Theory Assignment I

Introduction to Quantum Information and Computation Spring 2023, IIIT Hyderabad

January 19, 2024

Total Marks: 30 points — Due date: **26/01/24 11:59 pm**

General Instructions:

Make sure to be clear in your arguments. Vague arguments shall not be given full credit. You can discuss amongst your peers, but all written work must be done individually and must include the names of all collaborators. Start early as some questions might seem tricky since the notation is new!

Notation:
$\mathscr{H}$: The Hilbert Space under consideration
$\mathbb{C}$: The set of complex numbers

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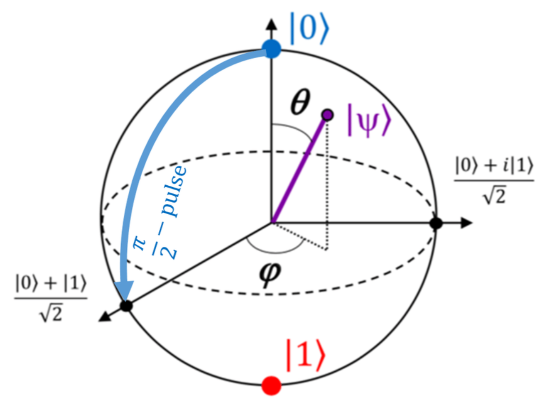
1. [3 points] Prove the following -
   Any n+1 vectors belonging to an n-dimensional vector space must be linearly dependent.

2. [2 points] Find the square root of matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

   Note that the square root of a matrix A is some matrix (denoted by $\sqrt{A}$) such that $\sqrt{A}\sqrt{A} = A$. (recall how to calculate the function of some operator A)

3. [3 points] Prove that $tr(A \otimes B) = tr(A) \times tr(B)$, where A and B are $m \times m$ *and* $n \times n$ matrices respectively
   - $tr(.)$ represents the trace of a matrix

4. [5 points] A Bloch sphere is a geometric representation of a 2-dimensional pure state space (qubit). Any pure state $|\psi\rangle$ can be represented on the Bloch sphere as mentioned below.



$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$$

Show that orthogonal quantum states are found at polar extremes of the Bloch sphere. In other words, the states corresponding to the points that lie on the diametrically opposite ends of the Bloch sphere are orthogonal.

Ponder: A qubit is represent using 2 complex numbers. Each complex number can be represeneted using 2 real numbers, one each for representing the real and imaginary part. Therefore a qubit can be represented by 4 real numbers.
But while plotting on the bloch sphere, we represent the 4 real numbers by a point in a 3D-space! **Why is this possible?**

5. [4 points] Consider a quantum state $|\psi\rangle \in \mathscr{H}$, such that $|\psi\rangle = \Sigma_{i=1}^{n} \alpha_i |u_i\rangle$ for some basis set $\{|u_i\rangle\}_{i=1}^{n}$, and probability amplitudes $\alpha_i \in \mathbb{C}$. Using Born Rule, show that the probability that the state $|\psi\rangle$ will collapse to some state $|u_k\rangle$ after measurement in the basis $\{|u_i\rangle\}_{i=1}^{n}$ is $|\alpha_k|^2$.

   Note: Using the Born rule you can see why $\Sigma_{i=1}^{n} |\alpha|^2 = 1$. This is because $\Sigma_{i=1}^{n} |\alpha|^2 = 1$ is the total probability that the state collapses to one of the basis after performing the measurement, and the total probability is 1 (as it has to collapse to one of the basis states).

6. [3 + 2 points] You have seen 2 types of formalisms in class. The first is the state-vector formalism for pure states, where a quantum state can be represented as a column vector $|\psi\rangle \in \mathscr{H}$. The second is the density matrix formalism which can represent pure states as well as mixed states. So the same pure state $|\psi\rangle$ can be represented as a density matrix $\rho = |\psi\rangle\langle\psi|$.

   Recall: Born Rule for State Vector Formalism:

$$Pr[state\ collapsing\ to\ |i\rangle\ ] \ = \ |\langle i|\psi\rangle|^2$$

   Born Rule for Density Matrix Formalism:

$$Pr[state\ collapsing\ to\ |i\rangle\langle i|] \ = \ Tr[\Pi^i \rho]$$

   Where $\rho = |\psi\rangle\langle\psi|$ in this case, and Π^i is the projection operator $\Pi^i = |i\rangle\langle i|$

   a For $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ verify that the probability of measuring $|\psi\rangle$ in the basis $\{|0\rangle, |1\rangle\}$ and getting $|1\rangle$ is the same via the Born Rule for both the formalisms.

   b For the same state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, what will be the probability of getting $|+i\rangle$ when measuring in the basis $\{|+i\rangle, |-i\rangle\}$? Here $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$.

7. [2 + 2 + 1 + 3 points] Consider some state $|\Psi-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \in \mathscr{H}^2 \otimes \mathscr{H}^2$. Also let $\mathbb{B}$ be some set of measurement operators: $\mathbb{B} = \{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$.

   a Is $|\Psi-\rangle$ separable or entangled? Explain why.

   b What is the probability of state $|\Psi-\rangle$ collapsing to $|1\rangle \otimes |0\rangle$ when measured in $\mathbb{B}$?

   c What is the probability of state $|\Psi-\rangle$ collapsing to $|0\rangle \otimes |0\rangle$ when measured in $\mathbb{B}$?

   d What is the expectation value of operator $O = Z \otimes Z$ for the state $|\Psi-\rangle$?
   Recall that the expectation value of some operator O for some state $|\psi\rangle$ is the quantity $\langle O\rangle = \frac{\langle\psi|O|\psi\rangle}{\langle\psi||\psi\rangle}$