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1 The Green's function for the IVP

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{, with } y(0) = 0 \text{ and } y'(0) = 0 \tag{1}$$

is given by

$$G(x,s) = \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{W[y_1,y_2](s)} \tag{2}$$

where y_1 and y_2 are independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

(a) Functions

$$\tilde{y}_1(x) = ay_1(x) + by_2(x)$$
$$\tilde{y}_2(x) = cy_1(x) + dy_2(x)$$

are also solutions of (3). Give a condition on a, b, c and d which make $\tilde{y}_1(x)$ and $\tilde{y}_2 x)$ independent solutions. Differentiating the equations for $\tilde{y}_1(x)$ and $\tilde{y}_2 x)$ we obtain

$$\tilde{y}_1(x) = ay_1'(x) + by_2'(x)$$
$$\tilde{y}_2(x) = cy_1'(x) + dy_2'(x)$$

Substituting this into $W[\tilde{y}_1, \tilde{y}_2]$ gives after a small amount of algebra

$$W[\tilde{y}_1^{;}\tilde{y}_2^{]} = (ad - bc)(y_1y_2' - y_2y_1') = (ad - bc)W[y_1, y_2]$$

Since y_1 and y_2 are independent solutions we know that $W[y1, y2](x) \neq 0$. Hence

$$W[\tilde{y}_1, \tilde{y}_2] \neq 0 \iff ad - bc \neq 0$$

Note that this is simply the condition that the linear transformation relating $\tilde{y}_1', \tilde{y}_2'$ to y_1, y_2 has non-zero determinant and is therefore invertible.

(b) Using the above equations we calculate that

$$\tilde{y}_1(s)\tilde{y}_2(x) - \tilde{y}_1(x)\tilde{y}_2(s) = (ad - bc)y_2(s)y_2(x) - y_1(x)y_2(s)$$

We already know that

$$W[\tilde{y}_1, \tilde{y}_2](s) = (ad - bc)W[y_1, y_2](s)$$

So that using formula (2) but with $\tilde{y}_1$ and

$$\tilde{y}_2$$

gives:

$$\begin{aligned}
\frac{\tilde{y}_1(s)\tilde{y}_2(x) - \tilde{y}_1(x)\tilde{y}_2(s)}{W[\tilde{y}_1, \tilde{y}_2]}(s) &= \frac{(ad - bc)y_1(s)y_2(x) - y_1(x)y_2(s)}{(ad - bc)W[y_1, y_2](s)} \\
&= \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{W[y_1, y_2](s)} \quad \text{Since}(ad - bc) \neq 0 \\
&= G(x, s)
\end{aligned}$$

So that the Green's function constructed from $\tilde{y}_1$ and $\tilde{y}_2$ is identical to that constructed from y_1 and y_2.

(c) To Green's function for the IVP

$$y'' - 4y' + 4y = r(x) \qquad \text{with } y(0) = 0 \text{ and } y\prime(0) = 0$$

and use this to obtain the solution to the above problem when $r(x) = xe^{2x}$. The homogeneous equation is

$$y'' - 4y' + 4y = 0$$

with characteristic equation

$$m^2 - 4m + 4 = 0$$

This is just $(m - 2)^2 = 0$ so that the solution is $m = 2$ (twice). We may therefore take $y_1(x) = e^{2x}$ and $y2(x) = xe^{2x}$ as independent solutions of the homogeneous equation. The Wronskian is given by

$$W[y_1, y_2](x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

Hence substituting into equation (2) gives:

$$G(x, s) = \frac{e^{2s}xe^{2x} - e^{2x}se^{2s}}{e^{4s}} = xe^{2x}e^{-2s} - e^{2x}se^{-2s}$$

Note (although we will not use this fact below) that we can write the above as $G(x, s) = (x - s)e^{2(x-s)}$ so that $G(x, s)$ can be written as a function of $(x - s)$. To solve the equation

$y'' - 4y' + 4y = xe^{2x}$ with $y(0) = 0$ and $y\prime(0) = 0$ we use the Green's function calculated above and $r(x) = xe^{2x}$

$$\begin{aligned}
y(x) &= \int_{s=0}^{s=x} G(x,s)r(s)ds \\
&= \int_{s=0}^{s=x} (xe^{2x}e^{-2s} - e^{2x}se^{-2s})se^{2s}ds \\
&= xe^{2x}\int_{s=0}^{s=x} sds - e^{2x}\int_{s=0}^{s=x} s^2ds \\
&= xe^{2x}(\frac{x^2}{2}) - e^{2x}(\frac{x^3}{3}) \\
&= (\frac{x^3}{6})e^{2x}
\end{aligned}$$

So that the required solution is $y(x) = (\frac{x^3}{6})e^{2x}$.

Exercise 2

(a) Given a boundary value problem,

$$y'' = \lambda y, \quad y(0) = 0, \quad y'(1) = 0, \quad x \in [0,1]. \tag{4}$$

In this subsection, required is to show that

$$\int_0^1 (y_m(x)y_n''(x) - y_n(x)y_m''(x))dx = 0 \tag{5}$$

We must prove equation 5 without calculating $y_n(x)$. Using integration by parts, and letting $u_1(x) = y_m(x)$, $u_2(x) = y_n(x)$, $v_1' = y_n''(x)$ and $v_2' = y_m''(x)$.
This therefore means that, $u_1'(x) = y_m'(x)$, $u_2'(x) = y_n'(x)$, $v_1 = y_n'(x)$ and $v_2 = y_m'(x)$. So, using the fact that $\int uv'dx = uv - \int vu'dx$, equation 5 becomes;

$$[y_m(x)y_n'(x) - y_n(x)y_m'(x)]_0^1 + \int_0^1 (y_n'(x)y_m'(x) - y_m'(x)y_n'(x))dx \tag{6}$$

The term integrated over $[0,1]$ vanishes because it's equal to zero. This leaves equation 6 as,

$$\Big[y_m(x)y_n'(x) - y_n(x)y_m'(x)\Big]_0^1$$

Evaluating this gives, $[y_m(1)y_n'(1) - y_n(1)y_m'(1)] - [y_m(0)y_n'(0) - y_n(0)y_m'(0)]$. Using boundary conditions stated in equation 4 implies that 5 is true.

Therefore, $\int_0^1 (y_m(x)y_n''(x) - y_n(x)y_m''(x))dx = 0$

Hence part

Using the fact that $y_n'' = \lambda_n y_n$ and $y_m'' = \lambda_m y_m$. Equation 5 then becomes.

$$\int_0^1 (\lambda_n y_m(x) y_n(x) - \lambda_m y_m(x) y_n(x))dx = 0$$

This can be written as, $(\lambda_n - \lambda_m) \int_0^1 y_m(x) y_n(x)dx = 0$. For n$\neq$ m, $\implies \int_0^1 y_m(x) y_n(x)dx = 0$

(b) In this subsection, required is to find the eigenvalues and eigenfunctions for the BVP given in equation 4. Since $\lambda \in \mathbb{R}$ we shall proceed by looking at the possible cases. That is, when $\lambda = 0, \lambda > 0$ and $\lambda < 0$.

Case (i)

When $\lambda = 0$; the ODE in equation 4 then becomes $y'' = 0$ with a solution given $y(x) = Ax + B$ Where A and B are constants. Then, $y'(x) = A$. Applying BV conditions, $y(0) = 0 \implies B = 0$ and $y'(1) = 0 \implies A = 0$. This implies that for $\lambda = 0$ the solution is trivial $\forall$ x.

Case (ii)

When $\lambda > 0$; if we define $\lambda := \mu^2$, then the general solution of the ODE in 4 becomes

$$y(x) = Ae^{\mu x} + Be^{-\mu x},$$

Then

$$y'(x) = \mu(Ae^{\mu x} - Be^{-\mu x}),$$

Applying BV conditions, $y(0) = 0 \implies A + B = 0 \iff A = -B$ and for $y'(1) = 0$ implies $\mu(Ae^{\mu} - Be^{-\mu}) = 0$ Since $A = -B$, then $\mu A(e^{\mu} - e^{-\mu}) = 0$. Because $\mu > 0$, this means that $A = B = 0$ since $e^{\mu} - e^{-\mu} \neq 0$. There for the solution is trivial $\forall$ x.

Case (iii)

When $\lambda < 0$; we define $\lambda := -\mu^2$ The ODE in equation 4 then has a solution of the form $y(x) = A\cos(\mu x) + B\sin(\mu x)$. Applying boundary conditions; For $y(0) = 0 \implies A = 0$. This then means that our solution is, $y(x) = B\sin(\mu x)$ and so $y'(x) = B\mu\cos(\mu x)$; For $y'(1) = 0$, this means that $B\mu\cos(\mu) = 0$. A trivial solution is obtained when $B = 0$.
When $B \neq 0$, a non-trivial solution is obtained. This is true only if

$$\cos(\mu) = 0 \iff \mu = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2} + 2\pi, \cdot\cdot$$

This means that for the solution to be non-trivial, $\mu_k = \frac{\pi}{2} + k\pi \quad ; k \in \mathbb{Z}$.

For **eigenvalues**, we recall that $\lambda = -\mu^2 \implies \lambda_n = -\left(\frac{(2n+1)\pi}{2}\right)^2 \quad$; n$\in \mathbb{N}$

Given the non-trivial solutions, $y(x) = B\sin(\mu x)$, the **eigenfunctions** are given by;

$$y_n(x) = B\sin\left(\frac{(2n+1)\pi}{2}x\right) \quad ; n \in \mathbb{N}$$

The set of eigenvalues used to deduce the eigenfunctions if picked from $\mathbb{N}$ to attain a set of distinct eigenvalues with a minimum eigenvalue as asserted by Sturm-Liouville theory.

For the orthogonality condition, we recall that $\int_0^1 y_m(x)y_n(x)dx = 0$. Substituting for $y_m(x)$ and $y_n(x)$, the eigenfunctions designed at the m^{th} and n^{th} eigenvalues respectively with $B = 1$ gives,

$$\int_0^1 sin\left(\frac{(2n+1)\pi}{2}x\right)\sin\left(\frac{(2m+1)\pi}{2}x\right)dx = 0$$

(c). We are required to calculate the Green's function for the IVP.

$$y'' - 4y' + 4y = r(x) \quad with \quad y(0) = 0, \quad and \quad y'(0) = 0 \tag{7}$$

and hence to use it to obtain the solution to the above problem when $r(x) = xe^{2x}$.

Firstly we need to find the Solution for homogeneous part

$$y'' - 4y' + 4y = 0$$

Setting

$$y = e^{rx} \Longrightarrow y' = re^{rx} \Longrightarrow y'' = r^2e^{rx}$$

Thus the auxiliary x-tics equation for the problem is

$$r^2 - 4r + 4 = 0 \Longleftrightarrow (r-2)(r-2) = 0 \Longrightarrow r = 2 \tag{8}$$

The problem has repeated root ($r = 2$). and therefore the solutions to the problem are

$$y_1(x) = e^{2x} \quad and \quad y_2(x) = xe^{2x}$$

.

y_1 and y_2 are the independent homogeneous equation's solution and from this two solutions we can obtain the Wroskian after getting the derivatives of y_1 and y_2.

$$y_1(x) = e^{2x} \Longrightarrow y_1'(x) = 2e^{2x}$$

Also

$$y_2(x) = xe^{2x} \Longrightarrow y_2'(x) = e^{2x} + 2xe^{2x}$$

Therefore;

$$\begin{aligned}W[y_1, y_2](x) &= y_1 y_2' - y_1' y_2 \\ &= e^{2x}(e^{2x} + 2xe^{2x}) - 2e^{2x} x e^{2x} \\ &= e^{4x} + 2xe^{4x} - 2xe^{4x} \\ &= e^{4x}\end{aligned} \tag{9}$$

Therefore to calculate the Green's function, we refer from equation **??**, which results to;

$$G(x, s) = \frac{e^{2s} x e^{2x} - e^{2x} s e^{2s}}{e^{4s}}$$

$$= xe^{2x} e^{-2s} - se^{2x} e^{-2s}$$

$$G(x, s) = (x - s)e^{2x} e^{-2s}$$

Therefore, the Green's function for the IVP is

$$G(x, s) = (x - s)e^{2(x-s)}$$

• We also supposed to find the solution of 7 when $r(x) = xe^{2x}$ using the obtained Green function; From

$$\begin{aligned}y(x) &= \int_{s=0}^{x} G(x, s) r(s) ds \\ &= \int_{s=0}^{x} (x - s)e^{2x} e^{-2s} s e^{2s} ds \\ &= \int_{s=0}^{x} (xe^{2x} - se^{2x}) s ds \\ &= \int_{s=0}^{x} xe^{2x} s ds - \int_{s=0}^{x} s^2 e^{2x} ds \\ &= xe^{2x} \int_{s=0}^{x} s ds - e^{2x} \int_{s=0}^{x} s^2 ds \\ &= xe^{2x} \left[\frac{s^2}{2}\right]_0^x - e^{2x} \left[\frac{s^3}{3}\right]_0^x \\ &= xe^{2x} \frac{x^2}{2} - e^{2x} \frac{x^3}{3} \\ &= \frac{x^3}{2} e^{2x} - \frac{x^3}{3} e^{2x} = \frac{x^3}{6} e^{2x}\end{aligned} \tag{10}$$

$\therefore$ The solution to the equation (7) is

$$y(x) = \frac{x^3}{6} e^{2x}$$

Exercise 3

(a) Graphical representation

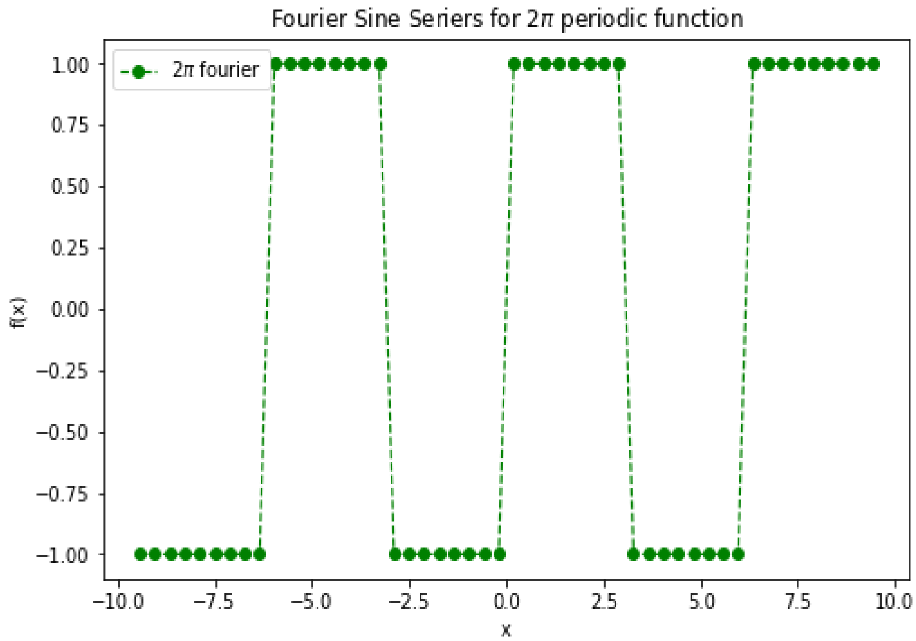

Figure 1: Graph of the function $f(x)$ for $-3\pi \leq x \leq 3\pi$

(b) Given is a 2π-periodic function, $f(x)$ defined on the interval $-\pi \leq x \leq \pi$.
Required is to explain why this function $f(x)$ has a Fourier Sine series.
By definition, the general Fourier Series is given by;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\} \tag{11}$$

In order to show tst this is true, we compute the value of the constant, a_n and show that $a_n = 0 \; \forall n \in \mathbb{N}$
By defiinition, a_n is given by;

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \tag{12}$$

For our piece wise function, we have

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos(nx)dx + \int_{0}^{\pi}(1)\cos(nx)dx\right] \\
&= \frac{1}{\pi}\left[-\frac{\sin(0)}{n} + \frac{\sin n(-\pi)}{n} + \frac{\sin(n\pi)}{n} - \frac{\sin(0)}{n}\right] \\
&= 0
\end{aligned}$$

Also, one can see this minus computing the integral. for an odd function, $f(-x) = -f(x)$. but $\cos(x)$ is an **even function** ($\cos(-nx) = \cos(nx)$), so the integral of $f(x)\cos(nx)$ is an **odd function**. This means that the integral over our piece wise intervals (-$\pi \leq x \leq 0$ and $0 \leq x \leq \pi$) will cancel out to leave a_n as zero.
This therefore proves the fact that $f(x)$ has a Fourier Sine series.r $\forall n \in \mathbb{N}$

(c) In this section, required is to calculate the Fourier series for $f(x)$. Since its known to be a Fourier sine series, it suffices to find the value of the constant b_n. By definition, it's known that,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
\implies b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin(nx)dx + \int_{0}^{\pi}(1)\sin(nx)dx\right] \\
&= \frac{1}{\pi}\left[\frac{\cos(nx)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]_{0}^{\pi} \\
&= \frac{1}{\pi n}\left[1 - (-1)^n - (-1)^n + 1\right] \\
&= \frac{2}{\pi n}\left[1 - (-1)^n\right]
\end{aligned}$$

The above result is true since $\cos(0) = 1$ and $\cos(-nx) = \cos(nx) = (-1)^n$.
With b_n known, equation (11) then becomes;

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\sin(nx)\right]$$

Which is the Fourier sine series for the piece wise function in question.