SCS - 2206.

Name of the lecturer: Dr. W. Ramasinghe

Course code : SCS-2206

Course Name : Mathematical Methods II

Lecture Time : 08.00-10.00 am every Monday

Evaluation Criteria : Two Inclass Assignments 30%
Final Exam 70%

Assigment Plan : Inclass Assignment I
During 6th working week

: Inclass Assignment II
During 14th working week.

Material Covers Assignment I : Sequences and Series, Nonlinear Equations, Interpolation, Curve fitting and approximating functions.

Material Covers Assignment 2 : Numerical Integration, Numerical Solution to ODEs, Groups

Comprehensive Final Exam.

SCS - 2206

Mathematical Methods II.

Let us begin from a very brief discussion on real numbers.

We use the letter $\mathbb{R}$ to denote the set of real numbers. The letter $\mathbb{Q}$ is used to denote the set of rational numbers. Also we use the letter $\mathbb{Z}$ to denote the set of whole integers. The letter $\mathbb{N}$ is used to denote the set of positive integers. Natural number is a synonym for a positive integer. 0 is a whole integer. 0 is not a positive integer. 0 is not a negative integer, either. Rational number is a real number which can be expressed in the form $\frac{p}{q}$ where p is a whole integer and q is a natural number. The empty set is denoted by $\emptyset$.

It is clear that $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.
Since $-2 \in \mathbb{Z}$ and $-2 \notin \mathbb{N}$, $\mathbb{N} \subsetneq \mathbb{Z}$. Also $\mathbb{Z} \subsetneq \mathbb{Q}$ Because $\frac{1}{2} \in \mathbb{Q}$ and $\frac{1}{2} \notin \mathbb{Z}$. As you know $\mathbb{Q} \subsetneq \mathbb{R}$ since $\sqrt{2} \in \mathbb{R}$ and $\sqrt{2} \notin \mathbb{Q}$.

Definition: A real number which is not a rational number is called an irrational number.

Example. $\sqrt{2}$ is an irrational number. (You have seen the proof of this in A/L math classes.)

Definition: Absolute Value of a real number. Let $x \in \mathbb{R}$. The absolute value of x is denoted by $|x|$ and is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Examples: $|4| = 4$, $|\sqrt{3}| = \sqrt{3}$, $|-7| = -(-7) = 7$, $|-1| = -(-1) = 1$.

Proposition. Let $x, y \in \mathbb{R}$. Then the following holds.

(i) $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$.

(ii) $|xy| = |x||y|$.

(iii) $|x+y| \leq |x| + |y|$.

Definition: Let $x \in \mathbb{R}$ and assume that $x \geq 0$. Then $\sqrt{x}$ denotes the real number such that $\sqrt{x} \geq 0$ and $(\sqrt{x})^2 = x$.

Examples. Since $2 \geq 0$ and $2^2 = 4$, we have $\sqrt{4} = 2$. However, $\sqrt{4} \neq -2$ since $-2 \not\geq 0$.

Definition: Let A be a nonempty subset of $\mathbb{R}$. We say that A is bounded above if there exists $\lambda \in \mathbb{R}$ such that for each $a \in A$, $a \leq \lambda$.

Examples. The open intervals $(0, 1)$, the closed interval $[2, 4]$ are bounded above. Also the interval $(-\infty, 3)$ is bounded above. However, the interval $(1, \infty)$ is not bounded above.

Definition: Let A be a nonempty subset of $\mathbb{R}$. We say that A is bounded below if there exists $\mu \in \mathbb{R}$ such that for each $a \in A$, $\mu \leq a$.

Examples. The intervals $[0, 2]$, $(1, \infty)$ are bounded below. However, the interval $(-\infty, 4]$ is not bounded below.

Definition: Let A be a nonempty subset of $\mathbb{R}$. We say that A is bounded if A is both bounded above and bounded below.

Thus A is bounded if there exist

$\lambda, \mu \in \mathbb{R}$ such that for each $a \in A$, $\mu \le a \le \lambda$.
In this situation λ is called an upper Bound of A and μ is called a lower Bound of A.

Examples $(-3, 4]$, $[0, 1)$, $[-10, 6]$, $(7, 9)$ are Bounded subsets of $\mathbb{R}$. However, $(0, \infty)$ $(-\infty, 3]$, $\mathbb{N}, \mathbb{Q}, \mathbb{Z}$ are unbounded subsets of $\mathbb{R}$.

Definition: (Maximum)
Let A Be a nonempty subset of $\mathbb{R}$ and let $M \in \mathbb{R}$. We say that M is the maximum of of A if M is an upper Bound of A and $M \in A$. Maximum of A (if it exists) is denoted by $\max A$.

Examples. Notice that $\max (0, 1] = 1$. Observe that $(2, 3)$ does not have a maximum Because 3 is an upper Bound of $(2, 3)$ and $3 \notin (2, 3)$ and any number between 2, and 3 is not an upperBound of $(2, 3)$.

Definition. (Minimum)
Let A Be a nonempty subset of $\mathbb{R}$ and let $m \in \mathbb{R}$. We say that m is the minimum of A if m is a lower Bound of A and $m \in A$.

Minimum of A (if it exists) is denoted by $\min A$.

Examples: Observe that $\min[2, 3) = 2$. Notice that $(0, 2]$ does not have a minimum because 0 is a lower bound of $(0, 2]$, $0 \notin (0,2]$ and any number between 0 and 2 is not a lower bound of $(0, 2]$.

Did you observe that 3 is the smallest upper bound of $(2, 3)$? We say 3 is the supremum of $(2, 3)$. Also did you observe that 0 is the largest lower bound of $(0, 2]$? We say that 0 is the infimum of $(0, 2]$. We do not go to the rigour of supremum and infimum in this course. It is enough for us to keep in mind that when A is a nonempty subset of $\mathbb{R}$ which has a smallest upper bound λ, we call it the supremum of A and is denoted by $\sup A = \lambda$. Similarly when A is non empty subset of $\mathbb{R}$ which has a greatest lower bound μ, we call it the infimum of A and is denoted by $\inf A = \mu$.

Example. Notice that $\sup(0,2) = 2$, $\inf(0,2] = 0$
$\sup[0,1] = 1 = \max[0,1]$ and $\inf[0,1] = 0 = \min[0,1]$.
Also notice that $\sup(0,\infty)$ does not exist and
$\inf(-\infty, 2]$ does not exist.

The following is very important
about subsets of $\mathbb{R}$.

Completeness Axiom. Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound (sup) in $\mathbb{R}$.

Theorem. Every nonempty subset of $\mathbb{R}$ that is bounded below has a largest lower bound (inf) in $\mathbb{R}$.

Proof. We are not interested in the proofs in this course.

Example. Let $A = \{x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 3\}$.
Then $A \neq \emptyset$, $A \subseteq \mathbb{R}$, and A is bounded above by 2. By completeness axiom $\sup A$ exists in $\mathbb{R}$. $\sup A$ may or may not belong to A.

Example. Let $A = \{x \in \mathbb{Q} : 0 < x \text{ and } x^2 > 5\}$.
Show that 2 is a lower bound of A. Deduce that $\inf A$ exists in $\mathbb{R}$.

The following is an extremely important fact about $\mathbb{N}$. We state it without a proof.

Theorem: Archimedean Property

The set $\mathbb{N}$ of natural numbers is not bounded above.

Remark: You may think it is a trivial result and can be taken for granted. But in mathematics, nothing is taken for granted without a proof.

Sequences

Definition: A sequence of real numbers is an ordered, unending list of real numbers.

Since a sequence is an ordered list, it has a first term, second term, third term, fourth term etc. Since a sequence is unending every term has a successor.

Thus a sequence can be denoted by

$(x_1, x_2, x_3, \dots, x_n, x_{n+1}, \dots)$.

Examples (i) $(1, 2, 3, \dots, n, n+1, \dots)$

(ii) $(1^2, 2^2, 3^2, \dots, n^2, (n+1)^2, \dots)$

(iii) $(1, 0, 1, 0, 1, 0, \dots, 1, 0, \dots)$ are sequences.

Notation: In this course we use the notation $\langle x_n \rangle$ to denote the sequence $(x_1, x_2, x_3, \ldots, x_n, x_{n+1}, \ldots)$.

Thus $\langle x_n \rangle = (x_1, x_2, x_3, \ldots, x_n, x_{n+1}, \ldots)$.

Remark: Although the sequence $\langle x_n \rangle$ is usually written as $(x_1, x_2, x_3, \ldots)$ just by listing 3 or 4 terms, you cannot write a sequence. In other words $(1, 2, 3, \ldots)$, $(4, 4^2, 4^3, \ldots)$ cannot be used to give sequences. This is because we do not know the fourth, fifth, ... terms of the above lists. To the contrary, the list of positive integers in the increasing order, starting from 1 is a sequence. Also the list of powers of 4 in the increasing order of powers starting from 1 is a sequence.

It is possible that a formula can be given for the n th term of a sequence.

Example: $\left\langle \frac{1}{n^3} \right\rangle = \left(\frac{1}{1^3}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots, \frac{1}{n^3}, \frac{1}{(n+1)^3}, \ldots \right)$.

Also it is possible that it is extremely difficult to compute the n th term of a sequence.

Example. The sequence $\langle p_n \rangle$ where p_n is the n th prime number in the increasing order of primes, is one such example. First 6 terms of $\langle p_n \rangle$ in the order are 2, 3, 5, 7, 11, 13. However, it is very difficult to find p_n when n is very large. Indeed the world does not know very large primes.

Given a sequence $\langle x_n \rangle$ of real numbers, it induces a function from $\mathbb{N}$ to $\mathbb{R}$. defined by $f: \mathbb{N} \to \mathbb{R}$, $f(n) = x_n$. Hence a sequence can be regarded as a function from $\mathbb{N}$ to $\mathbb{R}$.

Examples: (i) $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = \sin n$.
(ii) $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = \frac{n}{n+1}$.

Algebra of Sequences.

Two sequences can be added, subtracted in the usual way:

$$\langle x_n \rangle + \langle y_n \rangle \equiv \langle x_n + y_n \rangle$$

$$\langle x_n \rangle - \langle y_n \rangle \equiv \langle x_n - y_n \rangle.$$

A sequence $\langle x_n \rangle$ can be multiplied by a constant c in the usual way:

$c\langle x_n\rangle \equiv \langle c x_n\rangle$.

Two sequence $\langle x_n\rangle$ and $\langle y_n\rangle$ can be multiplied in the usual way:

$\langle x_n\rangle \cdot \langle y_n\rangle \equiv \langle x_n y_n\rangle$.

A sequence $\langle x_n\rangle$ can be divided by a sequence $\langle y_n\rangle$ of which each term is nonzero.

$$\frac{\langle x_n\rangle}{\langle y_n\rangle} \equiv \left\langle \frac{x_n}{y_n}\right\rangle.$$

Examples: Let $\langle x_n\rangle = \langle 2^n\rangle$ and $\langle y_n\rangle = \langle \sin n\rangle$

Then $\langle x_n\rangle + \langle y_n\rangle = \langle 2^n\rangle + \langle \sin n\rangle = \langle 2^n + \sin n\rangle$,

$\langle x_n\rangle - \langle y_n\rangle = \langle 2^n\rangle - \langle \sin n\rangle = \langle 2^n - \sin n\rangle$,

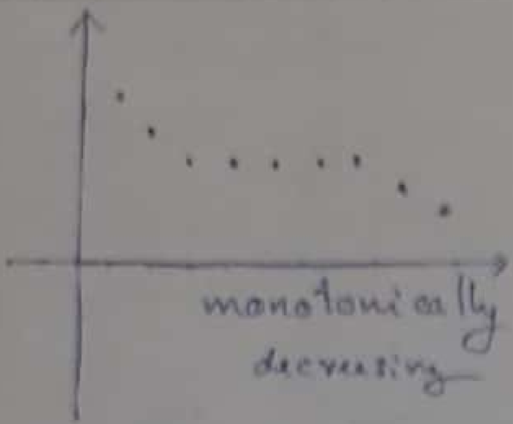
$5\langle x_n\rangle = 5\langle 2^n\rangle = \langle 5.2^n\rangle$,

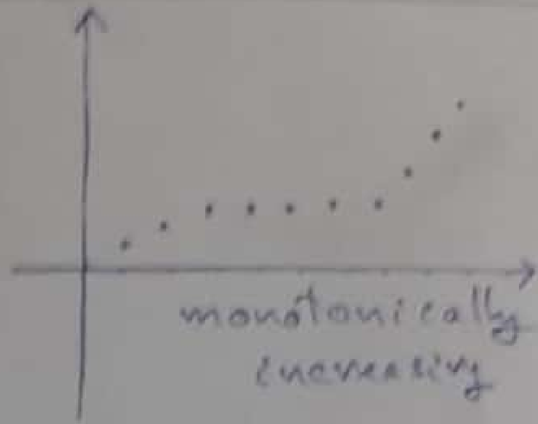
$\langle x_n\rangle \cdot \langle y_n\rangle = \langle 2^n \cancel{\times} \sin n\rangle = \langle 2^n \sin n\rangle$,

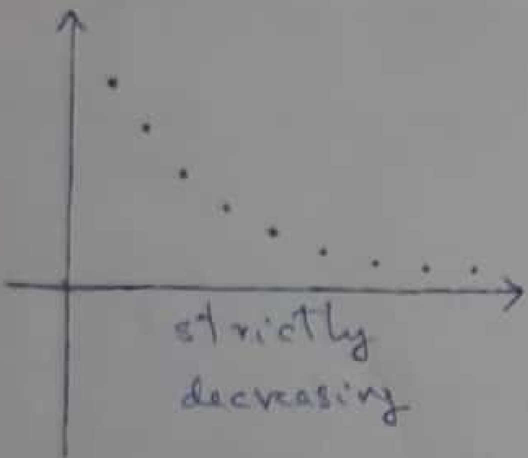
$$\frac{\langle x_n\rangle}{\langle y_n\rangle} = \frac{\langle 2^n\rangle}{\langle \sin n\rangle} = \left\langle \frac{2^n}{\sin n}\right\rangle.$$

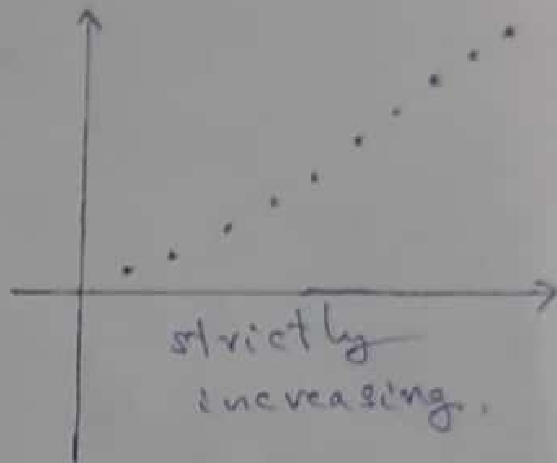
Remark: It can be shown that for each $n \in \mathbb{N}$, $\sin n \neq 0$. Show this.

Definition: A sequence $\langle x_n\rangle$ is said to be monotonically decreasing, monotonically increasing, strictly decreasing, strictly increasing if for each $n \in \mathbb{N}$, $x_{n+1} \leq x_n$, $x_{n+1} \geq x_n$, $x_{n+1} < x_n$, $x_{n+1} > x_n$ respectively.









Examples:

(i) $\left\langle \frac{1}{2^{(n-1)(n-2)}} \right\rangle$ is monotonically decreasing.

(ii) $\langle (n-1)(n-2) \rangle$ is monotonically increasing.

(iii) $\langle \frac{1}{n} \rangle$ is strictly decreasing.

(iv) $\langle 2^n \rangle$ is strictly increasing.

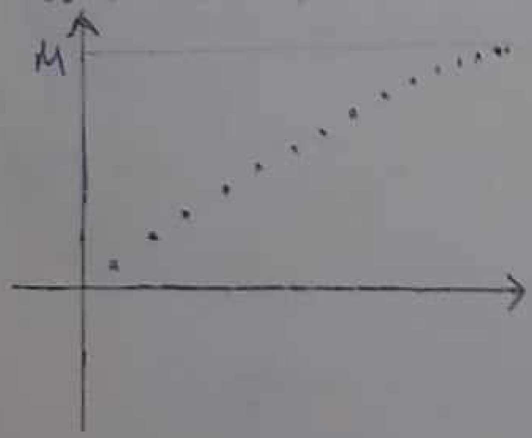
Definition: A sequence $\langle x_n \rangle$ is said to be constant if there exists $k \in \mathbb{R}$ such that for each $n \in \mathbb{N}$, $x_n = k$.

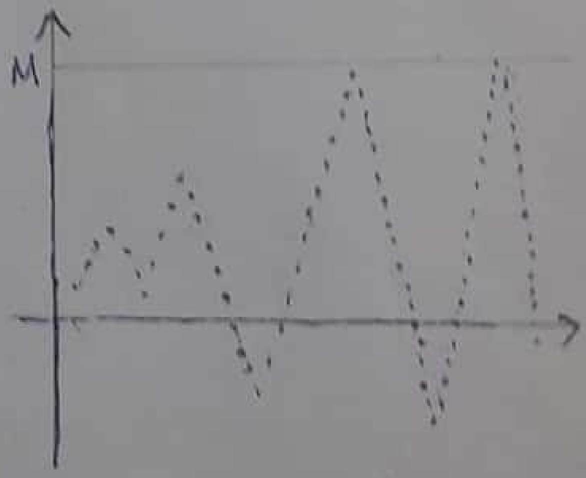
Example. Then sequence $\langle \sin^2 n + \cos^2 n \rangle$ is constant because for each $n \in \mathbb{N}$, $\sin^2 n + \cos^2 n = 1$.

Definition: A sequence $\langle x_n \rangle$ is said to be monotomic if $\langle x_n \rangle$ is monotonically increasing or monotonically decreasing or strictly increasing or strictly decreasing.

Examples. $\langle \frac{1}{n} \rangle$, $\langle 2^n \rangle$, $\langle (n-1)(n-2) \rangle$ are monotomic sequences. $\langle (-1)^n \rangle$ is not a monotomic sequence. Constant sequences are monotomic sequences.

Definition: A sequence $\langle x_n \rangle$ is said to be Bounded above if $\{x_n : n \in \mathbb{N}\}$ is bounded above (i.e, there exists $M \in \mathbb{R}$ such that for each $n \in \mathbb{N}$, $x_n \leq M$.).





Examples: (i) The sequence $\langle \frac{n+1}{n+2} \rangle$ is Bounded above Because for each $n \in \mathbb{N}$, $\frac{n+1}{n+2} \leq 1$.

(ii) The sequence $\langle \sin n \rangle$ is bounded above Because for each $n \in \mathbb{N}$, $\sin n \leq 1$.

(iii) The sequence $\langle n^2 \rangle$ is not bounded above

Because if $\{n^2 : n \in \mathbb{N}\}$ is Bounded above then that $\{n : n \in \mathbb{N}\} = \mathbb{N}$ is bounded above which contradicts the Archimedean Property.

Definition: A sequence $\langle x_n \rangle$ is said to be Bounded below if $\{x_n : n \in \mathbb{N}\}$ is Bounded Below (i.e. there exists $M \in \mathbb{R}$ such that for each $n \in \mathbb{N}$, $x_n \geq M$.).

Ex: Sketch a diagram to show this.

Examples.

(i) The sequence $\langle \frac{1}{n} \rangle$ is Bounded below Because for each $n \in \mathbb{N}$, $\frac{1}{n} \geq 0$.

(ii) The sequence $\langle \cos n \rangle$ is bounded below Because for each $n \in \mathbb{N}$, $\cos n \geq -1$.

(iii) the sequence $\langle (-1)^{n+1} n \rangle = (1, -2, 3, -4, 5, -6, \cdots)$ is not Bounded below. Because $\mathbb{N}$ is not Bounded above. (Prove this.)

Definition: A sequence $\langle x_n \rangle$ is said to be Bounded if $\langle x_n \rangle$ is both Bounded above and Bounded below (i.e $\{x_n : n \in \mathbb{N}\}$ is Bounded or there exists $M \in \mathbb{R}$ such that for each $n \in \mathbb{N}$, $|x_n| \leq M$.).

Examples:

(i) the sequence $\langle \sin n \rangle$ is Bounded Because for each $n \in \mathbb{N}$, $|\sin n| \leq 1$.

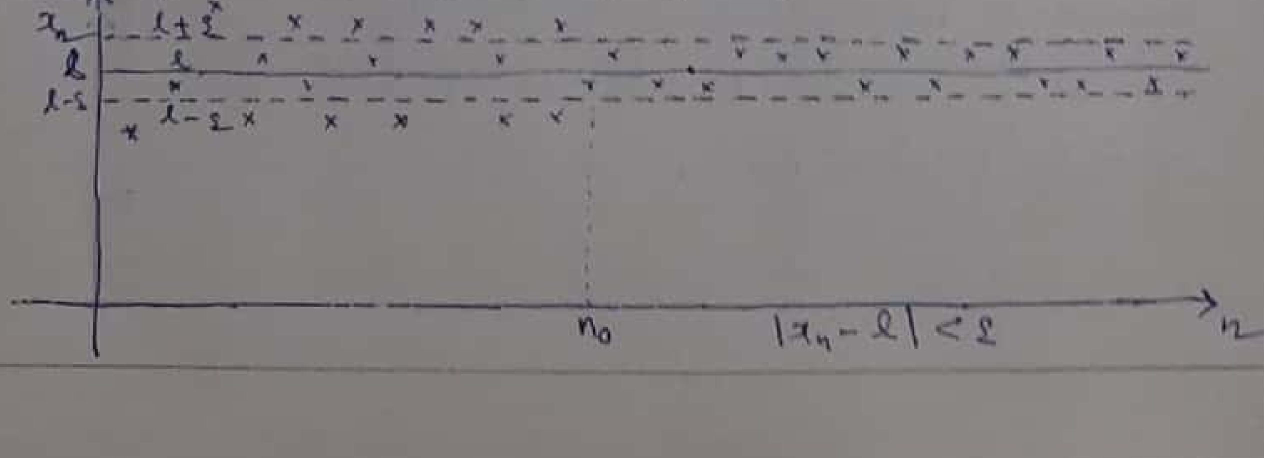
(ii) The sequence $\langle (-1)^n n \rangle$ is not Bounded Because it is not bounded above (and not Bounded below).

Convergent Sequences.

Definition. Let $\langle x_n \rangle$ Be a sequence of real numbers and let l Be a real number. We say that $\langle x_n \rangle$ converges to l if $|x_n - l|$ can Be made as small as one desires for each integer n after some positive integer n_0.

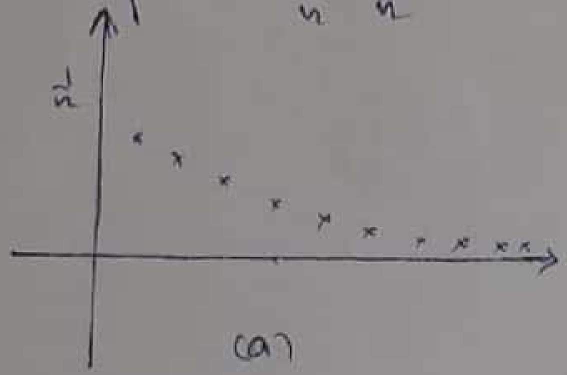
In other words we say that $\langle x_n \rangle$ converges to l if for each $\varepsilon > 0$, $|x_n - l|$ can Be made smaller than ε for each positive integer n after some positive integer n_0.

That is we say that $\langle x_n \rangle$ converges to l if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n > n_0$, $|x_n - l| < \varepsilon$ (i.e $l - \varepsilon < x_n < l + \varepsilon$).



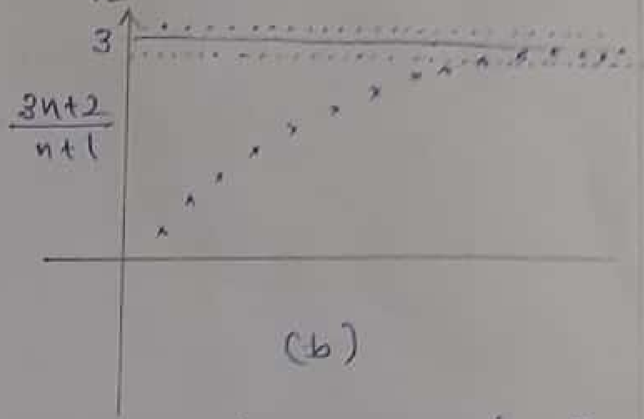
Notation: If $\langle x_n \rangle$ converges to l, we write $\lim_{n\to\infty} x_n = l$, or $\lim_n x_n = l$, or $x_n \to l$ as $n \to \infty$, or $x_n \to l$.

Examples. (a) $\lim_n \frac{1}{n} = 0$ (b) $\lim_n \frac{3n+2}{n+1} = 3$.



(a)

$\langle \frac{1}{n} \rangle$ converges to 0



(b)

$\langle \frac{3n+2}{n+1} \rangle$ converges to 3.

Example. $\langle (-1)^n \rangle$ does not converge. The terms of $\langle (-1)^n \rangle$ starts from -1, then go to $+1$, then again comes back to -1, then again goes back to $+1$ and continue this procedure. Hence the terms of $\langle x_n \rangle$ do not get closer to any real number. Thus $\langle (-1)^n \rangle$ does not converge.

Theorem. If a sequence $\langle x_n \rangle$ is convergent then $\lim_n x_n$ is unique.

Theorem: If a sequence $\langle x_n \rangle$ is convergent then $\langle x_n \rangle$ is bounded.

Theorem: If a sequence $\langle x_n \rangle$ is not bounded then $\langle x_n \rangle$ is not convergent.

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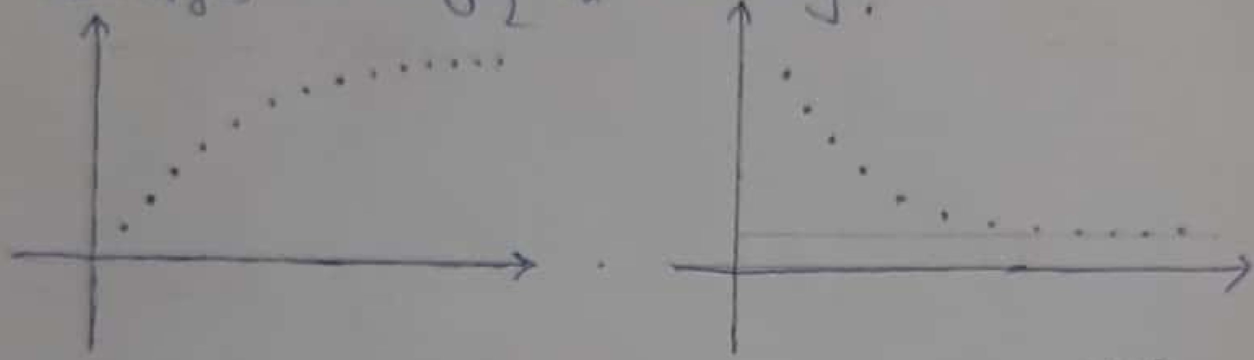
Theorem: Every increasing (monotonically or strictly) and bounded above sequence $\langle x_n \rangle$ converges to $\sup\{x_n : n \in \mathbb{N}\}$.

Theorem: Every decreasing (monotonically or strictly) and bounded below sequence $\langle x_n \rangle$ converges to $\inf\{x_n : n \in \mathbb{N}\}$.



Example. Consider the sequence $\left\langle \left(1+\frac{1}{n}\right)^n \right\rangle$.
It can be shown that $\left\langle \left(1+\frac{1}{n}\right)^n \right\rangle$ is strictly increasing and bounded above. Then $\left\langle \left(1+\frac{1}{n}\right)^n \right\rangle$ converges. We define $\lim_n \left(1+\frac{1}{n}\right)^n = e$.

Algebraic Properties of Convergent Sequences.

Theorem. Let $\langle x_n \rangle, \langle y_n \rangle$ be convergent sequences.
Then (i) $\langle x_n + y_n \rangle, \langle x_n - y_n \rangle$ are convergent and $\lim_n (x_n + y_n) = \lim_n x_n + \lim_n y_n$ and $\lim_n (x_n - y_n) = \lim_n x_n - \lim_n y_n$.

(ii) $\langle x_n y_n \rangle$ is convergent and $\lim_n (x_n y_n) = \lim_n x_n \lim_n y_n$.

(iii) Assume for each $n \in \mathbb{N}$, $y_n \neq 0$. Then $\left\langle \frac{x_n}{y_n} \right\rangle$ is convergent and $\lim_n \left(\frac{x_n}{y_n}\right) = \dfrac{\lim_n x_n}{\lim_n y_n}$.

Proposition: If $\langle x_n \rangle$ converges to l then $\langle |x_n| \rangle$ converges to $\langle |l| \rangle$.

Proof: Follows from $\big| |x_n| - |l| \big| \leq |x_n - l|$ for each n.

Proposition: If $\langle x_n \rangle$ converges to l then $\langle x_n^p \rangle$ converges to l^p for each $p \in \mathbb{N}$.

Theorem. Let $\langle x_n \rangle$ be a convergent sequence and k be a real number.

(a) If for each $n \in \mathbb{N}$, $x_n \geq k$, then $\lim_n x_n \geq k$.

(b) If for each $n \in \mathbb{N}$, $x_n \leq k$, then $\lim_n x_n \leq k$.

Theorem: Let $\langle x_n \rangle$, $\langle y_n \rangle$ be convergent sequences. Then

(a) If for each $n \in \mathbb{N}$, $x_n \leq y_n$, then $\lim_n x_n \leq \lim_n y_n$.

(b) If for each $n \in \mathbb{N}$, $x_n \geq y_n$, then $\lim_n x_n \geq \lim_n y_n$.

Theorem (Sandwich Theorem). Let $\langle x_n \rangle, \langle y_n \rangle, \langle z_n \rangle$ be sequences such that for each $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$ and $\lim_n x_n = \lim_n z_n$. Then $\langle y_n \rangle$ converges and

$$\lim_n y_n = \lim_n x_n = \lim_n z_n.$$

Example. Show that $\lim_n 2^{1/n} = \lim_n \sqrt[n]{2} = 1$.

Proof. Clearly for each $n \in \mathbb{N}$, $2^{1/n} > 1$. Thus for each $n \in \mathbb{N}$, there exists $t_n > 0$ such that $2^{1/n} = 1 + t_n$.

Hence $2 = (1+t_n)^n = 1 + nt_n + \frac{n(n-1)}{1.2}t_n^2 + \dots + t_n^n > \frac{n(n-1)}{2}t_n^2$.

Thus for each $n > 2$, $0 < t_n < \sqrt{\frac{4}{n(n-1)}}$. Since $\lim_n 0 = \lim_n \sqrt{\frac{4}{n(n-1)}} = 0$, by Sandwich theorem $\lim_n t_n = 0$,

Hence $\lim_n 2^{1/n} = \lim_n (1+t_n) = 1 + \lim_n t_n = 1 + 0 = 1$.

Cauchy Criterion For Convergent Sequences.

Let $\langle x_n \rangle$ be a convergent sequence that converges to l. Thus for very large n, m the terms x_n, x_m are very close to l. Hence for very large n, m the terms x_n, x_m are very close to each other.

Definition: Let $\langle x_n \rangle$ be a sequence of real numbers. We say that $\langle x_n \rangle$ is Cauchy if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $n, m > N$, $|x_n - x_m| < \varepsilon$.

Theorem. Every convergent sequence is Cauchy.

Proof: Let $\langle x_n \rangle$ be a convergent sequence that converges to $l \in \mathbb{R}$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for each $n > N$, $|x_n - l| < \varepsilon/2$. Then for each $n, m > N$, $|x_n - x_m| = |x_n - l + l - x_m| \leq |x_n - l| + |x_m - l| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This completes the proof.

Theorem: Every Cauchy sequence converges.

Thus $\langle x_n \rangle$ converges if and only if $\langle x_n \rangle$ is Cauchy.