# Lecture 7 - Fourier Series

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Lectures on Partial Differential Equations

# Fourier Series definition

Let $f(x)$ be a $2\pi$ periodic function so that

$$f(x + 2\pi) = f(x) \tag{1}$$

Note that equation (1) implies that

$$f(x + 2k\pi) = f(x) \quad \text{where } k \text{ is an integer}$$

The aim is to write such a $2\pi$ periodic function $f(x)$ in terms of a sum of simple periodic functions involving $\sin$ and $\cos$. More precisely we want to write

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

with $a_n, b_n$ constants.

The Fourier Series consists of simple functions and so is easy to manipulate.

# Fourier Series restrictions - periodicity

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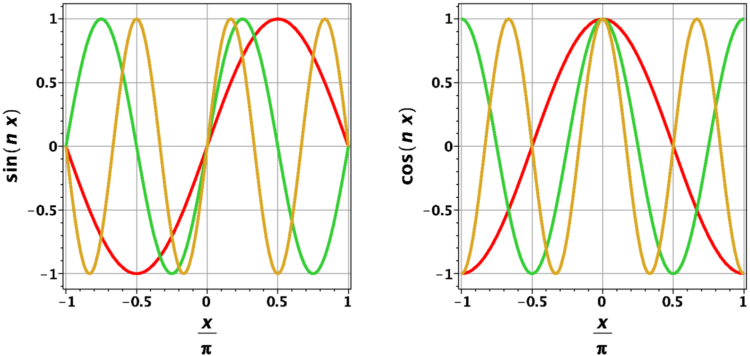



The Fourier Series

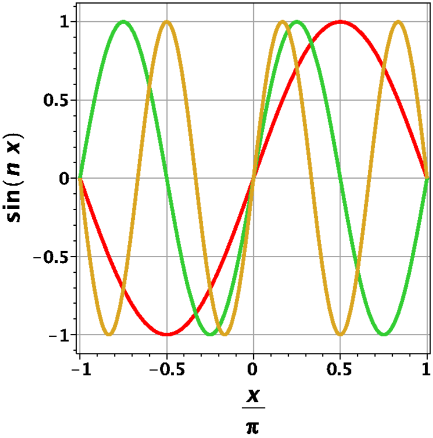
$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

contains terms with period $2\pi$, so $f$ must have period $2\pi$.

# Basic identities: Sine

As we are using trigonometric functions, the following are essential knowledge:

$$\sin(n\pi) = 0,$$
$$\sin\left(\left(n+\tfrac{1}{2}\right)\pi\right) = (-1)^n.$$

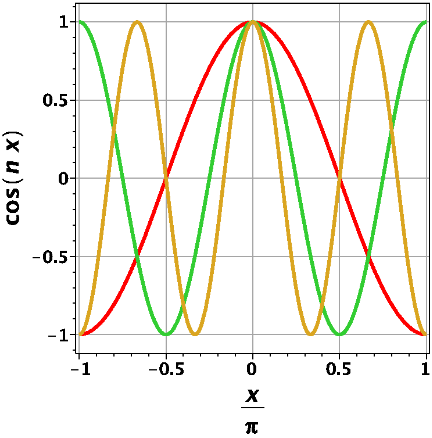


# Basic identities: Cosine



As we are using trigonometric functions, the following are essential knowledge:

$$\cos(n\pi) = (-1)^n,$$
$$\cos\left(\left(n + \tfrac{1}{2}\right)\pi\right) = 0.$$



# Computing a Fourier Series: I



$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

To find the Fourier series we need to compute $a_m, b_m$. We do this using the following key identities

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \pi\delta_{mn},$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn},$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0.$$

Here

$$\delta_{mn} = \begin{cases} 1 & m = n, \\ 0 & m \neq n \end{cases}.$$

# Computing a Fourier Series: II

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These identities which show *orthogonality* of $\sin(mx)$ and $\cos(nx)$ follow from the following trig formulas: Suppose $m \neq n$ then:

$$
\begin{aligned}
2\sin(mx)\sin(nx) &= \cos((m-n)x) - \cos((m+n)x)\,, \\
2\cos(mx)\cos(nx) &= \cos((m-n)x) + \cos((m+n)x)\,, \\
2\sin(mx)\cos(nx) &= \sin((m-n)x) - \sin((m+n)x)\,.
\end{aligned}
$$

Integrating these over $\pi$ to $\pi$ gives the above identities

In the case where $m = n$ these formulas do not hold. Instead we have

$$\sin^2(mx) = \frac{1}{2}(1 - \cos(2mx))\,.$$

Integrating over $-\pi$ to $\pi$ the cosine term will vanish and the integral of $1/2$ over this range gives $\pi$

# Computing a Fourier Series: II



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# Computing Fourier Series

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Prove $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = 0 \quad m \neq n$

Use $2\sin(mx)\sin(nx) = \cos\left((m-n)x\right) - \cos\left((m+n)x\right)$

# Computing Fourier Series



Prove $\displaystyle\int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi$

Use $\sin^2(nx) = \frac{1}{2}\,(1 - \cos(2nx))$.

# Euler formulas: I

Take our Fourier Series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \qquad (2)$$

and the two identities involving sines,

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \pi\delta_{mn}, \quad \int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0.$$

If we multiply equation (2) by $\sin(mx)$ and integrate between $-\pi$ and $\pi$ we get

$$\pi b_m = \int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x$$

for *one, single* term *m*.

# Euler formulas: I



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# Euler formulas: II



$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Similar results for the cosine terms give the full *Euler formulas*

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) \, \mathrm{d}x,$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, \mathrm{d}x.$$

These hold for all terms in the Fourier Series, including the $a_0$ term (hence the factor of $1/2$ in the definition!).

# Euler formulas: II

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# Summary

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- Fourier Series are just another way of representing a function.
- The representation is in terms of periodic functions; therefore, to be correct everywhere, the function you are representing must be periodic.
- Some trig identities are very important to know.
- The Euler formulas (orthogonality) give the Fourier Series coefficients.