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Exercise 1

(a) $\delta(t) = 0.05 + 0.04t, \qquad 0 \leq t \leq \frac{1}{2}$
$\delta(t) = 0.07 - 0.04\left(t - \frac{1}{2}\right)^2, \qquad \frac{1}{2} \leq t \leq 1$
$C = £10,000$
Withdraw $= £10,000$

$$u\left(0, \frac{1}{4}\right) = e^{\int_0^{\frac{1}{4}}(0.05+0.04t)dt} \quad = 1.0138$$
$$u\left(\frac{1}{4}, \frac{1}{2}\right) = e^{\int_{\frac{1}{4}}^{\frac{1}{2}}(0.05+0.04t)dt} \quad = 1.01638$$
$$u\left(\frac{1}{2}, \frac{3}{4}\right) = e^{\int_{\frac{1}{2}}^{\frac{3}{4}}\left(0.07-0.04\left(t-\frac{1}{2}\right)62\right)dt} \quad = 1.0174$$
$$u\left(\frac{3}{4}, 1\right) = e^{\int_{\frac{3}{4}}^{1}\left(0.07-0.04\left(t-\frac{1}{2}\right)62\right)dt} \quad = 1.01617$$

The about she earned at the first 3 months is:

$$\begin{aligned}
&= £10,000u\left(0, \frac{1}{4}\right) - 1000 \\
&= £10,000.0138 - £1000 \\
&= £9138 \times 1.01638 - £1000 \\
&= £8287.6804
\end{aligned}$$

The about she earned at the first 9 months is:

$$\begin{aligned}
&= 8287.6804u\left(\frac{1}{2}, \frac{3}{4}\right) - £1000 \\
&= 8287.6804.0138 - £1000 \\
&= 9138 \times 1.0174 - £1000 \\
&= £7431.88608
\end{aligned}$$

The about she earned over the whole year is:

$$\begin{aligned}
&= 7431.88608u\left(\frac{3}{4},1\right) \\
&= £7431.88608.01617 \\
&= £7552.059678
\end{aligned}$$

(b) Nominal Interest Rate
$u\,(t, t+h) = 1 + hi_h(t), \quad \text{where} i_h(t) = \frac{v(t,t+h)-1}{h}$
$h = \frac{1}{2}, \quad t = \frac{1}{2}$

$$\begin{aligned}
i_{\frac{1}{2}}(\frac{1}{2}) &= \frac{u\left(\frac{3}{4},1\right)-1}{\frac{1}{2}} \\
&= \frac{e^{\int_{\frac{1}{2}}^{1}\left(0.07-0.04\left(t-\frac{1}{2}\right)^2\right)}dt}{\frac{1}{2}} \\
&= \frac{e^{\left[0.08-\frac{0.04t^3}{3}-0.02t^2\right]_{\frac{1}{2}}^{1}-1}}{\frac{1}{2}} \\
&= \frac{e^{\frac{1}{75}}-1}{\frac{1}{2}} \\
&= \frac{1.0134-1}{\frac{1}{2}} \\
&= 0.069779
\end{aligned}$$

Exercise 2

(a)

$$u = 1 + i \tag{1}$$

$$v = \frac{1}{u}, \tag{2}$$

$$uv = 1 \tag{3}$$

(i)

$$\begin{aligned}
a_{\overline{n}|} &= \frac{1}{1+i}\left(\frac{1-v^n}{1-\frac{1}{1+i}}\right) \\
&= \frac{1}{1+i}\left(\frac{1-v^n}{\frac{1+i-1}{1+i}}\right) \\
&= \frac{1}{1+i}\left((1-v^n)\frac{1+i}{i}\right) \\
&= \frac{1+v^n}{i}
\end{aligned}$$

(ii)

$$
\begin{aligned}
s_{\overline{n}|} &= (1+i)^n a_n \\
&= \frac{v^n - 1}{i}
\end{aligned}
$$

From (1) $\implies i = u - 1$
$\therefore$

$$
\begin{aligned}
a_{\overline{n}|} &= \left(\frac{\frac{1}{v^n} - 1}{u-1}\right) \\
&= \left(\frac{\frac{1-v^n}{v^n}}{u-1}\right) \\
&= \left(\frac{1-v^n}{v^n}\frac{1}{u-1}\right) \\
&= \left(\frac{1-v^n}{i}\frac{i}{v^n}\right) \\
&= \left(\frac{1-v^n}{i}u^n\right) \\
&= (1=i)^n\, a_{\overline{n}|}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\frac{1}{a_{\overline{n}|}} - \frac{1}{s_{\overline{n}|}} &= \frac{i}{1-v^n} - \frac{1}{(1+i)^n}\frac{i}{1-v^n} \\
&= \frac{i}{1-v^n} - \frac{i}{u^n(1-v^n)} \\
&= \frac{iu^n - i}{(1-v^n)u^n} \\
&= \frac{i(u^n-1)}{u^n - (uv)^n} \quad \text{from equation (2) } uv = 1 \\
&= \frac{i(u^n-1)}{(u^n-1)} \\
&= i
\end{aligned}
$$

(b) If $a_n = 8.3064$ and $s_n = 14.2068$

$$
\begin{aligned}
i &= \frac{1}{a_n} - \frac{1}{s_n} \\
&= \frac{1}{8.3064} - \frac{1}{14.2068} \\
&= 0.05 \\
&= 5\%
\end{aligned}
$$

For n, we solve as follows:

$$
\begin{aligned}
s_n &= (1+i)^n \, a_n \\
(1+i)^n &= \frac{s_n}{a_b} \implies n \ln(1+i) \\
n &= \frac{\ln(s_n)}{\ln(1+i)} = 11
\end{aligned}
$$

Exercise 3

(a) Hint

$v^{m+1} + v^{m+2} + v^{m+3} + \cdots + v^{m+n} = a_{\overline{m+n}|} - a_{\overline{n}|}$

£15, 00 $\implies$ 15 years + £100 and £200

At interest rate $i = 4\%$ per annum $= 0.04$

The expression for the Present Value PV (initial amount of annual payment) is thus:

$$PV = 5,000 - Xa_{\overline{15}|} - (100(a_{\overline{15}|} - a_{\overline{5}|})) - (200(a_{\overline{15}|} - a_{\overline{10}|}))$$

And to solve for the the initial amount of the annual payment, we proceed as follows:

$$0 = 5,000 - Xa_{\overline{15}|} - (100(a_{\overline{15}|} - a_{\overline{5}|})) - (200(a_{\overline{15}|} - a_{\overline{10}|}))$$

But from equation (1),
$u = 1.04$
it follows that $v = \frac{1}{u}$ from equation (3)

$$
\begin{aligned}
\implies a_{\overline{n}|} &= \frac{1 - v^n}{i} \\
&= \frac{1 - (1.04)^{-n}}{0.04}
\end{aligned}
$$

$$
\begin{aligned}
a_{\overline{15}|} &= \frac{1 - v^{15}}{i} \\
&= \frac{1 - (1.04)^{-15}}{0.04} \\
&= 11.12
\end{aligned}
$$

$$
\begin{aligned}
a_{\overline{5}|} &= \frac{1 - v^{5}}{i} \\
&= \frac{1 - (1.04)^{-5}}{0.04} \\
&= 4.45
\end{aligned}
$$

$$
\begin{aligned}
a_{\overline{10|}} &= \frac{1-v^{10}}{i} \\
&= \frac{1-(1.04)^{-10}}{0.04} \\
&= 8.11
\end{aligned}
$$

$$
\begin{aligned}
0 &= 5,000 - 11.12x - (100(11.12 - 4.45)) - 200(11.12 - 8.11)) \\
x &= £335.52
\end{aligned}
$$

(b) To calculate the loan outstanding at the end of the 3rd year and hence calculate the interest paid in the 4th year, we proceed as follows:
We let Outstanding Loan be the variable OA, therefore, the Outstanding Loan for the 3rd year is:

$$
\begin{aligned}
OA_3 &= C(i+1)^n - (x(i+1)^{n-1} + x(i+1)^{n-2} + x) \quad \text{where x is the initial annual payment} \\
&= 5,000u^3 - (xu^2 + xu + x) \\
&= 5,000((1.04)^3 - 335.52((1.04)^2 + 1.04 + 1) \\
&= £4576.96
\end{aligned}
$$

The interest paid in the 4th year is:

$$
\begin{aligned}
I &= OA \times i \\
&= 4576.96 \times 0.04 \\
&= £183.0784
\end{aligned}
$$

(c) If at the end of the 7th year the investor requests that the loan be recalculated with level payments. the new amount of loan is thus:
First, we calculate the loan at the 7th year as stated below, where OL_7 stands for loan at the 7th year.

$$
\begin{aligned}
OL_7 &= 5,000u^7 - (xu^6 + xu^5 + xu^4 + xu^3 + xu^2 + (x+100)u + (x+100)) \\
&= £3725.6231
\end{aligned}
$$

$$
\begin{aligned}
z &= y a_{\overline{8|}} \\
&= y\frac{1-v^n}{i} \\
&= \frac{1-(1+i)^{-n}}{i} \\
\therefore y &= \frac{zi}{1-(1+i)^{-n}} \\
&= \frac{3725.6231 \times 0.04}{1-(1.04)^{-8}} \\
y &= £553.34 \quad \text{payable equally for the remaining 8 years}
\end{aligned}
$$

Exercise 4

(a) Given

$$150,000\bar{S_{\overline{2}|}} + 50,000\bar{S_{\overline{1}|}} = 25,000\int_0^{T-2} e^{-\delta t}dt$$

Proof

$$T = 2 - \frac{\ln[1 - 2\delta(1+i)(4-3i)]}{\delta}$$

It follows that

$$150,000u^2 + 50,000u = 25,000\int_0^{T-2} e^{-\delta t}dt \tag{4}$$

$$150,000u^2 + 50,000u25,000\int_0^{T-2} e^{-\delta t}dt \quad \text{dividing both sides by 25,000 we get}$$

$$6u^2 + 2u = \int_0^{T-2} e^{-\delta t}dt$$

We solve for

$$\begin{aligned}
&= 6u^2 + 2u \quad \text{from equation (1), where } u = 1+i \\
&= 6(1+i)^2 + 2(1+1) \\
&= 2(1+i)(3(1+i)+1) \\
&= 2(1+i_(4+3i)
\end{aligned}$$

$$6u^2 + 2u = 2(1+i_(4+3i) \tag{5}$$

Also for $\int_0^{T-2} e^{-\delta t}dt$

$$\begin{aligned}
\int_0^{T-2} e^{-\delta t}dt &= \left[\frac{-e^{-\delta t}}{\delta}\right]_0^{T-2} \\
&= \frac{-e^{-\delta(T-2)}}{\delta} + \frac{1}{\delta} \\
&= \frac{1 - e^{-\delta(T-2)}}{\delta}
\end{aligned}$$

$$\int_0^{T-2} e^{-\delta t}dt = \frac{1 - e^{-\delta(T-2)}}{\delta} \tag{6}$$

So we equate (5) and (6) as follows:

$$\begin{aligned}
2(1+i_(4+3i) &= \frac{1-e^{-\delta(T-2)}}{\delta} \\
1-e^{-\delta(T-2)} &= 2\delta(1+i)(4+3i) \\
e^{-\delta(T-2)} &= 1-2\delta(1+i)(4+3i)
\end{aligned}$$

We apply natural log (ln) to both sides

$$\begin{aligned}
-\delta(T-2) &= \ln(1-2\delta(1+i)(4+3i)) \\
T-2 &= \frac{-\ln\left(1-2\delta(1+i)(1+i)\right)}{\delta} \\
T &= 2-\frac{\ln\left(1-2\delta(1+i)(1+i)\right)}{\delta}
\end{aligned}$$

Given $\delta = \ln(1+i) = \ln(1.075)$

$$\begin{aligned}
T &= 2-\frac{(1-2\ln(1.075)(1.075)(4+3(0.075)))}{\ln(1.075)} \\
&= 44.44787 \text{ years} \\
&= 44.448 \text{ years}
\end{aligned}$$