Physics of the Early Universe (SC1.415)

IIIT-H, Semester Monsoon 24, Assignment 2:

Submission deadline: 29$^{\text{th}}$ September 2024

The problems are for you to practice. You may take help from any source to solve the problems.

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1. In a Cartesian system the coordinates are denoted as $x^i = (x, y, z)$, where $i = 1, 2, 3$. Consider a function $f(x)$. Using the Einstein summation convention (repeated indices are summed) write the expression of $df(x)/dx^i$ and $\partial f(x)/\partial x^i$.

2. Consider a Cartesian coordinate system $x^i = (x, y, z)$, where $i = 1, 2, 3$. In this frame

   (a) Write down the equations of motion of a free particle.

   (b) Suppose, we make a transformation to a new coordinate system $\chi^i = (\chi_1, \chi_2, \chi_3)$ which is not necessarily Cartesian, such that the relations between the old and new coordinates are $x^i = F_i(\chi^j)$ where $F_i(\chi^j)$ are some functions of the new coordinates. Express the equations of motion in the new coordinate system and in terms of F_i. It is the geodesic equation in the new coordinate system.

   (c) Suppose the new coordinate system is a spherical polar coordiante system, *i.e.,* $\chi^i = (r, \theta, \phi)$. Write down all the functions $F_i(r, \theta, \phi)$. Substitute in the previous equation you just obtained and show that the equations of motions in r, θ, ϕ are as

   $$\begin{aligned} \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta &= 0\,, \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta &= 0\,, \\ 2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta &= 0\,, \end{aligned}$$

   where first and 2nd over-dot mean first- and 2nd-order derivative of time.

   (d) Write down the metric, called η_{ij}, in the old system. Express the metric in the new coordiante, called g_{ij}, in terms of the function F_i and the components of the metric η_{ij}.

   (e) Calculate the affine connections Γ^i_{jk}.

   All the indices ij, k run from 1 to 3. Apart from differentiation and partial differentiation, the concepts of special and general relativity are not required to solve the steps. The purpose of the exercise is to show that the concept of geodesic equation, affine connections are very general concepts that emerges from coordinate transformation.

3. Start from the relation between a metric $g_{\mu\nu}$ in a new coordinate system x^μ, and a metric $\eta_{\mu\nu}$ in an old system ξ^μ, where $\mu, \nu = 0, 1, 2, 3$

   $$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}\,.$$

   Show that

   $$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = g_{\rho\nu}\Gamma^\rho_{\lambda\mu} + g_{\mu\rho}\Gamma^\rho_{\lambda\nu}\,. \tag{1}$$

   Using the result obtained show that the affine connection can be written as

   $$\Gamma^\sigma_{\mu\lambda} = \frac{g^{\nu\sigma}}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right) \tag{2}$$

4. During the 6th September class (Newtonian limit of geodesic equation, section 6.6.3) I wrote that in the Newtonian limit the metric of a spacetime is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \quad |h_{\mu\nu}| \ll 1\,,$$

where $h_{\mu\nu}$ is a tiny correction to the Minkowski metric $\eta_{\mu\nu}$. Show that if we neglect the 2nd order term of the perturbation (h^2) then

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}\,.$$

The relation was used in the class but not proven. *Hint: Make use of the fact that $g^{\mu\nu}$ and $g_{\mu\nu}$ are inverse of each other. Similarly $\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^{\mu}_{\nu}$.*

5. Using the transformation relation of a vector V^μ under a coordinate transformation $x \to x'$ show that $V^\mu V^\nu$ is a tensor. *Hint: Check transformation relation of tensors.*

6. The metric of our universe is given by the FRW metric in (t, r, θ, ϕ) as

$$\text{or} \quad g_{\mu\nu} \quad = \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a(t)^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & a(t)^2 r^2 \sin^2\theta \end{pmatrix}. \tag{3}$$

Calculate all the affine connections $\Gamma^{\mu}_{\lambda\nu}$ and the non-vanishing Ricci tensor $R_{\mu\nu}$. Also calculate the Ricci scalar R. *The exercise may be lengthy to do by hand, though it is recommended to calculate some of them, if not all. To calculate all of them write a code in Python. You must submit the code as well.*

7. Starting from the Bianchi identity prove that the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu}/2$ is divergenceless, *i.e.*,

$$\nabla^\mu G_{\mu\nu} = 0\,.$$