Lab 10 - Ridge Regression and the Lasso in Python

March 9, 2016

This lab on Ridge Regression and the Lasso is a Python adaptation of p. 251-255 of "Introduction to Statistical Learning with Applications in R" by Gareth James, Daniela Witten, Trevor Hastie and Robert Tibshirani. Adapted by R. Jordan Crouser at Smith College for SDS293: Machine Learning (Spring 2016).

1 6.6: Ridge Regression and the Lasso

```
In [95]: %matplotlib inline

         import pandas as pd
         import numpy as np
         import matplotlib.pyplot as plt

         from sklearn.preprocessing import scale
         from sklearn import cross_validation
         from sklearn.linear_model import Ridge, RidgeCV, Lasso, LassoCV
         from sklearn.metrics import mean_squared_error
```

We will use the `sklearn` package in order to perform ridge regression and the lasso. The main functions in this package that we care about are `Ridge`(), which can be used to fit ridge regression models, and `Lasso`() which will fit lasso models. They also have cross-validated counterparts: `RidgeCV`() and `LassoCV`(). We'll use these a bit later.

Before proceeding, let's first ensure that the missing values have been removed from the data, as described in the previous lab.

```
In [96]: df = pd.read_csv('Hitters.csv').dropna().drop('Player', axis=1)
         df.info()
         dummies = pd.get_dummies(df[['League', 'Division', 'NewLeague']])
```

```
<class 'pandas.core.frame.DataFrame'>
Int64Index: 263 entries, 1 to 321
Data columns (total 20 columns):
AtBat        263 non-null int64
Hits         263 non-null int64
HmRun        263 non-null int64
Runs         263 non-null int64
RBI          263 non-null int64
Walks        263 non-null int64
Years        263 non-null int64
CAtBat       263 non-null int64
CHits        263 non-null int64
CHmRun       263 non-null int64
CRuns        263 non-null int64
CRBI         263 non-null int64
```

```
CWalks       263 non-null int64
League       263 non-null object
Division     263 non-null object
PutOuts      263 non-null int64
Assists      263 non-null int64
Errors       263 non-null int64
Salary       263 non-null float64
NewLeague    263 non-null object
dtypes: float64(1), int64(16), object(3)
memory usage: 43.1+ KB
```

We will now perform ridge regression and the lasso in order to predict `Salary` on the `Hitters` data. Let's set up our data:

```
In [97]: y = df.Salary

         # Drop the column with the independent variable (Salary), and columns for which we created dum
         X_ = df.drop(['Salary', 'League', 'Division', 'NewLeague'], axis=1).astype('float64')

         # Define the feature set X.
         X = pd.concat([X_, dummies[['League_N', 'Division_W', 'NewLeague_N']]], axis=1)

         X.info()
```

```
<class 'pandas.core.frame.DataFrame'>
Int64Index: 263 entries, 1 to 321
Data columns (total 19 columns):
AtBat          263 non-null float64
Hits           263 non-null float64
HmRun          263 non-null float64
Runs           263 non-null float64
RBI            263 non-null float64
Walks          263 non-null float64
Years          263 non-null float64
CAtBat         263 non-null float64
CHits          263 non-null float64
CHmRun         263 non-null float64
CRuns          263 non-null float64
CRBI           263 non-null float64
CWalks         263 non-null float64
PutOuts        263 non-null float64
Assists        263 non-null float64
Errors         263 non-null float64
League_N       263 non-null float64
Division_W     263 non-null float64
NewLeague_N    263 non-null float64
dtypes: float64(19)
memory usage: 41.1 KB
```

2 6.6.1 Ridge Regression

The `Ridge()` function has an alpha argument (λ, but with a different name!) that is used to tune the model. We'll generate an array of alpha values ranging from very big to very small, essentially covering the full range of scenarios from the null model containing only the intercept, to the least squares fit:

```
In [98]: alphas = 10**np.linspace(10,-2,100)*0.5
         alphas
```

```
Out[98]: array([ 5.00000000e+09,   3.78231664e+09,   2.86118383e+09,
                 2.16438064e+09,   1.63727458e+09,   1.23853818e+09,
                 9.36908711e+08,   7.08737081e+08,   5.36133611e+08,
                 4.05565415e+08,   3.06795364e+08,   2.32079442e+08,
                 1.75559587e+08,   1.32804389e+08,   1.00461650e+08,
                 7.59955541e+07,   5.74878498e+07,   4.34874501e+07,
                 3.28966612e+07,   2.48851178e+07,   1.88246790e+07,
                 1.42401793e+07,   1.07721735e+07,   8.14875417e+06,
                 6.16423370e+06,   4.66301673e+06,   3.52740116e+06,
                 2.66834962e+06,   2.01850863e+06,   1.52692775e+06,
                 1.15506485e+06,   8.73764200e+05,   6.60970574e+05,
                 5.00000000e+05,   3.78231664e+05,   2.86118383e+05,
                 2.16438064e+05,   1.63727458e+05,   1.23853818e+05,
                 9.36908711e+04,   7.08737081e+04,   5.36133611e+04,
                 4.05565415e+04,   3.06795364e+04,   2.32079442e+04,
                 1.75559587e+04,   1.32804389e+04,   1.00461650e+04,
                 7.59955541e+03,   5.74878498e+03,   4.34874501e+03,
                 3.28966612e+03,   2.48851178e+03,   1.88246790e+03,
                 1.42401793e+03,   1.07721735e+03,   8.14875417e+02,
                 6.16423370e+02,   4.66301673e+02,   3.52740116e+02,
                 2.66834962e+02,   2.01850863e+02,   1.52692775e+02,
                 1.15506485e+02,   8.73764200e+01,   6.60970574e+01,
                 5.00000000e+01,   3.78231664e+01,   2.86118383e+01,
                 2.16438064e+01,   1.63727458e+01,   1.23853818e+01,
                 9.36908711e+00,   7.08737081e+00,   5.36133611e+00,
                 4.05565415e+00,   3.06795364e+00,   2.32079442e+00,
                 1.75559587e+00,   1.32804389e+00,   1.00461650e+00,
                 7.59955541e-01,   5.74878498e-01,   4.34874501e-01,
                 3.28966612e-01,   2.48851178e-01,   1.88246790e-01,
                 1.42401793e-01,   1.07721735e-01,   8.14875417e-02,
                 6.16423370e-02,   4.66301673e-02,   3.52740116e-02,
                 2.66834962e-02,   2.01850863e-02,   1.52692775e-02,
                 1.15506485e-02,   8.73764200e-03,   6.60970574e-03,
                 5.00000000e-03])
```

Associated with each alpha value is a vector of ridge regression coefficients, which we'll store in a matrix `coefs`. In this case, it is a 19×100 matrix, with 19 rows (one for each predictor) and 100 columns (one for each value of alpha). Remember that we'll want to standardize the variables so that they are on the same scale. To do this, we can use the `normalize = True` parameter:

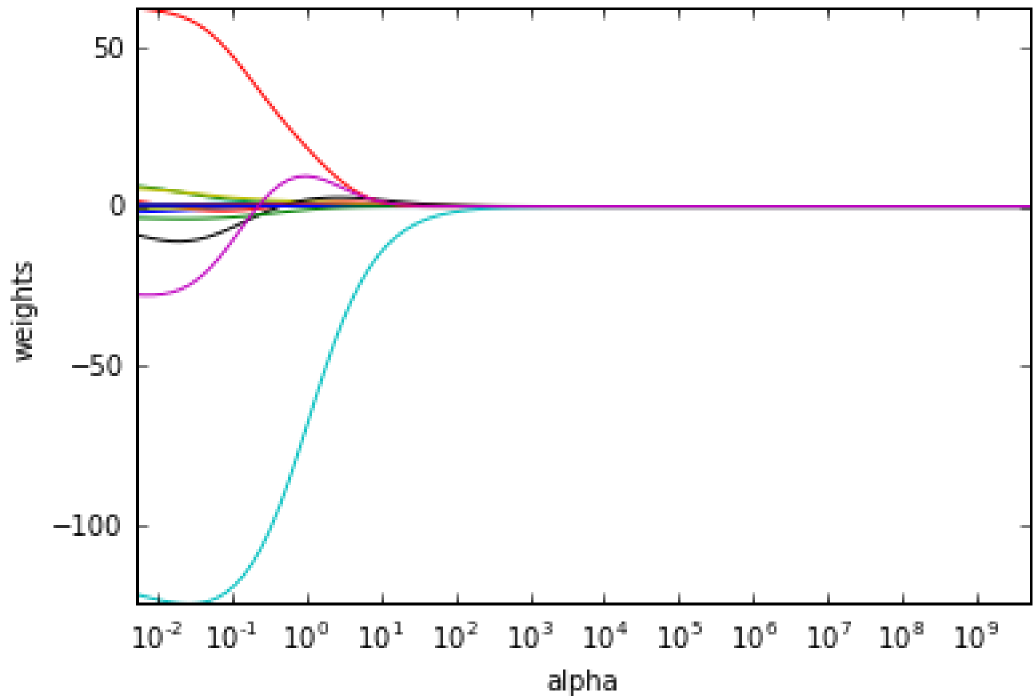
```
In [99]: ridge = Ridge(normalize=True)
         coefs = []

         for a in alphas:
             ridge.set_params(alpha=a)
             ridge.fit(X, y)
             coefs.append(ridge.coef_)

         np.shape(coefs)
```

```
Out[99]: (100, 19)
```

We expect the coefficient estimates to be much smaller, in terms of l_2 norm, when a large value of alpha is used, as compared to when a small value of alpha is used. Let's plot and find out:

```
In [100]: ax = plt.gca()
          ax.plot(alphas, coefs)
          ax.set_xscale('log')
          plt.axis('tight')
          plt.xlabel('alpha')
          plt.ylabel('weights')
```

```
Out[100]: <matplotlib.text.Text at 0x10ec3c5c0>
```



We now split the samples into a training set and a test set in order to estimate the test error of ridge regression and the lasso:

```
In [101]: # Use the cross-validation package to split data into training and test sets
          X_train, X_test , y_train, y_test = cross_validation.train_test_split(X, y, test_size=0.5, ran
```

Next we fit a ridge regression model on the training set, and evaluate its MSE on the test set, using $\lambda = 4$:

```
In [102]: ridge2 = Ridge(alpha=4, normalize=True)
          ridge2.fit(X_train, y_train)             # Fit a ridge regression on the training data
          pred2 = ridge2.predict(X_test)           # Use this model to predict the test data
          print(pd.Series(ridge2.coef_, index=X.columns)) # Print coefficients
          print(mean_squared_error(y_test, pred2))        # Calculate the test MSE
```

```
AtBat          0.098658
Hits           0.446094
HmRun          1.412107
Runs           0.660773
RBI            0.843403
Walks          1.008473
```

```
Years           2.779882
CAtBat          0.008244
CHits           0.034149
CHmRun          0.268634
CRuns           0.070407
CRBI            0.070060
CWalks          0.082795
PutOuts         0.104747
Assists        -0.003739
Errors          0.268363
League_N        4.241051
Division_W    -30.768885
NewLeague_N     4.123474
dtype: float64
106216.52238
```

The test MSE when alpha = 4 is 106216. Now let's see what happens if we use a huge value of alpha, say 10^{10}:

```
In [103]: ridge3 = Ridge(alpha=10**10, normalize=True)
          ridge3.fit(X_train, y_train)             # Fit a ridge regression on the training data
          pred3 = ridge3.predict(X_test)           # Use this model to predict the test data
          print(pd.Series(ridge3.coef_, index=X.columns)) # Print coefficients
          print(mean_squared_error(y_test, pred3))        # Calculate the test MSE
```

```
AtBat          1.317464e-10
Hits           4.647486e-10
HmRun          2.079865e-09
Runs           7.726175e-10
RBI            9.390640e-10
Walks          9.769219e-10
Years          3.961442e-09
CAtBat         1.060533e-11
CHits          3.993605e-11
CHmRun         2.959428e-10
CRuns          8.245247e-11
CRBI           7.795451e-11
CWalks         9.894387e-11
PutOuts        7.268991e-11
Assists       -2.615885e-12
Errors         2.084514e-10
League_N      -2.501281e-09
Division_W    -1.549951e-08
NewLeague_N   -2.023196e-09
dtype: float64
172862.235804
```

This big penalty shrinks the coefficients to a very large degree, essentially reducing to a model containing just the intercept. This over-shrinking makes the model more biased, resulting in a higher MSE.

Okay, so fitting a ridge regression model with alpha = 4 leads to a much lower test MSE than fitting a model with just an intercept. We now check whether there is any benefit to performing ridge regression with alpha = 4 instead of just performing least squares regression. Recall that least squares is simply ridge regression with alpha = 0.

```
In [104]: ridge2 = Ridge(alpha=0, normalize=True)
          ridge2.fit(X_train, y_train)             # Fit a ridge regression on the training data
```

```
pred = ridge2.predict(X_test)           # Use this model to predict the test data
print(pd.Series(ridge2.coef_, index=X.columns)) # Print coefficients
print(mean_squared_error(y_test, pred))         # Calculate the test MSE
```

```
AtBat          -1.821115
Hits            4.259156
HmRun          -4.773401
Runs           -0.038760
RBI             3.984578
Walks           3.470126
Years           9.498236
CAtBat         -0.605129
CHits           2.174979
CHmRun          2.979306
CRuns           0.266356
CRBI           -0.598456
CWalks          0.171383
PutOuts         0.421063
Assists         0.464379
Errors         -6.024576
League_N      133.743163
Division_W   -113.743875
NewLeague_N   -81.927763
dtype: float64
116690.468567
```

It looks like we are indeed improving over regular least-squares!

Instead of arbitrarily choosing alpha $ = 4$, it would be better to use cross-validation to choose the tuning parameter alpha. We can do this using the cross-validated ridge regression function, `RidgeCV()`. By default, the function performs generalized cross-validation (an efficient form of LOOCV), though this can be changed using the argument `cv`.

```
In [105]: ridgecv = RidgeCV(alphas=alphas, scoring='mean_squared_error', normalize=True)
          ridgecv.fit(X_train, y_train)
          ridgecv.alpha_

Out[105]: 0.57487849769886779
```

Therefore, we see that the value of alpha that results in the smallest cross-validation error is 0.57. What is the test MSE associated with this value of alpha?

```
In [106]: ridge4 = Ridge(alpha=ridgecv.alpha_, normalize=True)
          ridge4.fit(X_train, y_train)
          mean_squared_error(y_test, ridge4.predict(X_test))

Out[106]: 99825.648962927298
```

This represents a further improvement over the test MSE that we got using alpha $ = 4$. Finally, we refit our ridge regression model on the full data set, using the value of alpha chosen by cross-validation, and examine the coefficient estimates.

```
In [107]: ridge4.fit(X, y)
          pd.Series(ridge4.coef_, index=X.columns)

Out[107]: AtBat           0.055838
          Hits            0.934879
```

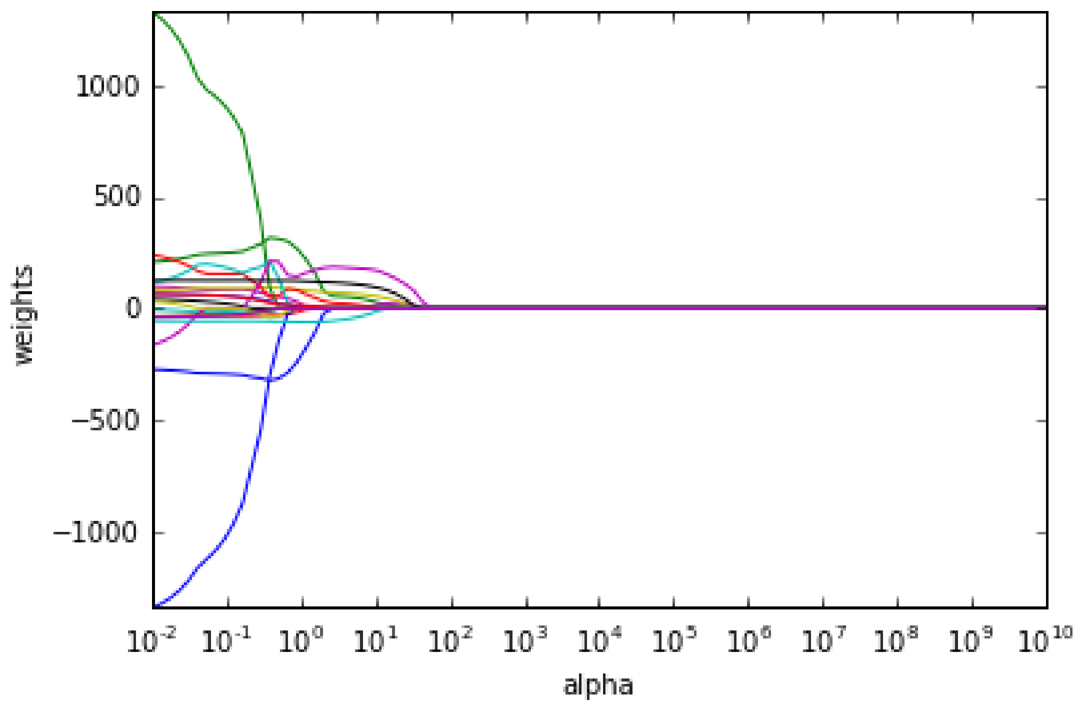
```
HmRun            0.369048
Runs             1.092480
RBI              0.878259
Walks            1.717770
Years            0.783515
CAtBat           0.011318
CHits            0.061101
CHmRun           0.428333
CRuns            0.121418
CRBI             0.129351
CWalks           0.041990
PutOuts          0.179957
Assists          0.035737
Errors          -1.597699
League_N        24.774519
Division_W     -85.948661
NewLeague_N      8.336918
dtype: float64
```

As expected, none of the coefficients are exactly zero - ridge regression does not perform variable selection!

3 6.6.2 The Lasso

We saw that ridge regression with a wise choice of alpha can outperform least squares as well as the null model on the Hitters data set. We now ask whether the lasso can yield either a more accurate or a more interpretable model than ridge regression. In order to fit a lasso model, we'll use the Lasso() function; however, this time we'll need to include the argument $\texttt{max}_{\texttt{i}}\texttt{ter} = 10000$. Other than that change, we proceed just as we did in fitting a ridge model:

```
In [109]: lasso = Lasso(max_iter=10000, normalize=True)
          coefs = []

          for a in alphas:
              lasso.set_params(alpha=a)
              lasso.fit(scale(X_train), y_train)
              coefs.append(lasso.coef_)

          ax = plt.gca()
          ax.plot(alphas*2, coefs)
          ax.set_xscale('log')
          plt.axis('tight')
          plt.xlabel('alpha')
          plt.ylabel('weights')

Out[109]: <matplotlib.text.Text at 0x10f062b38>
```



Notice that in the coefficient plot that depending on the choice of tuning parameter, some of the coefficients are exactly equal to zero. We now perform 10-fold cross-validation to choose the best alpha, refit the model, and compute the associated test error:

In [110]:
```
lassocv = LassoCV(alphas=None, cv=10, max_iter=100000, normalize=True)
lassocv.fit(X_train, y_train)

lasso.set_params(alpha=lassocv.alpha_)
lasso.fit(X_train, y_train)
mean_squared_error(y_test, lasso.predict(X_test))
```

Out[110]:
```
104960.65853895503
```

This is substantially lower than the test set MSE of the null model and of least squares, and only a little worse than the test MSE of ridge regression with alpha chosen by cross-validation.

However, the lasso has a substantial advantage over ridge regression in that the resulting coefficient estimates are sparse. Here we see that 13 of the 19 coefficient estimates are exactly zero:

In [111]:
```
# Some of the coefficients are now reduced to exactly zero.
pd.Series(lasso.coef_, index=X.columns)
```

Out[111]:
```
AtBat          0.000000
Hits           1.082446
HmRun          0.000000
Runs           0.000000
RBI            0.000000
Walks          2.906388
Years          0.000000
CAtBat         0.000000
CHits          0.000000
```

```
CHmRun          0.219367
CRuns           0.000000
CRBI            0.513975
CWalks          0.000000
PutOuts         0.368401
Assists        -0.000000
Errors         -0.000000
League_N        0.000000
Division_W    -89.064338
NewLeague_N     0.000000
dtype: float64
```

To get credit for this lab, post your responses to the following questions: - How do ridge regression and the lasso improve on simple least squares? - In what cases would you expect ridge regression outperform the lasso, and vice versa? - What was the most confusing part of today's class?

to Piazza: https://piazza.com/class/igwiv4w3ctb6rg?cid=38