3

Exercise 4.1.3 *Prove the following equality:*

$$\mathrm{Tr}\{A\} = \langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS}, \tag{4.22}$$

where A is a square operator acting on a Hilbert space $\mathcal{H}_S$, I_R is the identity operator acting on a Hilbert space $\mathcal{H}_R$ isomorphic to $\mathcal{H}_S$ and $|\Gamma\rangle_{RS}$ is the unnormalized maximally entangled vector from ($\boxed{3.233}$). This gives an alternate formula for the trace of a square operator A.

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Exercise 4.1.16 (Union Bound) *Prove a union bound for commuting projectors* Π_1 *and* Π_2 *where* $0 \leq \Pi_1, \Pi_2 \leq I$ *and for an arbitrary density operator* ρ *(not necessarily diagonal in the same basis as* Π_1 *and* Π_2*):*

$$\operatorname{Tr}\{(I - \Pi_1\Pi_2)\,\rho\} \leq \operatorname{Tr}\{(I - \Pi_1)\,\rho\} + \operatorname{Tr}\{(I - \Pi_2)\,\rho\}. \tag{4.77}$$

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Exercise 4.2.2 *Suppose we have an ensemble $\{p_X(x), \rho_x\}$ of density operators and a POVM with elements $\{\Lambda_x\}$ that should identify the states ρ_x with high probability, i.e., we would like $\mathrm{Tr}\{\Lambda_x \rho_x\}$ to be as high as possible. The expected success probability of the POVM is then*

$$\sum_x p_X(x) \,\mathrm{Tr}\{\Lambda_x \rho_x\} \,. \tag{4.99}$$

Suppose that there exists some operator τ such that

$$\tau \geq p_X(x)\rho_x \,, \tag{4.100}$$

where the condition $\tau \geq p_X(x)\rho_x$ is the same as $\tau - p_X(x)\rho_x \geq 0$ (i.e., that the operator $\tau - p_X(x)\rho_x$ is a positive semi-definite operator). Show that $\mathrm{Tr}\{\tau\}$ is an upper bound on the expected success probability of the POVM. After doing so, consider the case of encoding n bits into a d-dimensional subspace. By choosing states uniformly at random (in the case of the ensemble $\{2^{-n}, \rho_i\}_{i\in\{0,1\}^n}$), show that the expected success probability is bounded above by $d\ 2^{-n}$. Thus, it is not possible to store more than n classical bits in n qubits and have a perfect success probability of retrieval.

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Exercise 4.3.1 *Show that the purity $P(\rho_A)$ is equal to the following expression:*

$$P(\rho_A) = \mathrm{Tr}\left\{(\rho_A \otimes \rho_{A'}) F_{AA'}\right\}. \tag{4.106}$$

where system A' has a Hilbert space structure isomorphic to that of system A and $F_{AA'}$ is the swap operator that has the following action on kets in A and A':

$$\forall x, y \qquad F_{AA'}|x\rangle_A|y\rangle_{A'} = |y\rangle_A|x\rangle_{A'}. \tag{4.107}$$

(One can in fact show more generally that $\mathrm{Tr}\{f(\rho_A)\} = \mathrm{Tr}\{(f(\rho_A) \otimes I_{A'}) F_{AA'}\}$ for any function f on the operators in system A.)

Exercise 4.3.6 *Show that a parity measurement (defined in the previous exercise) of the state* $|\Phi^+\rangle_{AB}$ *returns an even parity result with probability one, and a parity measurement of the state* $\pi_A \otimes \pi_B$ *returns even or odd parity with equal probability. Thus, despite the fact that these states have the same local description, their global behavior is very different. Show that the same is true for the phase parity measurement, given by* y

$$\Pi^X_{\text{even}} \equiv \frac{1}{2}\left(I_A \otimes I_B + X_A \otimes X_B\right), \tag{4.134}$$

$$\Pi^X_{\text{odd}} \equiv \frac{1}{2}\left(I_A \otimes I_B - X_A \otimes X_B\right). \tag{4.135}$$

Exercise 4.3.18 *Show that performing a measurement with measurement operators* $\{\Lambda_A^j\}$ *on system* A *is the same as performing a measurement of the ensemble in* ($\boxed{4.173}$). *That is, show that* $\mathrm{Tr}\{\rho_A \Lambda_A^j\} = \mathrm{Tr}\{\rho_{XA}(I_X \otimes \Lambda_A^j)\}$, *where* ρ_A *is defined in* ($\boxed{4.175}$). 4

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Exercise 4.4.1 *Prove that a linear map $\mathcal{N}$ is completely positive if its corresponding Choi operator, as defined in Definition 4.4.4, is a positive semi-definite operator. (Hint: Use the fact that any positive semi-definite operator can be diagonalized, the fact that $\mathrm{id}_R \otimes \mathcal{N}$ is linear, and use something similar to (4.202)–(4.205)).*

Exercise 4.6.2 *Prove that a quantum channel* $\mathcal{N}_{A\to B}$ *is entanglement-breaking if* $(\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(\Phi_{RA})$ *is a separable state, where* Φ_{RA} *is a maximally entangled state. (Hint: You can use a trick similar to that which you used to solve Exercise 4.4.1. Alternatively, you can inspect the proof of Theorem 4.6.1 below.)* 5

Exercise 4.6.3 *Show that both a classical–quantum channel and a quantum–classical channel are entanglement-breaking—i.e., if we input the* A *system of a bipartite state* ρ_{RA} *to either of these channels, then the resulting state on systems* RB *is separable.* 5