AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

(AIMS RWANDA, KIGALI)

Name: Yusuf Brima Assignment Number: 2
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Exercise 1

Given a world in which there are only two risky assets S_1 and S_2, with respective expected returns.

$\bar{R}_1 = 0.1 = \frac{10}{100}, \quad \bar{R}_2 = 0.18 = \frac{18}{100}$

and variances and covariances given by

$\sigma_1^1 = 0.016 = \frac{16}{10000}, \quad \sigma_{12}^2 = 0.016 = \frac{16}{10000}, \quad \sigma_2 = 0.01 = \frac{1}{100}$

where the opportunity set in

$(\bar{R}, \sigma)$ space

The Market Price of Risk (**MPR**) is given by

$$\theta = \frac{\bar{R}_{AC} - R_B}{\sigma_{AC}} \tag{1}$$

From (1), hte MPR is thus:

$$\theta = \frac{\sqrt{49728}}{168}$$

$$\implies x_1 = \frac{13}{21}, \quad x_2 = \frac{8}{21}$$

And the Mean Return $\bar{R}_\pi$ is given in the equation

$$\bar{R}_\pi = \lambda_1 \bar{R}_1 + \lambda_2 \bar{R}_2 \text{ where } \lambda \text{ is the number of shares and } \bar{R} \text{ is the return} \tag{2}$$

Thus

$$\begin{aligned}
\bar{R}_\pi &= \lambda \bar{R}_1 + (1-\lambda)\bar{R}_2 \\
&= \frac{10}{100}\lambda + (1-\lambda)\frac{18}{100} \\
&= \frac{18}{100} - \frac{8\lambda}{100} \\
&= \frac{1}{50}(9-4\lambda)
\end{aligned}$$

The correlation between two asset investments is given by equation (3).

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2} \tag{3}$$

and the Risk of Investment is given by

$$\sigma_\pi^2 = \lambda_1^2\sigma_1^2 + 2\lambda_1\sigma_1\lambda_2\sigma_2\rho_{12} + \lambda_2^2\sigma_2^2 \tag{4}$$

From (4)

$$\begin{aligned}
\sigma_\pi^2 &= \lambda^2\left(\frac{16}{1000}\right) + 2\lambda(1-\lambda)\left(\frac{16}{1000}\right) + \lambda^2\left(\frac{1}{100}\right) \\
&= \frac{1}{100}\left[\frac{16\lambda^2}{100} + (2\lambda - 2\lambda^2)\left(\frac{16}{100}\right) + 1 - 2\lambda + \lambda^2\right] \\
&= \frac{1}{10000}\left[16\lambda^2 + 32\lambda - 32\lambda^2 + 100 - 200\lambda + 100\lambda^2\right] \\
&= \frac{1}{10000}\left[84\lambda^2 - 168\lambda + 100\right]
\end{aligned}$$

but

$$\lambda = \frac{-50\bar{R}_\pi + 9}{4}$$

$$\begin{aligned}
\therefore \sigma_\pi^2 &= \frac{1}{10000}\left[84\left(\frac{-50\bar{R}_\pi + 9}{4}\right) - 168\left(\frac{-50\bar{R}_\pi + 9}{4}\right) + 100\right] \\
&= \frac{1}{10000}\left[\frac{84}{16}\left(81 - 900\bar{R}_\pi^2 + 25000\bar{R}_\pi^2\right) - \frac{1512 + 8400\bar{R}_\pi + 400}{4}\right] \\
&= \frac{1}{4\times^4}\left[210000\bar{R}_\pi^2 - 42000\bar{R}_\pi + 2356\right] \\
\therefore \sigma_\pi^2 &= \frac{1}{400}\left[21,000\bar{R}_\pi^2 - 42,000\bar{R}_\pi + 2356\right]^{\frac{1}{2}}
\end{aligned}$$

Given $R_0 = 0.06 = \frac{6}{100}$ and from (1)

$$\begin{aligned}
\implies \theta &= \frac{12 - 8\lambda}{\sqrt{84\lambda^2 - 168\lambda + 100}} \\
&= (12-\lambda)(84\lambda^2 - 168\lambda + 100)^{\frac{1}{2}}
\end{aligned}$$

To minimize the risk, we differentiate θ w.r.t. λ.

$$\begin{aligned}\frac{d\theta}{d\lambda} &= (-8)(84\lambda^2 - 168\lambda + 100)^{\frac{-1}{2}} - \frac{1}{2}\left[(12-\lambda)(168\lambda - 168)(84\lambda^2 - 168\lambda + 100)^{-1}\right] = 0 \\ &= (84\lambda^2 - 168\lambda + 100)^{\frac{-1}{2}}\left[-8 - (6-4\lambda)(168\lambda - 168)(84\lambda^2 - 168\lambda + 100)^{-1}\right] = 0\end{aligned}$$

$$\begin{aligned}(84\lambda^2 - 168\lambda + 100)^{\frac{-1}{2}} &= 0 \quad \text{or} \\ -8 - (6-4\lambda)(168\lambda - 168)(84\lambda^2 - 168\lambda + 100)^{-1} &= 0\end{aligned}$$

$$\begin{aligned}-172\lambda^2 + 1344\lambda - 800 - 1008\lambda + 1008 + 672\lambda^2 - 672\lambda &= 0 \\ -336\lambda + 208 &= 0 \\ \lambda &= \frac{208}{336} \\ \lambda &= \frac{13}{31}\end{aligned}$$

$$\begin{aligned}\therefore \theta &= \frac{12-\lambda}{\sqrt{84\lambda^2 + 168\lambda + 100}} \\ &= \frac{12 - \frac{13}{21}}{\sqrt{84\left(\frac{13}{21}\right)^2 - 168\left(\frac{13}{21}\right) + 100}} \\ &= \frac{7.0476}{\sqrt{28.190476}} \\ &= \frac{7.0476}{5.3094} \\ &= 1.327367\end{aligned}$$

Thus

$$\begin{aligned}\bar{R} &= \theta\sigma + R_0 \\ &= 1.327367\sigma + 0.06\end{aligned}$$

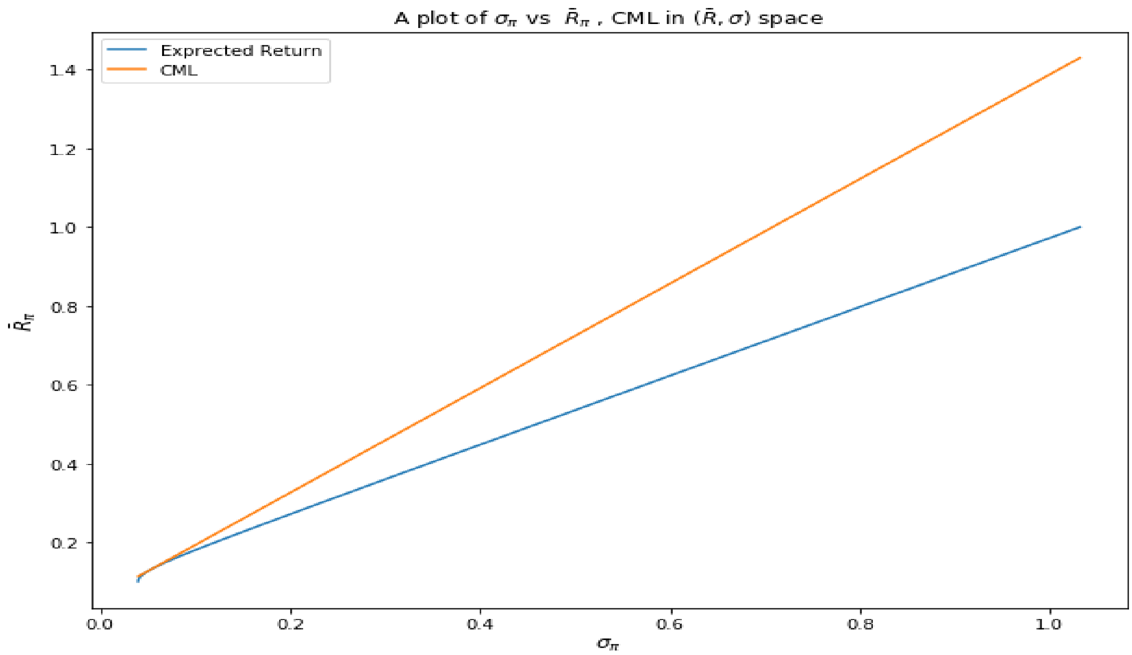

Figure 1: The opportunity set in $(\bar{R}, \sigma)$ space and the efficient frontier

Exercise 2

Considering a situation where there are three risky assets S_1, S_2 and S_3 with respective expected returns

$\bar{R}_1 = 0.09 = \frac{9}{100}, \quad \bar{R}_2 = 0.11 = \frac{11}{100}, \quad \bar{R}_3 = 0.17 = \frac{17}{100}$

whilst the variances and covariances are given by

$\sigma_1^1 = 0.016 = \frac{16}{10000}, \quad \sigma_{12}^2 = 0.016 = \frac{16}{10000}, \quad \sigma_{13}^2 = 0, \quad \sigma_2 = 0.01 = \frac{1}{100}, \quad \sigma_{23}^2 = 0.012 = \frac{12}{10000}, \quad \sigma_3^2 = 0.0144. = \frac{144}{10000},$

If we suppose that the risk free rate $R_0 = 0.05 = \frac{5}{100}$ and short selling and borrowing are allowed.

Therefore,

$$\sigma_{i,j} = \begin{pmatrix} 16 & 16 & 0 \\ 16 & 100 & 12 \\ 0 & 12 & 144 \end{pmatrix}$$

$$\begin{pmatrix} \sigma,1,1 & \sigma1,2 & \sigma1,3 \\ \sigma2,1 & \sigma2,2 & \sigma2,3 \\ \sigma3,1 & \sigma3,2 & \sigma3,3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \bar{R}_1 - R_0 \\ \bar{R}_2 - R_0 \\ \bar{R}_3 - R_0 \end{pmatrix}$$
$$\begin{pmatrix} 16 & 16 & 0 \\ 16 & 100 & 12 \\ 0 & 12 & 144 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \bar{R}_1 - R_0 \\ \bar{R}_2 - R_0 \\ \bar{R}_3 - R_0 \end{pmatrix}$$
$$\begin{pmatrix} 16 & 16 & 0 \\ 16 & 100 & 12 \\ 0 & 12 & 144 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix}$$

$$z_1 = \frac{79}{332}, \quad z_2 = \frac{1}{83}, \quad z_3 = \frac{41}{490}$$

Afterwards, we find the value of λ as follows

however,

$$z_1 + z_2 + z_3 = \lambda \tag{5}$$

from (5)

$$\lambda = \frac{79}{332} + \frac{1}{83} + \frac{41}{490}$$
$$= \frac{331}{996}$$

Then we proceed to find the values of x_i s.t. $\forall i \in \{3\}$

$$x_1 = \frac{z_1}{\lambda} = \frac{237}{331}$$
$$x_2 = \frac{z_2}{\lambda} = \frac{12}{331}$$
$$x_3 = \frac{z_3}{\lambda} = \frac{82}{331}$$

It therefore follows that:

$$x_1 + x_2 + x_3 = 1 \text{ Proved.}$$

The Mean of Return $\bar{R}_\pi$ is thus

$$\begin{aligned}\bar{R}_\pi &= x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 \\ &= \frac{237}{331}\frac{132}{3300} + \frac{12}{331}\frac{11}{100} + \frac{82}{331}\frac{17}{100} \\ &= 0.1105 = 11.05\%\end{aligned}$$

$$\begin{aligned}\sigma_\pi^2 &= x_1^2\lambda_1^2 + 2\lambda_1 x_1\lambda_2 x_2\rho_{12} + x_2^2\lambda_2^2 + 2x_2\lambda_2 x_3\lambda_3\rho_{23} + x_3^2\lambda_3^2 \\ &= 0.00082 + 0.00083 + 0.000013 + 0.000022 + 0.00088 \\ \therefore \sigma_\pi &= 0.04268 \quad \text{after taking square root of both sides}\end{aligned}$$

We find the Shape Ratio as follows

$$\theta = \frac{\bar{R}_\pi - R_0}{\sigma_\pi} \tag{6}$$

From (6)

$$\begin{aligned}\implies \theta &= \frac{0.1105 - 0.05}{\sqrt{0.0001822}} \\ &= 1.1475 \approx 1.418\end{aligned}$$

Exercise 3

A pension fund manager requires an expected return of $R\%$ with minimum risk on an investment in two risky assets S_1 and S_2 with respective expected returns

$\bar{R}_1 = 0.06 = \frac{6}{100}, \quad \bar{R}_2 = 0.08 = \frac{8}{100}$

with variances and covariances (scaled by 10^4) given by

$\sigma_1^2 = 1, \quad \sigma_{12}^2 = 2, \quad \sigma_2^2 = 2$

we note that

$$\begin{aligned}\lambda_1 + \lambda_2 &= 1 \\ \implies \lambda_1 + \lambda_2 - 1 &= 10\end{aligned}$$

Expected return is

$$\begin{aligned}\bar{R}_\pi &= x_1\lambda_1 + x_2\lambda_2 \\ &= 6\lambda_1 + 8\lambda_2 \\ &\implies 6\lambda_1 + 8\lambda_2 - R = 0\end{aligned}$$

Risk of Investment

$$\begin{aligned}\sigma_\pi^2 &= \lambda_1^2\sigma_1^2 + 2\lambda_1\sigma_1\lambda_2\sigma_2\rho_{12} + \lambda_2^2\sigma_2^2 \\ &= \lambda_1^2 + 4\lambda_1\lambda_2\sigma_{12} + 2\lambda_2^2\end{aligned}$$

$$\mathcal{L}\left(\lambda_1, \lambda_2, \alpha, \beta\right) = \lambda_1^2 + 4\lambda_1\lambda_2\sigma_{12} + 2\lambda_2^2 + \alpha\left(\lambda_1 + \lambda_2 - 1\right) + \beta\left(6\lambda_1 + 8\lambda_2 - R\right)$$

We Take the partial derivatives of $\lambda_1, \lambda_2, \alpha, \beta$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\lambda_1} &= 2\lambda_1 + 4\lambda_2 + \alpha + 6\beta = 0 \\ &= 2\lambda_1 + 4\lambda_2 = -\alpha - 6\beta\end{aligned}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\lambda_2} &= 4\lambda_1 + 4\lambda_2 + \alpha + 8\beta = 0 \\ &= 4\lambda_1 + 4\lambda_2 = -\alpha - 8\beta\end{aligned}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\alpha} &= \lambda_1 + \lambda_2 - 1 = 0 \\ &= \lambda_1 + \lambda_2 = 1\end{aligned}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\beta} &= 6\lambda_1 + 8\lambda_2 - R = 0 \\ &= 6\lambda_1 + 8\lambda_2 = R\end{aligned}$$

Using the partial derivatives $\frac{\partial\mathcal{L}}{\partial\lambda_1}, \frac{\partial\mathcal{L}}{\partial\lambda_2}$ we form the following system of equations.

$$2\lambda_1 + 4\lambda_2 = -\alpha - 6\beta \tag{7}$$
$$4\lambda_1 + 4\lambda_2 = -\alpha - 8\beta \tag{8}$$

where

$$\begin{aligned}&2\lambda_1 = -2\beta \implies \lambda_1 = -\beta \\ &\text{and} \\ &4\lambda_2 = -\alpha - 4\beta \\ &\lambda_2 = -\frac{1}{4}\alpha - \beta\end{aligned}$$

using the other system partial derivatives of α, β we derive the following system of linear simultaneous equations.

$$\lambda_1 + \lambda_2 = 1 \tag{9}$$
$$6\lambda_1 + 8\lambda_2 = R \tag{10}$$

$$-6\beta + 8(\frac{-1}{4}\alpha - \beta) = R$$
$$-2\alpha - 14\beta = R$$

from (7), we substitute for λ_1 and λ_2.

$$-\beta + \frac{-1}{4}\alpha - \beta = 1$$
$$\frac{-1}{4}\alpha - 2\beta = 1 - \alpha - 8\beta \qquad = 4$$
$$\therefore -2\alpha - 14\beta = R \quad \text{.........} \ \text{c}$$
$$-\alpha - 8\beta = 4 \quad \text{.........} \ \text{d}$$

For (c) - 2(d) $\implies$

$$2\beta = R - 8$$
$$\therefore \beta = \frac{R}{2} - 4$$

Also from (d)

$$-\alpha - \left(8\frac{R}{2} - 4\right) = 4$$
$$-\alpha - 4R + 32 = 4$$
$$-\alpha - 4R = -28$$
$$\alpha = 28 - 4R$$

$$\therefore \lambda_1 = -\beta$$
$$\implies \lambda_1 = -\left(\frac{R}{2} - 4\right)$$

For $\lambda_1 \geq 0 \quad \frac{R}{2} \leq 4 \quad \implies R \leq 8$

Also

$$\lambda_2 = -\beta - \frac{\alpha}{4}$$
$$= \left(\frac{\alpha}{2}\right) - \frac{1}{4}(28 - 4R)$$
$$= 4 - \frac{R}{2} - \frac{28}{4} + R$$
$$\lambda_2 = \frac{R}{2} - 3$$

For $\lambda_2 \geq 0, \quad \frac{R}{2} \geq 3, \quad R \geq 6.$

$\therefore 6 \leq R \leq 8$

The condition sufficiently satisfy the constraint that prohibits short selling.

Exercise 4

(a)

$$P(R \leq -t) = 1 - c \tag{11}$$

After substituting R in (11) we obtain the follow:

$$\begin{aligned}
P(Q \times X \leq -t) =& 1 - c \\
P(Q\{\mu + \sigma\mathcal{N}(0,1)\} \leq -t) =& 1 - c \\
P(\mu + \sigma\mathcal{N}[0,1] \leq -\frac{t}{Q}) =& 1 - c \\
P(\mathcal{N}(0,1) \leq -\frac{t}{Q\sigma} - \frac{\mu}{\sigma}) =& 1 - c \\
\Phi\Bigg(-\Bigg(\frac{t}{Q\sigma} + \frac{\mu}{\sigma}\Bigg)\Bigg) =& 1 - c \\
1 - \Phi\Bigg(\Bigg(\frac{t}{Q\sigma} + \frac{\mu}{\sigma}\Bigg)\Bigg) =& 1 - c \\
\Phi\Bigg(\Bigg(\frac{t}{Q\sigma} + \frac{\mu}{\sigma}\Bigg)\Bigg) =& c \\
\frac{t}{\sigma Q} + \frac{\mu}{\sigma} =& \Phi^{-1}(c) \\
\text{However} \quad \Phi^{-1}(-t) = 1 - \Phi(t) \; t + Q\mu =& Q\sigma\Phi^{-1}(c) \\
\therefore \qquad t =& Q\Bigg(\sigma\Phi^{-1}(c) - \mu\Bigg)
\end{aligned}$$

Thus proved.

The mean and variance for Netflix over the period (2017, 1, 1) to (2020, 1, 1) and the 1-day Value at Risk at the 95% confidence level of an investment of 1,000$.

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Value-at-Risk: $36.878877

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Mean is: $0.0015091

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Variance is: $0.023338

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