AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

(AIMS RWANDA, KIGALI)

Name: Yusuf Brima
Course: Partial Differential Equations

Assignment Number: 2
Date: January 17, 2021

Question 1

(a) To compute the full Fourier series representation of

$$f(x) = e^{ax}, \quad -\pi \leq x < \pi. \tag{1}$$

when extended as a 2π-periodic function.

The general form of a full Fourier series is of the form:

$$f(x) = \frac{a_0}{2} \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{2}$$

The Fourier coefficients a_0, a_n, b_n are solved as follows:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

By substituting the given function in (1) into the above expression we get:

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx \\
a_0 &= \frac{1}{\pi} \frac{1}{a} \left[e^{ax}\right]_{-\pi}^{\pi} \\
a_0 &= \frac{2}{a\pi} \frac{(e^{a\pi} - e^{-a\pi})}{2} \\
a_0 &= \frac{2}{a\pi} \sinh(a\pi)
\end{aligned}$$

For a_n

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

By substituting (1) into a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos(nx)dx$$

We let

$$P = \int_{-\pi}^{\pi} e^{ax}\cos(nx)dx$$

Using Integration by Parts method where:

$$\int uv = uv - \int vdu$$

Therefore $u = e^{ax}, \quad du = ae^{ax} \quad dv = \cos(nx)dx, \quad v = \frac{1}{n}\sin(nx)$
So

$$P = \frac{1}{n}e^{ax}\sin(nx) - \frac{a}{n}\int e^{ax}\sin(nx)dx$$

We further integrate

$$P_1 = \int e^{ax}\sin(nx)dx$$

Using Integration by parts as follows
where $u = e^{ax}, \quad du = ae^{ax} \quad dv = \sin(nx)dx, \quad v = \frac{-1}{n}\cos(nx)$

$$P_1 = \frac{-1}{n}e^{ax}\cos(nx) + \frac{a}{n}\int e^{ax}\cos(nx)dx$$

Therefore, we can rewrite P as follows:

$$\begin{aligned} P &= \frac{1}{n}e^{ax}\sin(nx) + \frac{a}{a^2}e^{ax}\cos(nx) - \left(\frac{a^2}{n^2}\right)P \\ P + \left(\frac{a^2}{n^2}\right)P &= \frac{a}{n^2}\left[\frac{1}{n}e^{ax}\sin(nx) + \frac{a}{a^2}e^{ax}\cos(nx)\right]_{-\pi}^{\pi} \\ P\left(1+\frac{a^2}{n^2}\right) &= \frac{a}{n^2}\left[e^{a\pi}\cos(n\pi) - e^{-a\pi}\cos(n\pi)\right] \\ P(n^2+a^2) &= 2a(-1)^n\left(\frac{e^{a\pi}-e^{-a\pi}}{2}\right) \\ P &= \frac{2a(-1)^n}{n^2+a^2}\sinh(a\pi) \end{aligned}$$

Finally, the solution for a_n is:

$$\begin{aligned} a_n &= P(\frac{1}{\pi}) \\ a_n &= \frac{1}{\pi}\frac{2a(-1)^n}{(n^2+a^2)}\sinh(a\pi) \end{aligned}$$

We solve for coefficient b_n as follows:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)$$

By similarly substituting (1) into the above expression

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin(nx)dx$$

By substitution, we let

$$P = \int_{-\pi}^{\pi} e^{ax}\sin(nx)dx$$

And using Method of Integration by Parts where $u = e^{ax}, \quad du = ae^{ax} \quad dv = \sin(nx)dx, \quad v = \frac{-1}{n}\cos(nx)$

$$\begin{aligned}
P &= \frac{-1}{n}e^{ax}\cos(nx) + \frac{a}{n}\int e^{ax}\cos(nx)dx \\
P &= \frac{-1}{n}e^{ax}\cos(nx) + \frac{a}{n}\left(\frac{1}{n}e^{ax}\sin(nx) - \frac{a}{n}P\right) \\
P &= \frac{-1}{n}e^{ax}\cos(nx) + \frac{a}{n^2}e^{ax}\sin(nx) - \frac{a^2}{n^2}P \\
P\left(1+\frac{a^2}{n^2}\right) &= \left[\frac{-1}{n}e^{ax}\cos(nx)\right]_{-\pi}^{\pi} \\
P\left(1+\frac{a^2}{n^2}\right) &= \frac{-1}{n}\left(e^{a\pi}\cos(n\pi) - e^{-a\pi}\cos(n\pi)\right) \\
P\left(1+\frac{a^2}{n^2}\right) &= \frac{-2}{n}(-1)^n\sinh(a\pi) \\
P &= \frac{-2n(-1)^n}{a^2+n^2}\sinh(a\pi)
\end{aligned}$$

Therefore, the solution for coefficient b_n is thus:

$$b_n = \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)}\sinh(a\pi)$$

Finally, all the coefficients into equation (2), we therefore get the Fourier solution thus:

$$f(x) = \frac{1}{a\pi}\sinh(a\pi) + \sum_{n=1}^{\infty}\frac{1}{\pi}\frac{2a(-1)^n}{(a^2+n^2)}\sinh(a\pi)\cos(nx) + \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)}\sinh(a\pi)\sin(nx)$$

(b) By using the result of equation (1), to determine the full Fourier series expansion for the function

$$g(x) = \sinh(x), \quad \text{where } -\pi \le x < \pi \tag{3}$$

The function in (3) can be restated as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

We proceed by setting $a = 1$, and therefore find the Fourier Series for e^x in the expression below.

$$e^x = \frac{1}{\pi}\sinh(\pi) + \sum_{n=1}^{\infty} \frac{1}{\pi}\frac{2(-1)^n}{n^2+1}\sinh(\pi)\cos(nx) + \frac{2n(-1)^{n+1}}{\pi(n^2+1)}\sinh(\pi)\sin(nx)$$

And for e^{-x}, we get:

$$e^{-x} = \frac{1}{\pi}\sinh(\pi) + \sum_{n=1}^{\infty} \frac{1}{\pi}\frac{2(-1)^n}{n^2+1}\sinh(\pi)\cos(nx) - \frac{2n(-1)^{n+1}}{\pi(n^2+1)}\sinh(\pi)\sin(nx)$$

Finally,

$$g(x) = \frac{e^x - e^{-x}}{2} = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{\pi(n^2+1)}\sinh(\pi)\sin(nx)$$

Question 2

The vibrations $u(x,t)$ of air in an open pipe of unit length satisfy the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad 0 \le x \le 1, \quad \text{and } t > 0 \tag{4}$$

given the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(1,t) = 0, \quad for \quad all \quad 0 \le x \le 1 \tag{5}$$

And the air is initially at rest so that

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad for \quad all \quad 0 \le x \le 1 \tag{6}$$

(a) We solve the Wave Equation (4) by Separation of Variables using the Boundary Conditions (5) and Initial Condition (6) to show that the solution is:

$$y(x,t) = \sum_{n=0}^{\infty} A_n \cos(n\pi x)\cos(cn\pi t)$$

as follows:
By using the the ansatz $u(x,t) = X(x)T(t)$ we get the following expression.

$$\frac{1}{c^2}X\ddot{T} = TX''$$

we divide both sides by XT, which results in

$$\frac{\ddot{T}}{c^2T} = \frac{X''}{X} = \lambda$$

where λ is the eigenvalue constants.
We get the following system of differential equations

$$\ddot{T} - c^2\lambda T = 0$$
$$X^{''} - \lambda X = 0$$

Therefore,

$$\frac{\partial u}{\partial x}(0,t) = X^{'}(0)T(t) = 0$$
$$\frac{\partial u}{\partial x}(1,t) = X^{'}(1)T(t) = 0$$

The non-trivial solution $X^{'}(0) = 0$,$X^{'}(0) = 0$

So we let, $\lambda_n = -(n\pi)^2$
Solving for $X^{''} - \lambda x = 0$, the characteristic equation is thus:

$$m^2 + (n\pi)^2 = 0$$
$$m = \pm(n\pi i)$$

$\Longrightarrow$

$$X_n(x) = A_n \cos(n\pi x) + B_n \sin(n\pi x)$$
$$X_n(x) = A_n \cos(n\pi x)$$

Solving for $\ddot{T} - c^2\lambda T = 0$ characteristic equation is as follows:

$$m^2 + c^2(n\pi)^2 = 0$$
$$m = \pm(icn\pi)$$

$\Longrightarrow$

$$T_n(t) = A_n \cos(cn\pi t) + B_n \sin(cn\pi t)$$
$$\dot{T}_n(t) = -cn\pi A_n \sin(cn\pi t) + cn\pi B_n \cos(cn\pi t)$$
$$\dot{T}_n(0) = 0$$

Given that

$$\dot{T}_n(0) = 0 \Rightarrow B_n = 0$$
$$T_n(t) = A_n \cos(cn\pi t)$$

Therefore

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos(n\pi x)\cos(cn\pi t)$$

(b) We find the **unique** solution that satisfies the initial condition :

$$u(x,0) = \begin{cases} 1 & for \quad 0 \leq x \leq \frac{1}{2} \\ 0 & for \quad \frac{1}{2} < x \leq 1 \end{cases}$$

as follows:

Given that the function is **even** where $b_n = 0 \implies$

$$\begin{aligned} u(x,0) = p(x) &= \sum_{n=0}^{\infty} A_n \cos(n\pi x) \\ &= A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \end{aligned}$$

We solve for the coefficient A_0

$$\begin{aligned} A_0 &= \frac{2}{2} \int_0^1 p(x)dx = \int_0^1 u(x,0)dx \\ &= \int_0^{\frac{1}{2}} 1dx + \int_{\frac{1}{2}}^1 0dx \\ A_0 &= \frac{1}{2} \end{aligned}$$

And for the coefficient A_n

$$\begin{aligned} A_n &= 2 \int_0^1 p(x)\cos(n\pi x)dx \\ &= 2 \left[\int_0^1 \cos(n\pi x)dx + \int_{\frac{1}{2}}^1 0dx \right] \\ &= \frac{2}{n\pi} \left[\sin\left(\frac{\pi}{2}n\right) \right] \end{aligned}$$

It follows that

$$A_n = \begin{cases} \frac{2}{n\pi}(-1)^n, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Finally,

$$u(x,t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{2n-1} \cos(n\pi x)\cos(cn\pi t)$$

Question 3

The function $f(x)$ is defined as

$$f(x) = 1 \quad 0 < x < \pi \tag{7}$$

(a) Sketch the odd extension and show that the Fourier sine series expansion is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{\sin((2n-1)x}{2n-1} \tag{8}$$

We determine the coefficient b_n as thus:

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \sin(nx) \\ &= \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} \\ &= \frac{2}{n\pi}\left(1-(-1)^n\right) \\ b_n &= \begin{cases} \frac{4}{n\pi}, & \text{n is odd} \\ 0, & \text{n is even} \end{cases} \end{aligned}$$

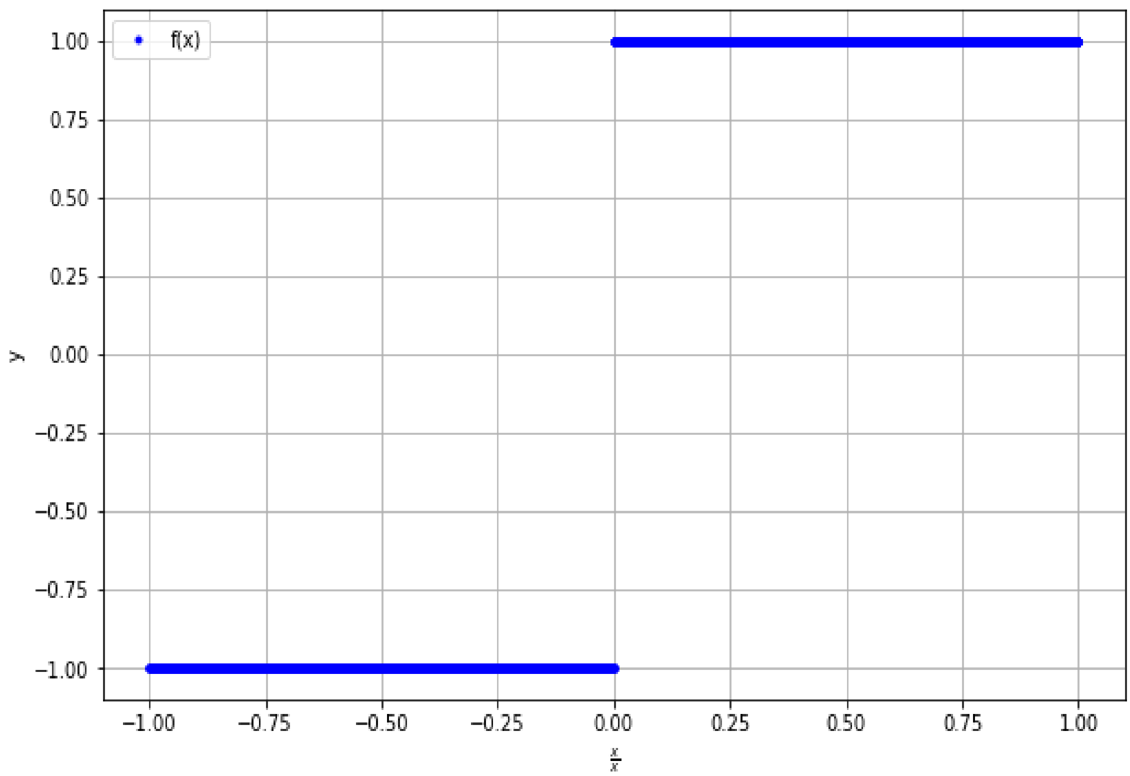

Figure 1: Odd Extension of the $f(x) = 1$ from $-\pi \leq x \leq \pi$

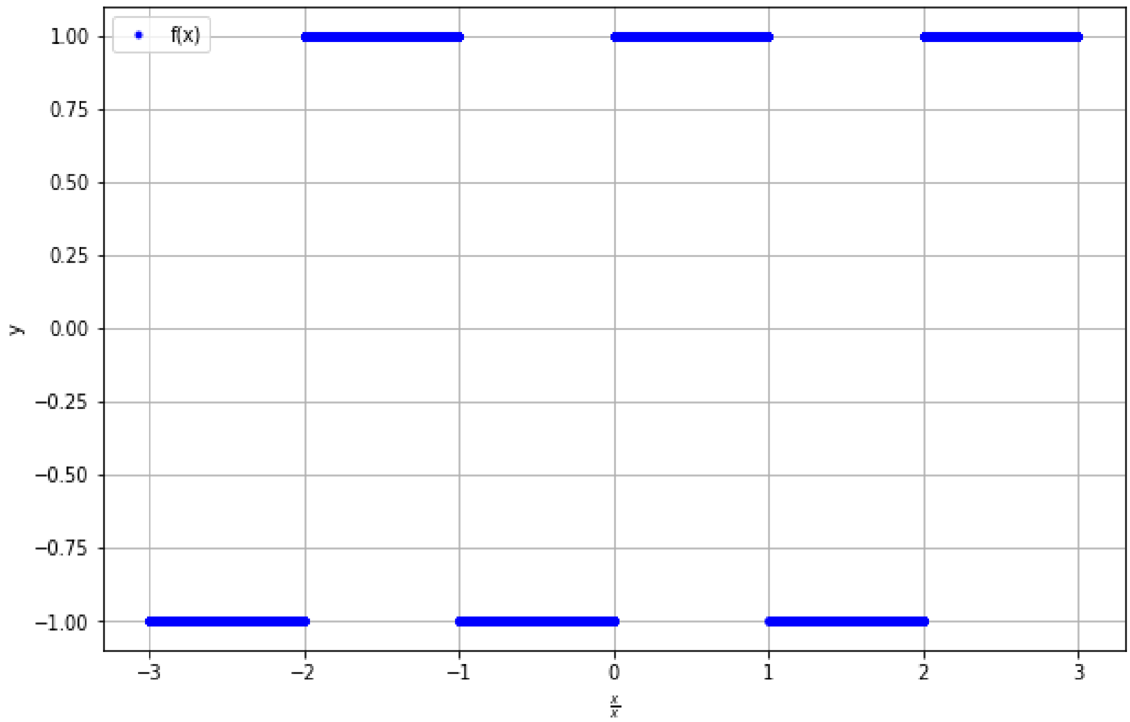

Figure 2: Odd Extension of the $f(x) = 1$ from $-3\pi \leq x \leq 3\pi$

(b) The graphs of the partial sum plotted in Python are for equation (9) for the cases when $N = 5, 10, 20$, over the range $0 < x < \pi$.. In addition plot the series with $N = 20, 50, 500$ terms over the range $0 < x < 0.1$.

$$f_N(x) = \frac{4}{\pi} \sum_{n=1}^{N} \frac{\sin((2n-1)x)}{2n-1} \tag{9}$$

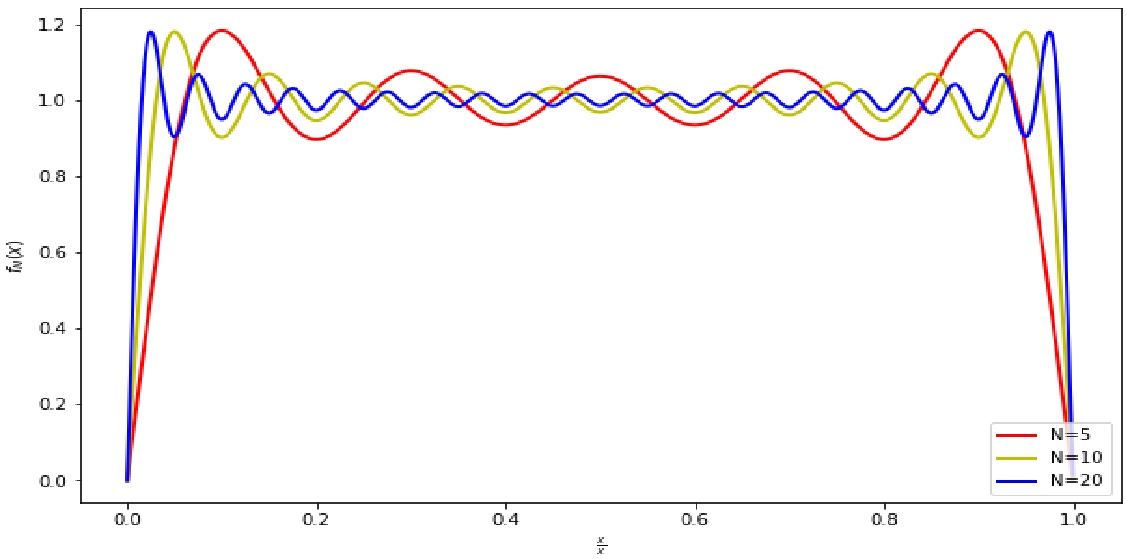

Figure 3: Partial Sums of the Fourier Series from $0 \leq x \leq \pi$

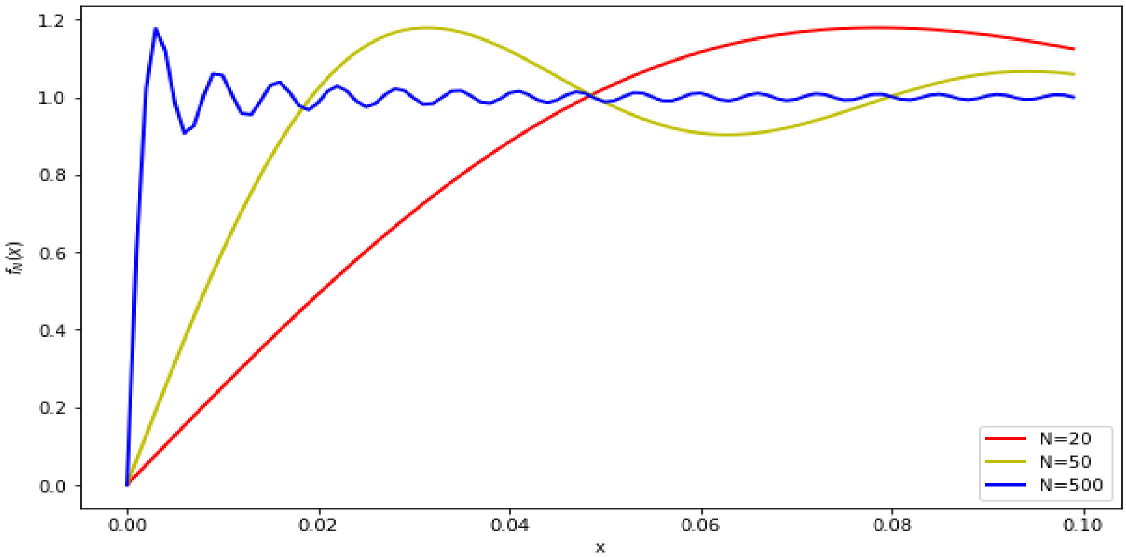

Figure 4: Partial Sums of the Fourier Series from $0 \le x \le \pi$

(c) Show that the partial sum in equation 9 may be written as:

$$f_N(x) = \frac{2}{\pi}\int_0^x \frac{\sin(2Nt)}{\sin(t)}dt$$

The following expression can be rewritten as:

$$\sin\frac{(2n-1)x}{2n-1} = \int_0^x \cos((2n-1)t)dt$$

Therefore, we get the following expression:

$$f_N(x) = \frac{4}{\pi}\left(\sum_{n=1}^{N}\int_0^t \cos((2n-1)t)\right)dt \tag{10}$$

Now when we differential equation (10), we get the following

$$f_N'(x) = \frac{4}{\pi}\sum_{n=1}^{N}\cos((2n-1)x) \tag{11}$$

when we multiply $\sin(x)$ to the above expression in 11 we get

$$\sin(x)f_N'(x) = \frac{4}{\pi}\sum_{n=1}^{N}\sin(x)\cos((2n-1)x)$$

Let $P = \cos((2n-1)x)$

$$P\sin(x) = \sin(x)\cos((2n-1)x)$$

we proceed by using Simpson Formulation

$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

$\therefore$

$$\begin{aligned}
2P\sin(x) &= \sin(x + (2n-1)x) + \sin(x - (2n-1)x) \\
&= \sin(2nx) - \sin(-2nx + 2x) \\
&= \sin(2nx) - \sin((2-2n)x) \\
P\sin(x) &= \frac{1}{2}\left(\sin(2nx) - \sin((2-2n)x)\right)
\end{aligned}$$

We apply summation on both sides as follows $\sin(x)f_N'(x)$

$$\begin{aligned}
\sum_{n=1}^{N} \sin(x)P &= \frac{1}{2}\sin(2Nx) \\
\sum_{n=1}^{N} P &= \frac{1}{2}\frac{\sin(2Nx)}{\sin(x)} \\
\frac{\pi}{4}f_N'(x) &= \frac{1}{2}\frac{\sin(2Nx)}{\sin(x)}
\end{aligned}$$

Integrating both sides gives

$$f_N(x) = \frac{2}{\pi}\int_0^x \frac{\sin(Nt)}{\sin t} \quad \text{hence proved.}$$

(d)