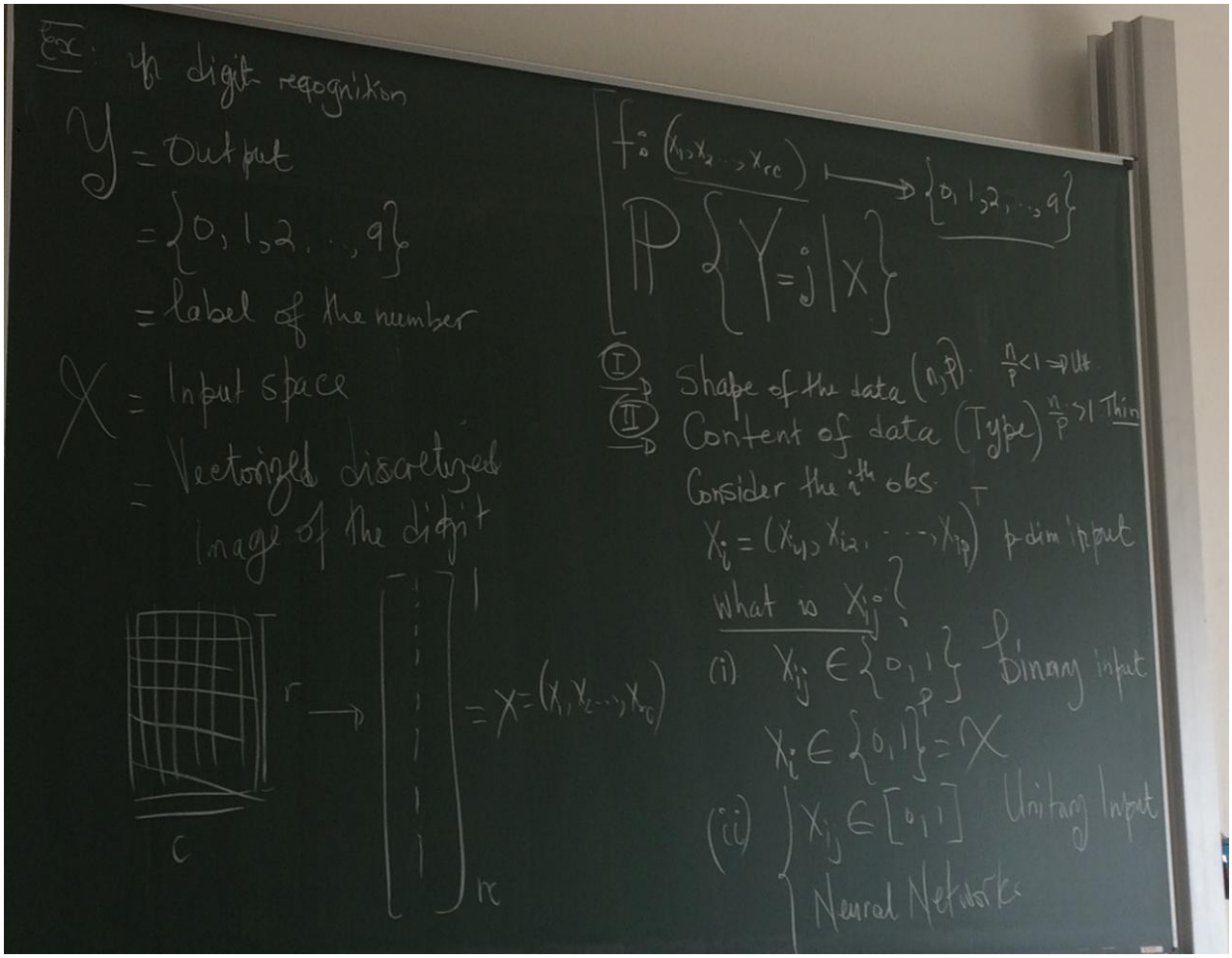
Bayes in learning
(2) Bayes for estimation
θ or $\eta(X)$
instead of $\hat{\theta}_{MLE} = \arg\max\{\log l(\theta)$
Construct $p(\theta|D) = \frac{p(\theta)p(D|\theta)}{p(D)}$


Ex: digit recognition
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= label of the number
X = Input space
= Vectorized discretized
= Image of the digit
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rc
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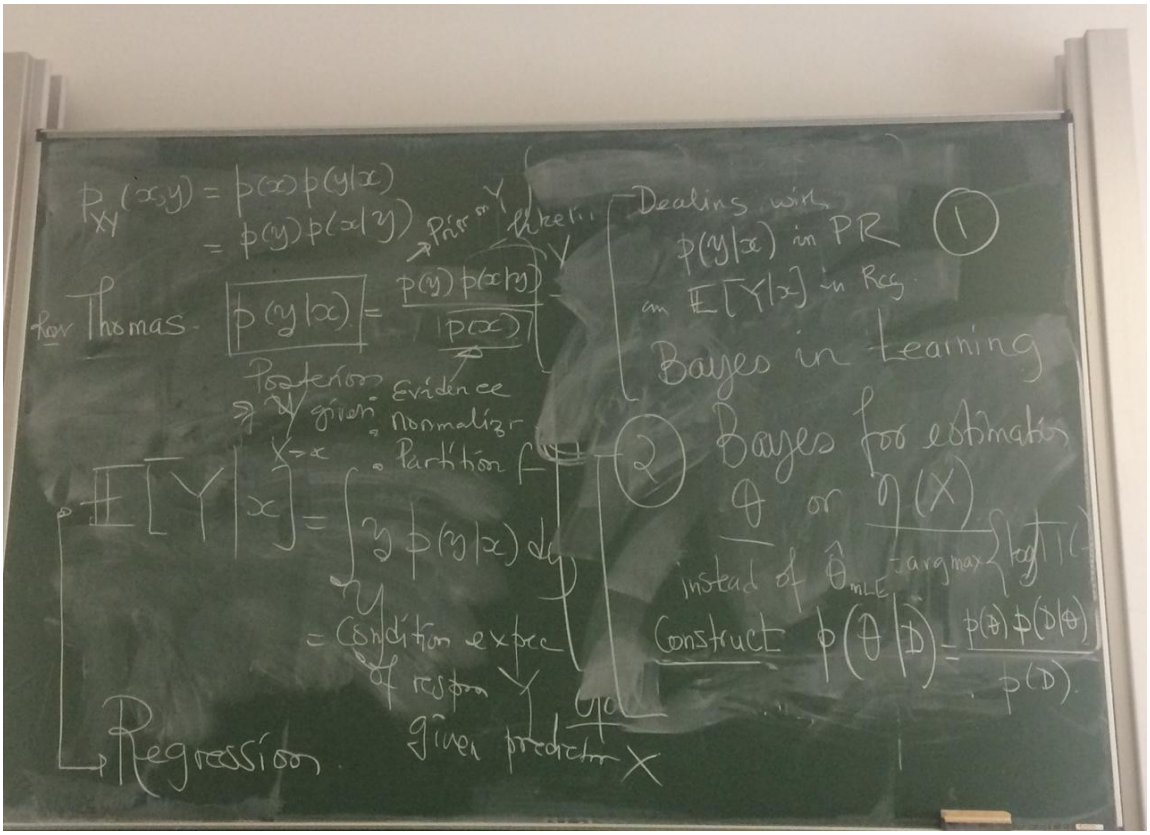
Neural Networks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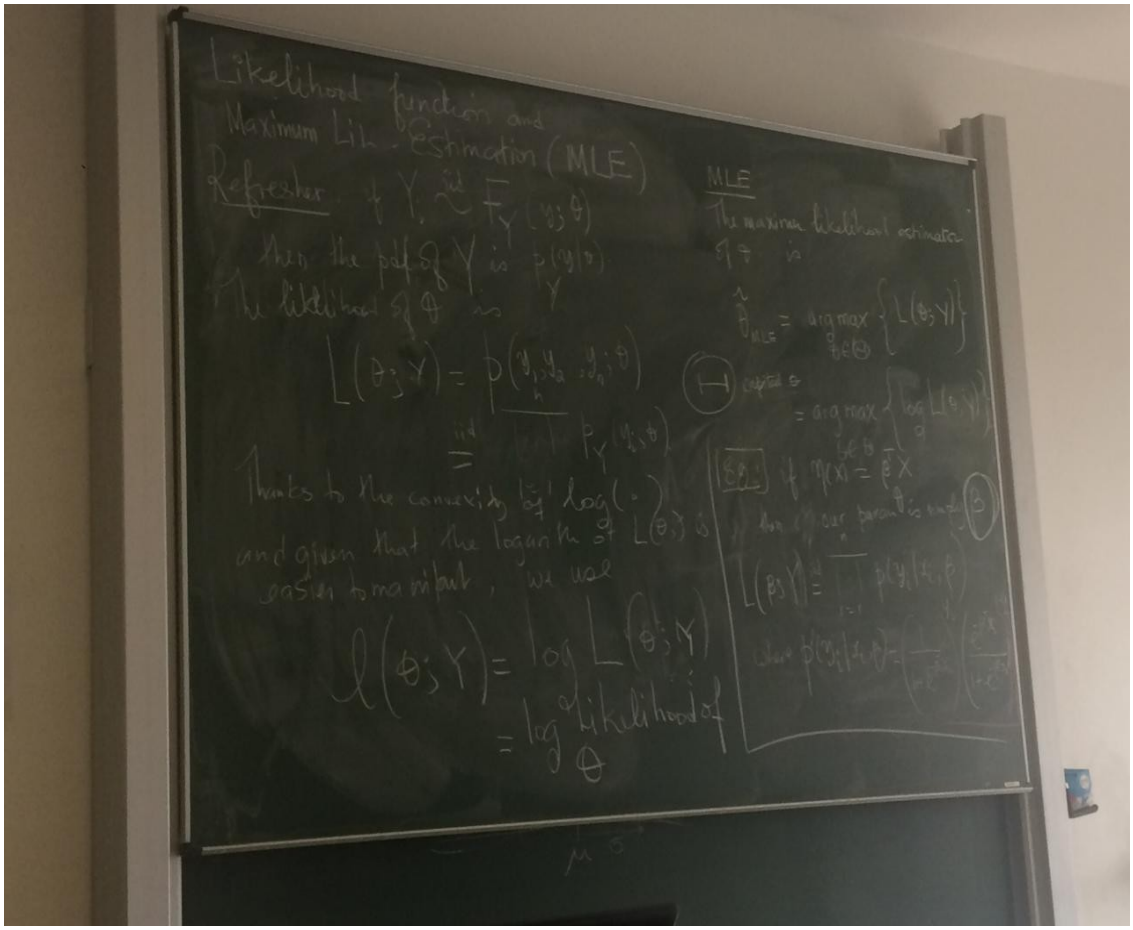
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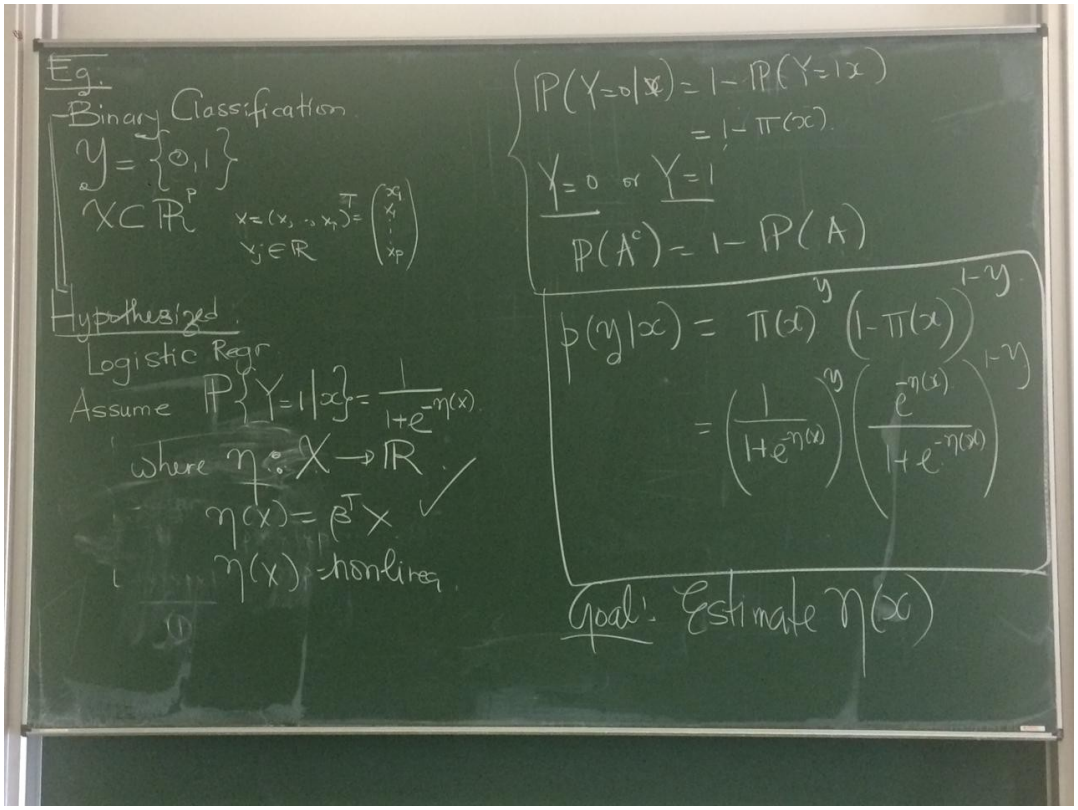
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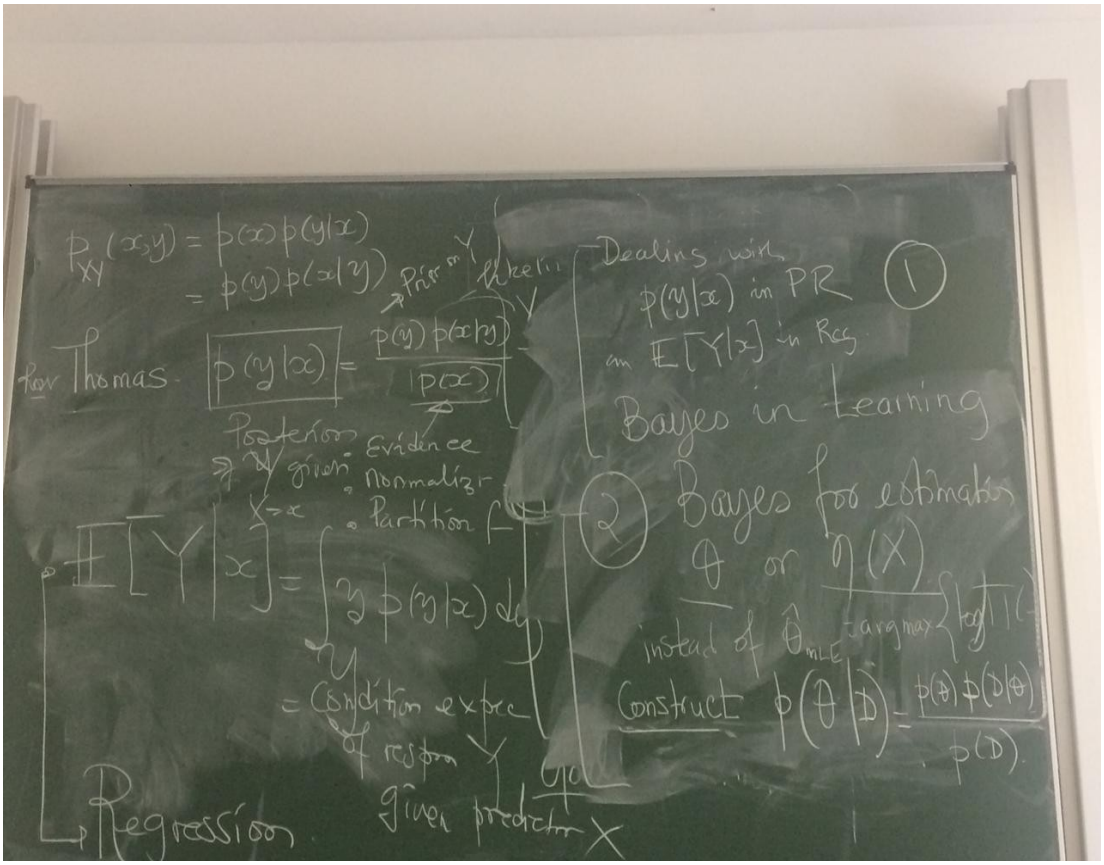
Goal: Estimate $\eta(x)$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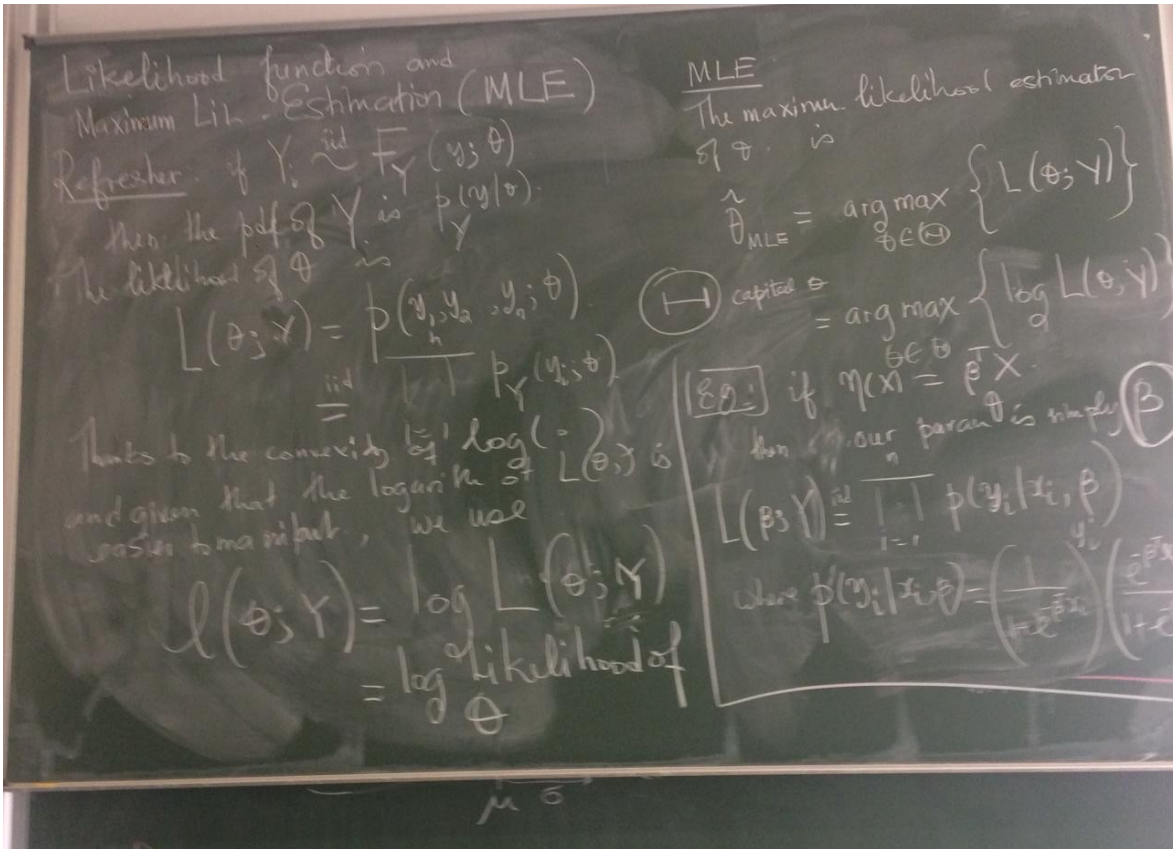
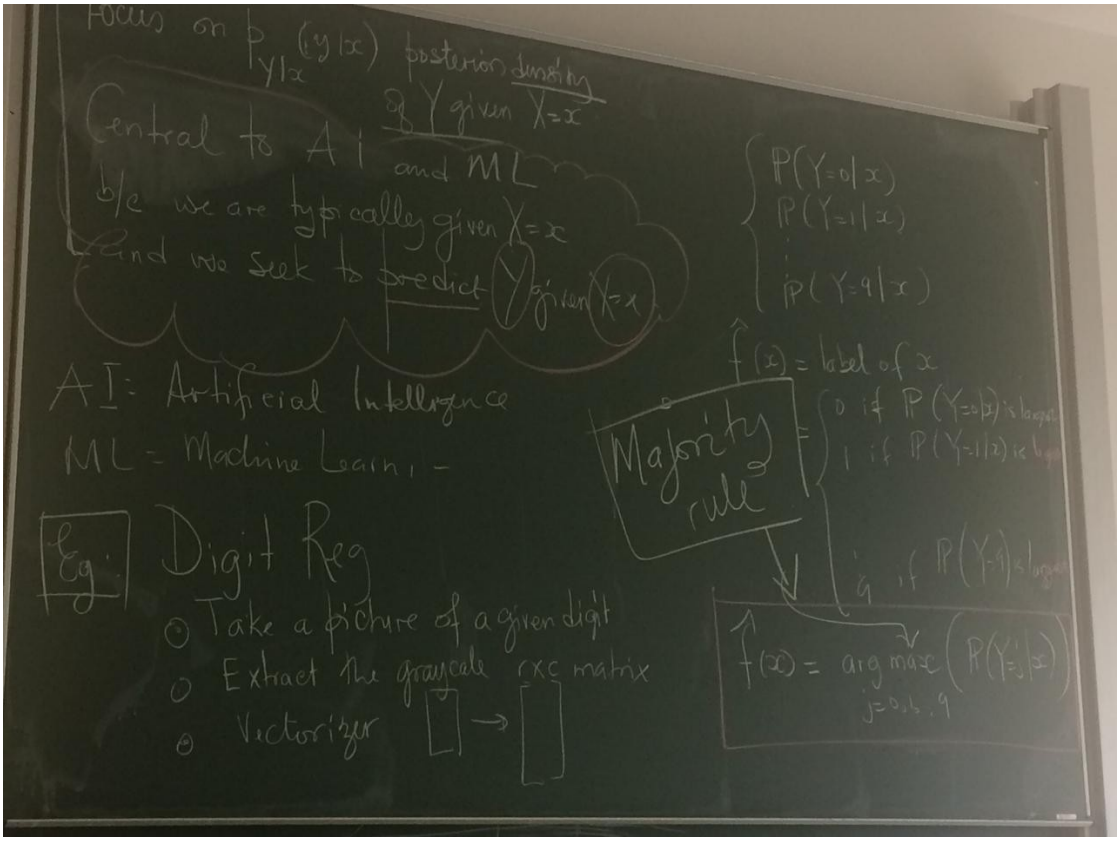

Likelihood function and Maximum Lik. Estimation (MLE)
Refresher: if $Y_i \overset{iid}{\sim} F_Y(y;\theta)$
Then the pdf of Y is $p_Y(y|\theta)$
The likelihood of θ is
$L(\theta;Y) = p(y_1, y_2, \dots, y_n; \theta)$
$\overset{iid}{=} \prod_{i=1}^{n} p_Y(y_i;\theta)$
Thanks to the convexity of $\log(\cdot)$ and given that the log of $L(\theta;Y)$ is easier to manipulate, we use
$\ell(\theta;Y) = \log L(\theta;Y)$
$= \log$ likelihood of θ
MLE
The maximum likelihood estimator of θ is
$\hat{\theta}_{MLE} = \arg\max_{\theta \in \Theta} \{L(\theta;Y)\}$
Θ capital θ
$= \arg\max_{\theta \in \Theta} \{\log L(\theta;Y)\}$
E.G.: if $\eta(x) = \beta^T x$
then our param θ is simply β
$L(\beta;Y) \overset{iid}{=} \prod_{i=1}^{n} p(y_i|x_i,\beta)$
where $p(y_i|x_i,\beta) = \left(\frac{1}{1+e^{\beta^T x_i}}\right)^{y_i} \left(\frac{e^{\beta^T x}}{1+e^{\beta^T x}}\right)$


Focus on $p_{Y|x}(y|x)$ posterior density of Y given $X=x$
Central to AI and ML
b/c we are typically given $X=x$
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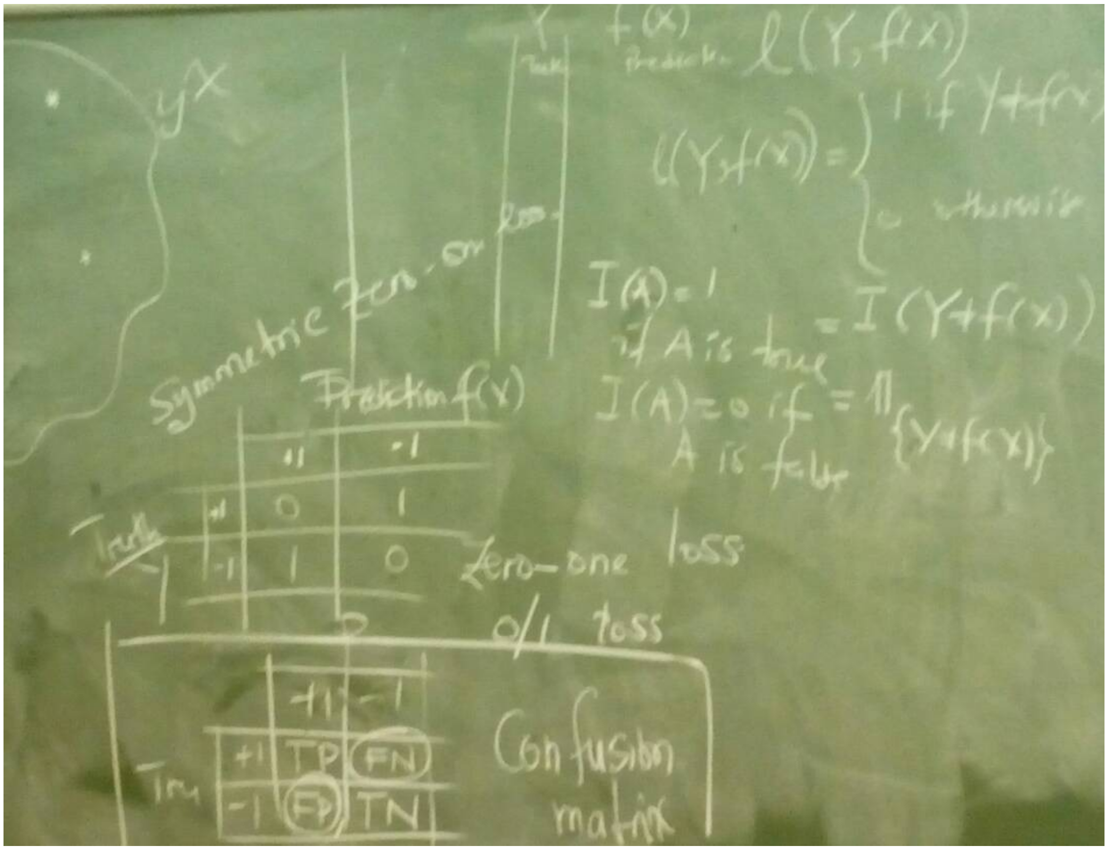
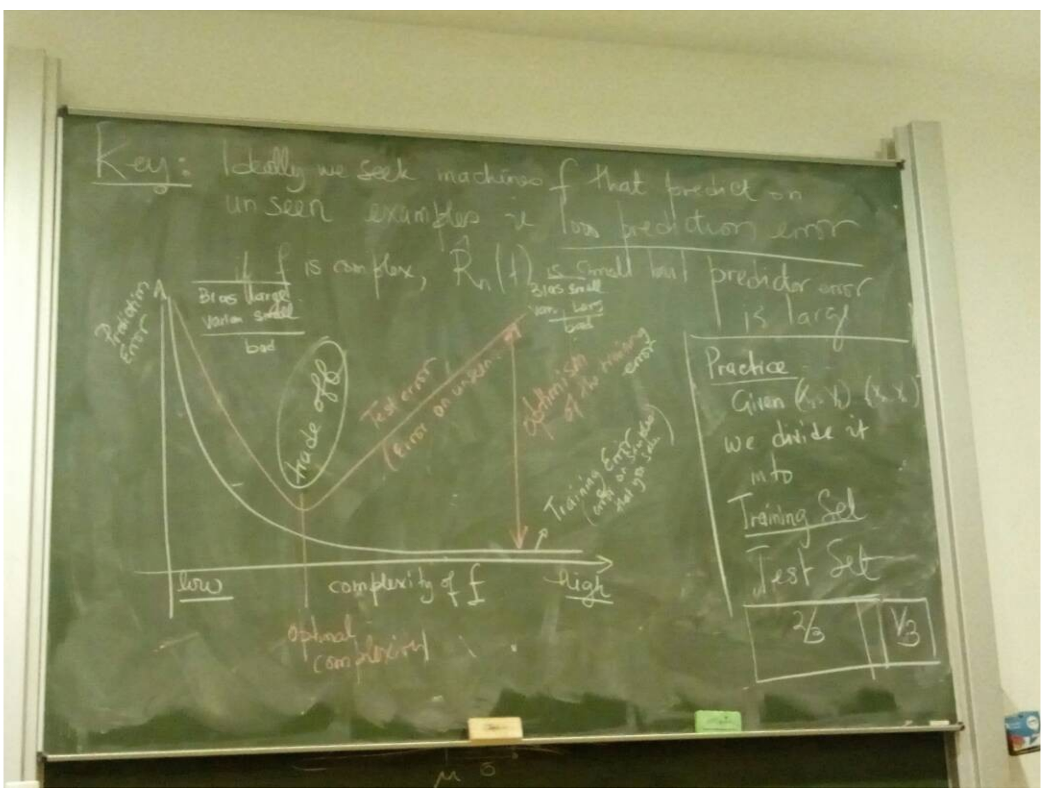
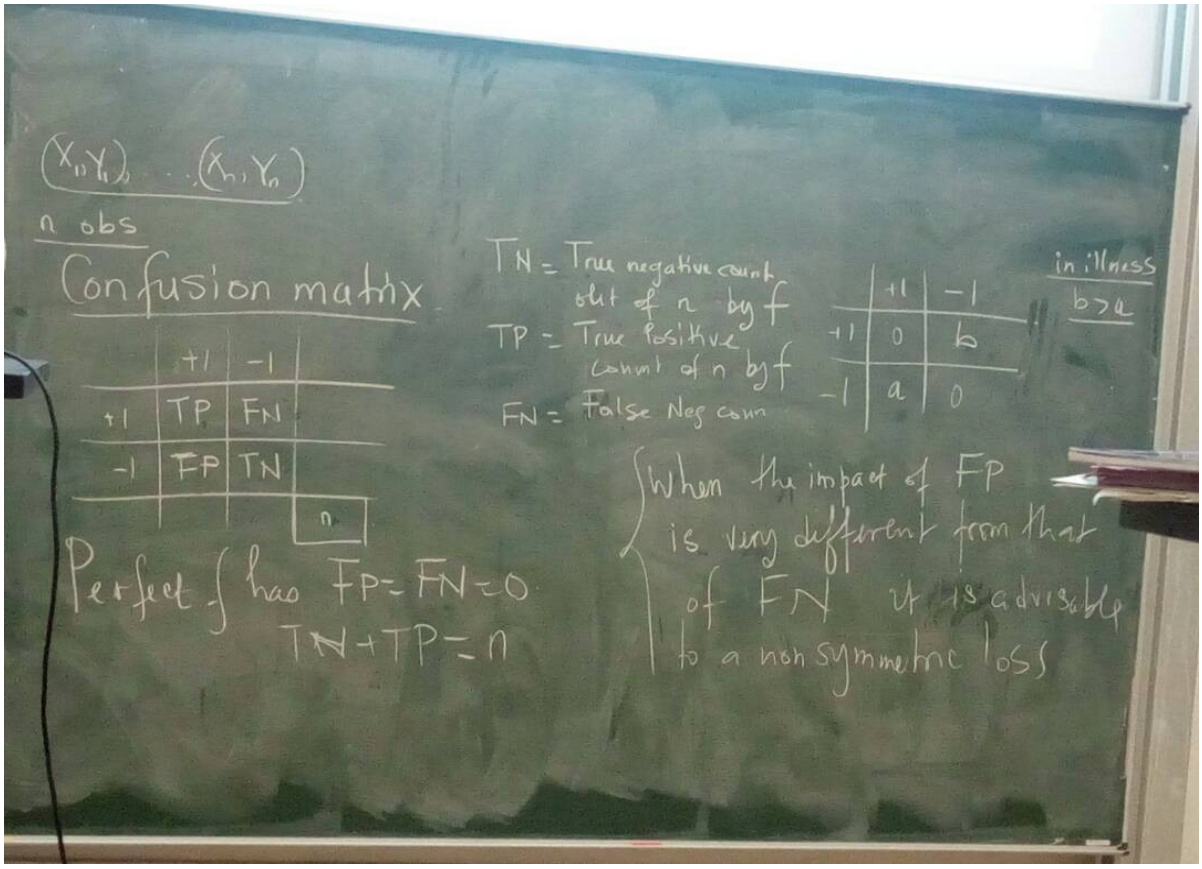
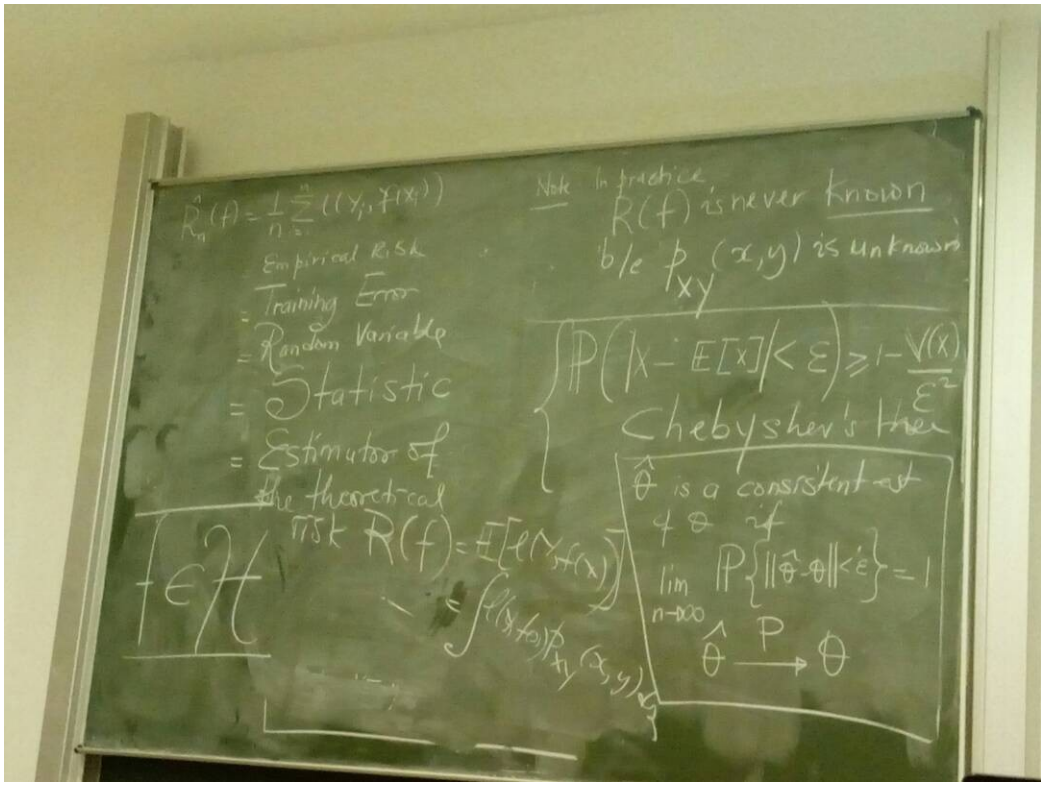
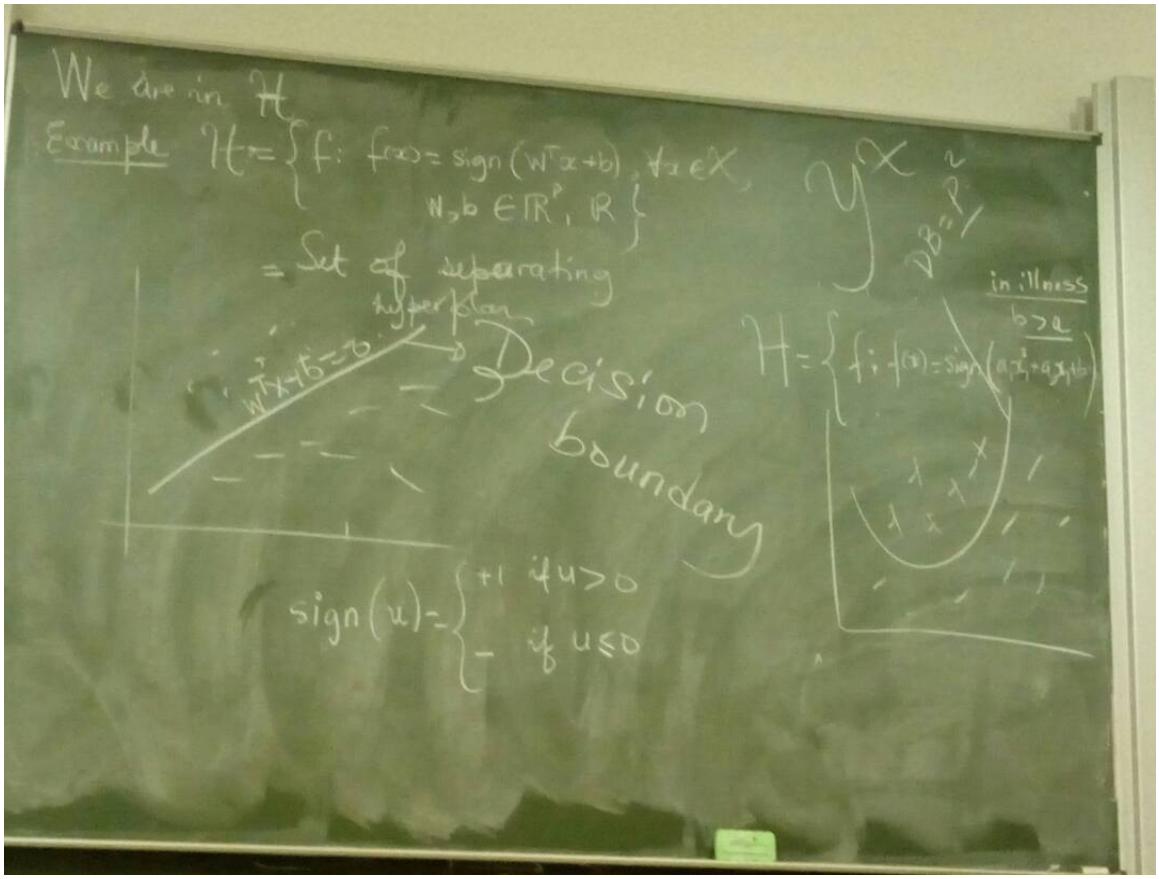
AI: Artificial Intelligence
ML: Machine Learning
Eg: Digit Reg
Take a picture of a given digit
Extract the grayscale rxc matrix
Vectorize [ ] → [ ]
$\begin{cases} P(Y=0|x) \\ P(Y=1|x) \\ \vdots \\ P(Y=9|x) \end{cases}$
$\hat{f}(x)$ = label of x
Majority rule
$= \begin{cases} 0 \text{ if } P(Y=0|x) \text{ is largest} \\ 1 \text{ if } P(Y=1|x) \text{ is largest} \\ \vdots \\ 9 \text{ if } P(Y=9|x) \text{ is largest} \end{cases}$
$\hat{f}(x) = \arg\max_{j=0,1,\dots,9} \left(P(Y=j|x)\right)$

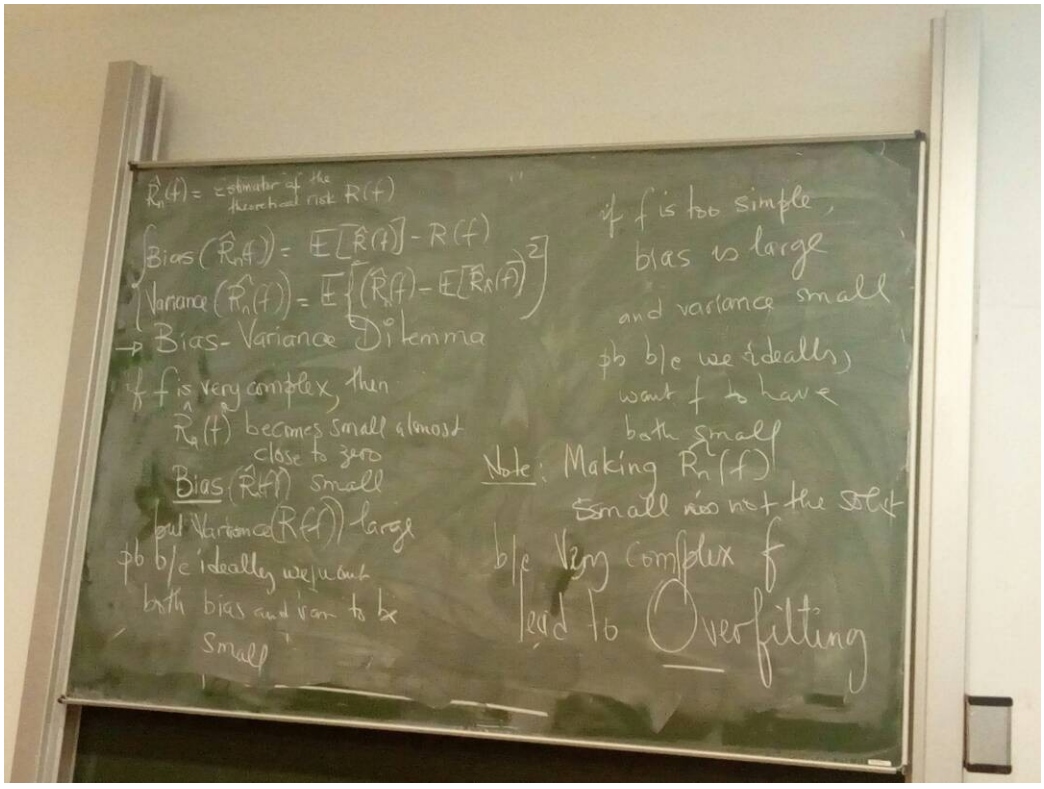

Symmetric zero-one loss
Prediction f(x)
Truth
I(A) = 1 if A is true
I(A) = 0 if A is false
otherwise
Zero-one loss
0/1 loss
+1 -1
TP FN
FP TN
Confusion matrix

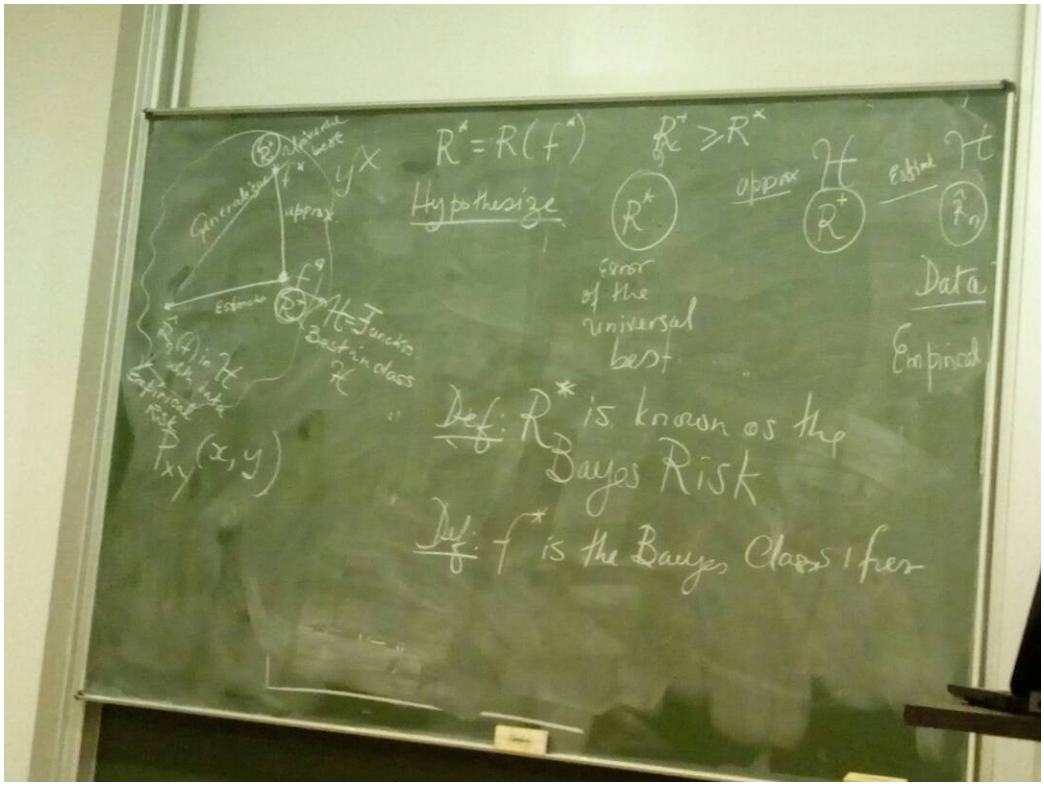

Key: Ideally we seek machines f that predict on unseen examples i.e. low prediction error
is complex
but predictor error is large
Prediction Error
Bias large
bad
trade off
Test error
Training Error
low
complexity of f
high
optimal complexity
Practice
we divide it into
Training Set
Test Set
2/3
1/3

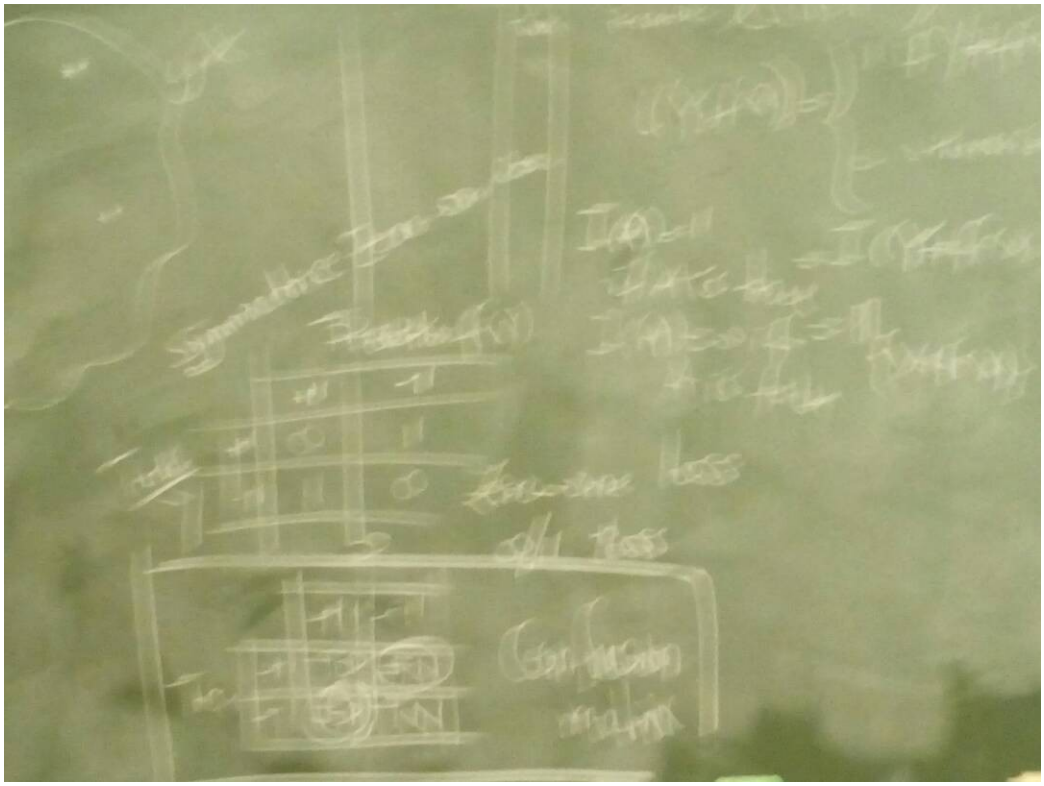

$(X_1, Y_1) \dots (X_n, Y_n)$
n obs
Confusion matrix
TN = True negative count out of n by f
TP = True Positive count of n by f
FN = False Neg coun
+1 -1
+1 TP FN
-1 FP TN
n
Perfect f has FP = FN = 0
TN + TP = n
+1 -1
+1 0 b
-1 a 0
in illness
b > a
When the impact of FP is very different from that of FN it is advisable to a non symmetric loss

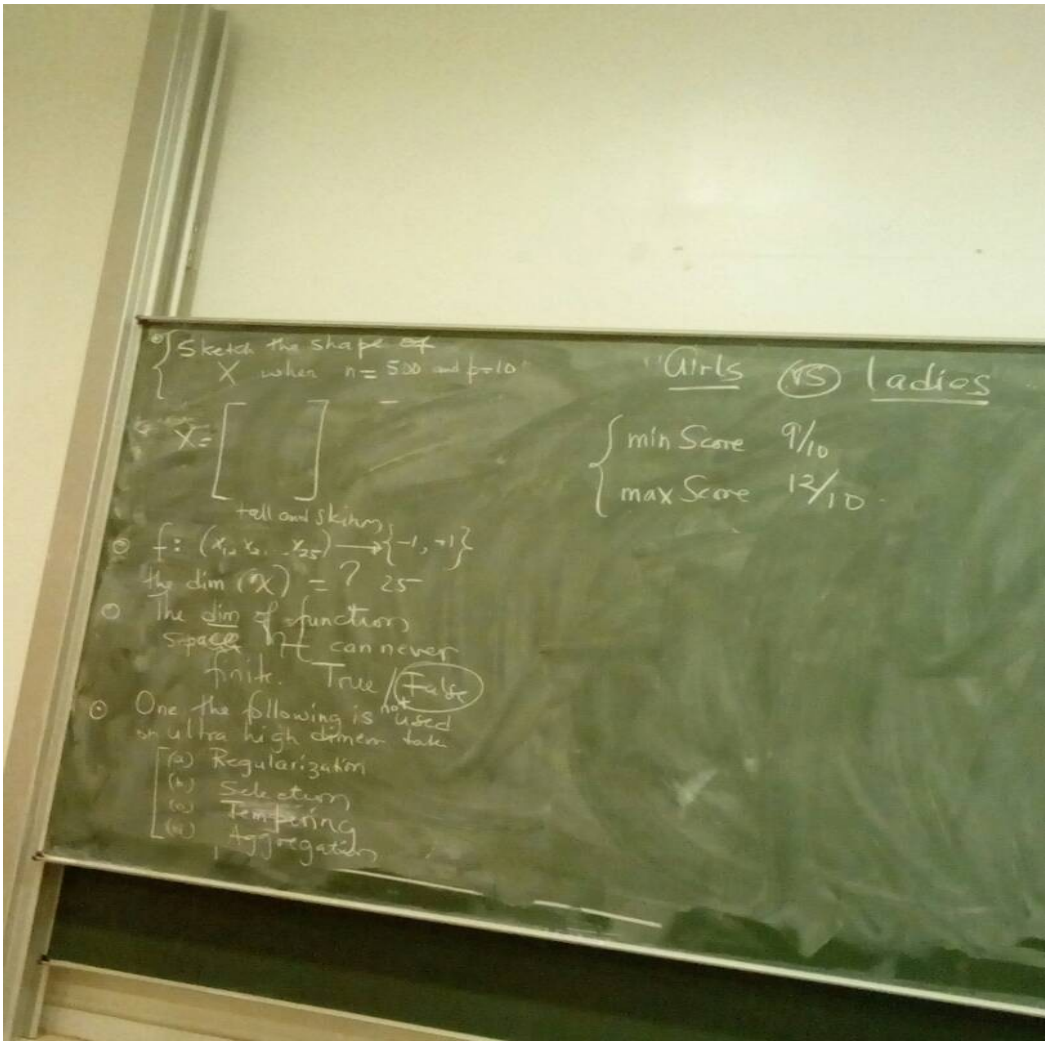

$\hat{R}_n(f) = \frac{1}{n}\sum_{i=1}^{n} L(Y_i, f(X_i))$
Empirical Risk
= Training Error
= Random Variable
= Statistic
= Estimator of the theoretical
$f \in \mathcal{H}$
risk $R(f) = E[L(Y, f(X))]$
$= \int L(y, f(x)) P_{XY}(x, y)$
Note In practice $R(f)$ is never known
ble $P_{XY}(x, y)$ is unknown
$P(|X - E[X]| < \varepsilon) \geq 1 - \frac{V(X)}{\varepsilon^2}$
Chebyshev's the
$\hat{\theta}$ is a consistent est
of θ if
$\lim_{n \to \infty} P\{\|\hat{\theta} - \theta\| < \varepsilon\} = 1$
$\hat{\theta} \xrightarrow{P} \theta$


We are in $\mathcal{H}$
Example $\mathcal{H} = \{f : f(x) = \text{sign}(w^T x + b), \forall x \in \mathcal{X}, w, b \in \mathbb{R}^2, \mathbb{R}\}$
= Set of separating hyperplan
$w^T x + b = 0$
Decision boundary
$\text{sign}(u) = \begin{cases} +1 & \text{if } u > 0 \\ - & \text{if } u \leq 0 \end{cases}$
$\mathcal{X}$
$\mathcal{Y}$
DB = ?
in illness
$b > a$
$H = \{f : f(x) = \text{sign}(ax_1^2 + ax_2 + b)$


$\hat{R}_n(f)$ = estimator of the theoretical risk $R(f)$
$\text{Bias}(\hat{R}_n(f)) = E[\hat{R}(f)] - R(f)$
$\text{Variance}(\hat{R}_n(f)) = E\left[\left(\hat{R}_n(f) - E[\hat{R}_n(f)]\right)^2\right]$
→ Bias-Variance Dilemma
if f is very complex, then
$\hat{R}_n(f)$ becomes small almost close to zero
Bias $(\hat{R}(f))$ small
but Variance $(\hat{R}(f))$ large
pb b/c ideally we would want both bias and vari to be small
if f is too simple, bias is large and variance small
pb b/c we ideally want f to have both small
Note: Making $\hat{R}_n(f)$ small is not the best b/c Very complex f lead to Overfitting


$R^* = R(f^*)$
$R_{\mathcal{H}}^* \geq R^*$
Hypothesize
Upper
Estim
Data
Empirical
Error of the universal best
Def: R^* is known as the Bayes Risk
Def: f^* is the Bayes Classifier
Generalization
Approx
Estimation
Empirical Risk
$R_{xy}(x,y)$




Sketch the shape of X when n = 500 and p = 10
X =
tall and skinny
f: (x1, x2, ... x25) → {-1, +1}
dim(X) = ? 25
The dim of function space H can never finite. True False
One the following is not used in ultra high dimen. task
(a) Regularization
(b) Selection
(c) Tempering
(d) Aggregation
Girls VS ladies
min Score 9/10
max Score 12/10