Lecture 6 - Sturm-Liouville Theory

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Lectures on partial Differential Equations

Sturm-Liouville Form

We start by considering a generalisation of the previous eigenvalue problem by looking at an ODE of the form

$$y'' + b(x)y' + c(x)y = \lambda d(x)y$$

If we multiply this equation by $p(x) := e^{\int b(x)dx}$ we may write this in the *Sturm-Liouville* form

$$(p(x)y')' + q(x)y + \lambda w(x)y = 0$$

where $q(x) = p(x)c(x)$ and $w(x) = -p(x)d(x)$. We say a Sturm-Liouville problem is *regular* on an interval $[x_0, x_1]$ if $p(x)$ and $w(x)$ are positive functions. $w(x)$ is called the *weight function*
The equation can be written as an *eigenvalue* problem

$$L[y] = \lambda y \tag{1}$$

where $L[y]$ is the *linear differential operator* given by

$$y(x) \mapsto L[y](x) = \frac{-1}{w(x)}[(p(x)y'(x))' + q(x)y(x)]$$

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Self-adjoint operators

We are often interested in solving ODEs on some interval $x_0 \leq x \leq x_1$ with boundary conditions at the ends.
Given two functions f and g defined on this interval we define an *inner product* with respect to the weight function $w(x)$ by

$$< f, g >= \int_{x_0}^{x_1} f(x)g(x)w(x)dx$$

We say a linear differential operator is *self-adjoint* on the interval if and only if for all pairs of functions y_A and y_B satisfying the appropriate boundary conditions we have:

$$< L[y_A], y_B >=< y_A, L[y_B] >$$

Or in terms of integrals

$$\int_{x_0}^{x_1} L[y_A]y_B w(x)dx - \int_{x_0}^{x_1} y_A L[y_B]w(x)dx = 0$$

Using the definition of L this gives

$$\int_{x_0}^{x_1} w\left(y_A L y_B - y_B L y_A\right) \mathrm{d}x = \int_{x_0}^{x_1} \left[\left(p y_B'\right)' y_A + q y_A y_B - \left(p y_A'\right)' y_B - q y_B y_A\right] \mathrm{d}x$$

$$= \left[p y_A y_B' - p y_B y_A'\right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \left[-p y_B' y_A' + p y_A' y_B'\right] \mathrm{d}x$$

$$= \left[p y_A y_B' - p y_B y_A'\right]_{x_0}^{x_1}.$$

So that the equation is self-adjoint if, and only if

$$\left[p y_A y_B' - p y_B y_A'\right]_{x_0}^{x_1} = 0$$

Sturm-Liouville Boundary conditions

We say we have *Sturm-Liouville boundary conditions* if L is self-adjoint. This requires

$$\left[-py_A y_B' + py_B y_A'\right]_{x_0}^{x_1} = 0$$

This is guaranteed if **at the end points** x_0 and x_1 we have any of the following boundary conditions:

1. $y = 0$: *Dirichlet condition*.
2. $y' = 0$ *Neumann condition*.
3. $y + ky' = 0$ *Radiation condition*.
4. We can also have *periodic conditions* $y(x_0) = y(x_1)$, $y'(x_0) = y'(x_1)$, $p(x_0) = p(x_1)$.
5. Finally we can have *singular end point(s)* With $p = 0$ at the end point(s).

Solutions of equation (1) which satisfy (any combination of) these boundary conditions are *eigenfunctions* of the Sturm-Liouville problem.

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Properties of Sturm-Liouville eignevalue problems

- The eigenvalues λ_n are real.
- The eigenfunctions y_n are real.
- Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function. So that $< y_n, y_m >= \int_{x_0}^{x_1} w(x)y_n(x)y_m(x)dx = 0$ for $m \neq n$.
- There are infinitely many eigenvalues $\lambda_1 < \lambda_2 < \ldots$ that approach infinity.

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- The eigenfunctions are *complete* in the sense that for *any* bounded, piecewise continuous function $g(x)$ on $[x_0, x_1]$ one can find constants a_n such that the *error* ϵ_N

$$\epsilon_N = \int_{x_0}^{x_1} w(x) \left[g(x) - \sum_{n=1}^{N} a_n y_n(x) \right]^2 dx \tag{2}$$

  becomes arbitrarily small as $N \to \infty$.
- The eigenvalues are non-negative (ie. $\lambda_0 \geqslant 0$) provided that
  1. $q(x) \leq 0$ for $x \in [X_0, x_1]$
  2. $y'(x_0)y(x_0) - y'(x_1)y(x_1) \geqslant 0$ (this is automatic for boundary condtions(1)-(4))

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Summary

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- Solving eigenvalue problems means finding which values of the unknown constant λ allow solutions.
- Eigenvalue problems show up in a wide range of PDE problems as we shall see later.
- The rich theory of Sturm-Liouville problems outlined in the notes show that many eigenvalue problems have key features:
  - An infinite number of (positive) real, distinct eigenvalues $\lambda_1 < \lambda_2 < \ldots$;
  - Orthogonal eigenfunctions y_n

  As well as being Mathematically important, this has practical applications as the eigenfunctions correspond to the fundamental *modes* for the problem.