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Question 1

Solution

(a) Decision variables:
We let x_A denote the units of product A supposed to be produced.
x_B denote the units of product B supposed to be produced.

$$y_1 = \begin{cases} 1 & \text{if additional time is used} \\ 0 & \text{Otherwise} \end{cases}$$

$$y_2 = \begin{cases} 1 & \text{if more than 10 units of product A is produced} \\ 0 & \text{Otherwise} \end{cases}$$

(b) Model for the Mixed Integer Programming Problem

$$\begin{array}{ll} \text{Maximize} & z = 200x_A + 400x_B - 1200y_1 \\ \text{subject to} & \begin{array}{c} 2x_A + 3x_B \leq 40 + 8y_1 \text{ (limited time)} \\ y_2 \geq \frac{1}{100}(x_A - 10) \text{ (more than ten A product)} \\ x_B \geq 5y_2 \text{ (at least 5 B)} \\ x_A \geq 0, x_B \geq 0, integer; y_1, y_2 \in \{0, 1\} \end{array} \end{array} \tag{1}$$

Question 2

Consider the problem

$$\begin{array}{ll} \text{Minimize} & z = 3x_1 - x_2 + x_3 \\ \text{subject to} & \begin{array}{c} x_1 + 2x_2 \leq 4 \\ 2x_1 - x_2 + x_3 \geq 1 \\ x_1, x_2 \geq 0, x_3 \leq 0 \end{array} \end{array} \tag{2}$$

Solution
The dual problem is given by:

$$\begin{array}{ll} \text{Maximize} & w = 4v_1 + v_2 \\ & v_1 + 2v_2 \leq 3 \\ \text{subject to} & 2v_1 - v_2 \leq -1 \\ & v_1 + v_2 \geq 1 \\ & v_1 \leq 0, v_2 \geq 0 \end{array} \tag{3}$$

(a) Is the point $(x_1, x_2, x_3) = (\frac{6}{5}, \frac{7}{5}, 0)$ an optimal solution?
No. $x_1 = \frac{6}{5} \rightarrow \mathrm{v}_{s1} = 0$ and $x_2 = \frac{7}{5} \rightarrow \mathrm{v}_{s2} = 0$. This means that $v_1 = \frac{1}{5}$ and $v_2 = \frac{7}{5}$ which do not satisfy the dual constraints.

(b) Is the point $(x_1, x_2, x_3) = (\frac{1}{2}, 0, 0)$ an optimal solution?
Yes, $x_1 = \frac{1}{2} \rightarrow \mathrm{v}_{s1} = 0$ and $x_{s1} = \frac{7}{2} \rightarrow \mathrm{v}_1 = 0$.

Question 3

Study the following LP problem using duality concept.

$$\begin{array}{ll} \text{Maximize} & z = z = c_1x_1 + c_2x_2 \\ & -x_1 + x_2 \leq 1 \\ & x_1 + 2x_2 \leq 2 \\ \text{subject to} & 2x_1 + x_2 \geq 0 \\ & 2x_1 - 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array} \tag{4}$$

(a) Determine the primal solution where the dual variables are given by the vector $v = (0, 1/3, 0, 2/3)^{\mathrm{T}}$
By using Complementary Slackness (5)

$$\bar{W}^{\mathrm{T}}(B - A\bar{X}) = 0 \quad \text{or} \quad \bar{X}^{\mathrm{T}}(A^{\mathrm{T}}\bar{W} - C) = 0 \tag{5}$$

Therefore, $\bar{W}^{\mathrm{T}}(B - A\bar{X}) = 0$ is solved as follows:

$$\begin{aligned} 0(1 - (-x_1 + x_2)) &= 0 \\ \frac{1}{3}(2 - (x_1 + 2x_2) &= 0 \\ 0(0 - (2x_1 + x_2)) &= 0 \\ \frac{2}{3}(1 - (2x_1 - 2x_2)) &= 0 \end{aligned}$$

We therefore, get the following system of simultaneous equations as stated below.

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 2x_1 - 2x_2 &= 1 \end{aligned}$$

Solving the above simultaneous equation where $x_1 = 1$ and $x_2 = \frac{1}{2}$; the solution for the primal is thus: $\bar{X}^{\mathrm{T}} = (1, \frac{1}{2})^{\mathrm{T}}$

(b) Determine the values of c_1 and c_2 for which the primal solution is optimal.
New $\bar{X}^{\mathrm{T}}(A^{\mathrm{T}}\bar{W} - C) = 0$
To solve for c_1 and c_2, we first determine the dual to the Linear Programming Problem in (4) as follows:

$$\begin{array}{ll} \text{Minimize} & v = w_1 + 2w_2 + 0w_3 + w4 \\ & -w_1 + w_2 + 2w_3 + 2w_4 \geq c_1 \\ \text{subject to} & w_1 + 2w_2 + w_3 - 2w_4 \geq c_2 \\ & w_1, w_2 \geq 0, w_3 \leq 0, w_4 \geq 0 \end{array} \tag{6}$$

$\bar{X}^{\mathrm{T}}(A^{\mathrm{T}}W - C) = 0$
From (6), where therefore solve for c_1 and c_2 as follows:

$$\begin{aligned} (1)1(-w_1 + w_2 + 3w_3 + 2w_4 - c_1) &= 0 \\ (0 + \frac{1}{3} + 0 + \frac{4}{3} - c_1) &= 0 \\ c_1 &= \frac{5}{3} \end{aligned}$$

$$\begin{aligned} (2)\frac{1}{2}(w_1 + 2w_2 + w_3 - 2w_4 - c_2) &= 0 \\ (0 + \frac{2}{3} + 0 - \frac{4}{3} - c_2) &= 0 \\ c_2 &= \frac{-2}{3} \end{aligned}$$