Lecture 9 - Half Range Series and Convergence

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Lectures on partial Differential Equations

Half Range Series

The unknown coefficients a_n, b_n in the Fourier Series

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

are given by the Euler Formulas

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x, \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x.$$

In examples we have seen that often many of the terms vanish. These relate to whether the function f is *even* or *odd*.

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Even and odd functions

If a function $g(x)$ is *even* then $g(-x) = g(x)$. So

$$\int_{-L}^{L} g(x)\,\mathrm{d}x = \int_{-L}^{0} g(x)\,\mathrm{d}x + \int_{0}^{L} g(x)\,\mathrm{d}x$$
$$= 2\int_{0}^{L} g(x)\,\mathrm{d}x.$$

Even and odd functions

If a function $h(x)$ is *odd* then $h(-x) = -h(x)$. So

$$\int_{-L}^{L} h(x)\,\mathrm{d}x = \int_{-L}^{0} h(x)\,\mathrm{d}x + \int_{0}^{L} h(x)\,\mathrm{d}x$$
$$= 0.$$

Even and odd functions

We also have that the product of odd/even functions behaves as the product of odd/even integers, and that $\sin$ is odd and $\cos$ even, so we have that

$$g \text{ even} \Rightarrow \begin{cases} a_m & = \frac{2}{\pi}\int_0^\pi g(x)\cos(mx)\,\mathrm{d}x, \\ b_m & = 0, \end{cases}$$

$$h \text{ odd} \Rightarrow \begin{cases} a_m & = 0, \\ b_m & = \frac{2}{\pi}\int_0^\pi h(x)\sin(mx)\,\mathrm{d}x. \end{cases}$$

Half Range Series

Given a function f defined over only half the range; e.g., $0 \le x < \pi$. There are three possibilities:

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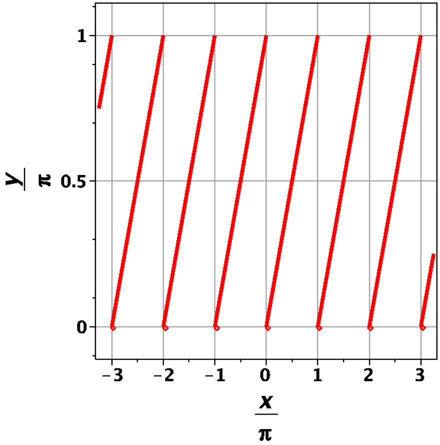
Periodic extension, period π
(NB **not** 2π hence factors of 2)

$$f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(2nx) + b_n \sin(2nx)$$

with general Euler Formulas

$$a_m = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \cos(2mx)\, \mathrm{d}x,$$

$$b_m = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \sin(2mx)\, \mathrm{d}x.$$



Half Range Series

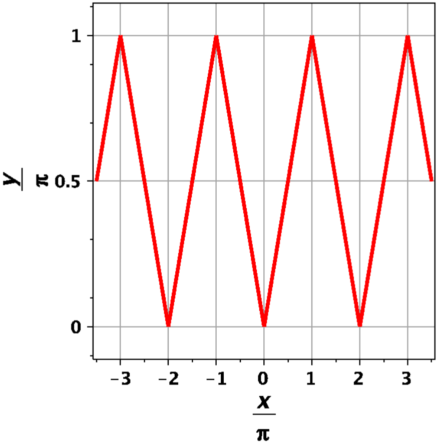
Given a function f defined over only half the range; e.g., $0 \leq x < \pi$. There are three possibilities:

Even extension. Fourier *Cosine* Series

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

with simplified Euler Formula

$$a_m = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(mx)\,\mathrm{d}x.$$



Half Range Series

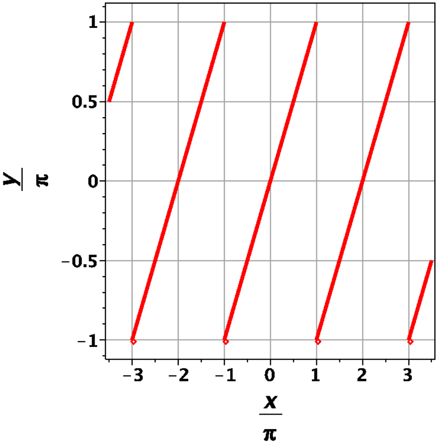
Given a function f defined over only half the range; e.g., $0 \leq x < \pi$.
There are three possibilities:

Odd extension. Fourier *Sine* Series

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

with simplified Euler Formulas

$$b_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(mx)\, dx.$$



Exercise

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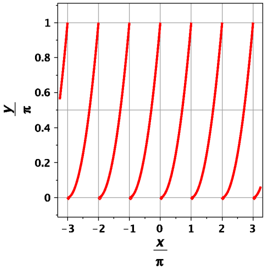

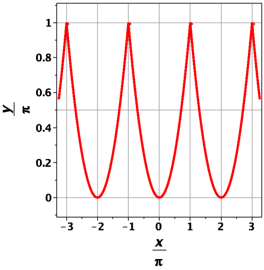

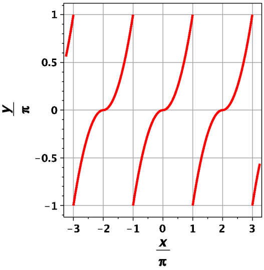

Check the extensions of $f = x^2$ with $0 \leq x < \pi$ are

$$\text{Periodic:} \quad a_m = \frac{2\left((-1)^m + 1\right)}{m^2}, \quad b_m = -\frac{\pi\left((-1)^m + 1\right)}{m},$$

$$\text{Even:} \quad a_m = \frac{4(-1)^m}{m^2} \qquad b_m = 0,$$

$$\text{Odd:} \quad a_m = 0, \qquad b_m = \frac{2\left(2\left((-1)^m - 1\right) - (-1)^m(\pi m)^2\right)}{\pi m^3}.$$

Fourier's Theorem

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The *Dirichlet conditions* are

1. $f(x)$ is bounded,
2. $f(x)$ is periodic,
3. $f(x)$ has a finite number of extrema and discontinuities on the period.

Fourier's Theorem essentially states that if f obeys the Dirichlet conditions then the Fourier series converges, and:

- where f is continuous the series converges to f;
- where f is discontinuous the series converges to the average value at the jump $\frac{1}{2}[f(x_-) + f(x_+)]$.

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Summary

- Even and odd functions have particularly simple Fourier Series.
- Half range expansions use these simple properties to get simple series appropriate for simple, but physically interesting, boundary conditions.
- The Dirichlet conditions
  1. $f(x)$ is bounded,
  2. $f(x)$ is periodic
  3. $f(x)$ has a finite number of extrema and discontinuities on the period.

  are sufficient to ensure convergence (in an average sense; note the behaviour at discontinuities).