UNIVERSITY OF Southampton
School of Mathematics

Lecture 8 - Fourier Series and Orthogonality

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Lectures on Partial Differential Equations

Orthogonality relations

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

To find the Fourier coefficients a_m, b_m we needed the (*orthogonality*) relations

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \pi\delta_{mn}, \qquad \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn},$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0.$$

Here

$$\delta_{mn} = \begin{cases} 1 & m = n, \\ 0 & m \neq n \end{cases}.$$

Orthogonality

We have see from direct calculation that the orthogonality relations

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follow from the trigonometric identities.

However there is an easier way

In general the eigenfunctions y_n of Sturm-Liouville problems

$$-\left(p(x)y'\right)' + q(x)y = \lambda w(x)y$$

with appropriate boundary conditions are orthogonal in the sense

$$\int_{x_0}^{x_1} w(x)y_m y_n\,\mathrm{d}x = C\delta_{mn}.$$

Fourier Series orthogonality follows from the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$$

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$$a_0 = 0,$$
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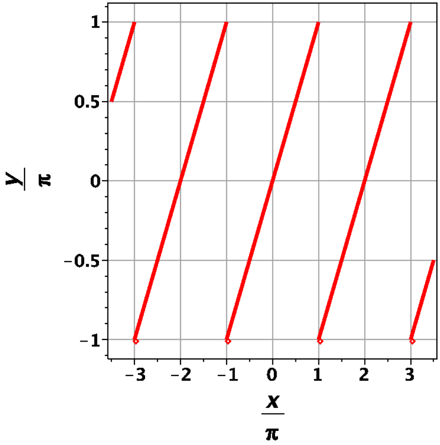
$$b_m = \frac{2(-1)^{m+1}}{m}.$$

We steadily see convergence inside the interval.



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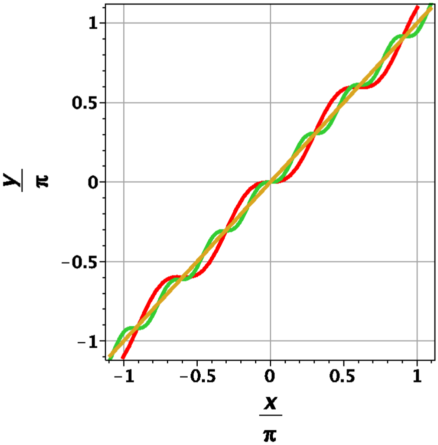

Example 2

For the *tent* function $f = |x|$ where $-\pi < x < \pi$ find a_n and b_n.

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$$= \begin{cases} -\frac{4}{\pi m^2} & m \text{ odd} \\ 0 & m \text{ even} \end{cases},$$

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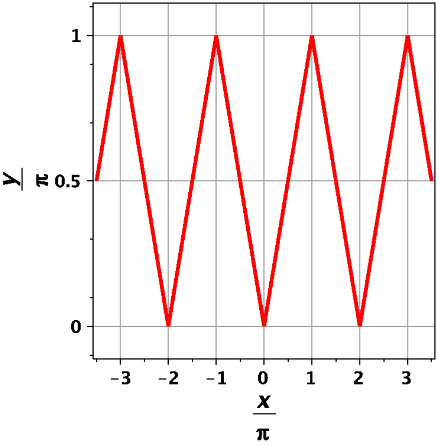

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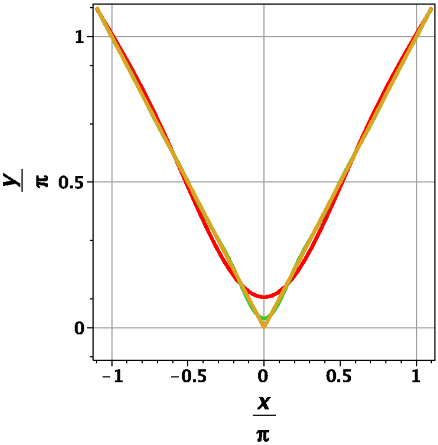

Getting more from Fourier Series

Later we will see the uses of Fourier Series for PDEs. But we can already use them to get mathematical results. Try evaluating the tent function example at $x = 0$:

$$0 = |0| = \frac{a_0}{2} + \sum_n a_n \cos(nx)\Big|_{x=0} = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(0)}{n^2},$$

from which it follows that

$$\frac{\pi^2}{8} = \sum_{n \text{ odd}} \frac{1}{n^2}.$$

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Summary

- The orthogonality relations that give the Euler formulas follow from general results of Sturm-Liouville theory.
- Practical Fourier Series calculations require lots of integration by parts.
- Convergence of Fourier Series is, in most cases, rapid.
- A number of arithmetic identities can be found from Fourier Series by evaluating them at a suitable point.