Lecture 5 - Boundary Value Problems

James Vickers

School of Mathematics,
University of Southampton, UK

Lectures on partial Differential Equations

Outline

Boundary value problems

University of Southampton
School of Mathematics

BVPs occur when the conditions to pick out a unique solution are imposed at different values of the independent variable x,
For example we wish to solve (1) with boundary condtion (2) at $x = 0$ and boundary condition (3) at $x = 1$

$$y'' + 2y' + y = 0, \tag{1}$$
$$y(0) + 2y'(0) = 1, \tag{2}$$
$$2y(1) - y'(1) = 0. \tag{3}$$

The general solution can found using whatever standard method works.
However, imposing the boundary conditions may lead to qualitatively different results.

Boundary value problems

BVPs occur when the conditions to pick out a unique solution are imposed at different values of the independent variable x,
For example we wish to solve (1) with boundary condtion (2) at $x = 0$ and boundary condition (3) at $x = 1$

$$y'' + 2y' + y = 0, \tag{1}$$
$$y(0) + 2y'(0) = 1, \tag{2}$$
$$2y(1) - y'(1) = 0. \tag{3}$$

The general solution can found using whatever standard method works.
However, imposing the boundary conditions may lead to qualitatively different results.

Boundary value problems

BVPs occur when the conditions to pick out a unique solution are imposed at different values of the independent variable x,
For example we wish to solve (1) with boundary condtion (2) at $x = 0$ and boundary condition (3) at $x = 1$

$$y'' + 2y' + y = 0, \tag{1}$$
$$y(0) + 2y'(0) = 1, \tag{2}$$
$$2y(1) - y'(1) = 0. \tag{3}$$

The general solution can found using whatever standard method works.
However, imposing the boundary conditions may lead to qualitatively different results.

The simplest example

The simple harmonic oscillator

$$y'' + y = 0$$

obviously has solution

$$y = c_1 \cos(x) + c_2 \sin(x).$$

The simplest example

The simple harmonic oscillator

$$y'' + y = 0$$

obviously has solution

$$y = c_1 \cos(x) + c_2 \sin(x).$$

If we impose boundary conditions

$$y(0) = 0, \qquad y\left(\frac{\pi}{2}\right) = 1$$

we get $c_1 = 0, c_2 = 1$ and the *unique solution*

$$y = \sin(x).$$

The simplest example

The simple harmonic oscillator

$$y'' + y = 0$$

obviously has solution

$$y = c_1 \cos(x) + c_2 \sin(x).$$

If we impose boundary conditions

$$y(0) = 0, \qquad y(\pi) = 1$$

we get $c_1 = 0, -c_1 = 1$. This is contradictory and hence there is no solution.

The simplest example

The simple harmonic oscillator

$$y'' + y = 0$$

obviously has solution

$$y = c_1 \cos(x) + c_2 \sin(x).$$

If we impose boundary conditions

$$y(0) = 0, \qquad y(\pi) = 0$$

we get $c_1 = 0, c_1 = 0$. This is not contradictory, but the conditions are not independent and hence there is a family of (∞ many!) solutions

$$y = c_2 \sin(x).$$

More complex example: I

The boundary value problem is

$$x^2y'' - 2xy' + 2y = 0, \qquad y(1) + y'(1) = 9, \quad y(2) - y'(2) = 3.$$

The general solution is

$$y = c_1x + c_2x^2 \quad \Rightarrow \quad y' = c_1 + 2c_2x.$$

The boundary conditions thus imply

$$2c_1 + 3c_2 = 9, \qquad c_1 = 3\,,$$

giving the unique solution

$$y = 3x + x^2.$$

More complex example: I

University of Southampton School of Mathematics

The boundary value problem is

$$x^2y'' - 2xy' + 2y = 0, \qquad y(1) + y'(1) = 9, \quad y(2) - y'(2) = 3.$$

The general solution is

$$y = c_1 x + c_2 x^2 \quad \Rightarrow \quad y' = c_1 + 2c_2 x.$$

The boundary conditions thus imply

$$2c_1 + 3c_2 = 9, \qquad c_1 = 3\,,$$

giving the unique solution

$$y = 3x + x^2.$$

More complex example: I

The boundary value problem is

$$x^2 y'' - 2xy' + 2y = 0, \qquad y(1) + y'(1) = 9, \quad y(2) - y'(2) = 3.$$

The general solution is

$$y = c_1 x + c_2 x^2 \quad \Rightarrow \quad y' = c_1 + 2c_2 x.$$

The boundary conditions thus imply

$$2c_1 + 3c_2 = 9, \qquad c_1 = 3\,,$$

giving the unique solution

$$y = 3x + x^2.$$

More complex example: II

The boundary value problem is

$$x^2y'' - 2xy' + 2y = 0, \qquad 4y(1) - 3y'(1) = 1, \quad 3y(2) - 4y'(2) = 3.$$

The general solution is

$$y = c_1 x + c_2 x^2 \quad \Rightarrow \quad y' = c_1 + 2c_2 x.$$

The boundary conditions thus imply

$$c_1 - 2c_2 = 1, \qquad 2c_1 - 4c_2 = 3.$$

The equations are dependent and contradictory, so there is no solution.

More complex example: II

The boundary value problem is

$$x^2 y'' - 2xy' + 2y = 0, \qquad 4y(1) - 3y'(1) = 1, \quad 3y(2) - 4y'(2) = 3.$$

The general solution is

$$y = c_1 x + c_2 x^2 \quad \Rightarrow \quad y' = c_1 + 2c_2 x.$$

The boundary conditions thus imply

$$c_1 - 2c_2 = 1, \qquad 2c_1 - 4c_2 = 3.$$

The equations are dependent and contradictory, so there is no solution.

Eigenvalue problems

We may have a boundary value problem containing an unknown constant; a simple example is

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

This particular problem arises from heat conduction in a bar, but pretty much any simple PDE will give a similar problem.

The approach is, for all values of λ, to

1. find the general solution for y;
2. check if the boundary conditions allow a non-trivial solution.

We typically find that only certain λ work.

Eigenvalue problems

We may have a boundary value problem containing an unknown constant; a simple example is

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

This particular problem arises from heat conduction in a bar, but pretty much any simple PDE will give a similar problem.

The approach is, for all values of λ, to

1. find the general solution for y;
2. check if the boundary conditions allow a non-trivial solution.

We typically find that only certain λ work.

Eigenvalue problems: case 1

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

Check the case $\lambda = 0$.

The general solution is

$$y = c_1 x + c_2.$$

The only solution is the trivial solution.

Eigenvalue problems: case 1

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

Check the case $\lambda = 0$.
The general solution is

$$y = c_1 x + c_2.$$

The only solution is the trivial solution.

Eigenvalue problems: case 1

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

Check the case $\lambda = 0$.
The general solution is

$$y = c_1 x + c_2.$$

The only solution is the trivial solution.

Eigenvalue problems: case 2

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

Check the case $\lambda = -\mu^2 < 0$.

The general solution is

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The only solution is the trivial solution.

Eigenvalue problems: case 2

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

Check the case $\lambda = -\mu^2 < 0$.
The general solution is

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The only solution is the trivial solution.

Eigenvalue problems: case 2

UNIVERSITY OF Southampton
School of Mathematics

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

Check the case $\lambda = -\mu^2 < 0$.
The general solution is

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The only solution is the trivial solution.

Eigenvalue problems: case 3

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

Check the case $\lambda = \mu^2 > 0$.

The general solution is

$$y = c_1 \sin(\mu x) + c_2 \cos(\mu x).$$

The first boundary condition gives $c_2 = 0$, but the second gives

$$c_1 \left(\mu \cos(\mu) + \sin(\mu)\right) = 0.$$

In this case the term in brackets may vanish, giving the nontrivial solution

$$y = c_1 \sin(\mu x)$$

where μ must satisfy

$$\mu \cos(\mu) + \sin(\mu) = 0.$$

Eigenvalue problems: case 3

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

Check the case $\lambda = \mu^2 > 0$.
The general solution is

$$y = c_1 \sin(\mu x) + c_2 \cos(\mu x).$$

The first boundary condition gives $c_2 = 0$, but the second gives

$$c_1 \left(\mu \cos(\mu) + \sin(\mu)\right) = 0.$$

In this case the term in brackets may vanish, giving the nontrivial solution

$$y = c_1 \sin(\mu x)$$

where μ must satisfy

$$\mu \cos(\mu) + \sin(\mu) = 0.$$

Eigenvalue problems: case 3

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0.$$

Check the case $\lambda = \mu^2 > 0$.
The general solution is

$$y = c_1 \sin(\mu x) + c_2 \cos(\mu x).$$

The first boundary condition gives $c_2 = 0$, but the second gives

$$c_1 \left(\mu \cos(\mu) + \sin(\mu)\right) = 0.$$

In this case the term in brackets may vanish, giving the nontrivial solution

$$y = c_1 \sin(\mu x)$$

where μ must satisfy

$$\mu \cos(\mu) + \sin(\mu) = 0.$$

Eigenvalue problem: solution

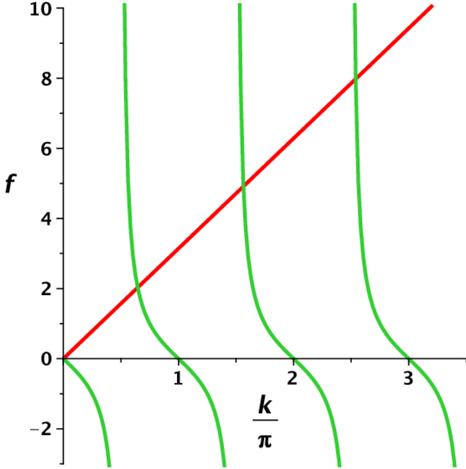
Our *eigenfunctions* are

$$y_n = \sin\left(\sqrt{\lambda_n} x\right)$$

with *eigenvalues* $\lambda_n = \mu_n^2$, where μ_n are solutions of

$$\begin{aligned} & 0 = \mu\cos(\mu) + \sin(\mu) \\ \Rightarrow \quad & \mu = -\tan(\mu). \end{aligned}$$

These cannot be found in closed form, but are obvious graphically.



Summary

- Solving BVPs is, in practice, just solving the DE and seeing if the boundary conditions are compatible.
- The theory of when solutions exist is outlined in the notes.
- Solving eigenvalue problems means finding which values of the unknown constant λ allow solutions.
- Eigenvalue problems show up in a wide range of PDE problems as we shall see later.
- The rich theory of Sturm-Liouville problems outlined in the notes show that many eigenvalue problems have key features:
  - An infinite number of real, distinct eigenvalues $\lambda_1 < \lambda_2 < \dots$;
  - Orthogonal eigenfunctions y_n which have $n - 1$ zeros inside the domain.

  Mathematically important, this has practical applications in stability theory.