AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

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Exercise 1

(i) Given the dynamical systems in (1), the 4^{th} order Runge-Kutta solution is plotted below.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix} \mathbf{x} \tag{1}$$

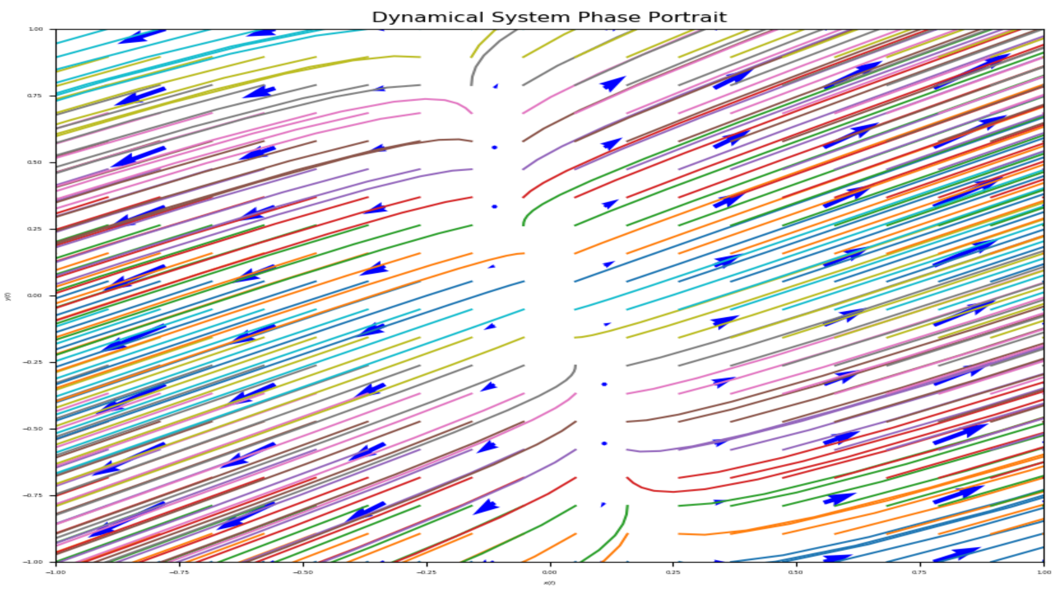

Figure 1: A plot of the dynamical system of ODEs using RK4 method

(ii) Given the dynamical systems in (2), the 4^{th} order Runge-Kutta solution is plotted below.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} \tag{2}$$

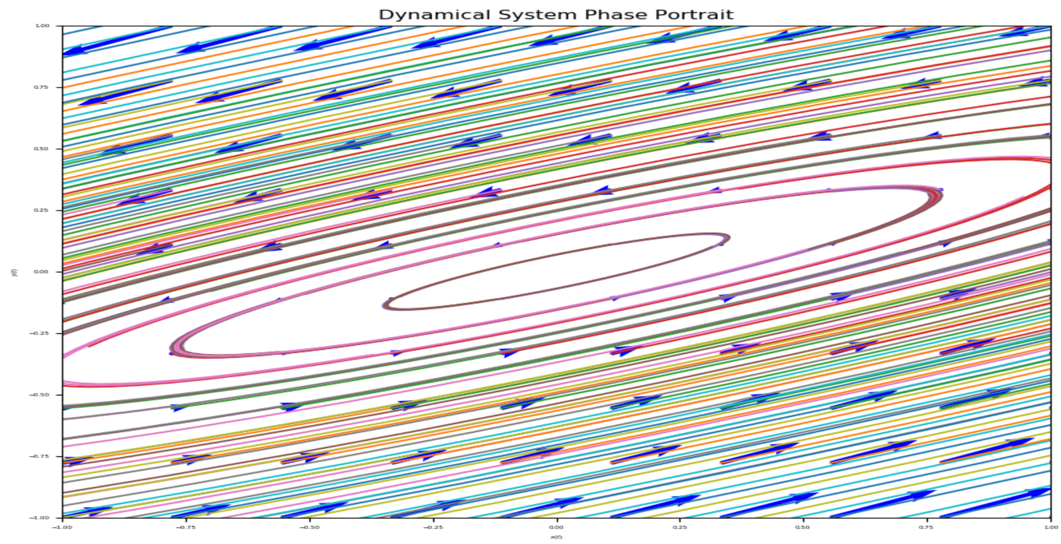

Figure 2: A plot of the dynamical system of ODEs using RK4 method

(iii) Given the dynamical systems in (3), the 4^{th} order Runge-Kutta solution is plotted below.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x} \tag{3}$$

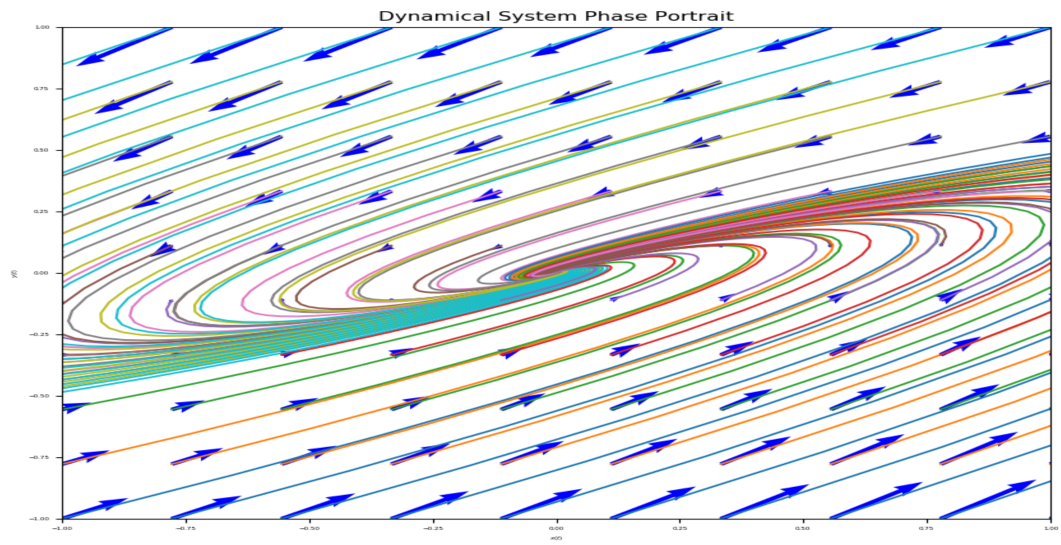

Figure 3: A plot of the dynamical system of ODEs using RK4 method

(iv) Given the dynamical systems in (4), the 4^{th} order Runge-Kutta solution is plotted below.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} \tag{4}$$

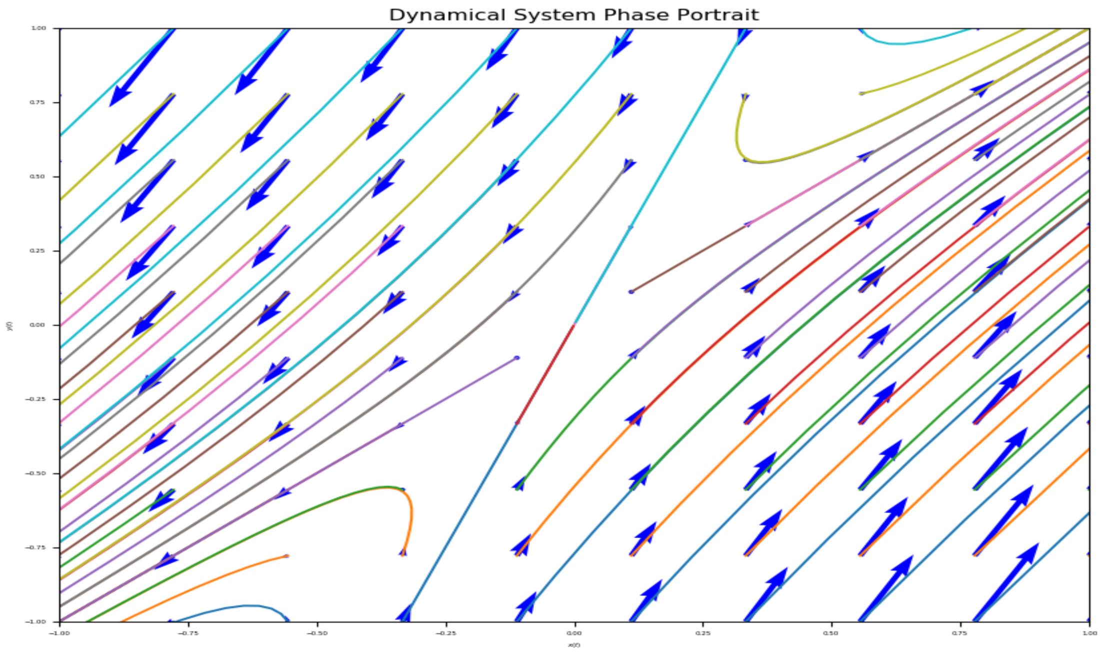

Figure 4: A plot of the dynamical system of ODEs using RK4 method

Exercise 2

Given

$$\frac{dx}{dt} = y - x^2, \quad \frac{dy}{dt} = x - 2 \tag{5}$$

(i) We calculate the equilibrium point as thus:

$$\frac{dx}{dt} = y - x^2 \tag{6}$$

$$\frac{dy}{dt} = x - 2 \tag{7}$$

To derive the equilibrium point of the above system, we set it equal to 0 as indicated below.

$$\frac{dx}{dt} = y - x^2 = 0$$

$$\frac{dy}{dt} = x - 2 = 0$$

$$\begin{aligned} y - x^2 = 0 \quad &\implies x = 2 \\ x - 2 = 0 \quad & y = 4 \end{aligned}$$

Thus the equilibrium points of the system is:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

(ii) We proceed as follows to linearize the dynamical system around the equilibrium in (5).

$$\begin{aligned} -cy + dxy &= 0 \\ y(-c + dx) &= 0 \quad \text{we divied through by } y \\ -c + +dx &= 0 \\ dx &= x \\ \therefore x &= \frac{c}{d} \end{aligned}$$

Let

$$f_1 = y - x^2 \tag{8}$$

and

$$f_2 = x - 2 \tag{9}$$

Taking a partial derivative of 8 and 9 w. r. t. x and y

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

and substitute in the matrix to get,

$$B = \begin{pmatrix} -2x & 1 \\ 1 & 0 \end{pmatrix} \tag{10}$$

solving the matrix (10) at (2,4), we have:

$$B = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{12}$$

Hence shown.

(iii) We find the eigenvalue and eigenvector of the linearized system, and the character of the equilibrium.

We are required to find Eigenvalues using the matrix we have found.

$$B = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} \tag{13}$$

$$\lambda^2 - tr(B)\lambda + det(B) = 0 \tag{14}$$

replacing for $tr(B) = -4$ and $det(B) = -1$ in equation (14),

$$\lambda^2 + 4\lambda - 1 = 0 \tag{15}$$

To find eigen values we solve (18) to get

$$\begin{aligned} \lambda_1 &= \sqrt{5} - 2 \\ \lambda_2 &= -(2 + \sqrt{5}) \end{aligned}$$

using $(B - \lambda I)\vec{x} = 0$ to get eigenvector
if $\lambda_1 = \sqrt{5} - 2$:

$$\begin{pmatrix} -2-\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{16}$$

and

$$(-2-\sqrt{5})x + y = 0 \Rightarrow y = (2+\sqrt{5})x.$$

Letting $x = \alpha$ so that $y = (2+\sqrt{5})\alpha$ and then

$$v_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2+\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 \\ 2+\sqrt{5} \end{pmatrix}$$

if $\alpha = 1$
If $\lambda_1 = -(2+\sqrt{5})$, we have

$$\begin{pmatrix} -2+\sqrt{5} & 1 \\ 1 & 2+\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{17}$$

and

$$(-2+\sqrt{5})x + y = 0 \Rightarrow y = (2-\sqrt{5})x.$$

Letting $x = \alpha$ so that $y = (2-\sqrt{5})\alpha$ and then

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2-\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\sqrt{5} \end{pmatrix}$$

if $\alpha = 1$
The general solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{\lambda_1 t} \vec{v_1} + C_2 e^{\lambda_2 t} \vec{v_2}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{(-2+\sqrt{5})t} \begin{pmatrix} 1 \\ 2+\sqrt{5} \end{pmatrix} + C_2 e^{(-2-\sqrt{5})t} \begin{pmatrix} 1 \\ 2-\sqrt{5} \end{pmatrix}$$

$$x(t) = C_1 e^{(-2+\sqrt{5})t} + C_2 e^{(-2-\sqrt{5})t} \tag{18}$$

and

$$y(t) = C_1(2+\sqrt{5}) e^{(-2+\sqrt{5})t} + C_2(2-\sqrt{5}) e^{(-2-\sqrt{5})t} \tag{19}$$

It is a saddle point and equilibrium point stability is unstable.

(iv) Given the dynamical systems above the 4^{th} order Runge-Kutta solution is plotted below.

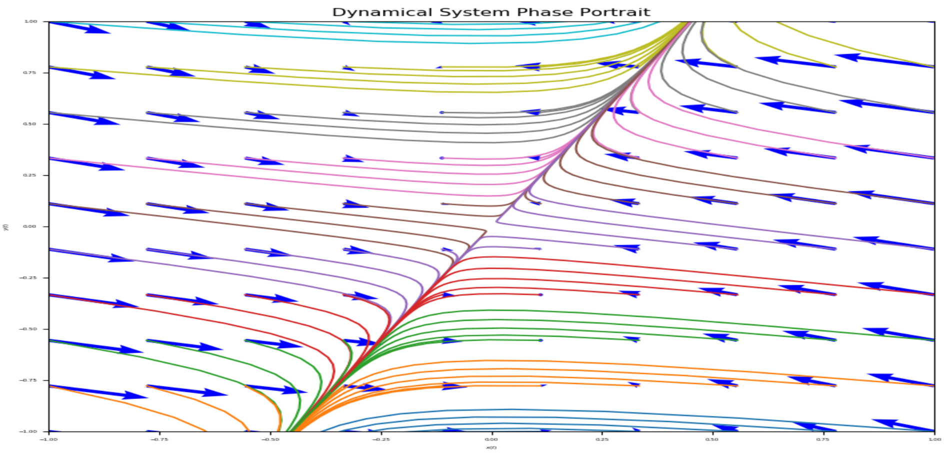

Figure 5: A plot of the dynamical system of ODEs using RK4 method