AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

(AIMS RWANDA, KIGALI)

Name: Yusuf Brima

Assignment Number: 1

Course: Fluid Mechanics

Date: February 20, 2021

Exercise 1

Given the 2nd order linear ODE

$$\frac{d^2z}{dt^2} + 3\frac{dz}{dt} + 3z = 0 \tag{1}$$

(a) The general solution.
We let

$$z(t) = e^{rt} \tag{2}$$

$$\therefore \frac{dz}{dt} = re^{rt} \tag{3}$$

$$\text{and } \frac{d^2z}{dt^2} = r^2e^{rt} \tag{4}$$

we substitute (2) , (3) and (4) into (1) which results in:

$$\begin{aligned} r^2e^{rt} + 3re^{rt} + 3e^{rt} &= 0 \\ (r^2 + 3r + 3)e^{rt} &= 0 \\ \therefore r^2 + 3r + 3 &= 0 \end{aligned}$$

$a = 1, b = 3, c = 3$

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^3 - 4(1 \times 3)}}{(2 \times 1)} \\ &= \frac{-3 \pm \sqrt{-3}}{(2)} \\ &= \frac{-3 + \sqrt{3}i}{(2)}, \frac{-3 - \sqrt{3}i}{(2)} \end{aligned}$$

$$
\begin{aligned}
r_1 &= \alpha = \frac{-3}{2}, \beta = \frac{\sqrt{3}}{2} \\
r_2 &= \alpha = \frac{-3}{2}, \beta = \frac{\sqrt{3}}{2} \\
z(t) &= e^{\alpha t}\left(A\cos(\beta t) + B\sin(\beta t)\right) \\
z(t) &= e^{\frac{-3}{2}t}\left(A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right)
\end{aligned}
$$

The general solution is thus:

$$
z(t) = e^{\frac{-3}{2}t}\left(A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right) \tag{5}
$$

(b) The particular solutions under the initial conditions,

$$
\begin{aligned}
z(0) &= 1 \\
\frac{dz}{dt}(0) &= 0
\end{aligned}
$$

From (5)

$$
\begin{aligned}
&z(0 = e^{0}\left(A\cos(0) + B\sin(0)\right) = 1 \\
&\therefore A = 1
\end{aligned}
$$

Also from (5)

$$
\frac{dz}{dt}(0) = \frac{-3}{2}A + frac{-3} - \sqrt{3}i(2)B = 0
$$

$$
\begin{aligned}
1 &= A \\
0 &= \frac{-3}{2}A + \frac{\sqrt{3}}{2}B \\
B &= \sqrt{3}
\end{aligned}
$$

The particular solution is thus:

$$
z(t) = e^{\frac{-3}{2}t}\left(\cos\left(\frac{3}{2}t\right) + \sqrt{3}\sin\left(\frac{3}{2}t\right)\right) \tag{6}
$$

(c) A plot of the trajectories.

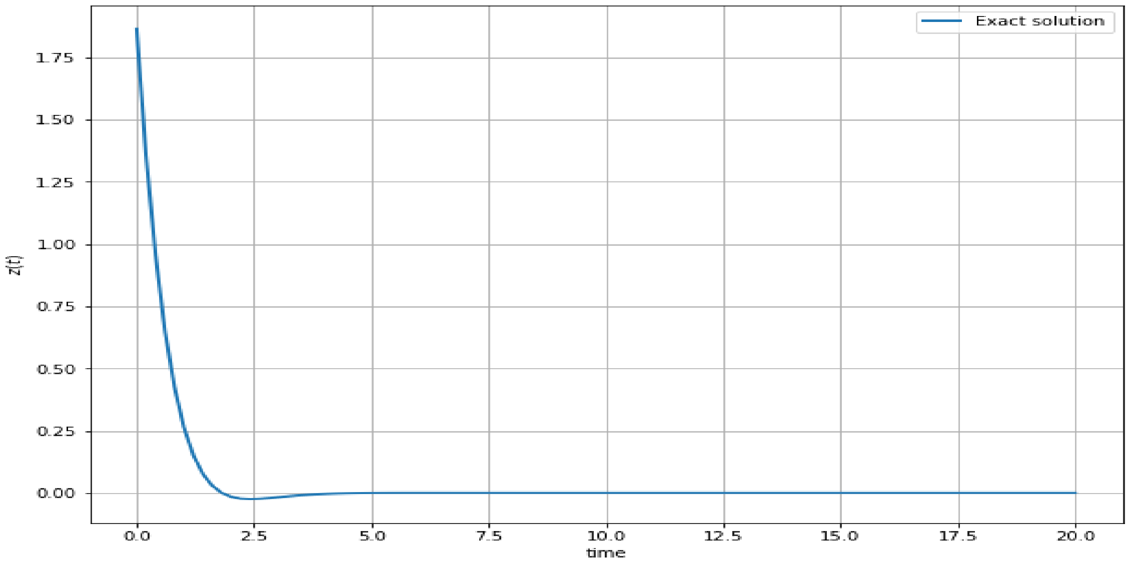

Figure 1: A plot of the particular solution of the 2^{nd} order linear ODE,

Exercise 2

The motion of a mass around an equilibrium position in a system of spring-mass-damper system can be described by the following 2nd order linear ODE

$$m\frac{d^t}{dt^2} + \gamma\frac{dz}{dt} + kz = 0 \quad \text{where } m > 0, \gamma > 0, k > 0 \tag{7}$$

Let $z(t) = e^{rt}$, $\frac{dz}{dt} = re^{rt}$, $\frac{d^z}{dt^2} = r^2e^{rt}$
Substituting the above expressions in (7), one gets

$$mr^2e^{rt} + \gamma re^{rt} + ke^{rt} = 0$$
$$\text{Discriminant } D = \gamma^2 - 4mk$$

The resulting the characteristic equation is thus:

$$mr^2 + \gamma r + k = 0$$
$$\text{with the discriminant } D = \gamma^2 - 4mk$$

we observe the 3 cases:

(1) $D < 0$ (This will be an **under-damped system** γ is small relative to $m$and k)
We have characteristic roots

$$r_1 = \frac{-\gamma}{2m} + \frac{\sqrt{\gamma^2 - 4mk}}{2m}, \quad r_2 = \frac{-\gamma}{2m} - \frac{\sqrt{\gamma^2 - 4mk}}{2m}$$

the general solution therefore is

$$z(t) = e^{\frac{-\gamma}{2m}t}\left(B_1\cos\left(\frac{\sqrt{\gamma^2 - 4mk}}{2m}t\right) + B_2\sin\left(\frac{\sqrt{\gamma^2 - 4mk}}{2m}t\right)\right)$$

(2) $D = 0$ (This will result in a **critically-damped system**, γ is just between over and under-damped)

$$\text{the characteristic root is} r = \frac{-\gamma}{2m}$$

Therefore, the general solution is:

$$z(t) = e^{\frac{-\gamma}{2m}t}\left((B_1 + B_2)\,t\right)$$

(3) $D > 0$ (This will result in an **over-damped system**, γ is large relative to m and k)

$$r = \frac{-\gamma \pm \sqrt{-\gamma^2 - 4mk}}{2m}$$

$$\text{where the characteristic roots are } r_1 = \frac{-\gamma + \sqrt{-\gamma^2 - 4mk}}{2m}, r_2 = \frac{-\gamma - \sqrt{-\gamma^2 - 4mk}}{2m}$$

Therefore, the general solution is:

$$z(t) = B_1 e^{r_1 t} + B_2 e^{r_2 t}$$

Exercise 3

(a)

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8}$$

Step 1: We find the eigenvalues

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$

$$\lambda^2 - trace(A)\lambda + det(A) = 0 \tag{9}$$

where A is the matrix of coefficients given system of equations.
trace(A)$= 3$ and det(A)$= -10$.

and we obtain the eigenvalues λ_1, λ_2 where $\lambda_1 = -2$ and $\lambda_2 = 5$.

Step 2: We find the eigenvectors for $\lambda_1 = -2$:

$$(A - \lambda I)\,\vec{x} = 0$$

where I is the identity matrix.

$$(A + 2I)\,\vec{x} = 0 \tag{10}$$

this gives the following resulting expression:

$$\begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The above system has linearly dependent basis vectors which results in a single equation and it is not possible to directly solve them at once.

$$x + y = 0$$

We let $y = \alpha$

$$\implies V_1 = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\therefore$ for v_1 using $\lambda_1 = -2$ is:

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ when } \alpha = 1$$

For $\lambda_2 = 5$:

$$(A - \lambda I)\,\vec{x} = 0$$

$$(A - 5I)\,\vec{x} = 0$$

which gives

$$\begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The above system has linearly dependent basis vectors which results in a single equation and it is not possible to directly solve them at once.

$$-4x + 3y = 0$$

We let $y = \alpha$

$$\implies V_2 = \begin{pmatrix} \frac{3}{4}\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}$$

$\therefore$ v_2 for $\lambda_2 = 5$ is:

$$V_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ when } \alpha = 4$$

The resulting general solution for the system of linear ODEs is thus:

$$x(t) = B_1 e^{-2t} + 3B_2 e^{5t}$$

$$y(t) = B_1 e^{-2t} + 4B_2 e^{5t}$$

where $x(t)$ and $y(t)$ are the two fundamental solution set.

Using Initial Conditions (ICs) in equation (8), one obtains

$$x(0) = -B_1 + 3B_2 = 1$$

and

$$y(0) = B_1 + 4B_2 = 1$$

The particular solution for the IVP problem is thus:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -\frac{1}{7}e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{2}{7}e^{5t}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

(b)

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{11}$$

Step 1: We first solve for the eigenvalues

$$A = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$$

$$\lambda^2 - trace(A)\lambda + det(A) = 0 \tag{12}$$

where A is the matrix of coefficients given,
trace(A)$= 3$ and det(A)$= -10$.

$$\lambda^2 - 1 = 0$$

This results in complex roots, $\lambda_1 = i$ and $\lambda_2 = -i$ which are the eigenvalues.

Step 2: We solve for the eigenvectors:

We let the eigenvector

$$v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$$

be the vector for $\lambda_1 = i$.

$$\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ai \\ b \end{pmatrix}$$

We derive the follow equation as follows

$$-2a - b = ia \Rightarrow -2a - ia = b$$

We suppose $a = \alpha$

$$\therefore v_1 = \alpha\begin{pmatrix} 1 \\ -2-i \end{pmatrix}$$

iff $\alpha = 1 \implies$

$$v_1 = \begin{pmatrix} 1 \\ -2-i \end{pmatrix}$$

We consider the case for $\lambda_2 = -i$

$$\implies v_2 = \begin{pmatrix} 1 \\ -2+i \end{pmatrix}$$

$$\therefore v = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The general solution for the differential equation is thus:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = B_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(t) - B_1 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin(t) + B_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(t) + B_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(t)$$

$$x(t) = B_1 \cos(t) + B_2 \sin(t)$$

and

$$y(t) = -2B_1 \cos(t) + B_1 \sin(t) - 2B_2 \sin(t) - B_2 \cos(t)$$

where $\{(x(t), y(t)\}$ are the fundamental solution set.

Using the initial conditions given in equation (11), we get:

$$x(0) = B_1 \tag{13}$$

$$y(0) = -2B_1 - B_2 \tag{14}$$

resulting in

$$\begin{aligned} B_1 &= 1 \\ B_2 &= -3 \end{aligned}$$

The particular solution for the given problem is as follows:

$$x(t) = \cos(t) - 3\sin(t) \tag{15}$$

$$y(t) = \cos(t) + 7\sin(t) \tag{16}$$

(c)

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{17}$$

Step 1: Firstly, we solve for the eigenvalues:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

where A is the matrix of coefficients in the given problem and trace(A)$= 4$ and det(A)$= 4$.

$$\lambda^2 - 4\lambda + 4 = 0$$

The resulting eigenvalues λ_1, λ_3 are $\lambda_1 = 2$ and $\lambda_2 = 2$.

Step 2: We proceed with finding the eigenvectors:
For $\lambda_1 = 2$:

$$(A - \lambda I)\,\bar{x} = 0$$

$$(A - 2I)\,\bar{x} = 0$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The above system has linearly dependent basis vectors which results in a single equation and it is not possible to directly solve them at once.

$$-x - y \;=\; 0$$

We let $y = \alpha$

$$\implies \vec{v_1} = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The resulting eigenvector v_1 for $\lambda_1 = 2$ is thus:

$$\vec{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ iff } \alpha = 1$$

We solve for the second eigenvector v_2 as follows:

$$(A - 2I)\,\vec{v_2} = \vec{v_1}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The above system has linearly dependent basis vectors which results in a single equation and it is not possible to directly solve them at once.

$$x + y \;=\; 1$$

We therefore suppose $y = \alpha$.

$$\implies \vec{v_2} = \begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The general solution for the system of ODEs is thus:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = B_1 e^{\lambda t}\vec{v_1} + B_2\left(te^{\lambda t}\vec{v_1} + e^{\lambda t}\vec{v_2}\right)$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = B_1 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + B_2\left(te^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$x(t) = -B_1e^{2t} - B_2te^{2t} + B_2e^{2t}$$

and

$$y(t) = B_1e^{2t} + B_2te^{2t}$$

where $\{x(t), y(t)\}$ are our fundamental solution set.

The particular solution is thus:

$$x(t) = -e^{2t} - 2te^{2t} + 2e^{2t}$$

and

$$y(t) = e^{2t} + 2te^{2t}$$

Exercise 4

A numerical solution to the equation of motion for different versions of a one-dimensional oscillator using the Verlet/leapfrog method with Damping where $\Delta t = 0.01$.

(a) Numerical solution for the harmonic oscillator equation with damping

$$\ddot{x} = -x - \beta\dot{x} \tag{18}$$

Using the initial conditions

$x(0) = 1$

$\dot{x}(0) = 1$

for several values of $\beta > 0$, under- damped ($\beta < 2$) and overdamped ($\beta > 2$) motions.

A plot $x(t)$ for under-damped and over-damped systems where $0 < \beta \leq 8$ and the phase diagram for the different β.

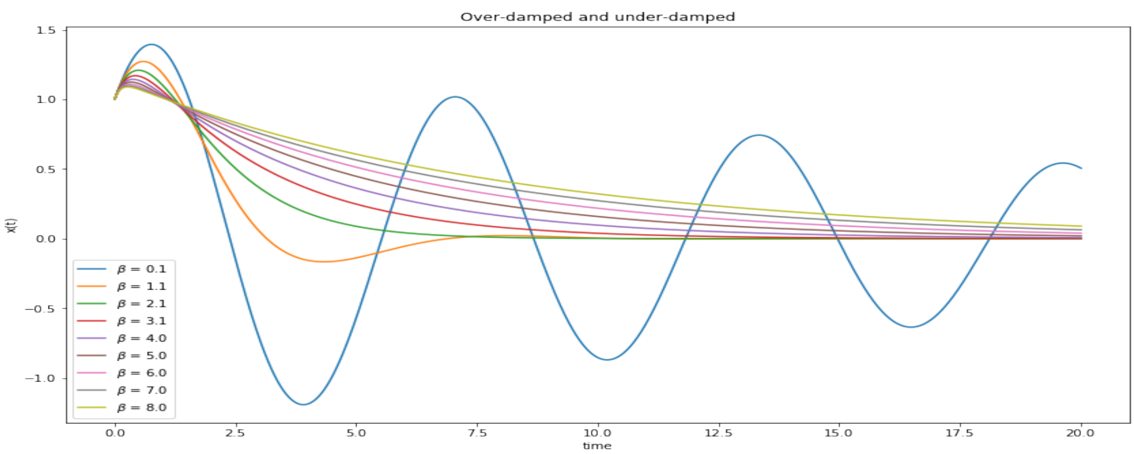

Figure 2: A plot of $x(t)$ for the under-damped system with varying β values

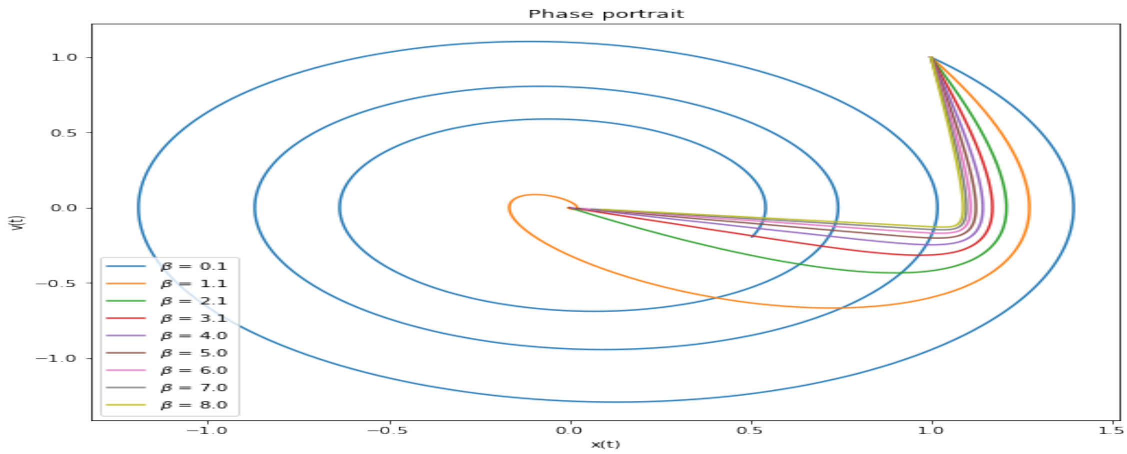

Figure 3: A plot of the phase portrait $x(t)$ vs $v(t)$ for the under-damped and over-damped systems with varying β values

(b) When we switch to a friction force that is more realistic under many circumstances, replacing βx with $-\beta \sin \dot{x}$. The case of the initial condition $\dot{x} = 0$, the friction force is directed against the returning force x and it is equal in magnitude to $\min(|x|, \beta)$.

(i) A Plot $x(t)$ under-damped and over-damped systems where $0 < \beta \leq 8$ and the phase diagram for the different β.

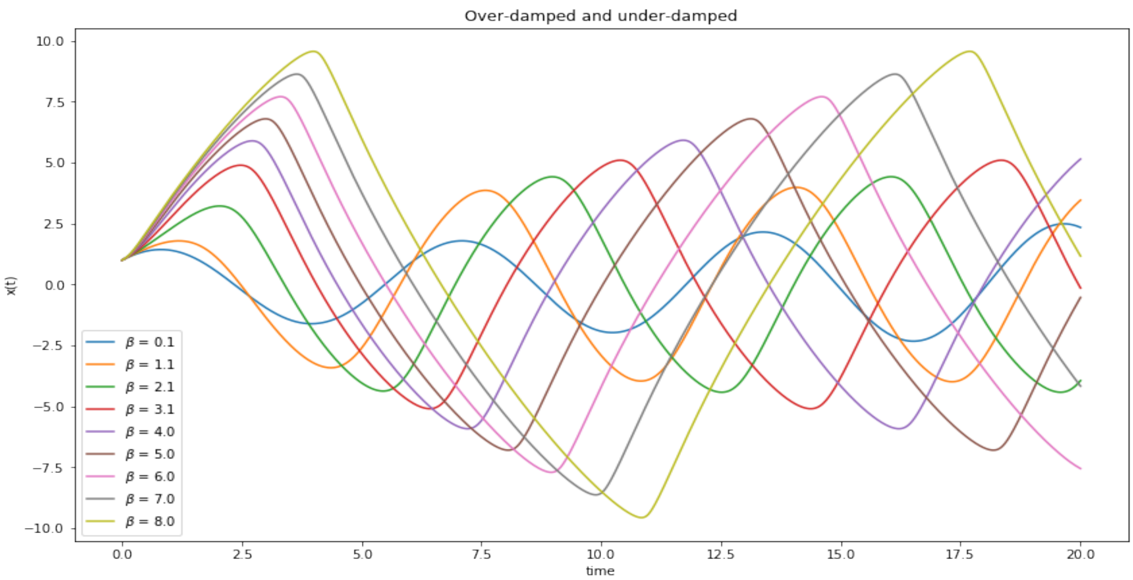

Figure 4: A plot of $x(t)$ with varying β values

(ii) A plot of the phase diagram for different values of β.

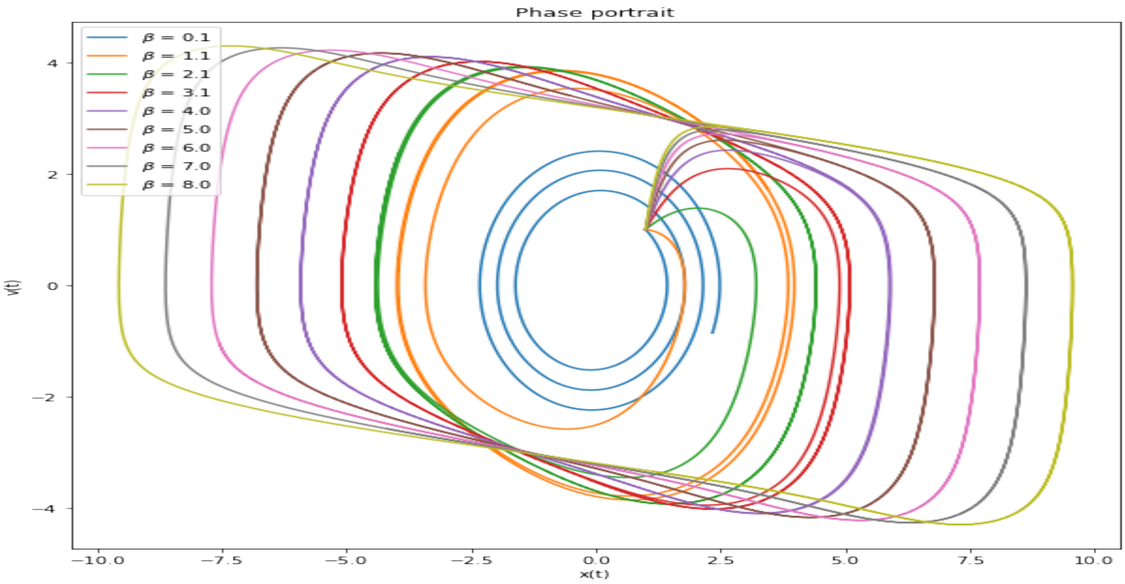

Figure 5: A plot of the phase portrait $x(t)$ vs $v(t)$ with varying β values