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Supervised models

Time Series Forecasting

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Thanks to Dr.Sridhar Pappu for the material

"Prediction is very difficult, especially if it's about the future."
--Niels Bohr, Nobel laureate in Physics

What is Time Series data?

- A sequence of data points in successive order, indexed by time.

$$y_t,\ y_{t-1}, y_{t-2}, y_{t-3}, y_{t-4}, \dots$$

- Eg: Population of the country listed year-wise, Temperature in the city listed by the hour, Number of iPhones sold listed for each quarter

Forecasting

- Factors needed to forecast the next month's stock price of Tata Motors ($\hat{y}_{t+1}$)
  - Current price (y_t)
  - Current Sales, Revenue and profit data (x_1)
  - Sales trend (x_2)
  - Level debt carried by the company (x_3)
  - Competition (x_4)
  - Import/export rules (x_5)
  - Interest rate environment (x_6)
  - US/INR exchange rate (x_7)
  - Tax rates (x_8)
  - Crack down on black money? (x_9)
  - Cost of steel? (x_{10})
  - Number of smart phones sold? (x_{11})

Forecasting

$$\hat{y}_{t+1} = g(t, x_1, x_2, x_3 \dots, y_t, y_{t-1}, y_{t-2}, \dots)$$

g might be some complex linear or nonlinear function.

Time series forecasting attempts to do same forecast just using the past data of y, without relying on any other external predictors (x_i).

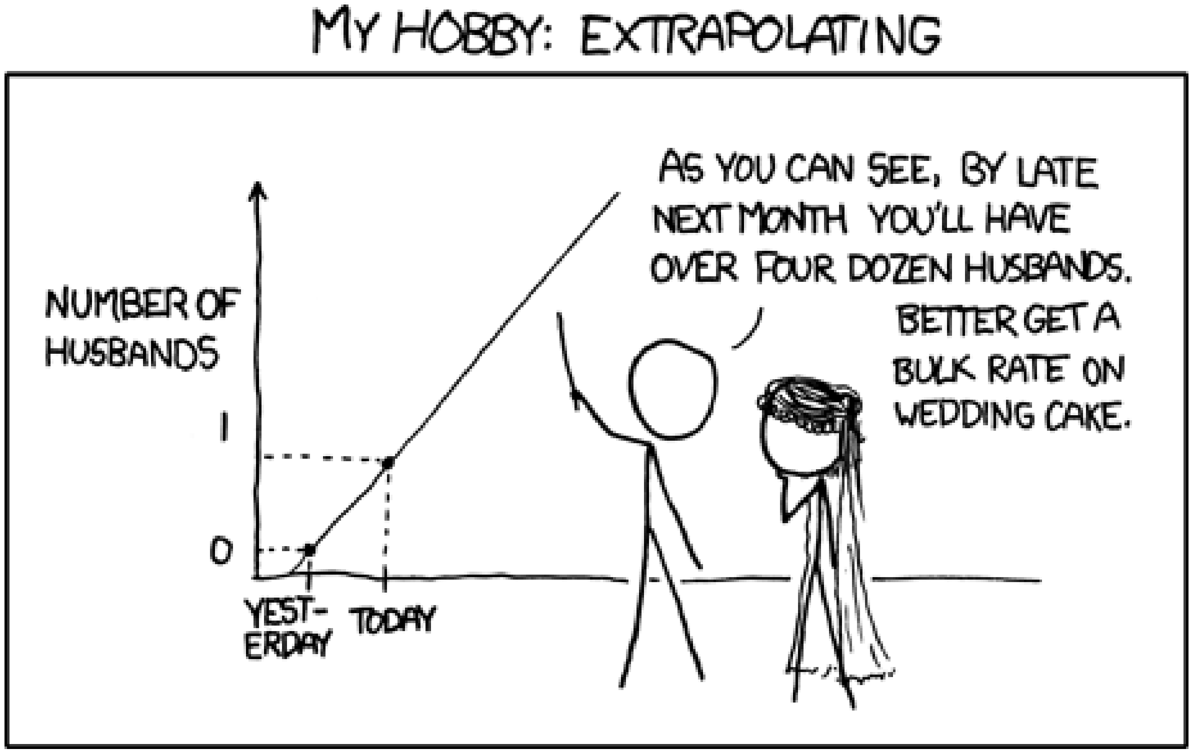
Typical Time Series

$$\hat{y}_{t+1} = f(t, y_t, y_{t-1}, y_{t-2} \cdots)$$

f can be linear or nonlinear function

Why Time Series forecasting?

- Causal independent variables are
  - Unknown to us
  - Not available
  - Might not fit the data well
  - Difficult to forecast



FORECASTING THROUGH TREND ANALYSIS

Regression with time

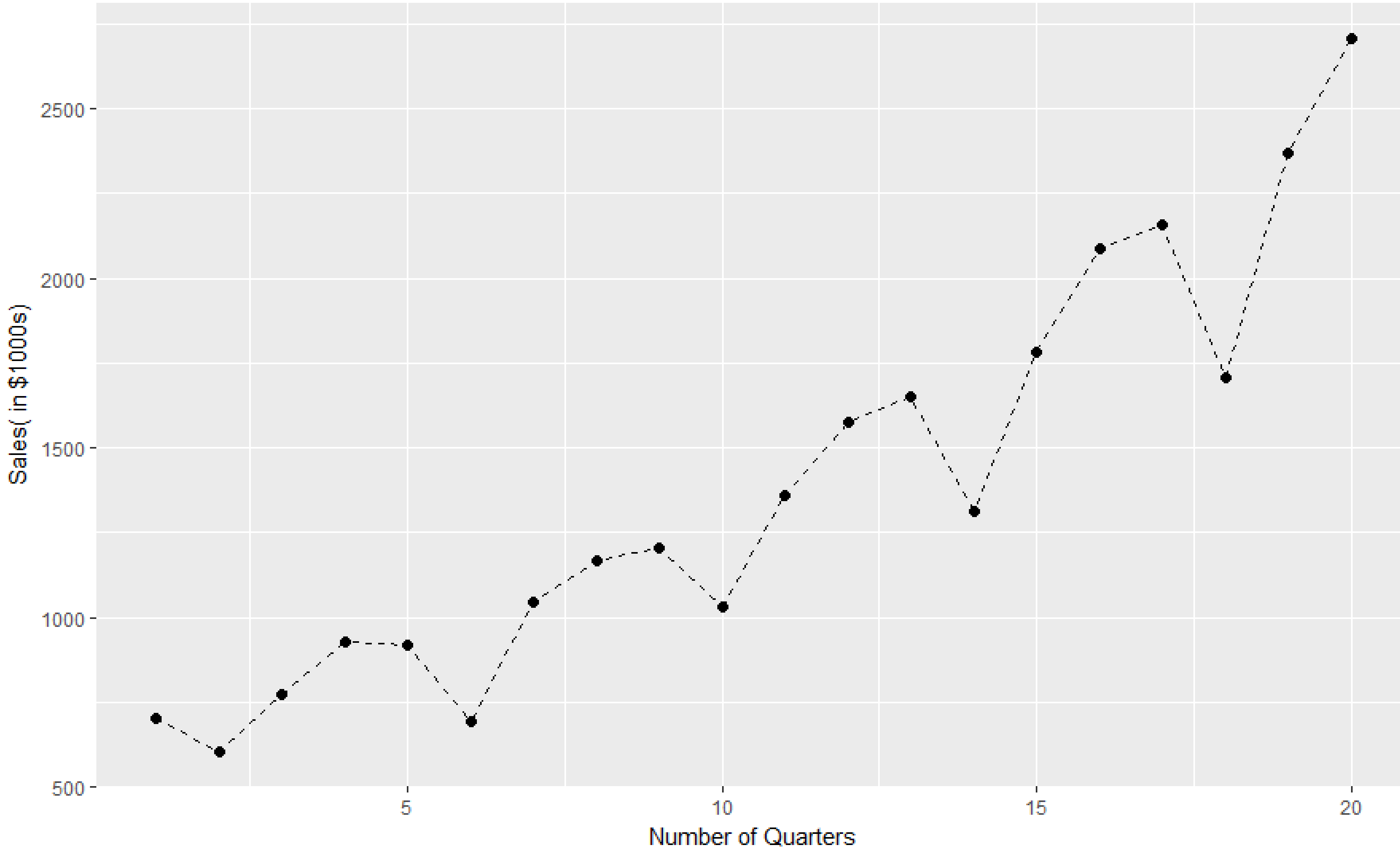
$$\hat{y}_{t+1} = f(t)$$

CSE 7202c

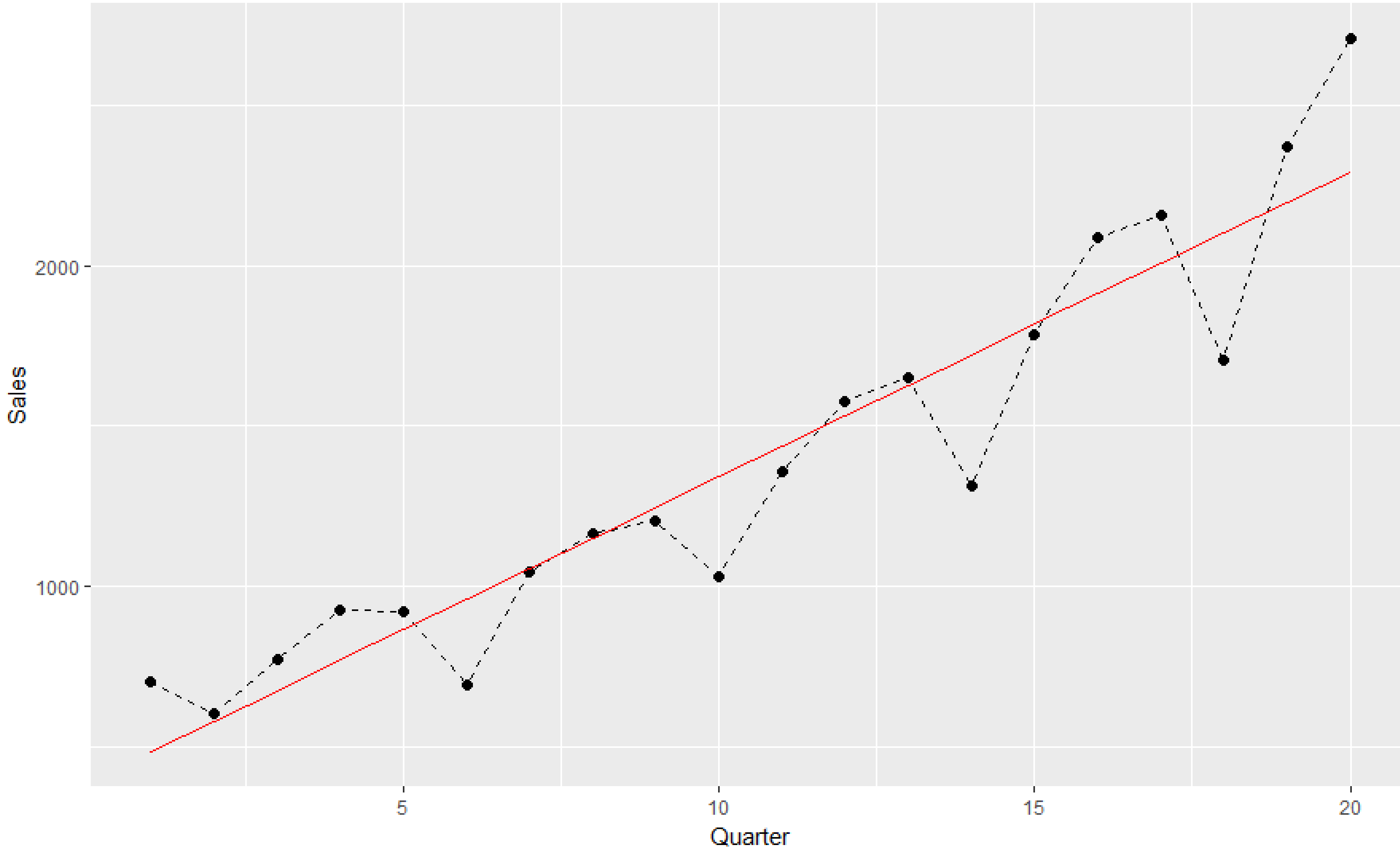
Regression on Time

- Use when trend is the most pronounced

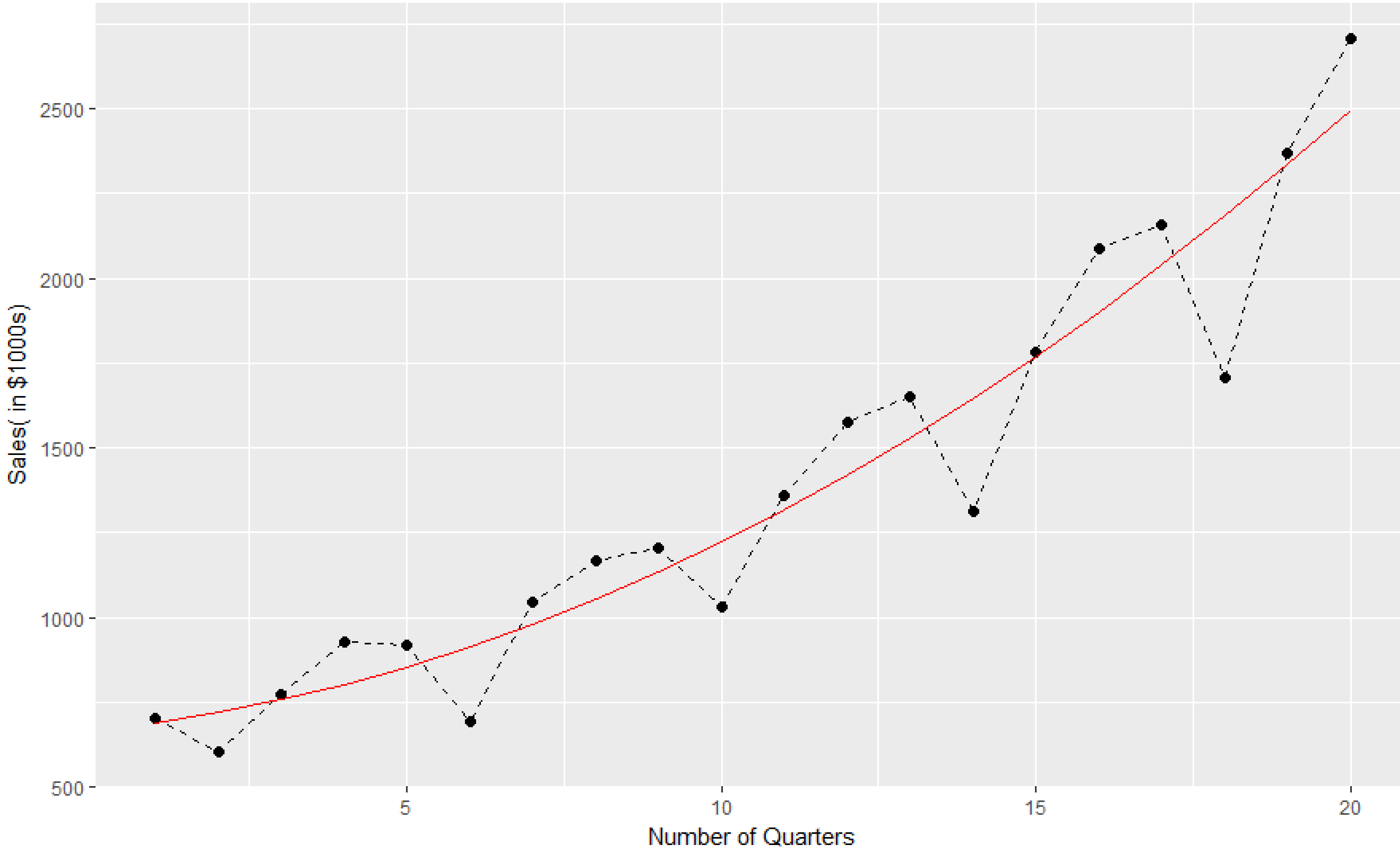
Seasonality



Regression Analysis – Linear fit



Quadratic Trend



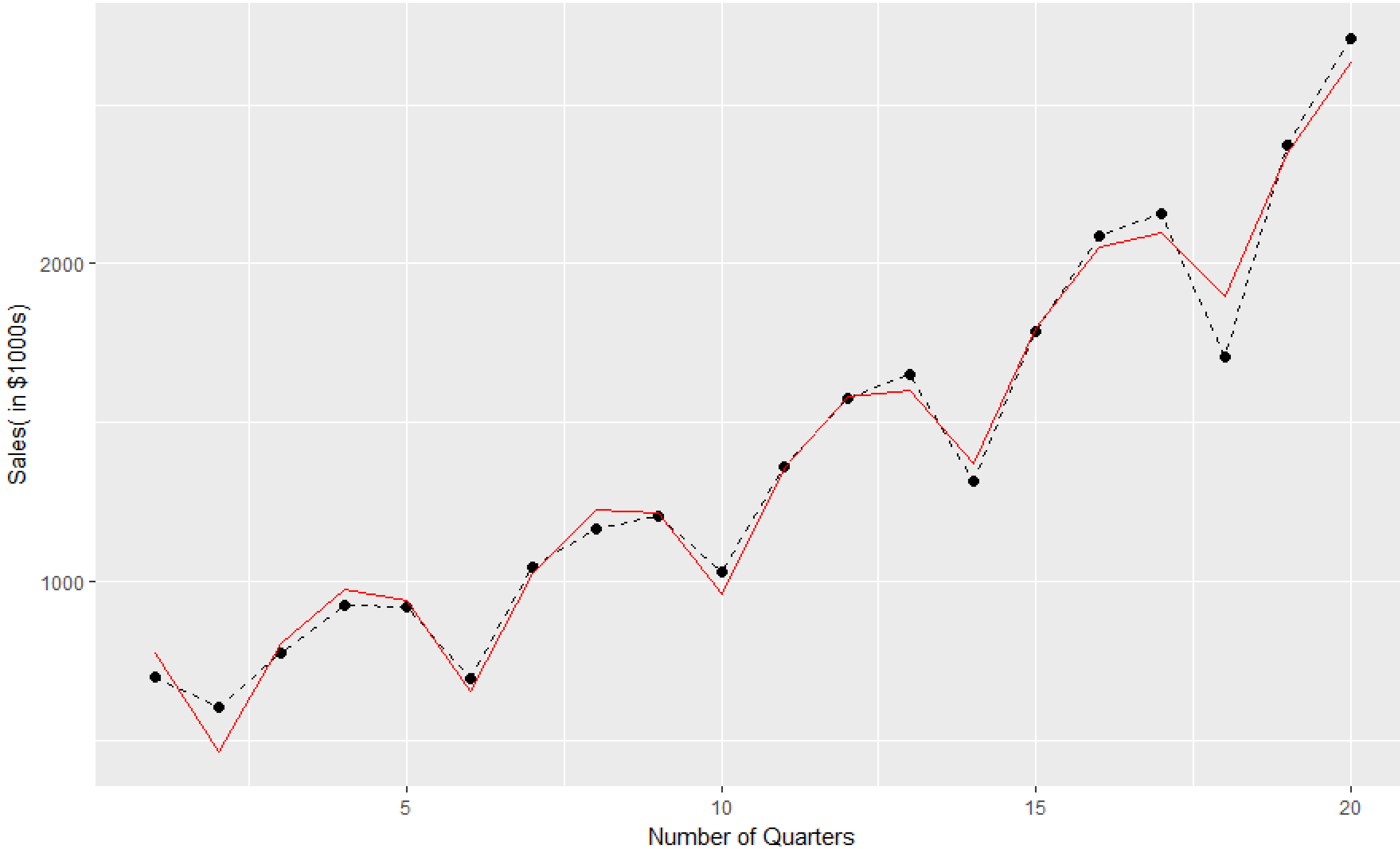
Seasonal Regression Models

	Value of		
Quarter	X_{3_t}	X_{4_t}	X_{5_t}
1	1	0	0
2	0	1	0
3	0	0	1
4	0	0	0

$$Y_t = \beta_0 + \beta_1 X_{1_t} + \beta_2 X_{2_t} + \beta_3 X_{3_t} + \beta_4 X_{4_t} + \beta_5 X_{5_t} + \varepsilon_t$$

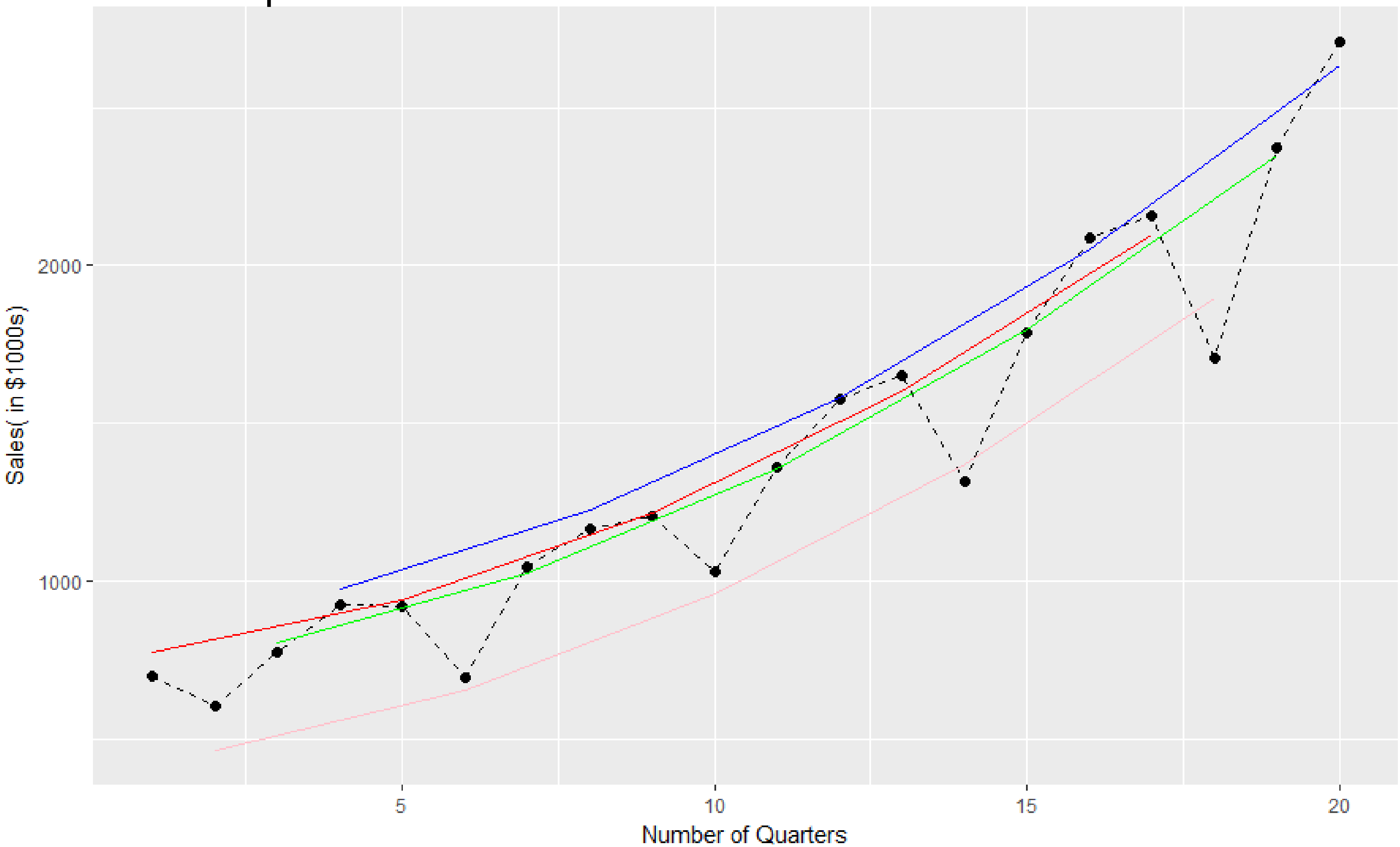
where, $X_{1t} = t$ and $X_{2t} = t^2$.

Seasonal Regression Models

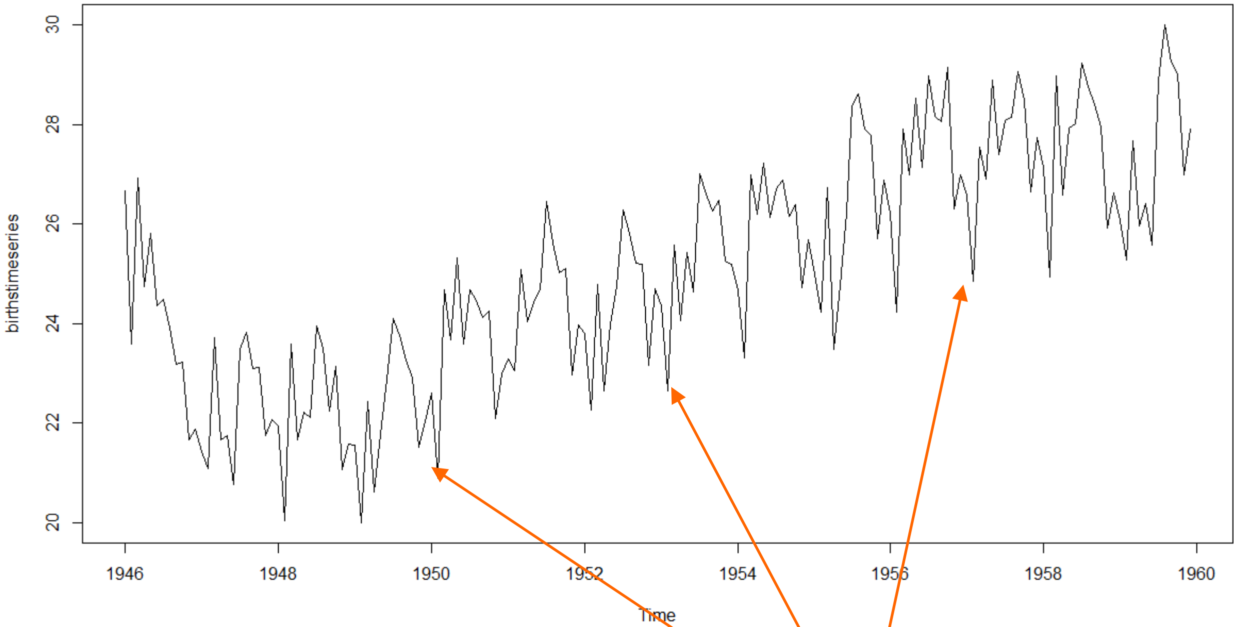


Quadratic fit with seasonality

Plotting the fitted data-points separately for each quarter, shows how R manages to do such a good fit. Its basically fitting a quadratic line for each quarter with a different intercept.



Births in NY



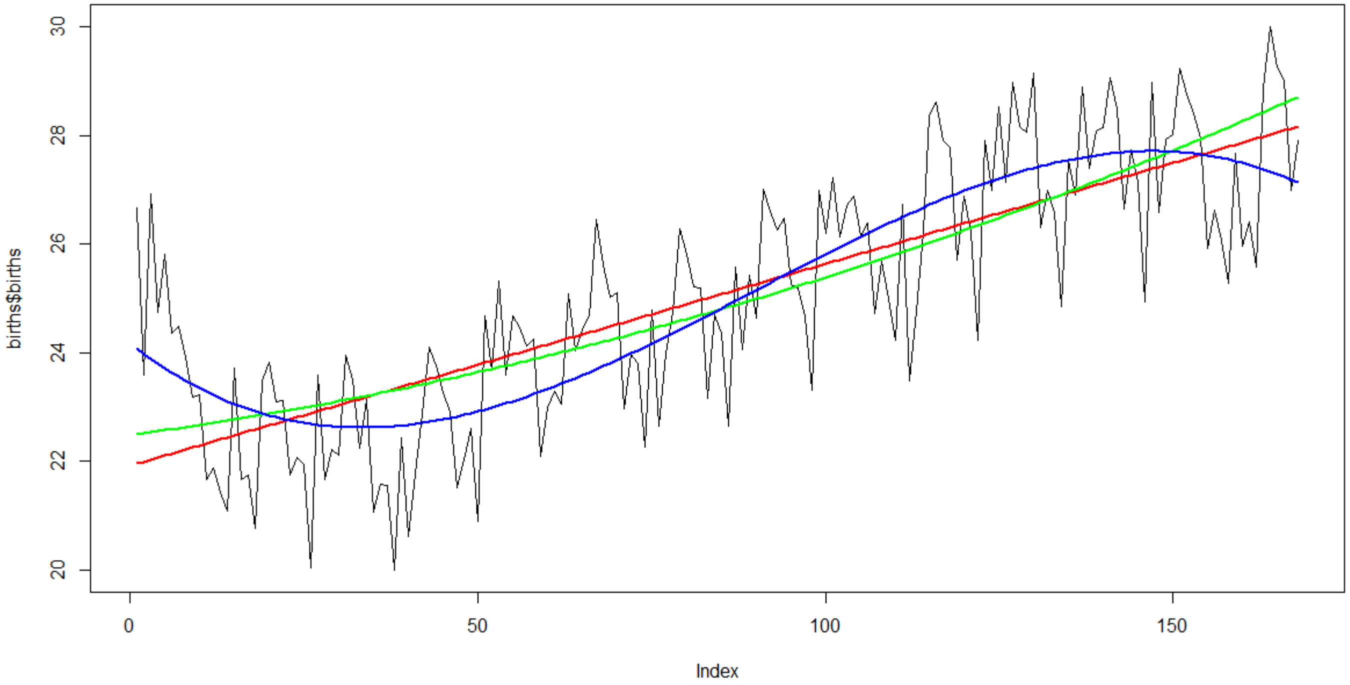

```
> birthstimeseries <- ts(births,
+                        frequency=12,
+                        start=c(1946,1))
> birthstimeseries
```

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1946	26.663	23.598	26.931	24.740	25.806	24.364	24.477	23.901	23.175	23.227	21.672	21.870
1947	21.439	21.089	23.709	21.669	21.752	20.761	23.479	23.824	23.105	23.110	21.759	22.073
1948	21.937	20.035	23.590	21.672	22.222	22.123	23.950	23.504	22.238	23.142	21.059	21.573
1949	21.548	20.000	22.424	20.615	21.761	22.874	24.104	23.748	23.262	22.907	21.519	22.025
1950	22.604	20.894	24.677	23.673	25.320	23.583	24.671	24.454	24.122	24.252	22.084	22.991
1951	23.287	23.049	25.076	24.037	24.430	24.667	26.451	25.618	25.014	25.110	22.964	23.981
1952	23.798	22.270	24.775	22.646	23.988	24.737	26.276	25.816	25.210	25.199	23.162	24.707
1953	24.364	22.644	25.565	24.062	25.431	24.635	27.009	26.606	26.268	26.462	25.246	25.180
1954	24.657	23.304	26.982	26.199	27.210	26.122	26.706	26.878	26.152	26.379	24.712	25.688
1955	24.990	24.239	26.721	23.475	24.767	26.219	28.361	28.599	27.914	27.784	25.693	26.881
1956	26.217	24.218	27.914	26.975	28.527	27.139	28.982	28.169	28.056	29.136	26.291	26.987
1957	26.589	24.848	27.543	26.896	28.878	27.390	28.065	28.141	29.048	28.484	26.634	27.735
1958	27.132	24.924	28.963	26.589	27.931	28.009	29.229	28.759	28.405	27.945	25.912	26.619
1959	26.076	25.286	27.660	25.951	26.398	25.565	28.865	30.000	29.261	29.012	26.992	27.897

```
>
```

Seasonal Regression Models

Seasonal Regression Models - Births

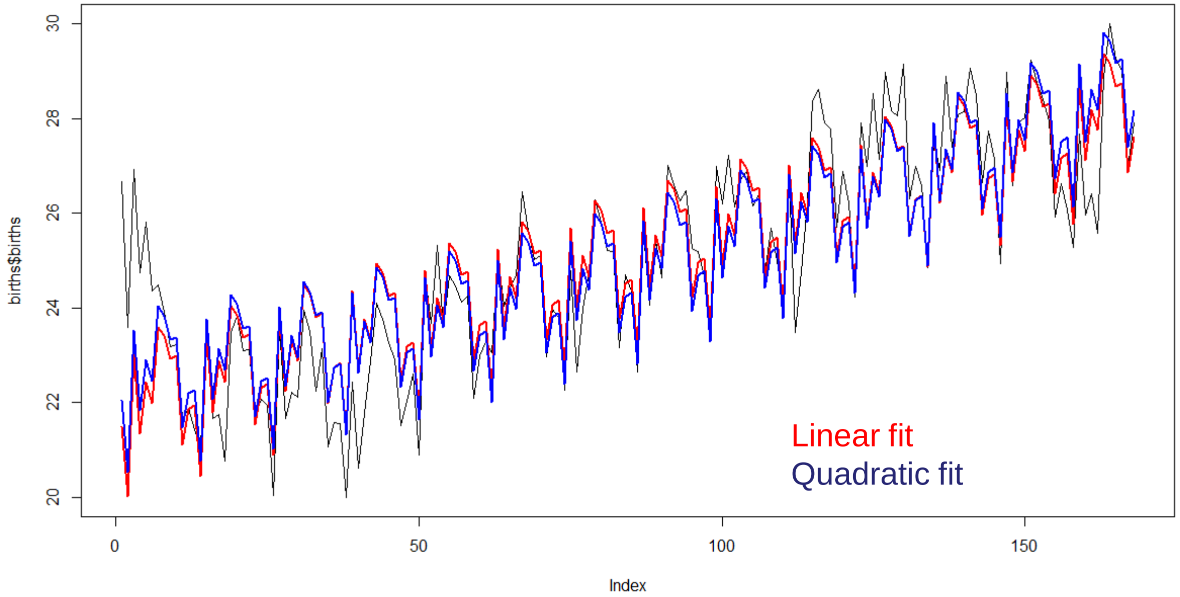


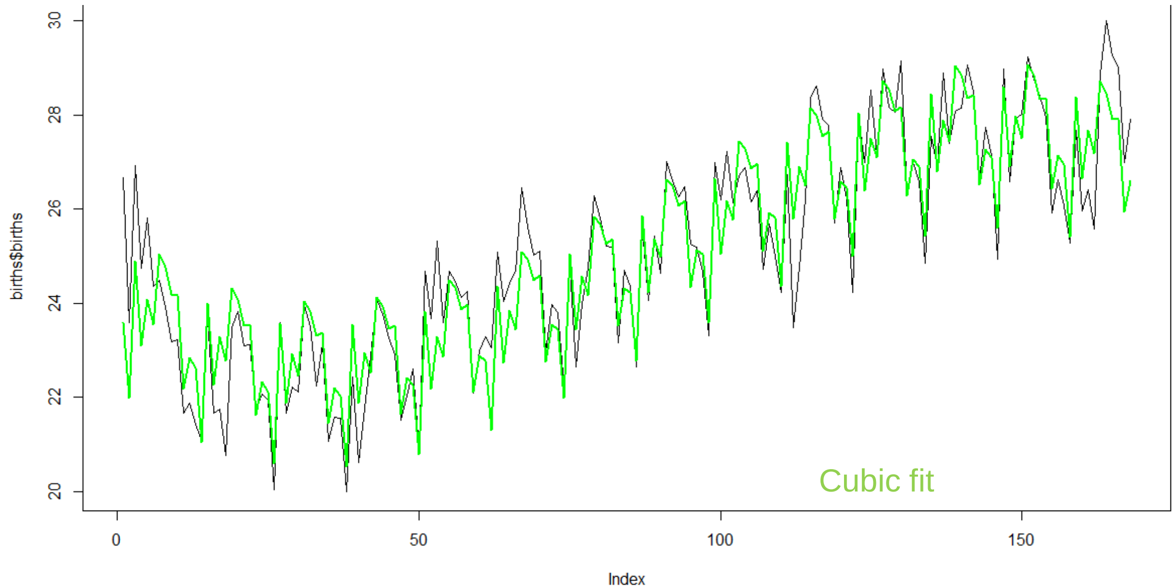
Data Editor

File Edit Help

	births	time	var3	var4	var5	var6
1	26.663	1				
2	23.598	2				
3	26.931	3				
4	24.74	4				
5	25.806	5				
6	24.364	6				
7	24.477	7				
8	23.901	8				
9	23.175	9				
10	23.227	10				
11	21.672	11				
12	21.87	12				
13	21.439	13				
14	21.089	14				
15	23.709	15				
16	21.669	16				
17	21.752	17				
18	20.761	18				
19	23.479	19				

Seasonal Regression Models - Births





Data Editor

File Edit Help

	births	time	seasonal	var4	var5
1	26.663	1	1		
2	23.598	2	2		
3	26.931	3	3		
4	24.74	4	4		
5	25.806	5	5		
6	24.364	6	6		
7	24.477	7	7		
8	23.901	8	8		
9	23.175	9	9		
10	23.227	10	10		
11	21.672	11	11		
12	21.87	12	12		
13	21.439	13	1		
14	21.089	14	2		
15	23.709	15	3		
16	21.669	16	4		
17	21.752	17	5		
18	20.761	18	6		
19	23.479	19	7		

Another Simple Way of Incorporating Seasonality

- Take the trend prediction and actual prediction.
- Depending on additive or multiplicative model compute the deviation and map it as seasonality effect for each prediction.
- Take averages of the seasonality value. Use this to make future predictions.

Case

Year	Quarter	Time variable (this is created)	Revenues (in $M)
2008	I	1	10.2
	II	2	12.4
	III	3	14.8
	IV	4	15
2009	I	5	11.2
	II	6	14.3
	III	7	18.4
	IV	8	18

What is the Regression equation?

$$y = 10.0393 + 0.9440x$$

```
Call:
lm(formula = y ~ x)

Residuals:
    Min      1Q  Median      3Q     Max 
-3.5595 -0.9384  0.4405  1.3265  1.9286 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  10.0393     1.5531   6.464  0.00065 ***
x             0.9440     0.3076   3.069  0.02196 *  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.993 on 6 degrees of freedom
Multiple R-squared: 0.6109,     Adjusted R-squared: 0.5461 
F-statistic: 9.422 on 1 and 6 DF,  p-value: 0.02196
```

Seasonality: Multiplicative

Time	Observed values TSI* (assuming no impact of cyclicality)	Predicted values (per the regression) T*	SI* = TSI/T
1	10.2	10.983	0.929
2	12.4	11.927	1.040
3	14.8	12.871	1.150
4	15.0	13.815	1.086
5	11.2	14.759	0.759
6	14.3	15.703	0.911
7	18.4	16.647	1.105
8	18.0	17.591	1.023

* T: Trend; S: Seasonal; I: Irregular

Quarterly Seasonality

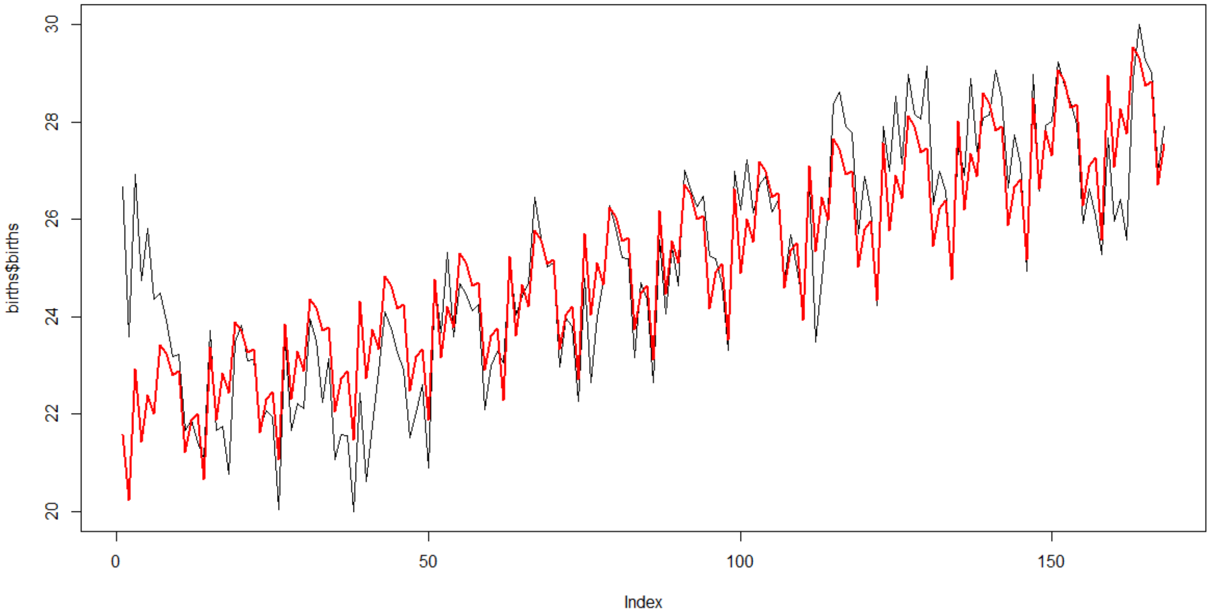
Time	Average seasonality factor
Q1	$0.844 \left(= \frac{0.929+0.759}{2}\right)$
Q2	0.975
Q3	1.127
Q4	1.054

Time	Observed values TSI* (assuming no impact of cyclicality)	Predicted values (per the regression) T*	SI* = TSI/T
1	10.2	10.983	0.929
2	12.4	11.927	1.040
3	14.8	12.871	1.150
4	15.0	13.815	1.086
5	11.2	14.759	0.759
6	14.3	15.703	0.911
7	18.4	16.647	1.105
8	18.0	17.591	1.023

Computations

- Trend Y_9 =10.039 + 0.944(9) = 18.535
- Corrected for seasonality and randomness: 18.535 * 0.844 = 15.643

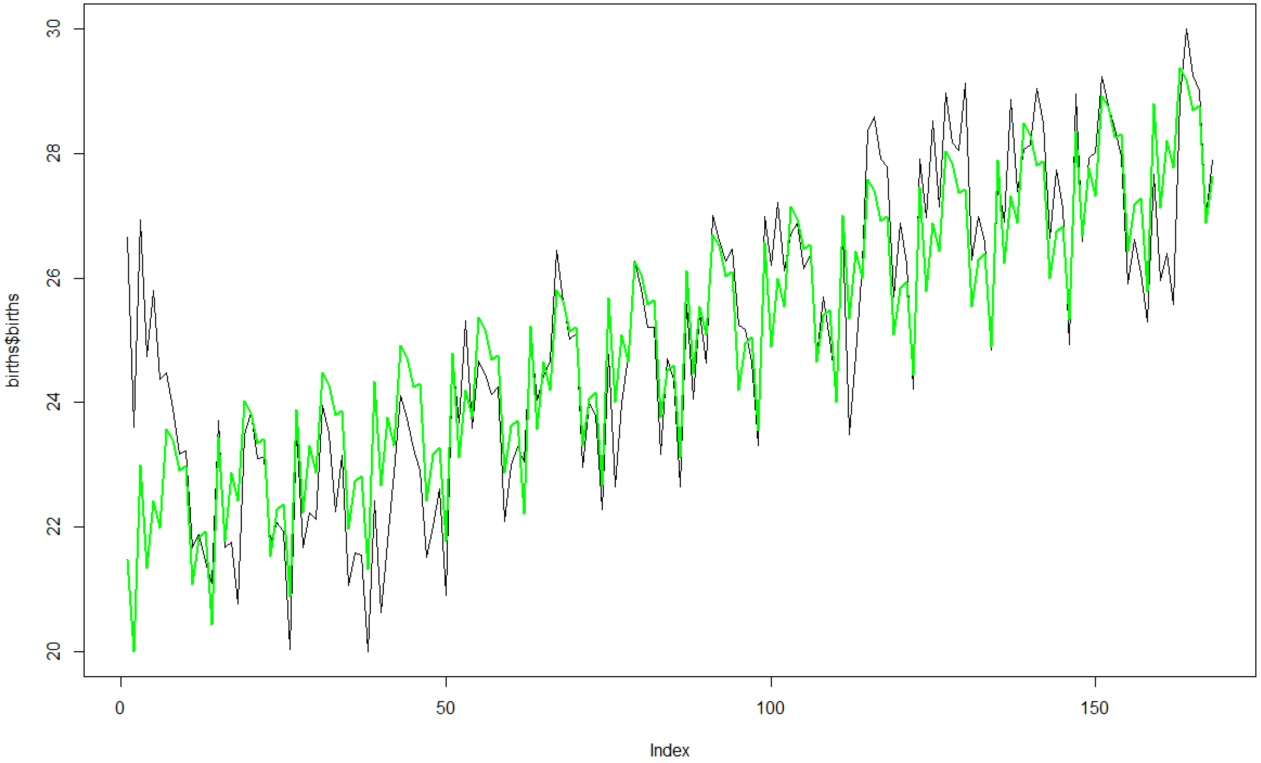
Seasonality: Multiplicative



births	time	seasonal	SeasonalFactor
26.663	1	1	1.2143042
23.598	2	2	1.0729008
26.931	3	3	1.2223736
24.740	4	4	1.1210359
25.806	5	5	1.1673742
24.364	6	6	1.1002941
24.477	7	7	1.1035459
23.901	8	8	1.0757752
23.175	9	9	1.0413570
23.227	10	10	1.0419543
21.672	11	11	0.9705802
21.870	12	12	0.9778208
21.439	13	1	0.9569611
21.089	14	2	0.9397800
23.709	15	3	1.0547879
21.669	16	4	0.9624398
21.752	17	5	0.9645349

```
> births$SeasonalFactor <- births$births/predict(lm1)
> seasonalAdustFactor <- tapply(births$SeasonalFactor,
+                          births$seasonal, mean)
> birthspr <- predict(lm1)*rep(seasonalAdustFactor,14)
> plot(births$births, type="l")
> points(births$time, birthspr,   type="l", col="red", lwd=2)
```

Seasonality: Additive



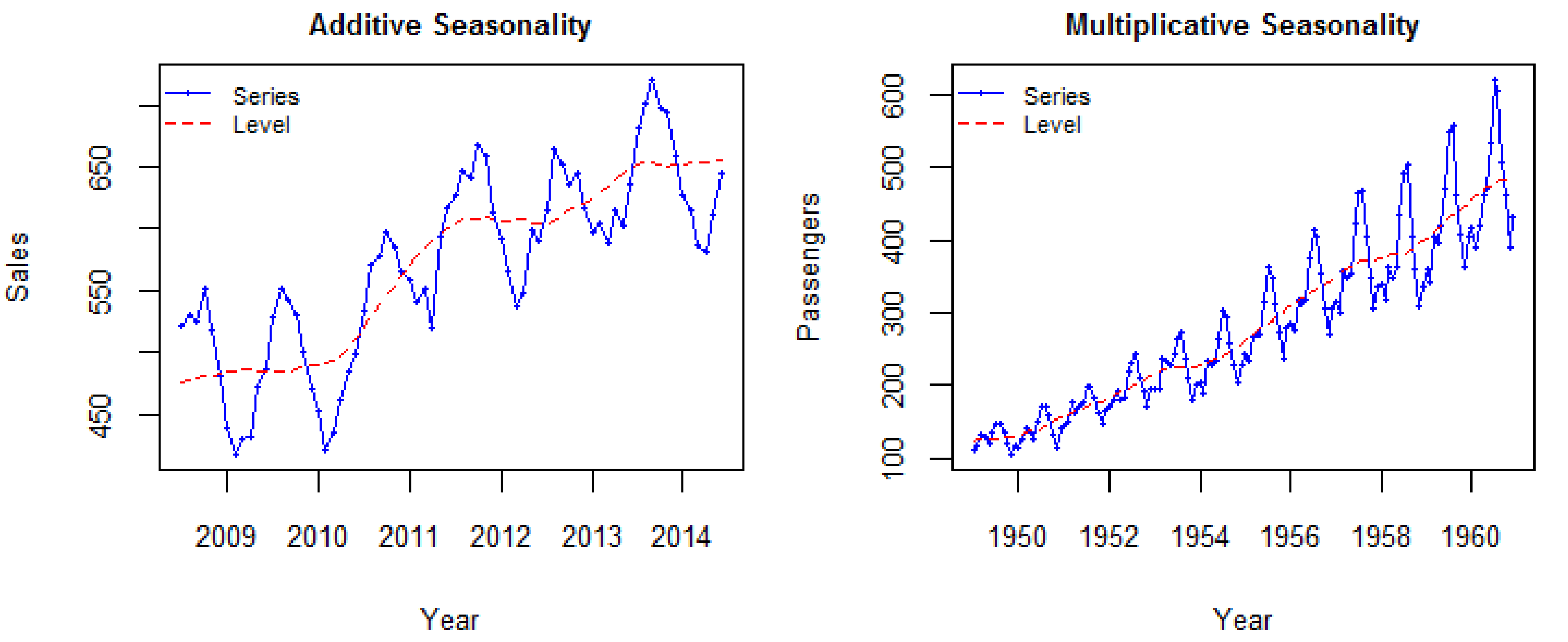
Data Editor

File Edit Help

	births	time	seasonal	mae
1	26.663	1	1	4.70557
2	23.598	2	2	1.603422
3	26.931	3	3	4.899274
4	24.74	4	4	2.671125
5	25.806	5	5	3.699977
6	24.364	6	6	2.220829
7	24.477	7	7	2.29668
8	23.901	8	8	1.683532
9	23.175	9	9	0.920384
10	23.227	10	10	0.9352357
11	21.672	11	11	-0.6569126
12	21.87	12	12	-0.4960608
13	21.439	13	1	-0.9642091
14	21.089	14	2	-1.351357
15	23.709	15	3	1.231494
16	21.669	16	4	-0.8456539
17	21.752	17	5	-0.7998021
18	20.761	18	6	-1.82795
19	23.479	19	7	0.8529014

```
> births$mae <- births$births-predict(lm1)
> seasonalAdd <- tapply(births$mae,
+                       births$seasonal, mean)
> birthspr <- predict(lm1)+rep(seasonalAdd,14)
> plot(births$births, type="l")
> points(births$time, birthspr, type="l", col="green", lwd=2)
>
```

Additive or Multiplicative



Source: http://www.forsoc.net/2014/11/11/can-you-identify-additive-and-multiplicative-seasonality/

Goodness of Fit

- MSE (Mean square error)
- MAE (Mean absolute error)
- RMSE (Root mean square error)

- MAPE (Mean absolute percent error)

- NMSE (Normalized mean square error)
- NMAE (Normalized mean absolute error)
- NMAPE (Normalized mean absolute percent error)

Issues with Regressing on Time

- It is too much of a curve fit for a statistician to sleep well!
- If there is no trend or if seasonality and fluctuations are more important than trend, then the coefficients behave weirdly

TIME SERIES: AUTO REGRESSIVE METHODS

Auto Regressive Methods

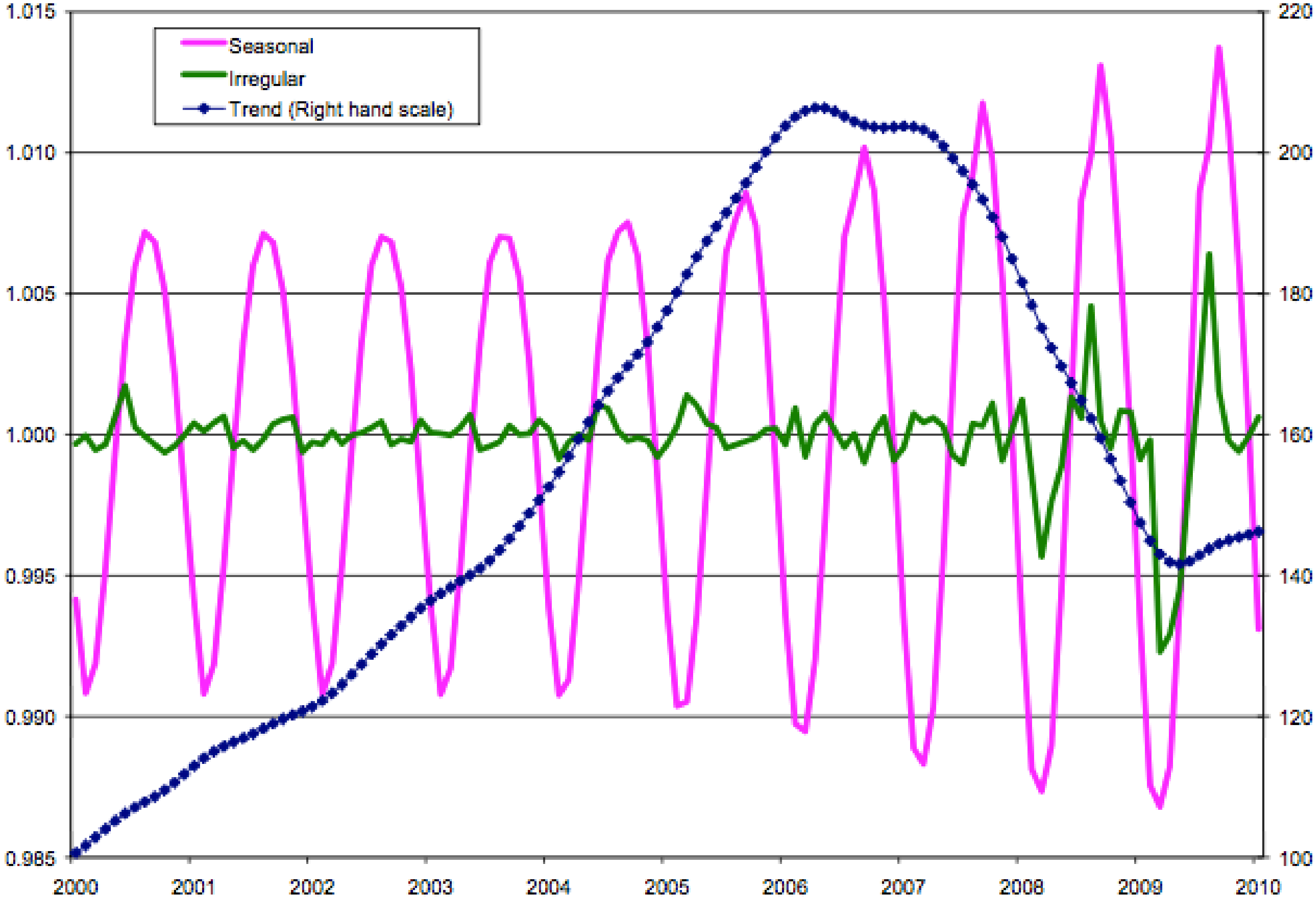
$$\hat{y}_{t+1} = f(y_t, y_{t-1}, y_{t-2} \cdots)$$

Components of time series

- We use different techniques for time series with different characterstics
  - Trend
  - Seasonal
  - Random stationary

- First we need to identify them

Trend, Seasonality and Randomness



Time Series Descriptive Statistics

- In descriptive statistics covered earlier (central tendencies, measures of variability, skewness, kurtosis, distributions, correlations, etc.), the order of observations in the data was of no consequence.

- In time series descriptive statistics, order of observations is of primary importance and so autocorrelations, etc. play a vital role in identifying the models and their characteristics.

- Autocorrelation is a metric that allows us to understand the strength of order in the time-series

AUTOCORRELATION AND PARTIAL AUTOCORRELATION

Autocorrelation (ACF) and Partial ACF (PACF)

- ACF: n^{th} lag of ACF is the correlation between a day and n days before that.

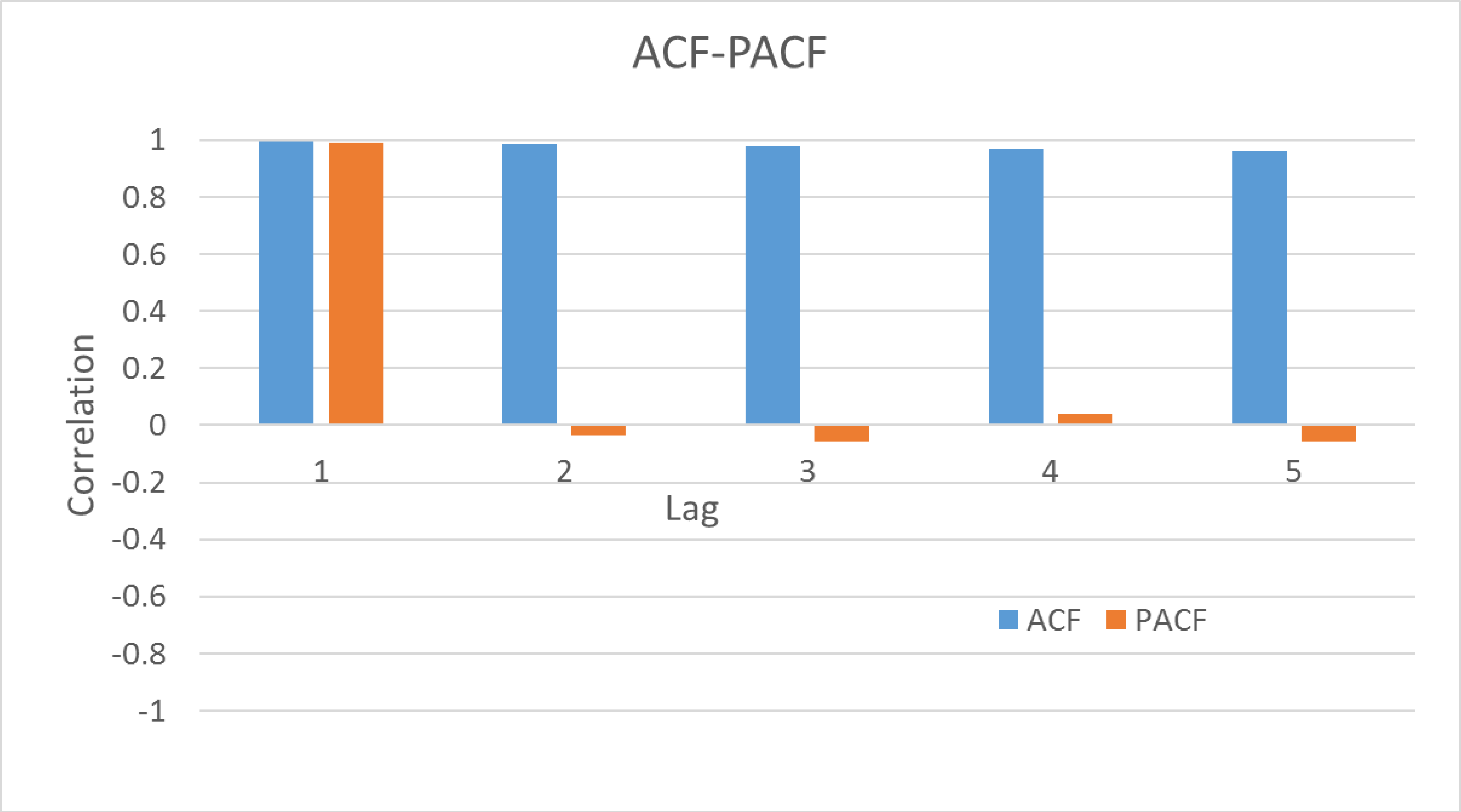
- PACF: The same as ACF with all intermediate correlations removed. It is the k_{th} coefficient of the ordinary least squares regression.

$$[y_t] = \beta_0 + \sum_{i=1}^{k} \beta_i [y_{t-i}] \text{ where}$$

$[y_t]$ is the input time series, k is the lag order and β_i is the i_{th} coefficient of the linear multiple regression.

EXCEL ACTIVITY

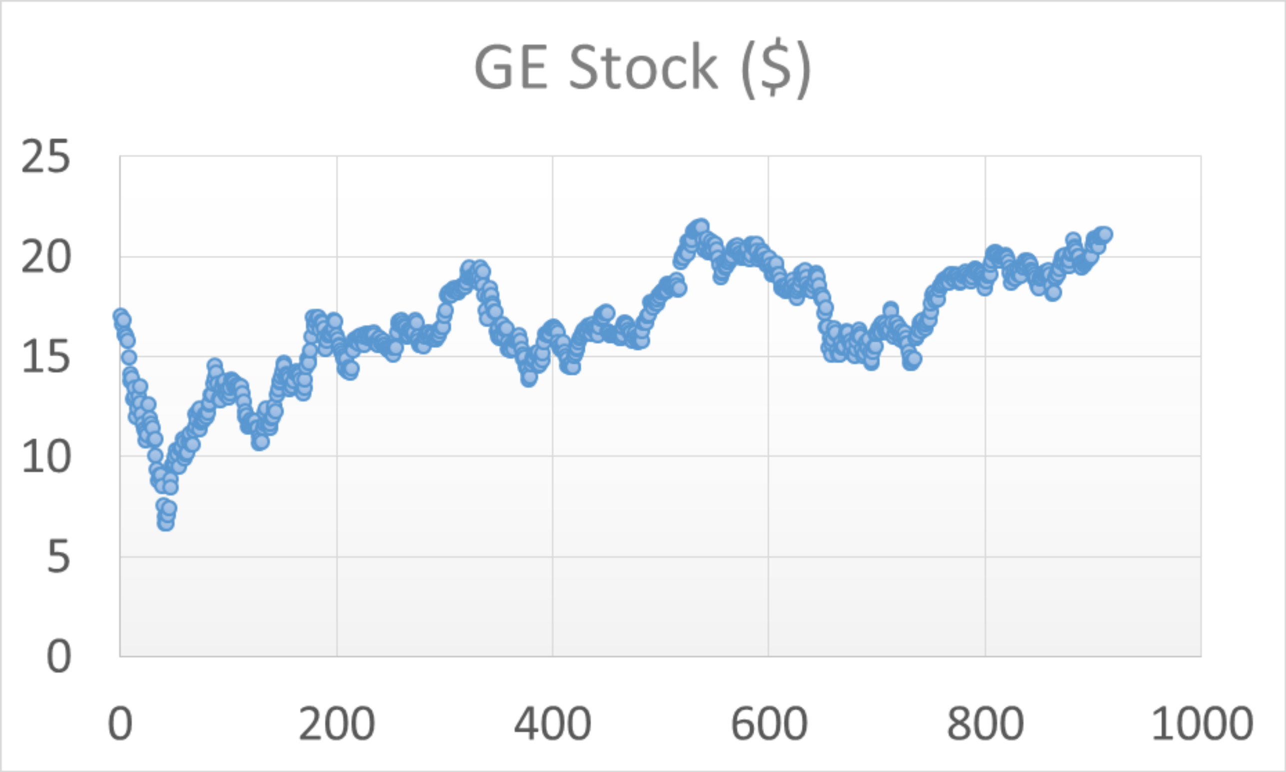
Autocorrelation (ACF) and Partial ACF (PACF)



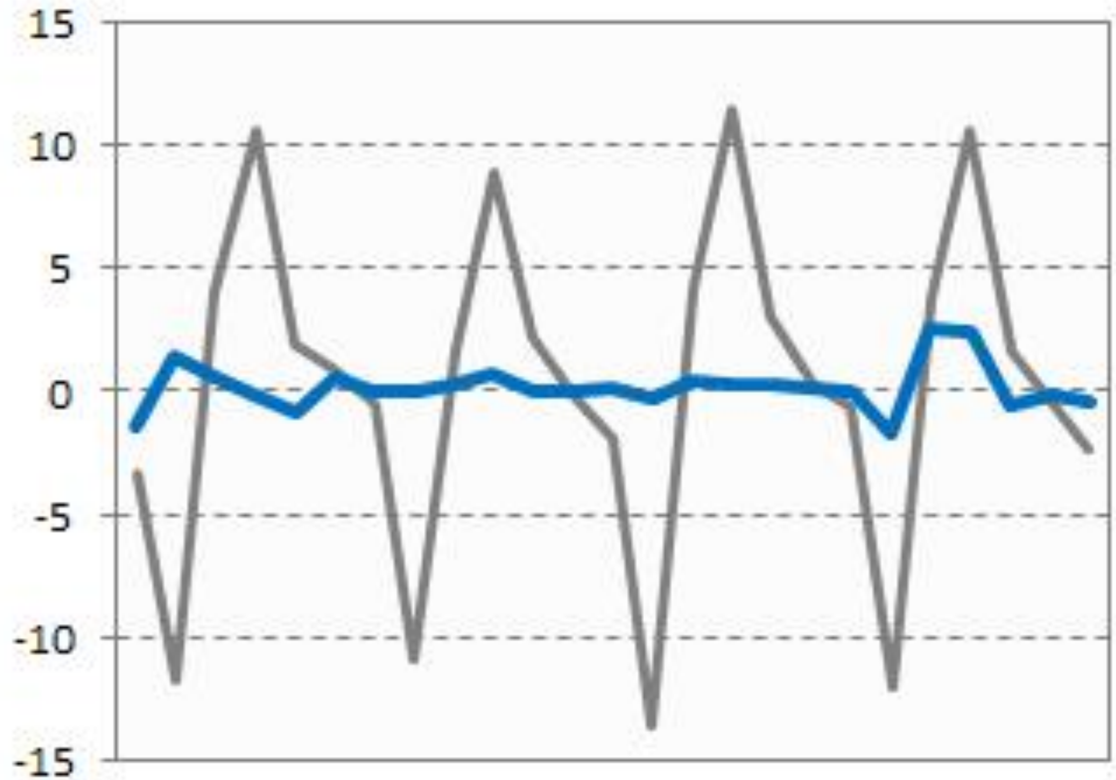
See the attached file 01Correlations.xlsx

Components of Time Series

- **Trend**
- Seasonality
- Random component



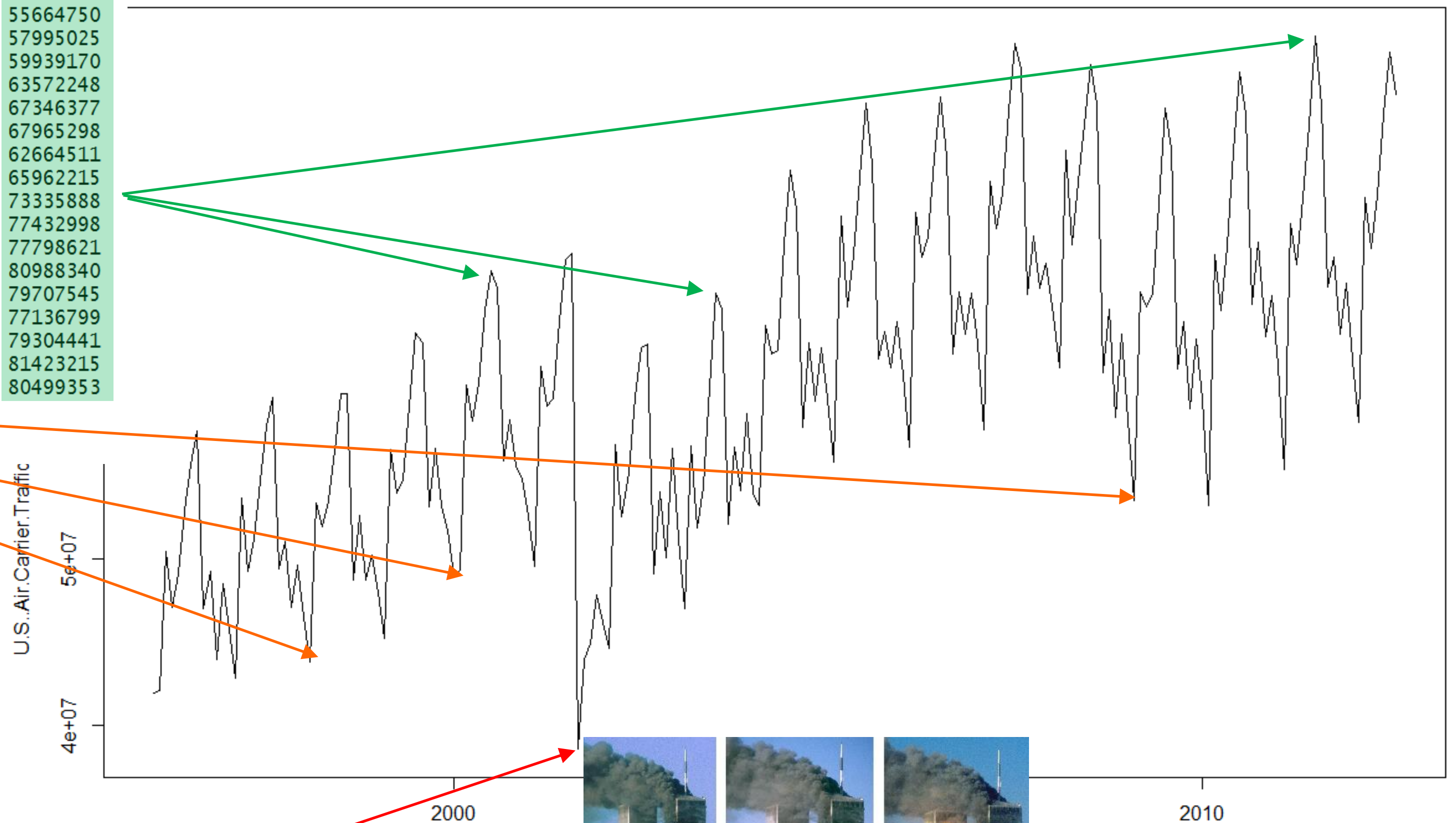
Seasonality



US Air Carrier Traffic – Revenue Passenger Miles ('000)

RPM

```
> milestimeseries <- ts(miles, frequency = 12, start = c(1996,1))
> milestimeseries
          Jan      Feb      Mar      Apr      May      Jun      Jul
1996 41972194 42054796 50443045 47112397 49118248 52880510 55664750
1997 45850623 42838949 53620994 49282817 51191842 54707221 57995025
1998 46514139 43769273 53361926 51968480 53515798 56460422 59939170
1999 47988560 45241211 56555731 53920855 54674958 59213000 63572248
2000 49045412 49306303 60443541 58286680 60533783 64903295 67346377
2001 52634354 49532578 61575055 59151645 59662416 64353323 67965298
2002 46224031 44615129 56897729 52542164 55116060 59745343 62664511
2003 51197175 47040806 56766580 51857453 54335598 60272900 65962215
2004 53979786 53179693 64035864 62340117 62530704 68866398 73335888
2005 59629608 55795165 70595861 65145552 68268899 72952959 77432998
2006 61035027 56729212 70799794 68120559 69352606 74099239 77798621
2007 63016013 57793832 72700241 69836156 71933109 76926452 80988340
2008 64667106 61504426 74575531 68906882 72725750 76162105 79707545
2009 58373786 53506580 66027341 65166300 65868254 71350227 77136799
2010 59651061 53240066 68307090 64953250 68850904 74474550 79304441
2011 61630362 55391206 70158268 67683558 71711448 76057910 81423215
2012 61940180 58243763 71696039 68669228 71887523 76760759 80499353
          Aug      Sep      Oct      Nov      Dec
1996 57723208 47035464 49263120 43937074 48539606
1997 59715433 49418190 51058879 47056048 49654209
```

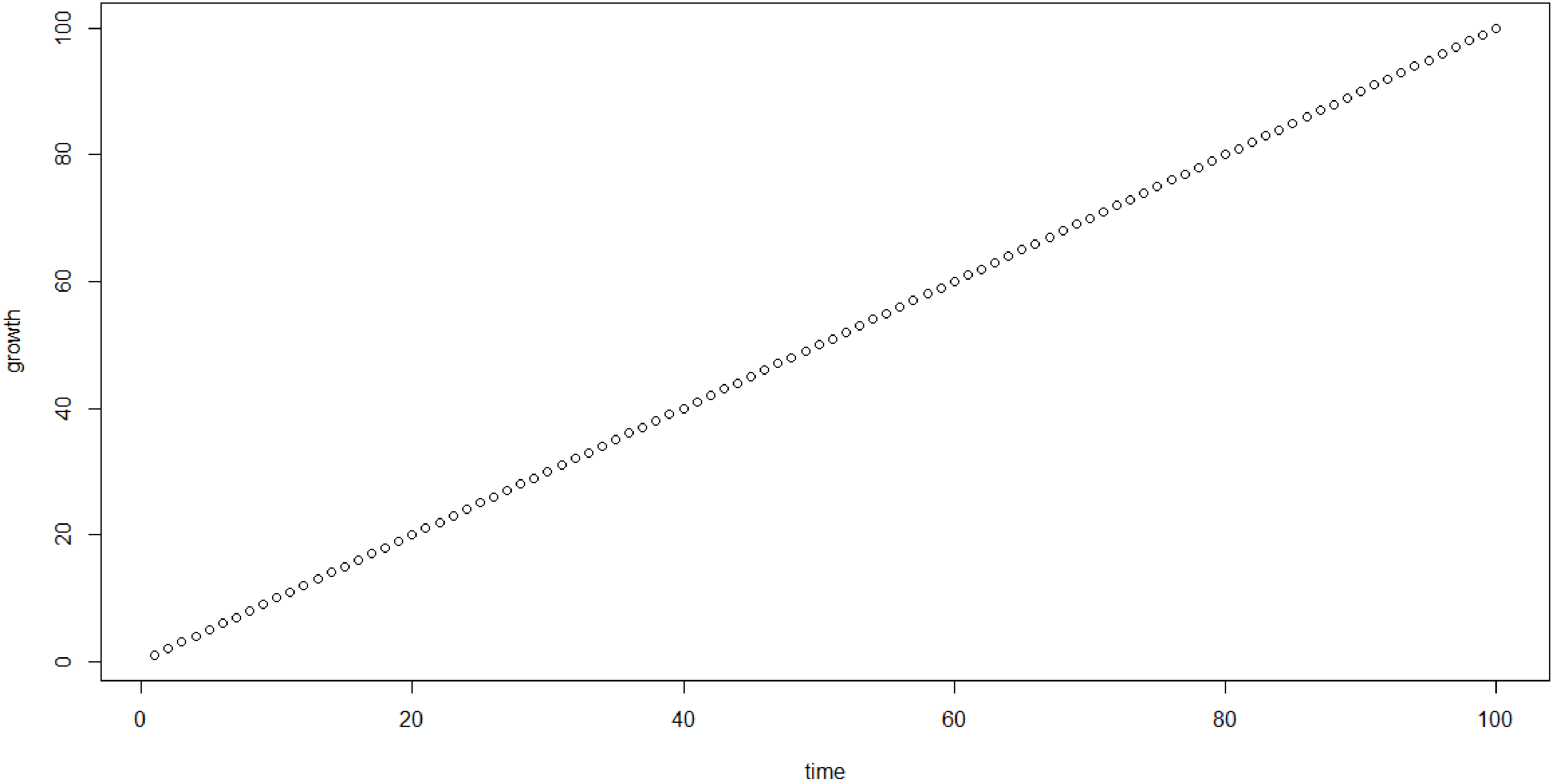


Data sources:
http://www.bts.gov/xml/air_traffic/src/index.xml
and https://datamarket.com/data/set/281x/us-air-carrier-traffic-statistics-revenue-passenger-miles
Last accessed: 31-Mar-2016

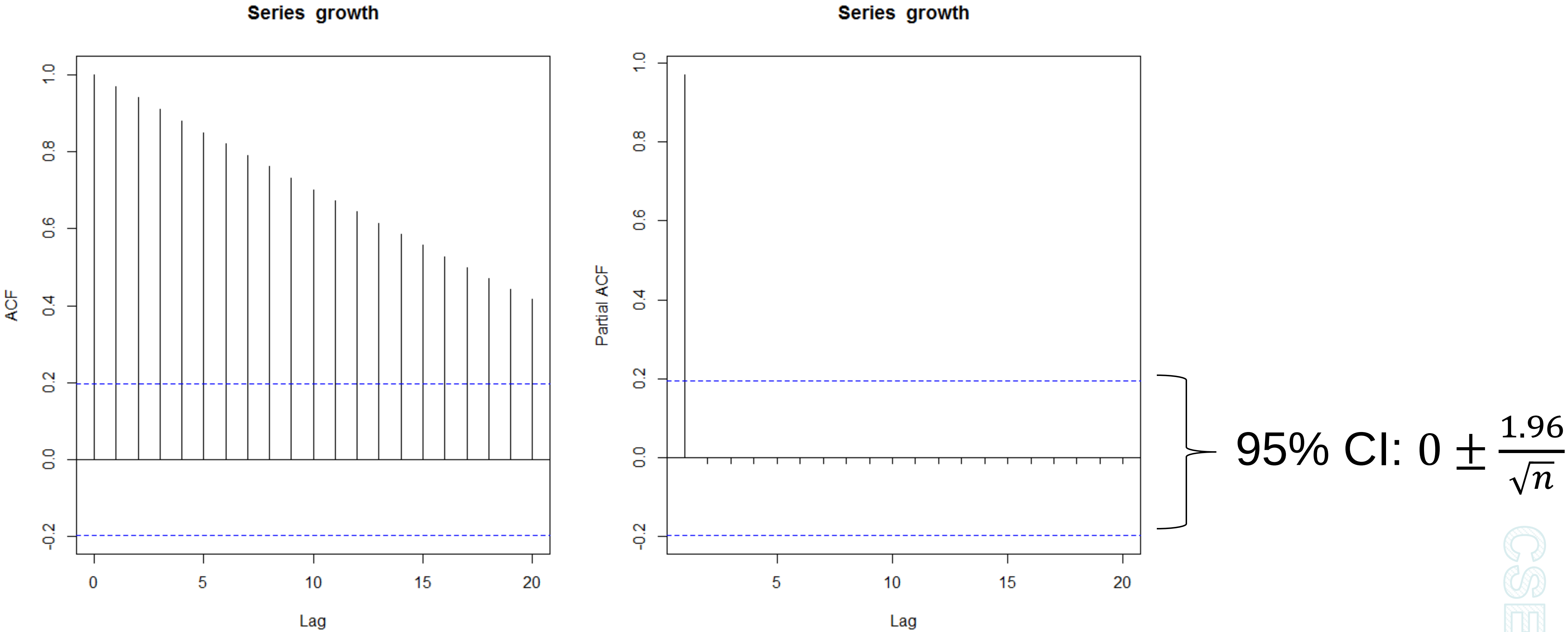
	Aug	Sep	Oct	Nov	Dec
1996	57723208	47035464	49263120	43937074	48539606
1997	59715433	49418190	51058879	47056048	49654209
1998	59927214	48751280	52578217	48734375	50208641
1999	63003663	53131972	56653901	53215500	51746821
2000	66256804	55900504	58373996	55590325	54822970
2001	68377080	$\boxed{38601868}$	43964788	44915764	47836501
2002	62944816	49096035	54019748	50106814	56656594
2003	64989766	52121480	56724551	54128776	58739845
2004	70961522	57881042	63021142	59453943	62680310

ACF and PACF – Idealized Trend, Seasonality and Randomness

ACF and PACF – Idealized Trend

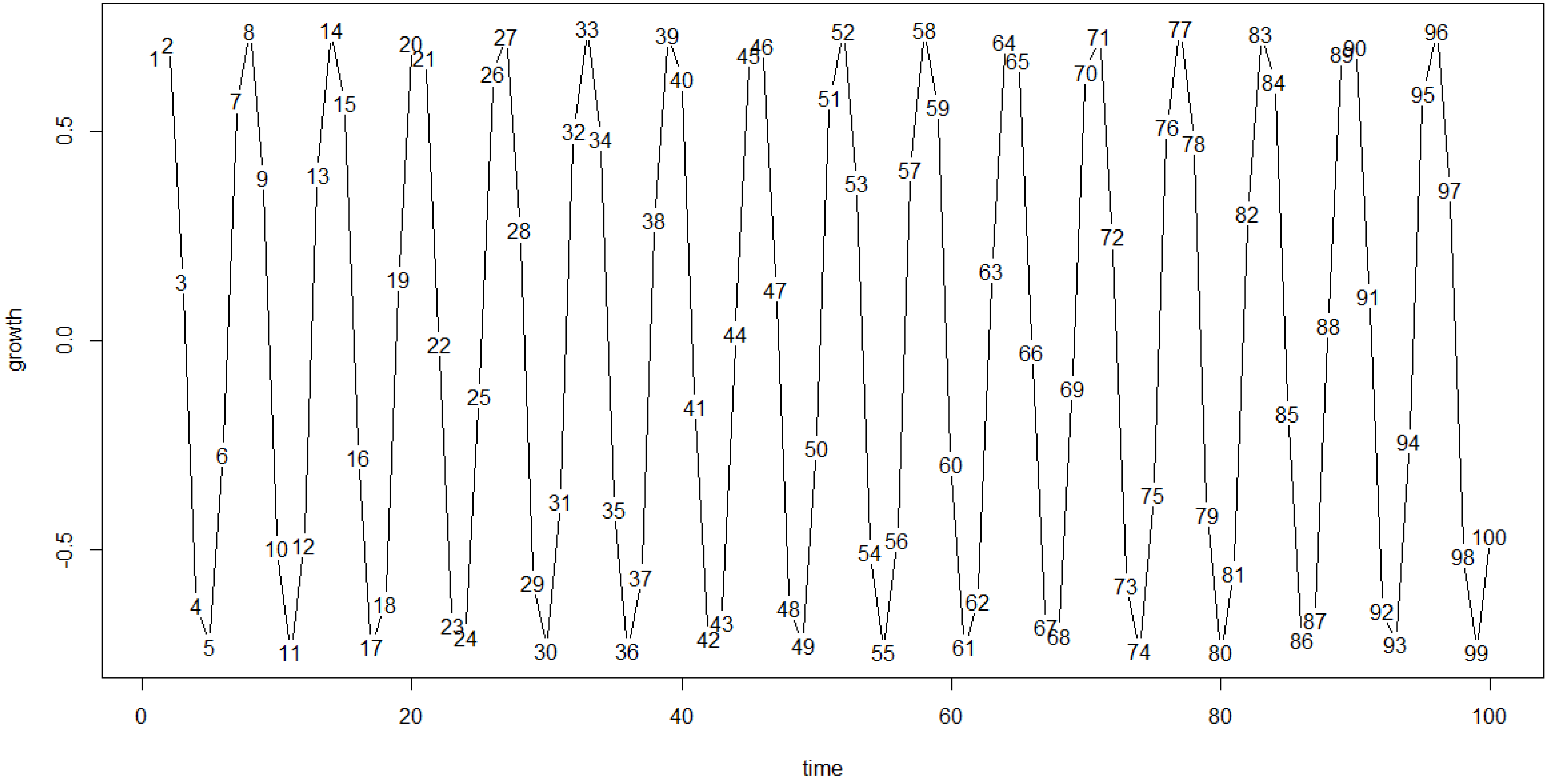


ACF and PACF – Idealized Trend

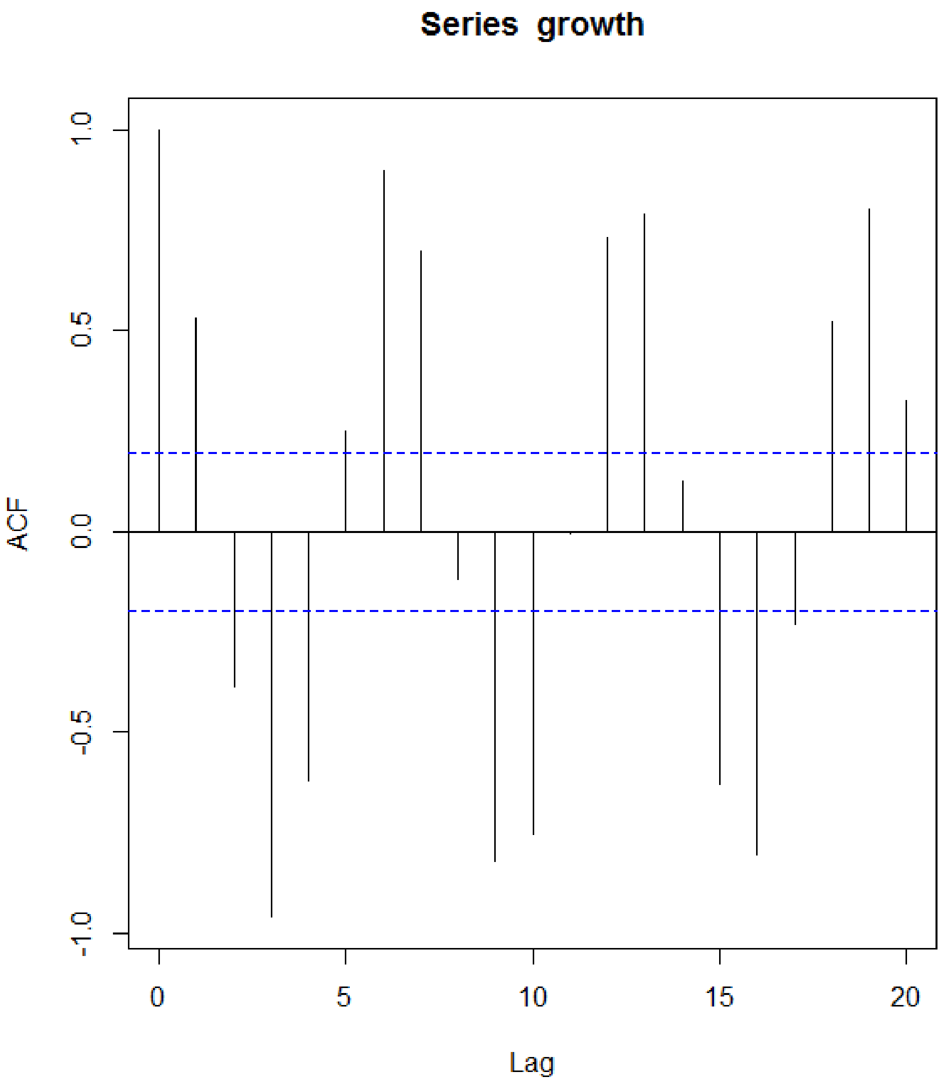


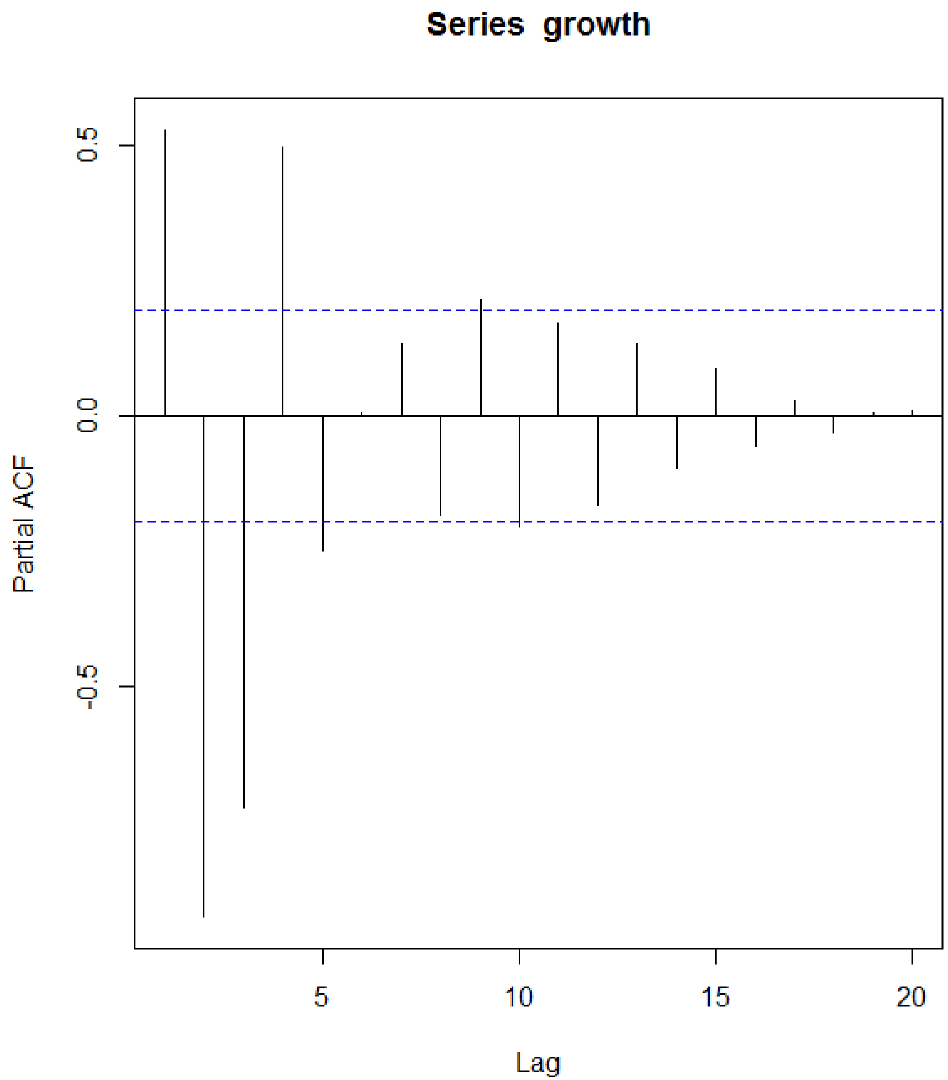
- ACF is a bar chart of correlation coefficients of the time series and its lags.
- PACF is a plot of the partial correlation coefficients of the time series and its lags.

ACF and PACF – Idealized Seasonality

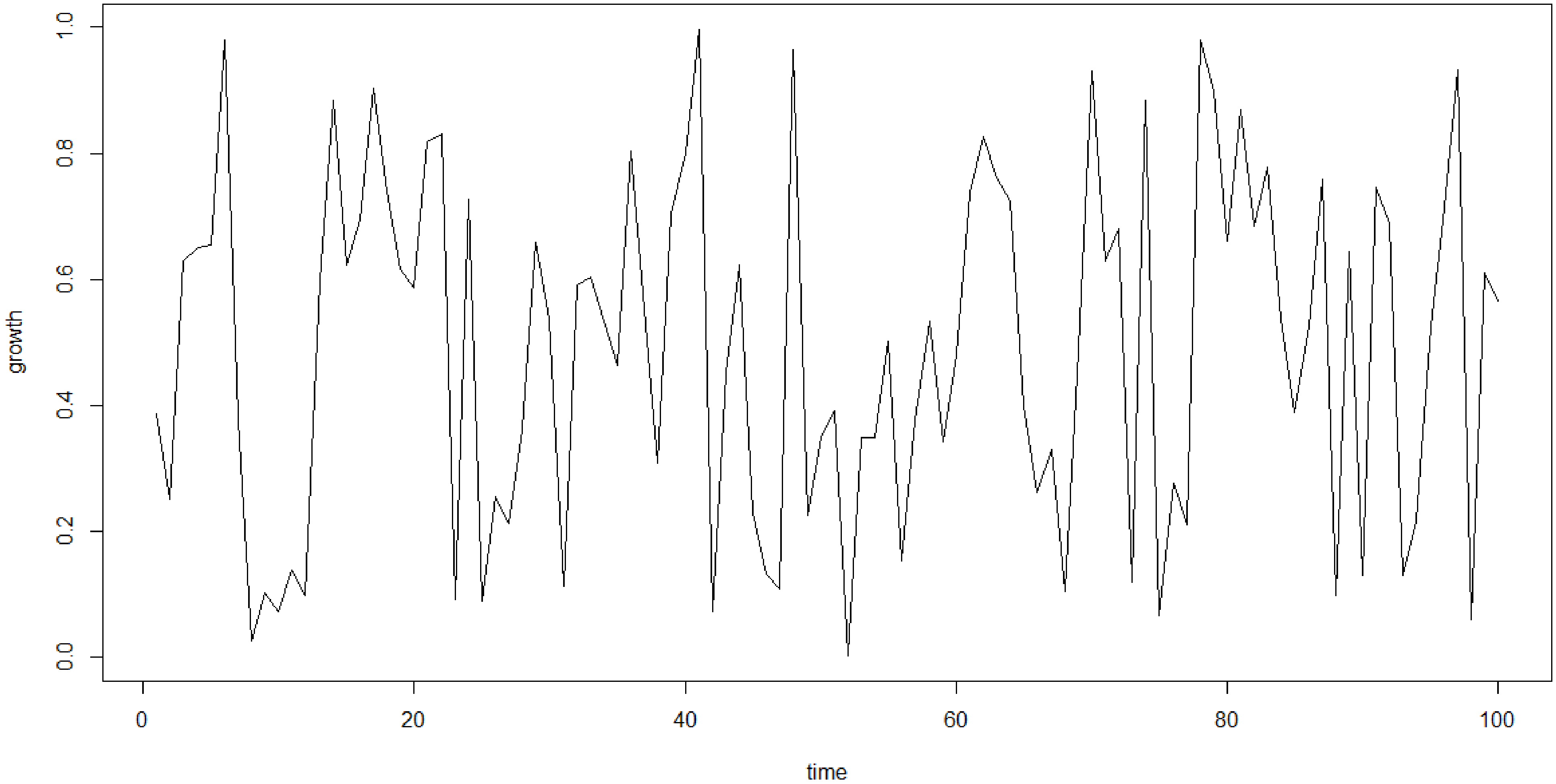


ACF and PACF – Idealized Seasonality

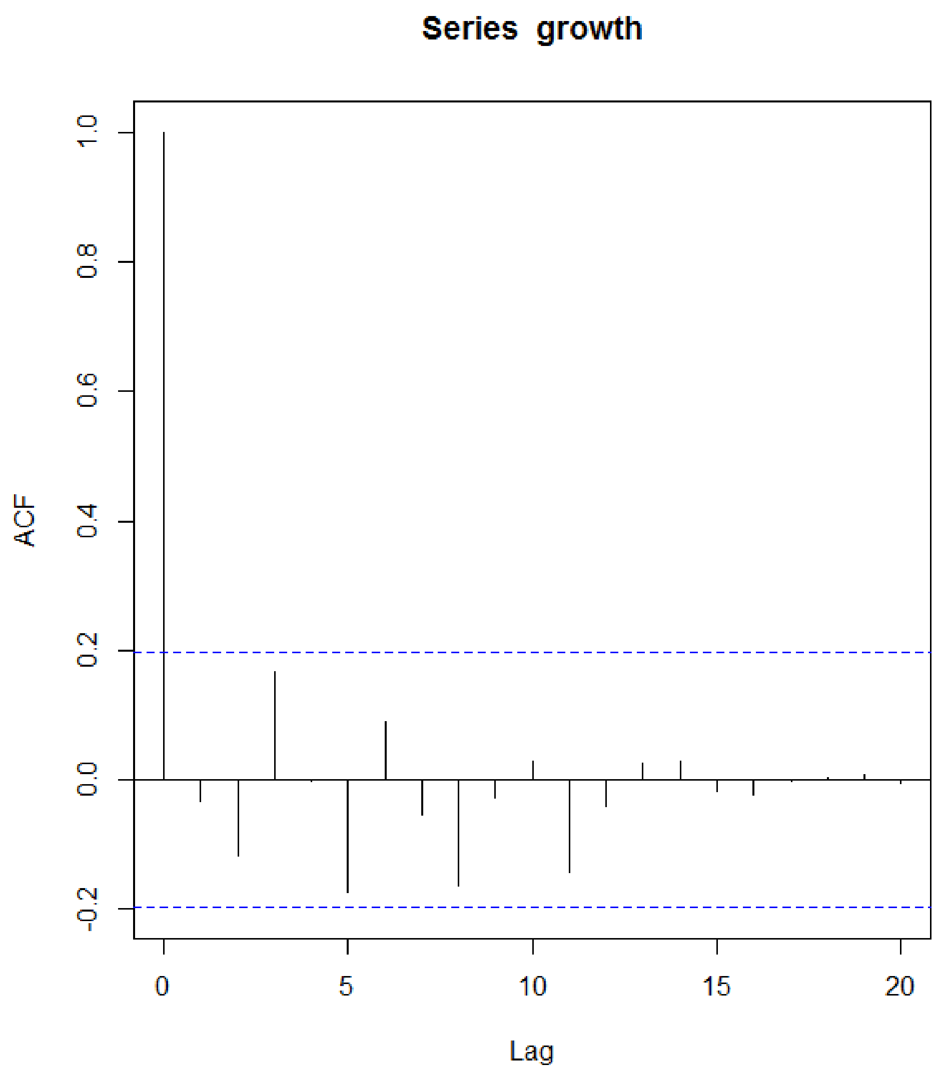


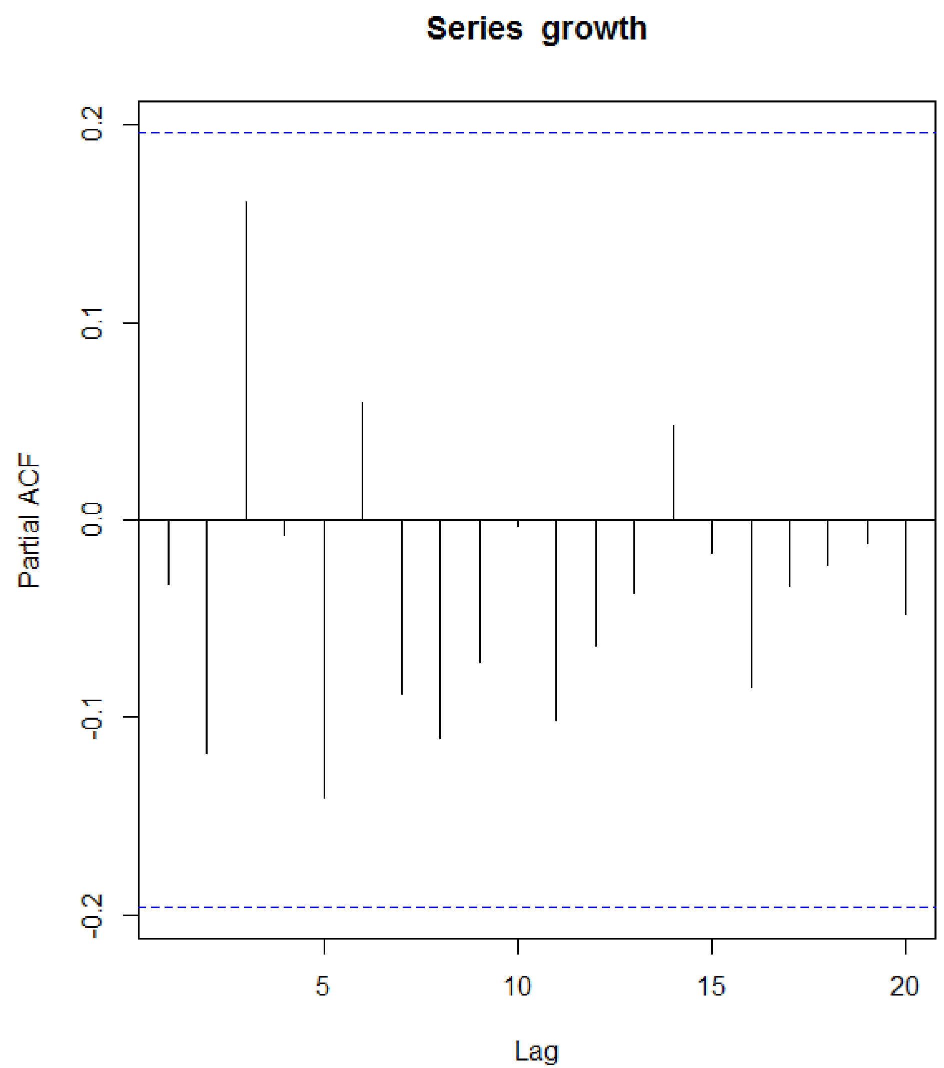


ACF and PACF – Idealized Randomness



ACF and PACF – Idealized Randomness



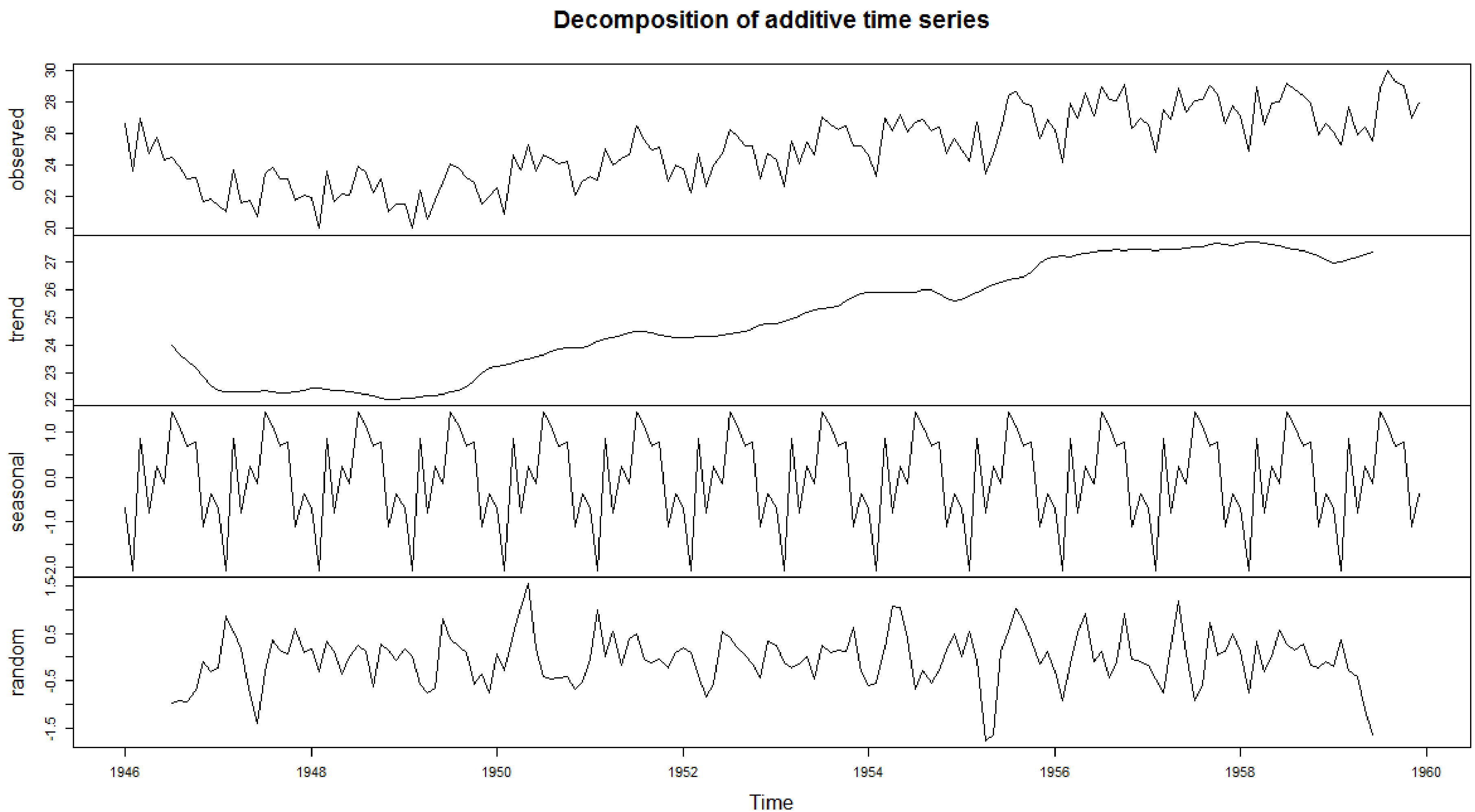


ACF and PACF – Idealized Trend, Seasonality and Randomness

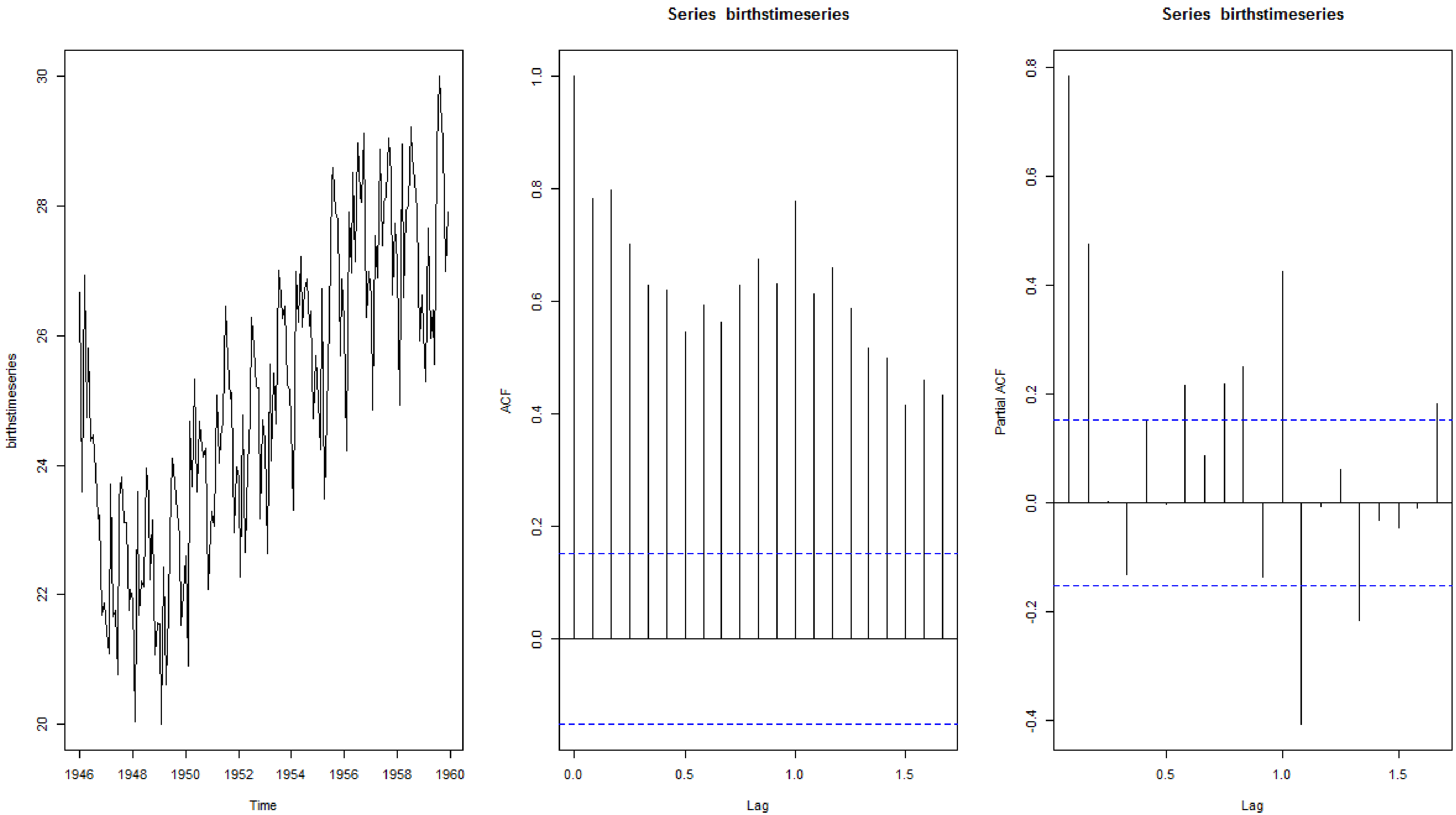
- Ideal Trend: Decreasing ACF and 1 or 2 lags of PACF
- Ideal Seasonality: Cyclicality in ACF and a few lags of PACF with some positive and some negative
- Ideal Random: A spike may or may not be present; even if present, magnitude will be small

ACF and PACF (Real-world): Decomposing Time Series into the 3 Components

ACF and PACF (Real-world): Decomposing Time Series into the 3 Components – Births in NY



ACF and PACF (Real-world): Decomposing Time Series into the 3 Components – Births in NY



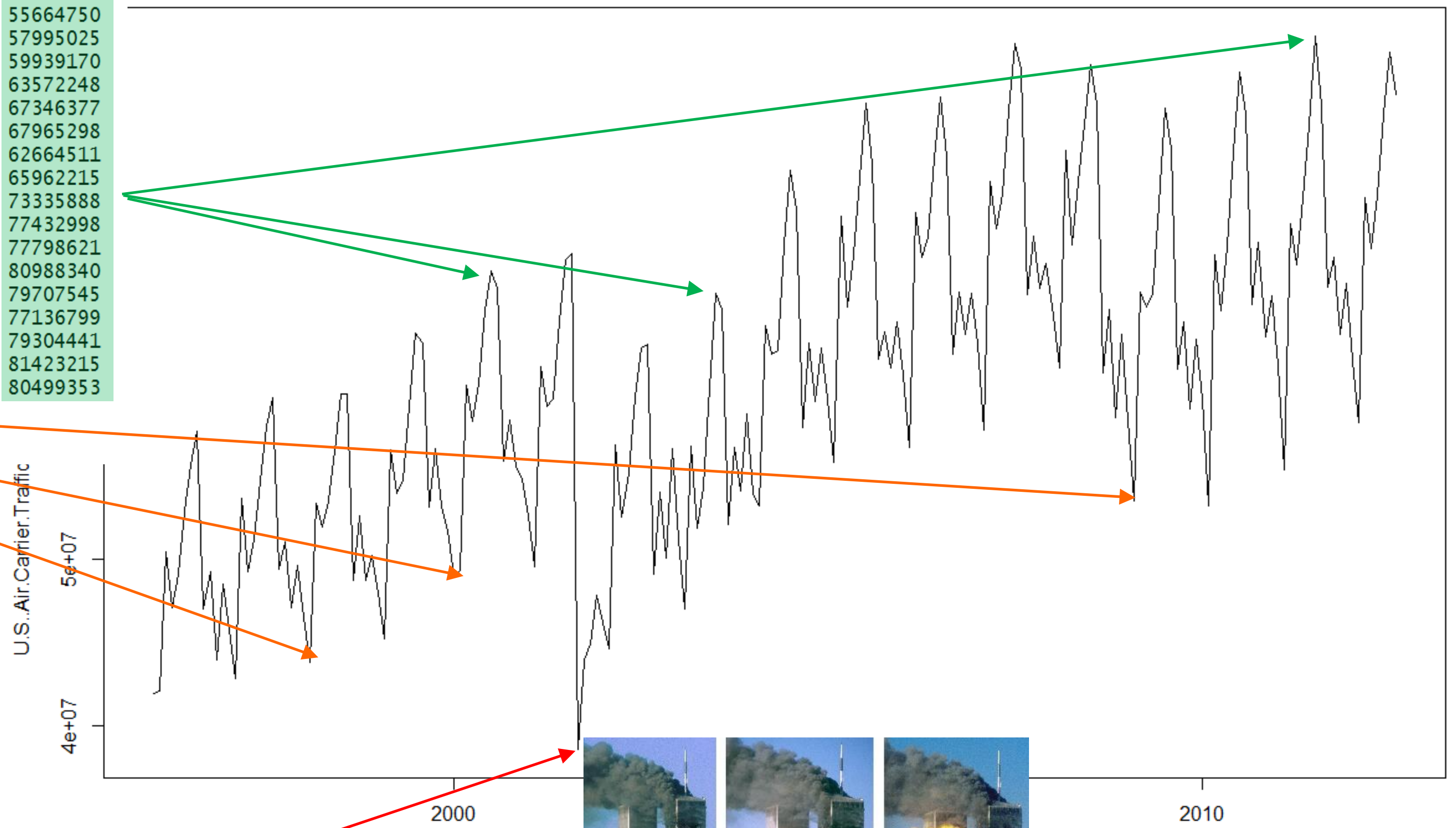
CSE 7202c

US Air Carrier Traffic – Revenue Passenger Miles ('000) *RPM*

```
> milestimeseries <- ts(miles, frequency = 12, start = c(1996,1))
> milestimeseries
```

	Jan	Feb	Mar	Apr	May	Jun	Jul
1996	41972194	42054796	50443045	47112397	49118248	52880510	55664750
1997	45850623	42838949	53620994	49282817	51191842	54707221	57995025
1998	46514139	43769273	53361926	51968480	53515798	56460422	59939170
1999	47988560	45241211	56555731	53920855	54674958	59213000	63572248
2000	49045412	49306303	60443541	58286680	60533783	64903295	67346377
2001	52634354	49532578	61575055	59151645	59662416	64353323	67965298
2002	46224031	44615129	56897729	52542164	55116060	59745343	62664511
2003	51197175	47040806	56766580	51857453	54335598	60272900	65962215
2004	53979786	53179693	64035864	62340117	62530704	68866398	73335888
2005	59629608	55795165	70595861	65145552	68268899	72952959	77432998
2006	61035027	56729212	70799794	68120559	69352606	74099239	77798621
2007	63016013	57793832	72700241	69836156	71933109	76926452	80988340
2008	64667106	61504426	74575531	68906882	72725750	76162105	79707545
2009	58373786	53506580	66027341	65166300	65868254	71350227	77136799
2010	59651061	53240066	68307090	64953250	68850904	74474550	79304441
2011	61630362	55391206	70158268	67683558	71711448	76057910	81423215
2012	61940180	58243763	71696039	68669228	71887523	76760759	80499353

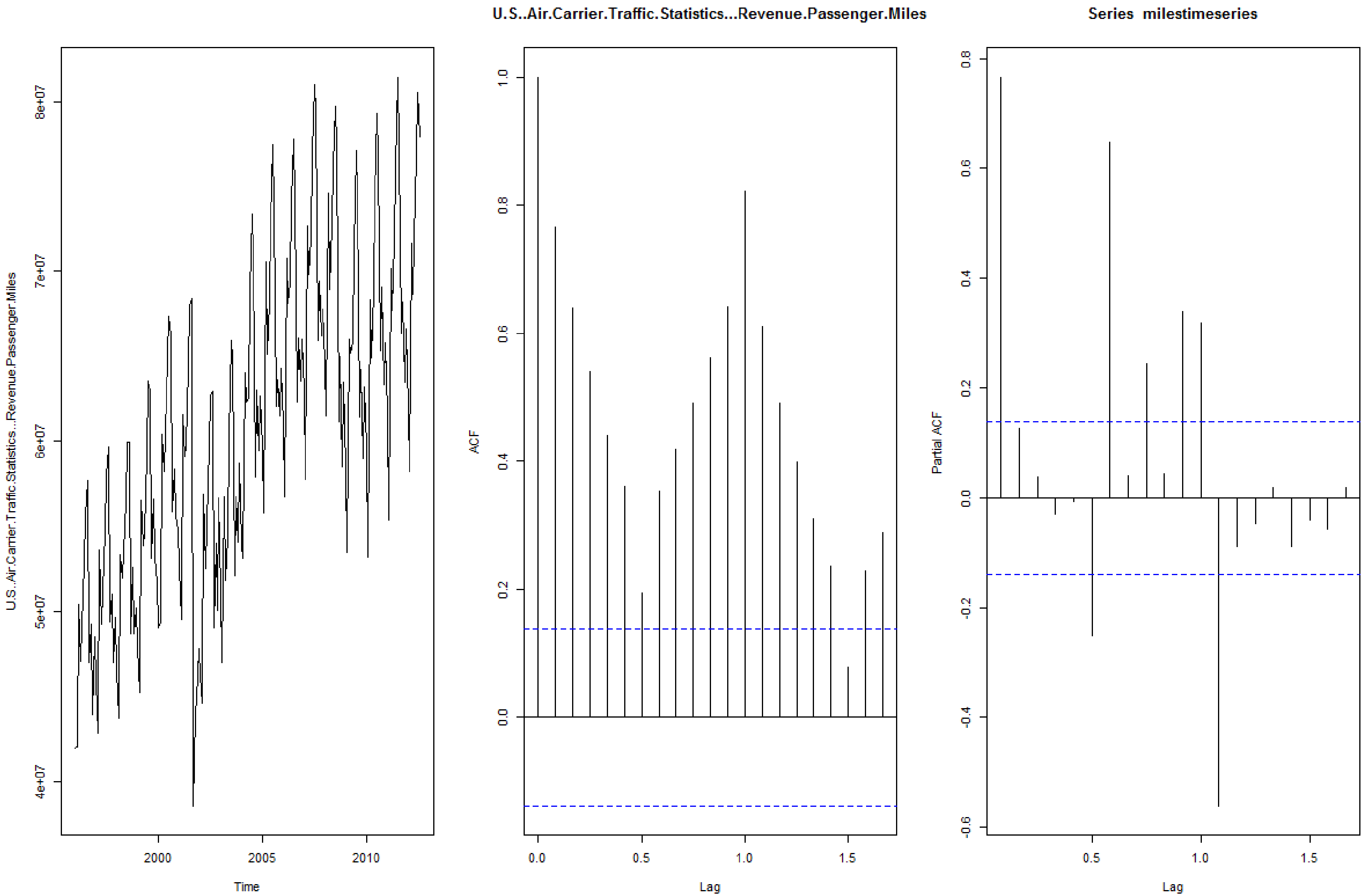
	Aug	Sep	Oct	Nov	Dec
1996	57723208	47035464	49263120	43937074	48539606
1997	59715433	49418190	51058879	47056048	49654209



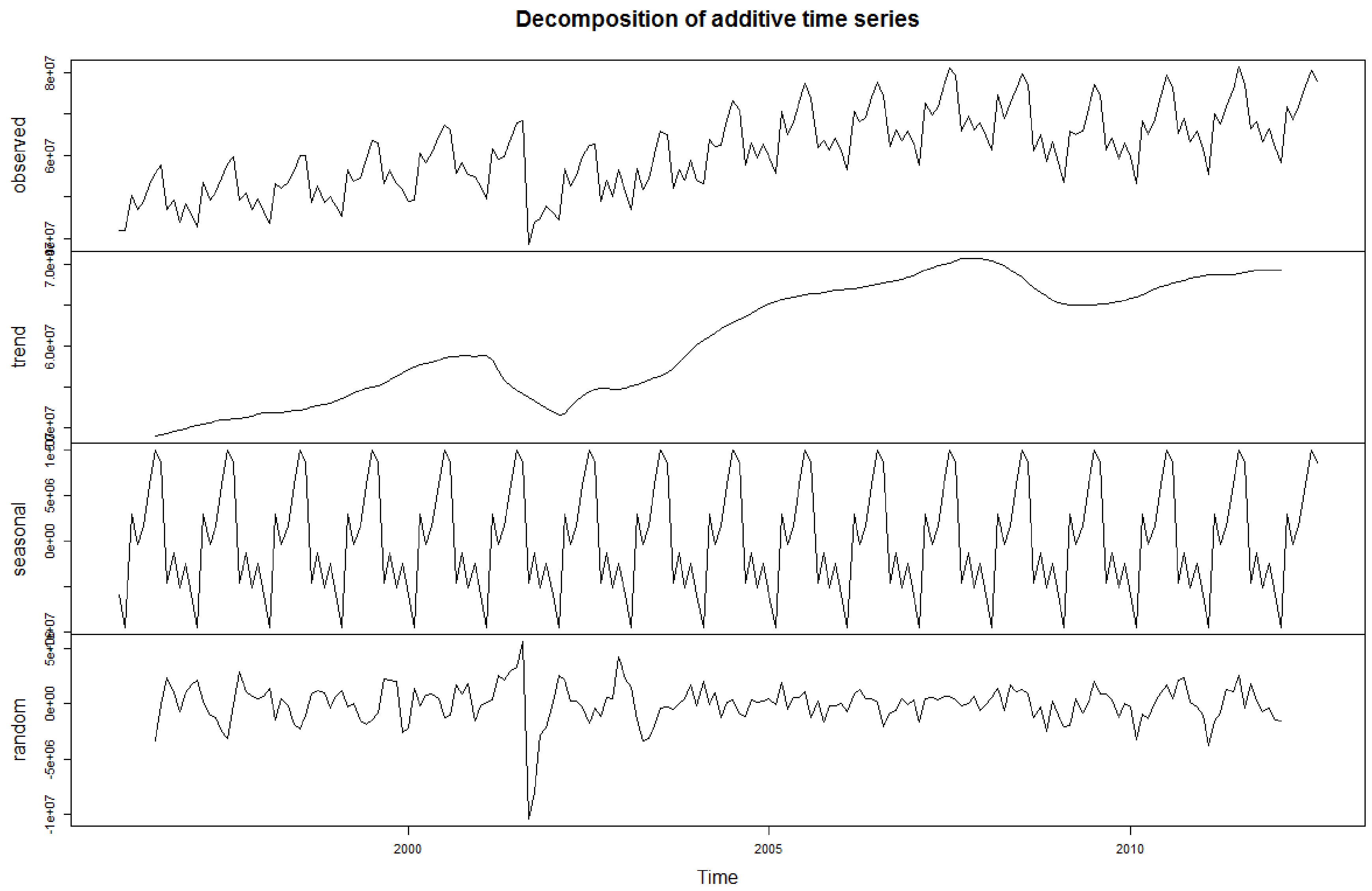
Data sources:
http://www.bts.gov/xml/air_traffic/src/index.xml
and https://datamarket.com/data/set/281x/us-air-carrier-traffic-statistics-revenue-passenger-miles
Last accessed: 31-Mar-2016

	Aug	Sep	Oct	Nov	Dec
1996	57723208	47035464	49263120	43937074	48539606
1997	59715433	49418190	51058879	47056048	49654209
1998	59927214	48751280	52578217	48734375	50208641
1999	63003663	53131972	56653901	53215500	51746821
2000	66256804	55900504	58373996	55590325	54822970
2001	68377080	$\boxed{38601868}$	43964788	44915764	47836501
2002	62944816	49096035	54019748	50106814	56656594
2003	64989766	52121480	56724551	54128776	58739845
2004	70961522	57881042	63021142	59453943	62680310

Revenue Passenger Miles: ACF and PACF



ACF and PACF (Real-world): Decomposing Time Series into the 3 Components – Revenue Passenger Miles (RPM)



Stationary and Non-Stationary

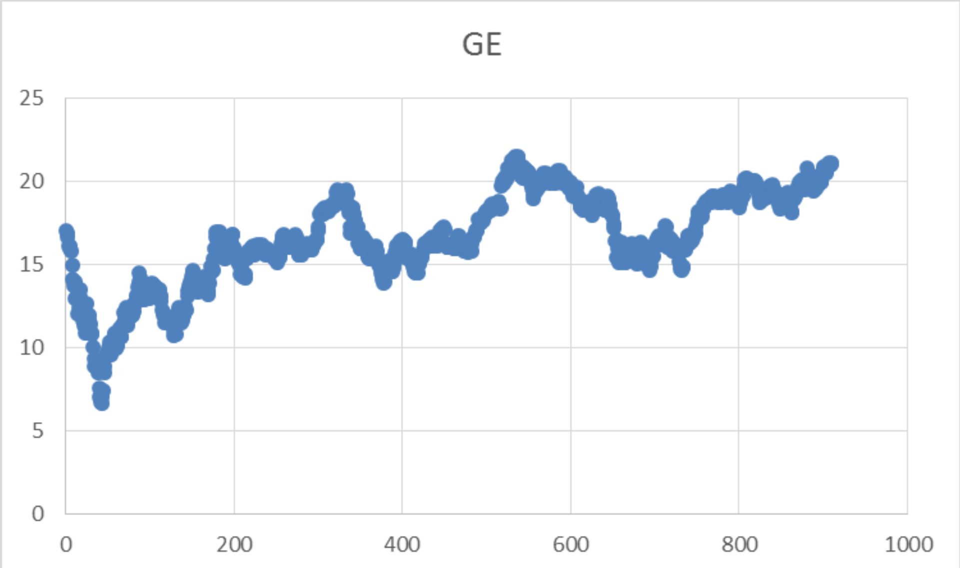
- Stationary data has constant statistical properties – mean, variance, autocorrelation, etc. – over time

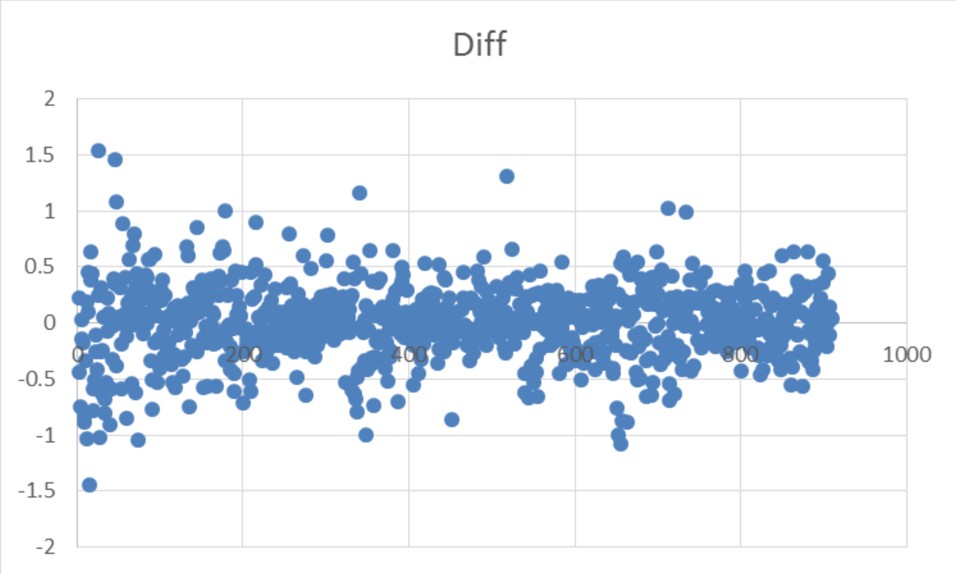
- If the data is stationary, forecasting is easier!

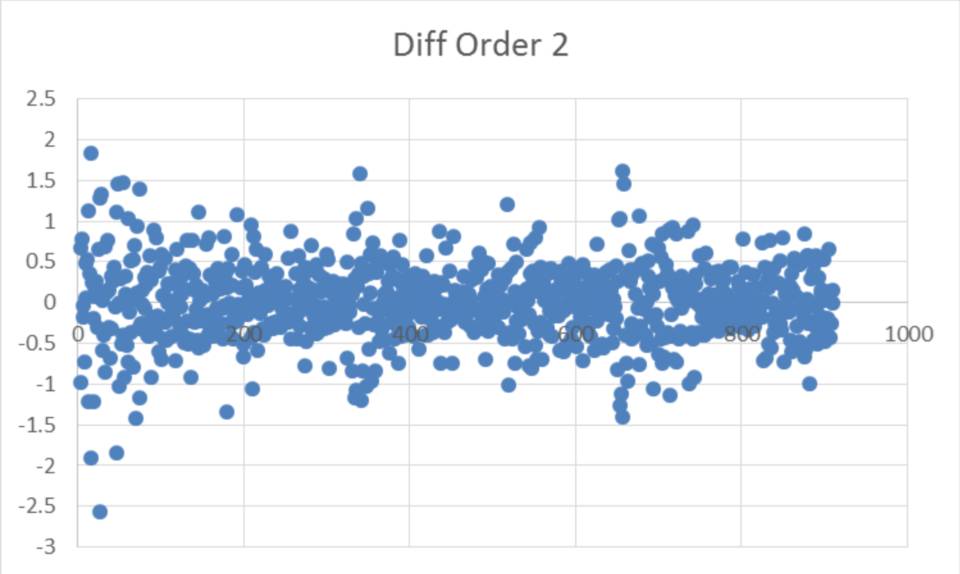
- Differencing to convert non-stationary to stationary

EXCEL ACTIVITY

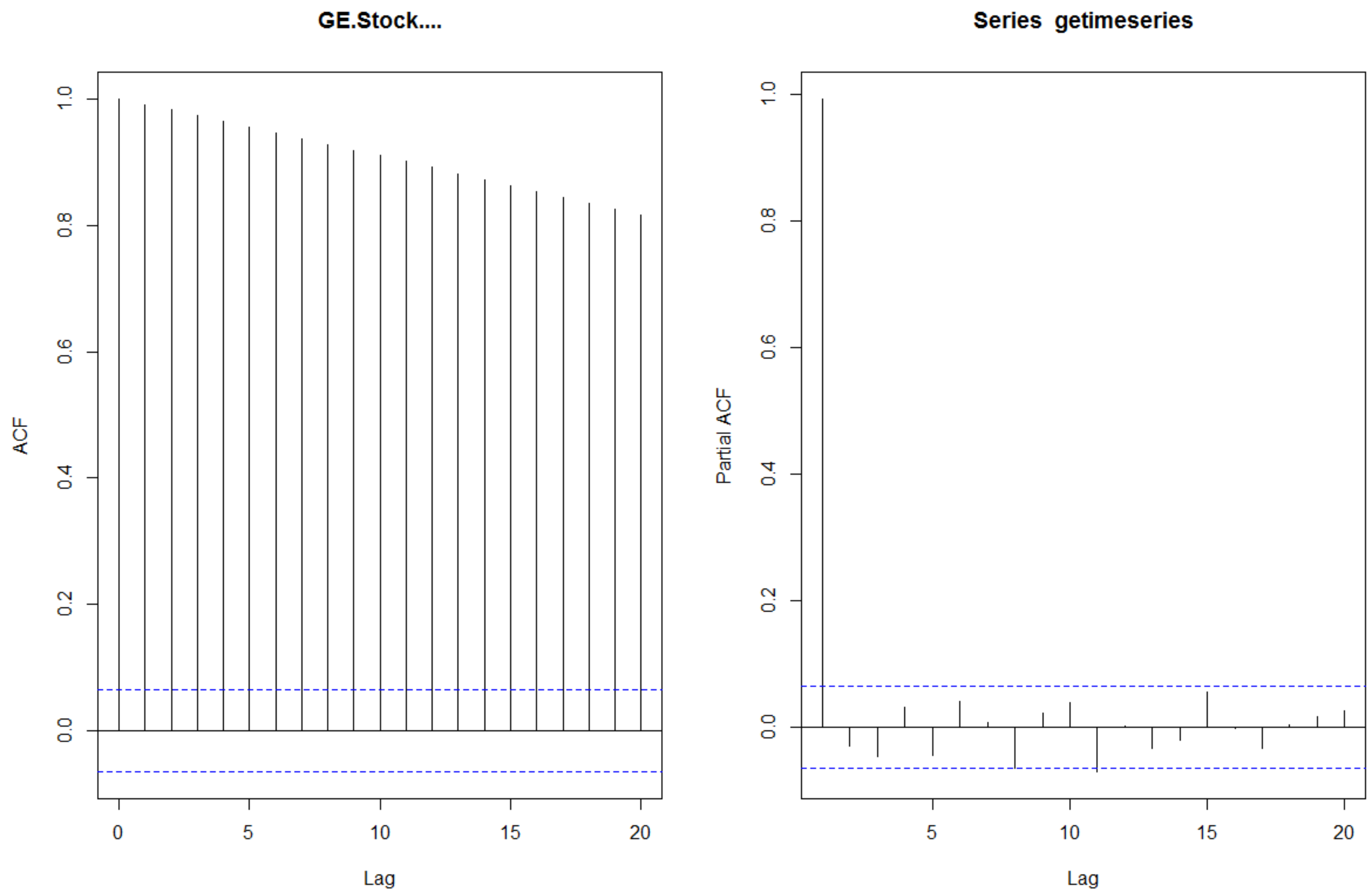
Removing Trend from Data





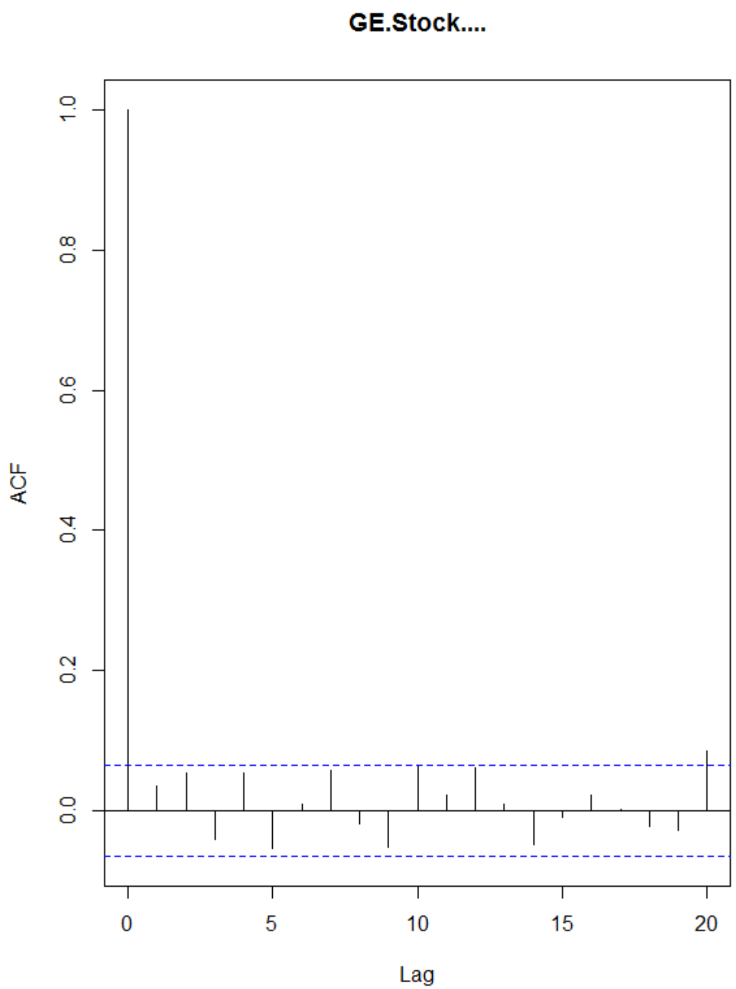


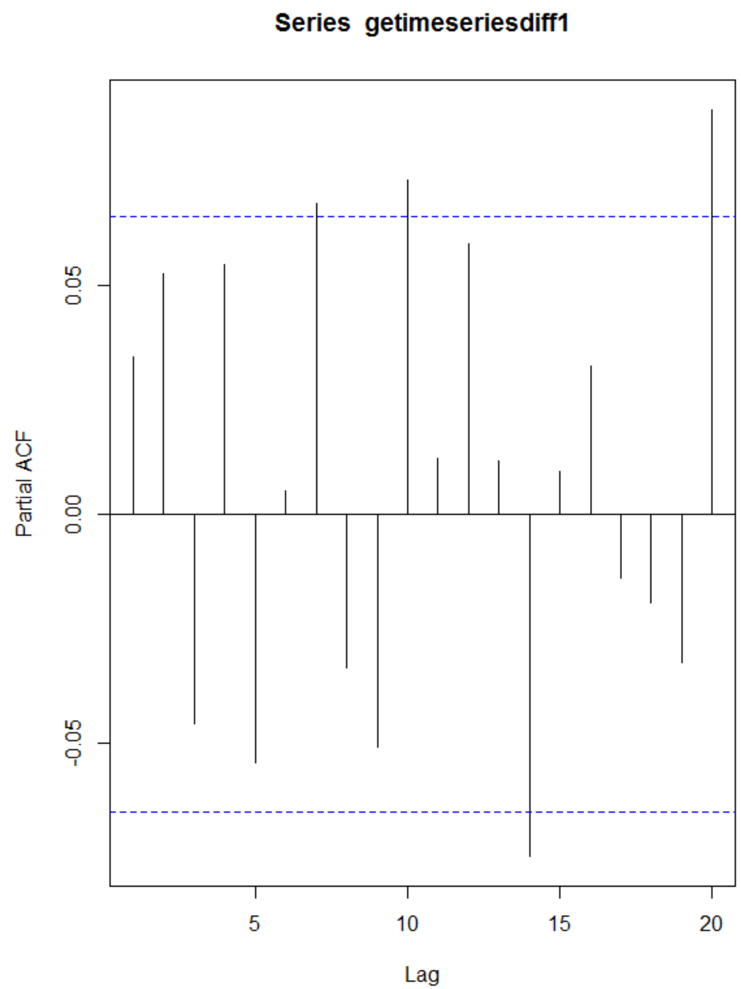
ACF and PACF of Stationary and Non-Stationary



Price of GE stock is highly correlated with the previous day's value.

ACF and PACF of Stationary and Non-Stationary





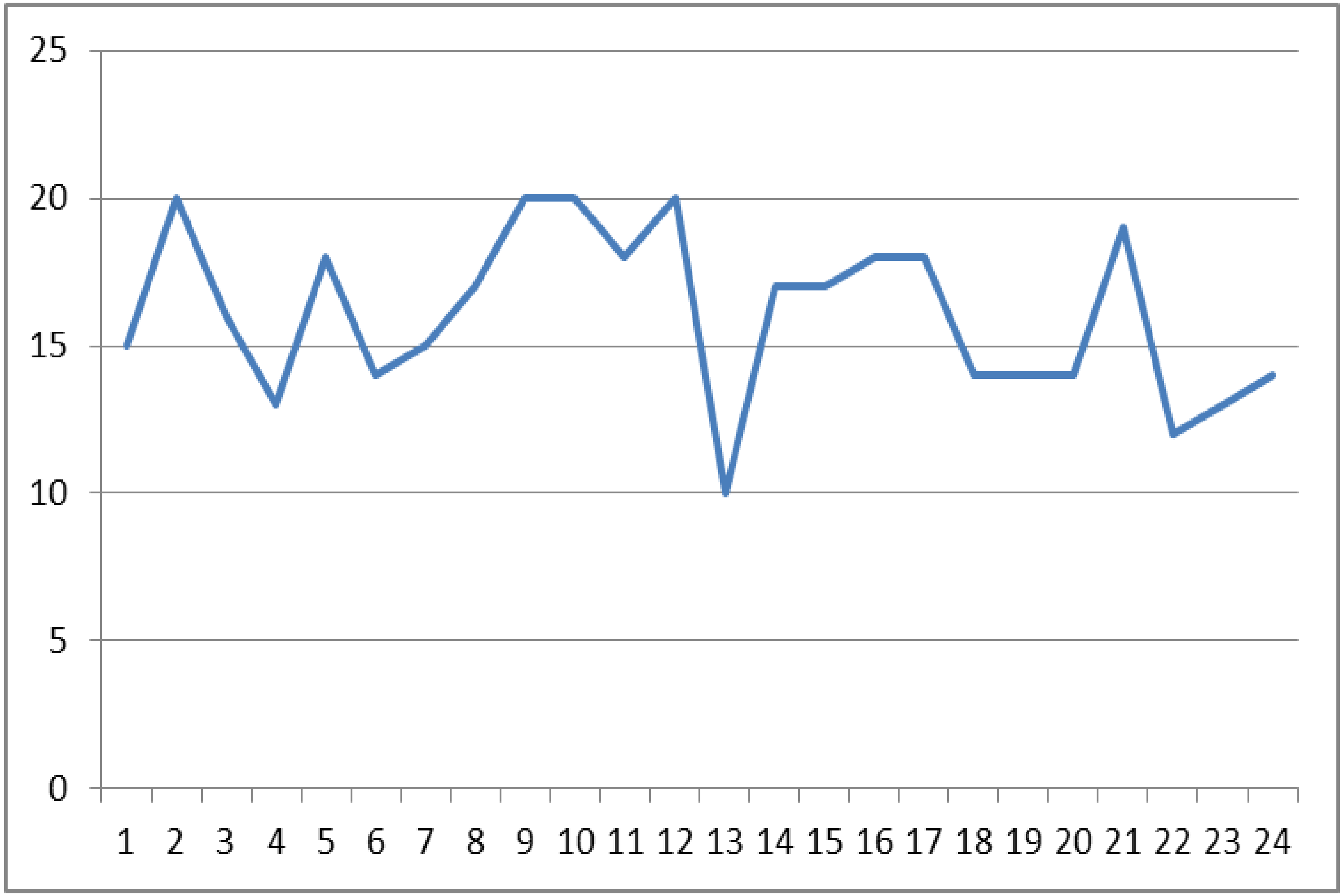
Daily changes in GE stock price are essentially random.

ACF and PACF of Stationary and Non-Stationary

- Non-stationary series have an ACF that remains significant for half a dozen or more lags, rather than quickly declining to zero.

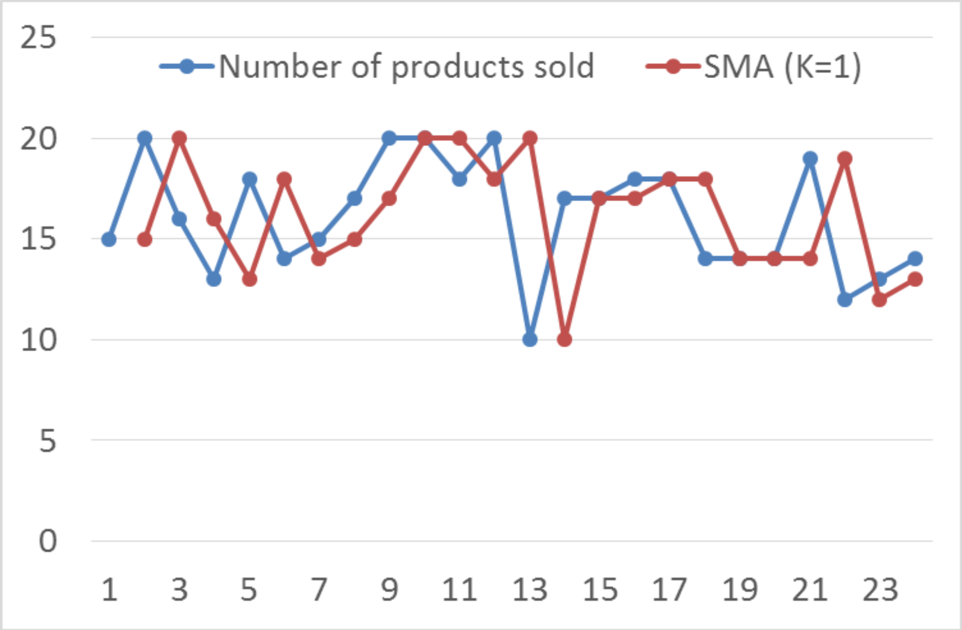
- You must difference such a series until it is stationary before you can identify the process.

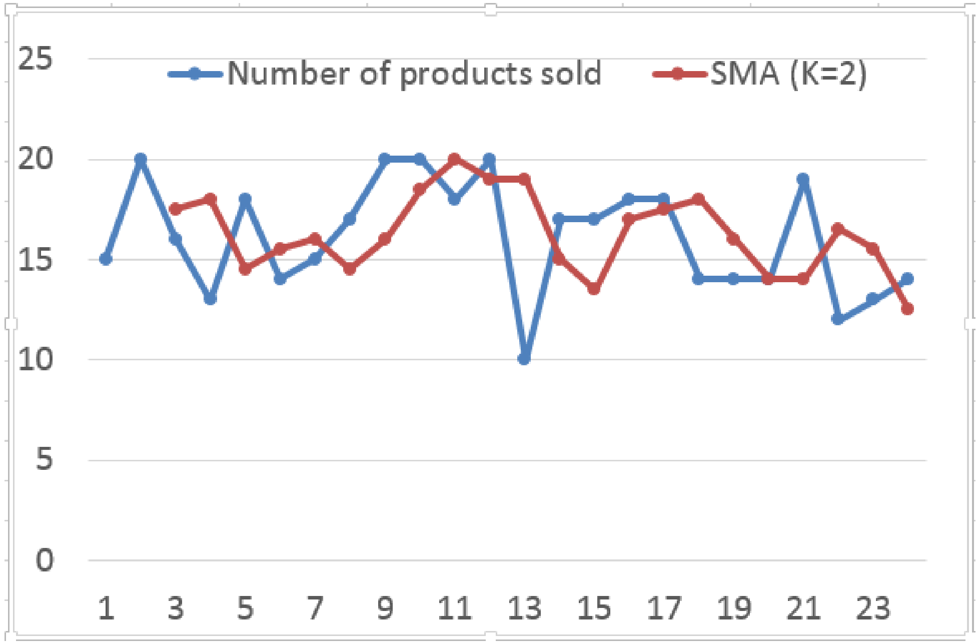
Stationary Model: Moving Averages

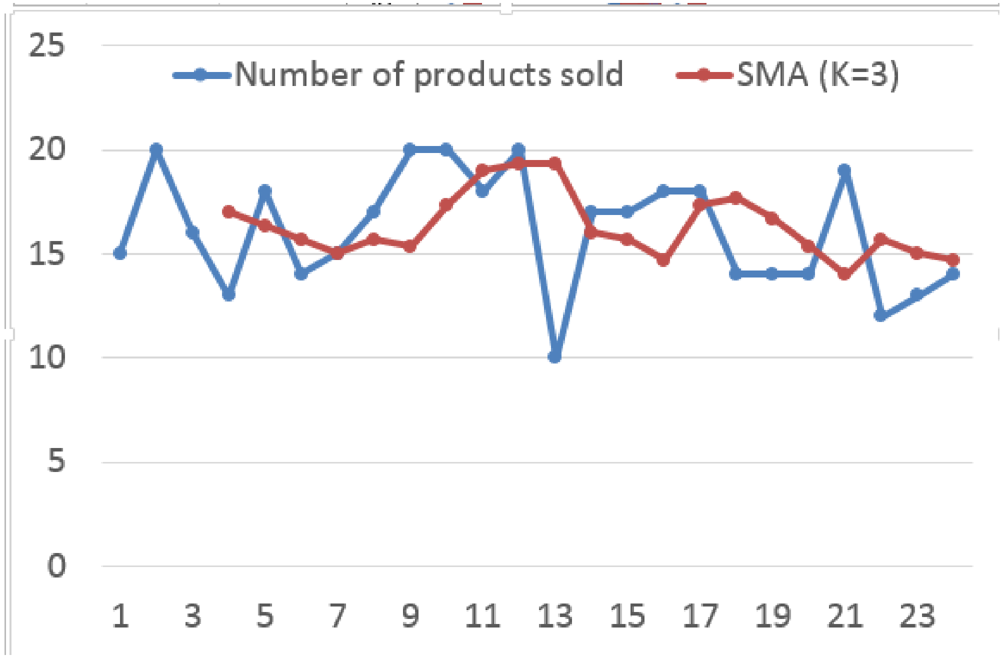


Stationary Model: Case 1 – Simple Moving Averages

Number of products sold	SMA (K=1)	Error	SMA (K=2)	Error	SMA (K=3)	Error
15						
20	15	5				
16	20	4	17.5	1.5		
13	16	3	18	5	17	4
18	13	5	14.5	3.5	16.333333	1.66667
14	18	4	15.5	1.5	15.666667	1.66667
15	14	1	16	1	15	0
17	15	2	14.5	2.5	15.666667	1.33333
20	17	3	16	4	15.333333	4.66667
20	20	0	18.5	1.5	17.333333	2.66667
18	20	2	20	2	19	1
20	18	2	19	1	19.333333	0.66667
10	20	10	19	9	19.333333	9.33333
17	10	7	15	2	16	1
17	17	0	13.5	3.5	15.666667	1.33333
18	17	1	17	1	14.666667	3.33333
18	18	0	17.5	0.5	17.333333	0.66667
14	18	4	18	4	17.666667	3.66667
14	14	0	16	2	16.666667	2.66667
14	14	0	14	0	15.333333	1.33333
19	14	5	14	5	14	5
12	19	7	16.5	4.5	15.666667	3.66667
13	12	1	15.5	2.5	15	2
14	13	1	12.5	1.5	14.666667	0.66667
		2.91304		**2.68182**		**2.49206**







Only decision point is K

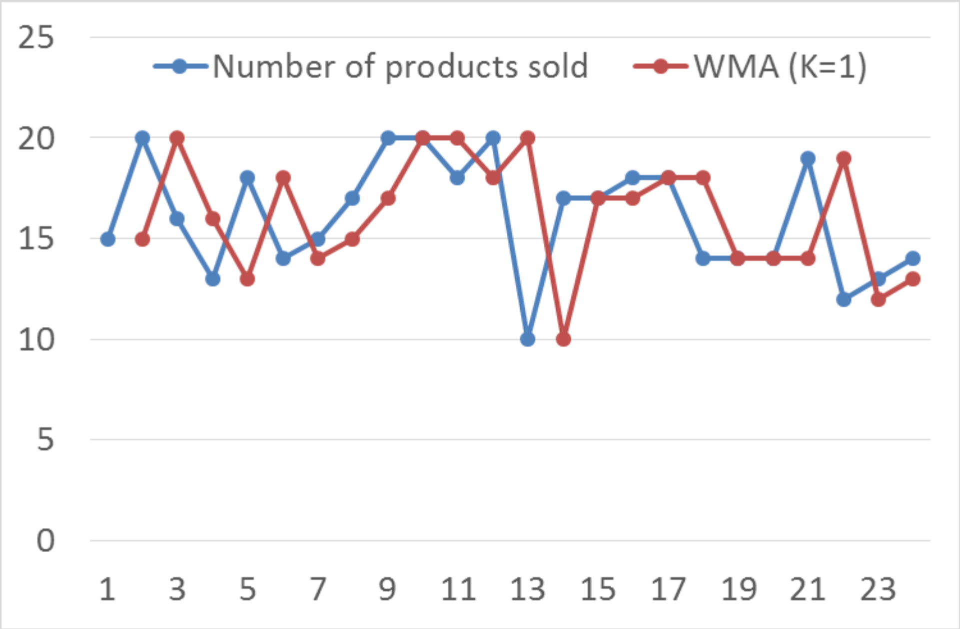
Stationary Model: Case 2 – Weighted Moving Averages

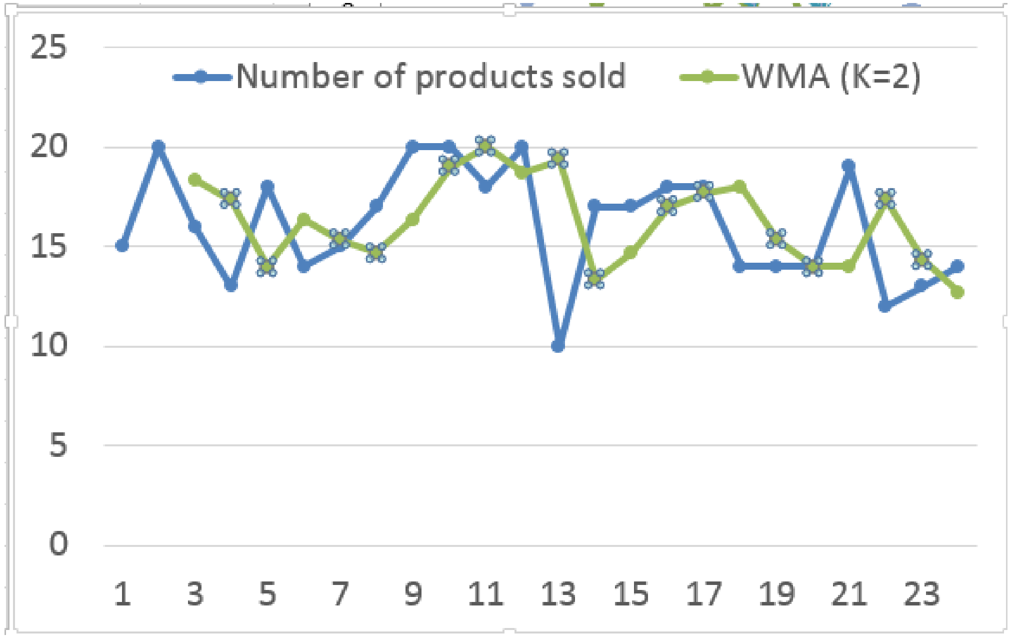
$$\hat{Y}_{t+1} = w_1 Y_t + w_2 Y_{t-1} + \cdots + w_k Y_{t-k+1}$$

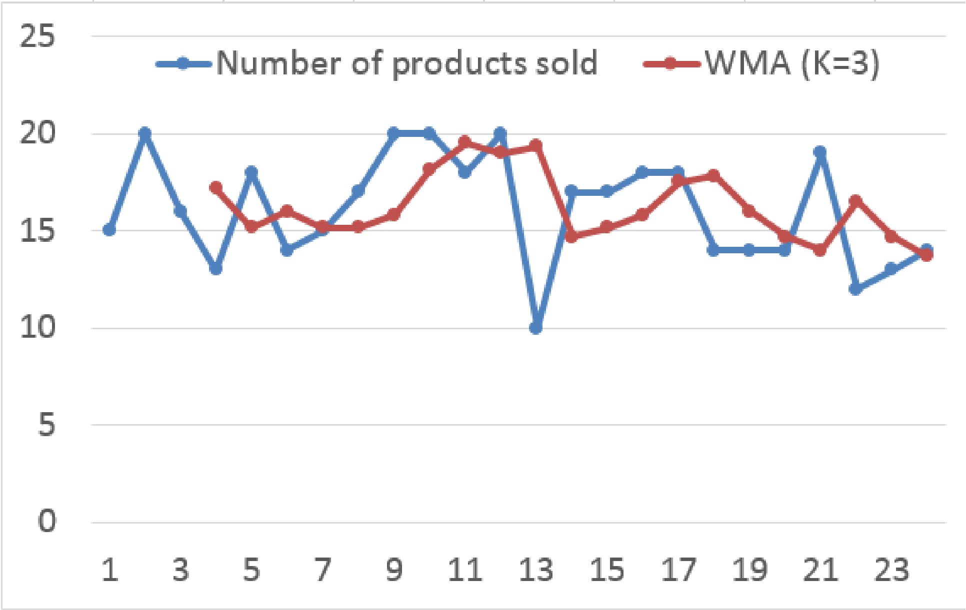
- Typically we choose a time period of moving average and weights are chosen such that the error is minimized

Stationary Model: Case 2 – Weighted Moving Averages

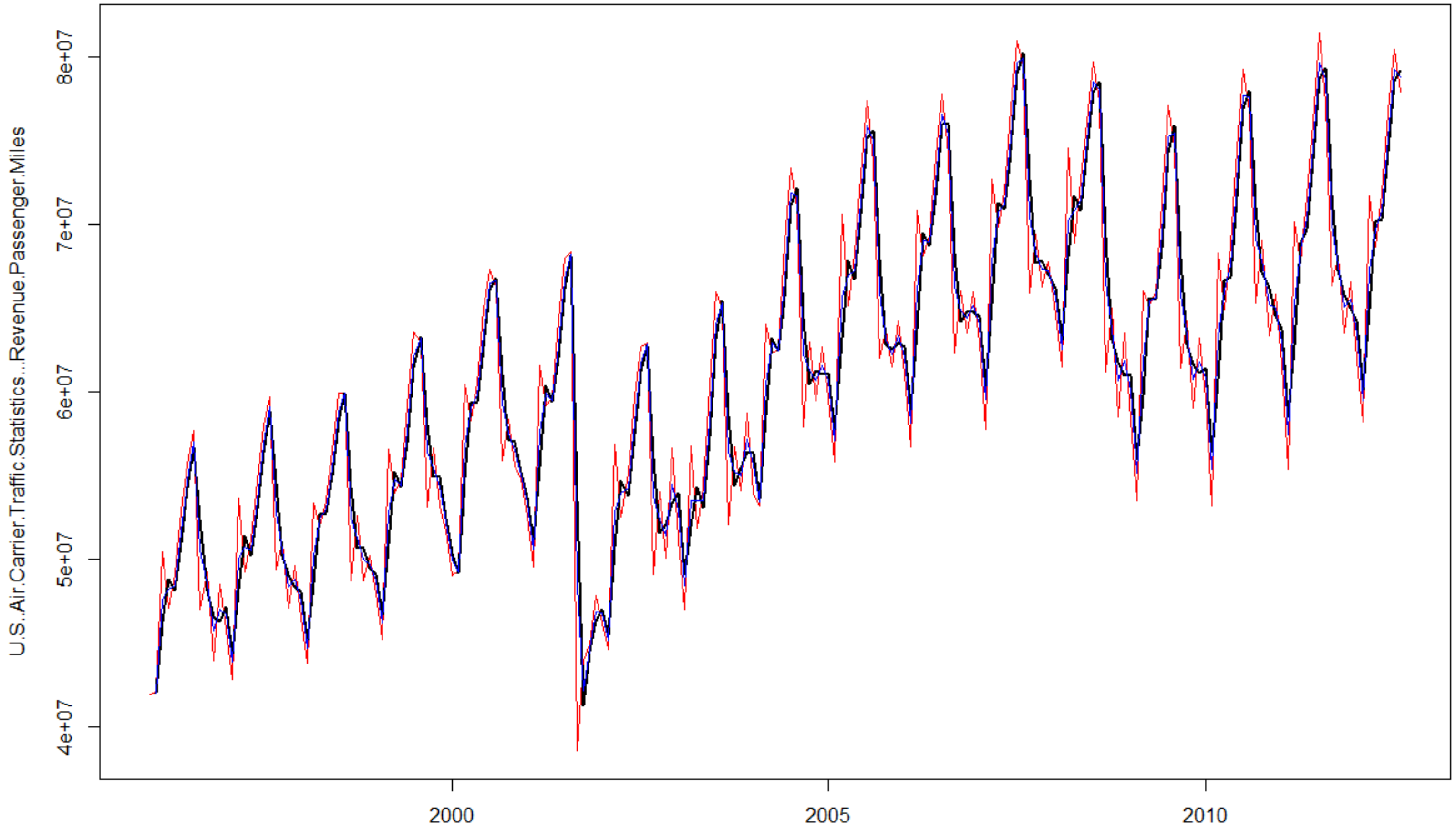
Number of products sold	WMA (K=1)	Error	WMA (K=2)	Error	WMA (K=3)	Error
15						
20	15	5				
16	20	4	18.3333333	2.33333333		
13	16	3	17.3333333	4.33333333	17.1666667	4.16666667
18	13	5	14	4	15.1666667	2.83333333
14	18	4	16.3333333	2.33333333	16	2
15	14	1	15.3333333	0.33333333	15.1666667	0.16666667
17	15	2	14.6666667	2.33333333	15.1666667	1.83333333
20	17	3	16.3333333	3.66666667	15.8333333	4.16666667
20	20	0	19	1	18.1666667	1.83333333
18	20	2	20	2	19.5	1.5
20	18	2	18.6666667	1.33333333	19	1
10	20	10	19.3333333	9.33333333	19.3333333	9.33333333
17	10	7	13.3333333	3.66666667	14.6666667	2.33333333
17	17	0	14.6666667	2.33333333	15.1666667	1.83333333
18	17	1	17	1	15.8333333	2.16666667
18	18	0	17.6666667	0.33333333	17.5	0.5
14	18	4	18	4	17.8333333	3.83333333
14	14	0	15.3333333	1.33333333	16	2
14	14	0	14	0	14.6666667	0.66666667
19	14	5	14	5	14	5
12	19	7	17.3333333	5.33333333	16.5	4.5
13	12	1	14.3333333	1.33333333	14.6666667	1.66666667
14	13	1	12.6666667	1.33333333	13.6666667	0.33333333
		2.91304348		**2.66666667**		**2.55555556**


25
20
15
10
5
0
Number of products sold
WMA (K=1)
1 3 5 7 9 11 13 15 17 19 21 23


25
20
15
10
5
0
Number of products sold
WMA (K=2)
1 3 5 7 9 11 13 15 17 19 21 23


25
20
15
10
5
0
Number of products sold
WMA (K=3)
1 3 5 7 9 11 13 15 17 19 21 23

SMA and WMA – Revenue Passenger Miles



> MAPE-SMA 4.093731 > MAPE-WMA 2.729154

Stationary Model: Case 3 – Exponential ~~Weighted Moving Averages~~ or Exponential Smoothing

Averaging over long periods dampens fluctuations, removing not only the noise but also trend and seasonality.

Moving averages over short recent periods maintains trend and seasonality but determining an optimum number for periods is tricky, even when using metrics like MAE. If averaged over too few periods, irregularities continue to remain and if averaged over long periods, dampening again becomes a problem.

Exponential smoothing **retains all older periods** while giving a greater weight to more recent periods (hence not a MOVING average).

Caution: It doesn't make any one method superior for all situations.

Stationary Model: Case 3 – Exponential ~~Weighted Moving Averages~~ or Exponential Smoothing

$$\hat{Y}_{t+1} = \hat{Y}_t + \alpha\left(Y_t - \hat{Y}_t\right)$$

Above equation indicates that the predicted value for time period *t*+1 ($\hat{Y}_{t+1}$) is equal to the predicted value for the previous period ($\hat{Y}_t$) plus an adjustment for the error made in predicting the previous period's value ($\alpha\left(Y_t - \hat{Y}_t\right)$).

The parameter α can assume any value between 0 and 1 ($0 \le \alpha \le 1$).

Exponential Smoothing in Other Ways

$\hat{Y}_{t+1} = \hat{Y}_t + \alpha(Y_t - \hat{Y}_t)$ can be rewritten variously as

$$= \alpha Y_t + (1-\alpha)\hat{Y}_t$$

$$= Y_t - (1-\alpha)\left(Y_t - \hat{Y}_t\right)$$

$$= \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \cdots + \alpha(1-\alpha)^n Y_{t-n} + \cdots$$

Exponential Smoothing

$$\hat{Y}_{t+1} = \hat{Y}_t + \alpha(Y_t - \hat{Y}_t) \qquad \alpha = \frac{2}{N+1}$$

- Forecasting at time t+1 requires both the forecasted value and the True Value at time t
- So if you want to forecast more than 1 time period into the future, the best you can do is to use the last available value
- All future predictions are same! This is in accordance with **stationary** assumption.

Exponential Smoothing

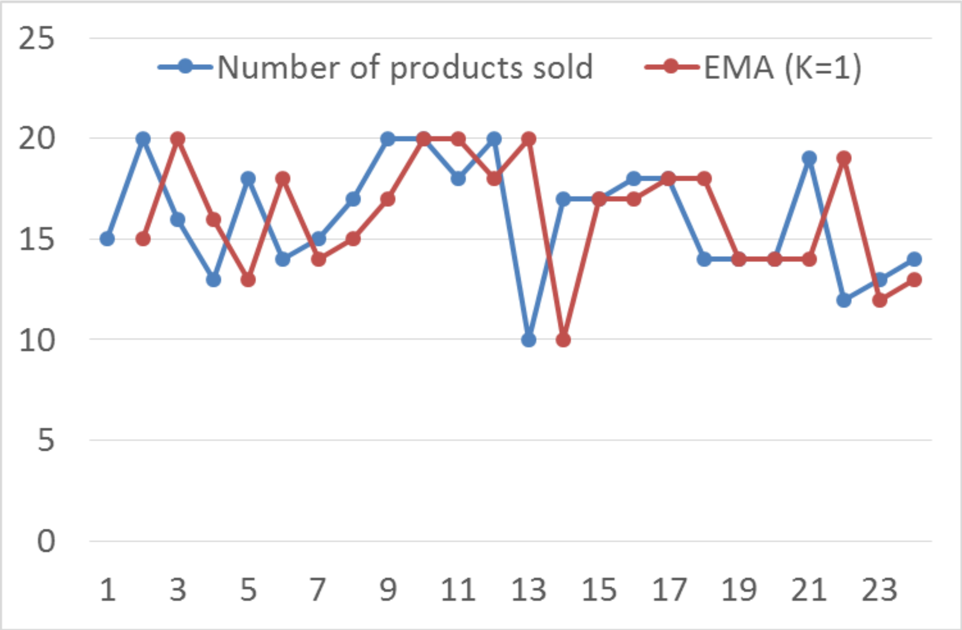
	A	B	C	D	E	F	G
1	Numbe	EMA (K=1)	Error	EMA (K=2)	Error	EMA (K=3)	Error
2	15						
3	20	=A2*1	=ABS(B3-A3)	=15			
4	16	=A3*K2+B3*L2	=ABS(B4-A4)	=A3*K3+D3*L3	=ABS(A4-D4)	=AVERAGE(A2:A3)	
5	13	=A4*K2+B4*L2	=ABS(B5-A5)	=A4*K3+D4*L3	=ABS(A5-D5)	=A4*K4+F4*L4	=ABS(A5-F5)
6	18	=A5*K2+B5*L2	=ABS(B6-A6)	=A5*K3+D5*L3	=ABS(A6-D6)	=A5*K4+F5*L4	=ABS(A6-F6)
7	14	=A6*K2+B6*L2	=ABS(B7-A7)	=A6*K3+D6*L3	=ABS(A7-D7)	=A6*K4+F6*L4	=ABS(A7-F7)
8	15	=A7*K2+B7*L2	=ABS(B8-A8)	=A7*K3+D7*L3	=ABS(A8-D8)	=A7*K4+F7*L4	=ABS(A8-F8)
9	17	=A8*K2+B8*L2	=ABS(B9-A9)	=A8*K3+D8*L3	=ABS(A9-D9)	=A8*K4+F8*L4	=ABS(A9-F9)
10	20	=A9*K2+B9*L2	=ABS(B10-A10)	=A9*K3+D9*L3	=ABS(A10-D10)	=A9*K4+F9*L4	=ABS(A10-F10)
11	20	=A10*K2+B10*L2	=ABS(B11-A11)	=A10*K3+D10*L3	=ABS(A11-D11)	=A10*K4+F10*L4	=ABS(A11-F11)
12	18	=A11*K2+B11*L2	=ABS(B12-A12)	=A11*K3+D11*L3	=ABS(A12-D12)	=A11*K4+F11*L4	=ABS(A12-F12)
13	20	=A12*K2+B12*L2	=ABS(B13-A13)	=A12*K3+D12*L3	=ABS(A13-D13)	=A12*K4+F12*L4	=ABS(A13-F13)
14	10	=A13*K2+B13*L2	=ABS(B14-A14)	=A13*K3+D13*L3	=ABS(A14-D14)	=A13*K4+F13*L4	=ABS(A14-F14)
15	17	=A14*K2+B14*L2	=ABS(B15-A15)	=A14*K3+D14*L3	=ABS(A15-D15)	=A14*K4+F14*L4	=ABS(A15-F15)
16	17	=A15*K2+B15*L2	=ABS(B16-A16)	=A15*K3+D15*L3	=ABS(A16-D16)	=A15*K4+F15*L4	=ABS(A16-F16)
17	18	=A16*K2+B16*L2	=ABS(B17-A17)	=A16*K3+D16*L3	=ABS(A17-D17)	=A16*K4+F16*L4	=ABS(A17-F17)
18	18	=A17*K2+B17*L2	=ABS(B18-A18)	=A17*K3+D17*L3	=ABS(A18-D18)	=A17*K4+F17*L4	=ABS(A18-F18)
19	14	=A18*K2+B18*L2	=ABS(B19-A19)	=A18*K3+D18*L3	=ABS(A19-D19)	=A18*K4+F18*L4	=ABS(A19-F19)
20	14	=A19*K2+B19*L2	=ABS(B20-A20)	=A19*K3+D19*L3	=ABS(A20-D20)	=A19*K4+F19*L4	=ABS(A20-F20)
21	14	=A20*K2+B20*L2	=ABS(B21-A21)	=A20*K3+D20*L3	=ABS(A21-D21)	=A20*K4+F20*L4	=ABS(A21-F21)
22	19	=A21*K2+B21*L2	=ABS(B22-A22)	=A21*K3+D21*L3	=ABS(A22-D22)	=A21*K4+F21*L4	=ABS(A22-F22)
23	12	=A22*K2+B22*L2	=ABS(B23-A23)	=A22*K3+D22*L3	=ABS(A23-D23)	=A22*K4+F22*L4	=ABS(A23-F23)
24	13	=A23*K2+B23*L2	=ABS(B24-A24)	=A23*K3+D23*L3	=ABS(A24-D24)	=A23*K4+F23*L4	=ABS(A24-F24)
25	14	=A24*K2+B24*L2	=ABS(B25-A25)	=A24*K3+D24*L3	=ABS(A25-D25)	=A24*K4+F24*L4	=ABS(A25-F25)
26			**=AVERAGE(C3:C25)**		**=AVERAGE(E3:E25)**		**=AVERAGE(G3:G25)**

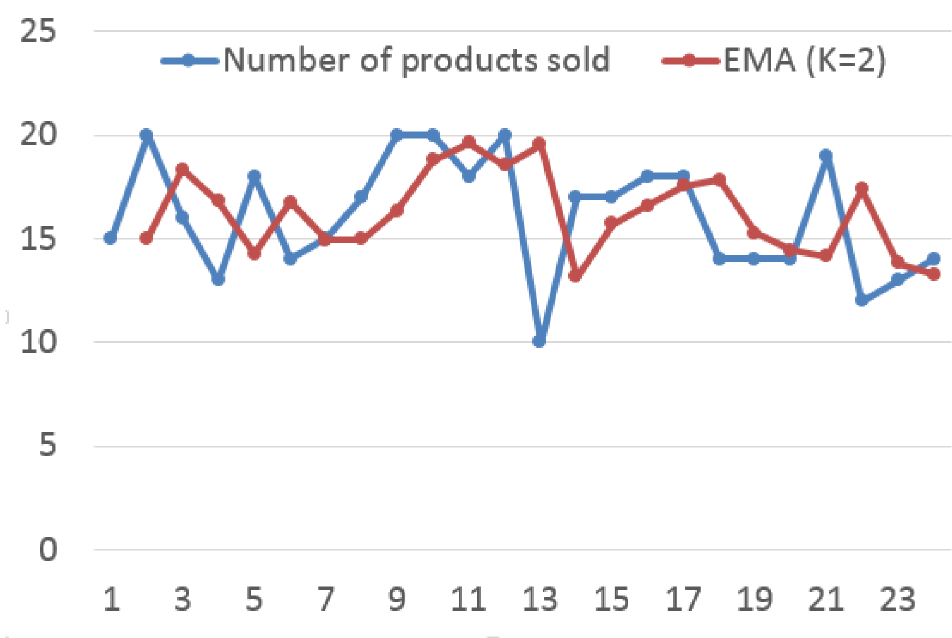
J	K	L
K	2/(K+1)	1-[2/(K+1)]
1	1	=1-K2
2	=2/(J3+1)	=1-K3
3	=2/(J4+1)	=1-K4
4	=2/(J5+1)	=1-K5

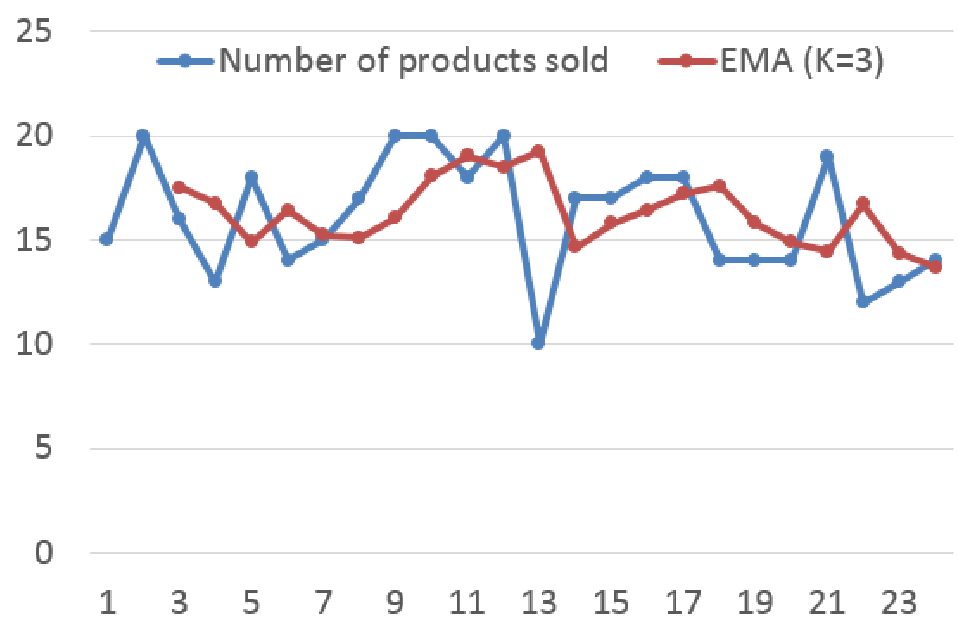
$$\hat{Y}_{t+1} = \alpha Y_t + (1-\alpha)\hat{Y}_t$$

Exponential Smoothing

	A	B	C	D	E	F	G	H	I
1	Number of	EMA (K=1)	Error	EMA (K=2)	Error	EMA (K=3)	Error	EMA (K=4)	Error
2	15								
3	20	15	5	15					
4	16	20	4	18.3333333	2.33333	17.5			
5	13	16	3	16.7777778	3.77778	16.75	3.75	17	
6	18	13	5	14.2592593	3.74074	14.875	3.125	15.4	2.6
7	14	18	4	16.7530864	2.75309	16.4375	2.4375	16.44	2.44
8	15	14	1	14.9176955	0.0823	15.21875	0.21875	15.464	0.464
9	17	15	2	14.9725652	2.02743	15.109375	1.890625	15.2784	1.7216
10	20	17	3	16.3241884	3.67581	16.054688	3.945313	15.96704	4.03296
11	20	20	0	18.7747295	1.22527	18.027344	1.972656	17.580224	2.41978
12	18	20	2	19.5915765	1.59158	19.013672	1.013672	18.548134	0.54813
13	20	18	2	18.5305255	1.46947	18.506836	1.493164	18.328881	1.67112
14	10	20	10	19.5101752	9.51018	19.253418	9.253418	18.997328	8.99733
15	17	10	7	13.1700584	3.82994	14.626709	2.373291	15.398397	1.6016
16	17	17	0	15.7233528	1.27665	15.813354	1.186646	16.039038	0.96096
17	18	17	1	16.5744509	1.42555	16.406677	1.593323	16.423423	1.57658
18	18	18	0	17.524817	0.47518	17.203339	0.796661	17.054054	0.94595
19	14	18	4	17.8416057	3.84161	17.601669	3.601669	17.432432	3.43243
20	14	14	0	15.2805352	1.28054	15.800835	1.800835	16.059459	2.05946
21	14	14	0	14.4268451	0.42685	14.900417	0.900417	15.235676	1.23568
22	19	14	5	14.1422817	4.85772	14.450209	4.549791	14.741405	4.25859
23	12	19	7	17.3807606	5.38076	16.725104	4.725104	16.444843	4.44484
24	13	12	1	13.7935869	0.79359	14.362552	1.362552	14.666906	1.66691
25	14	13	1	13.264529	0.73547	13.681276	0.318724	14.000144	0.00014
26			**2.913043**		**2.56867**		**2.49091**		**2.3539**



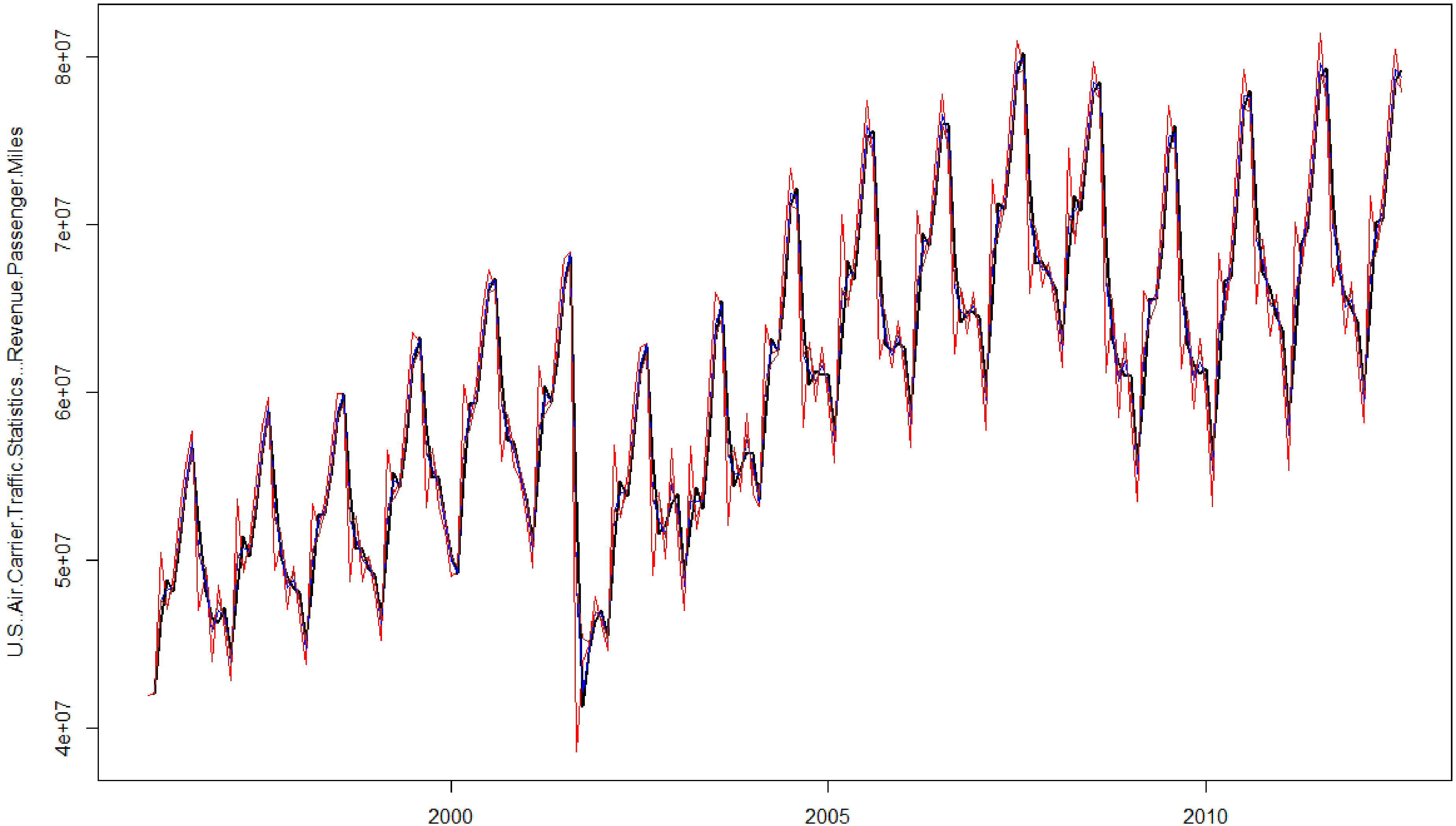




$$\hat{Y}_{t+1} = \hat{Y}_t + \alpha(Y_t - \hat{Y}_t)$$

$$\alpha = \frac{2}{K+1}$$

SMA, WMA and Exponential Smoothing – RPM




```
> MAPESMA 4.093731 > MAPEWMA 2.729154 > MAPEEMA 2.541979
```

ADDING TREND AND SEASONALITY TO MOVING AVERAGE PROCESSES

Holt-Winters Method

- This method separates the forecast into 3 components – Base Level + Trend + Seasonal

- It updates each of the level using a Exponential smoothing with a different smoothing factor.

- The algorithm computes the best possible smoothing factors based on the given data and does subsequent predictions using fitted smoothing constants.

Holt-Winters Method

Additive Seasonality

$$\hat{Y}_t = E_{t-1} + T_{t-1} + S_{t-p}$$

$$\hat{Y}_{t+n} = E_t + nT_t + S_{t+n-p}$$

Multiplicative Seasonality

$$\hat{Y}_t = (E_{t-1} + T_{t-1})S_{t-p}$$

$$\hat{Y}_{t+n} = (E_t + nT_t)S_{t+n-p}$$

The 3 smoothing equations are:

Additive:

$$E_t = \alpha\left(Y_t - S_{t-p}\right) + (1-\alpha)(E_{t-1} + T_{t-1})$$

$$T_t = \beta(E_t - E_{t-1}) + (1-\beta)T_{t-1}$$

$$S_t = \gamma(Y_t - E_t) + (1-\gamma)S_{t-p}$$

Multiplicative:

$$E_t = \alpha\frac{Y_t}{S_{t-p}} + (1-\alpha)(E_{t-1} + T_{t-1})$$

$$T_t = \beta(E_t - E_{t-1}) + (1-\beta)T_{t-1}$$

$$S_t = \gamma\frac{Y_t}{E_t} + (1-\gamma)S_{t-p}$$

Holt-Winters Method

- 3 components – Trend (T), Seasonality(S) and Base Level/Expectation(E).
- 3 weights – smoothing parameters – are used to update components at each period.
- Initial values for Base Level/Expectation and trend components are obtained using linear regression on time.
- Initial values for seasonal component are obtained from a dummy variable regression using de-trended data.
- In the Expectation equation, the series is seasonally adjusted by subtracting the seasonal component.

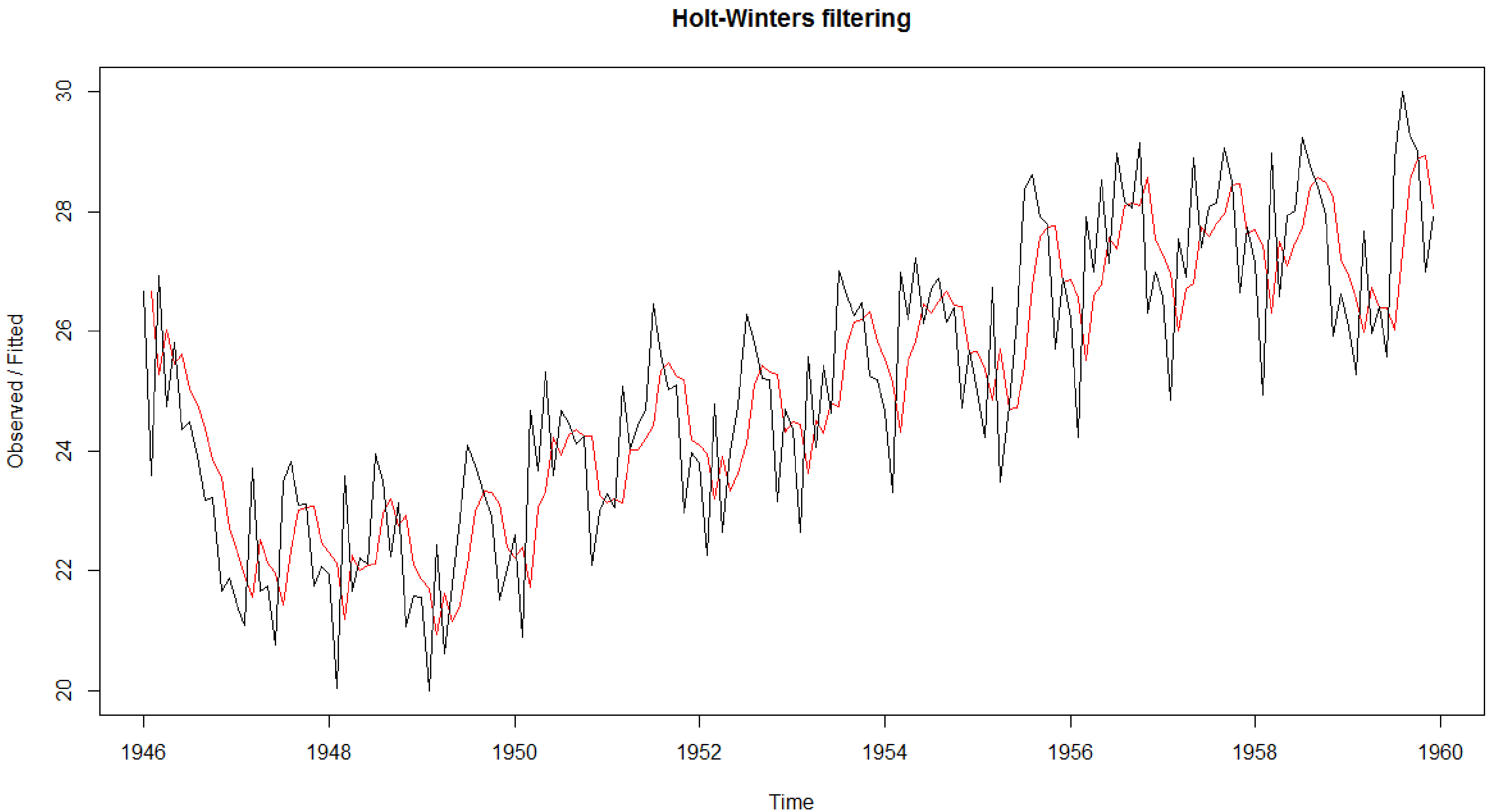
Holt-Winters Additive Seasonal Model

SSE = 103627

Holt-Winters Multiplicative Seasonal Model

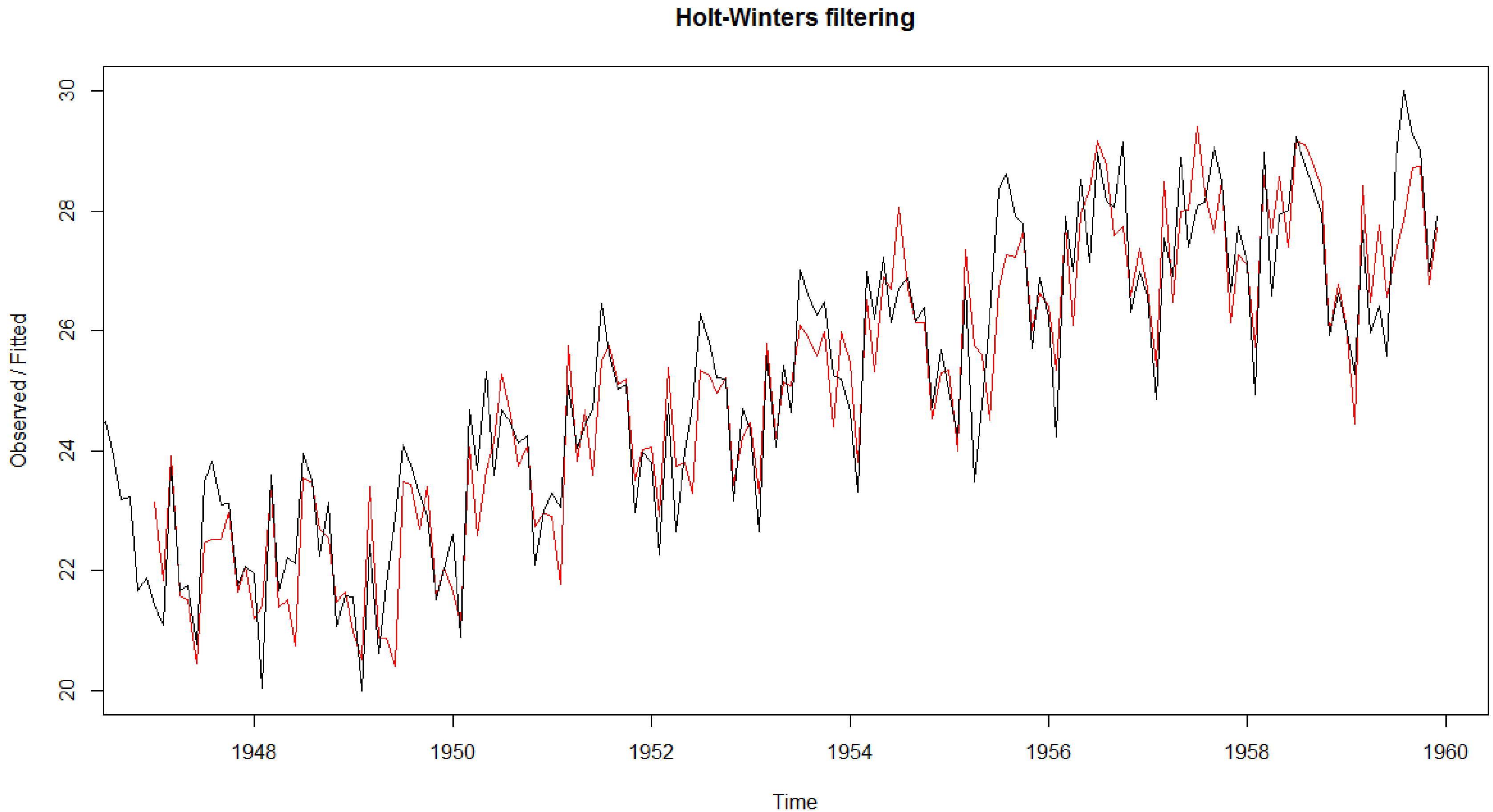
SSE = 49816

Holt-Winters Method: Only Randomness




```
> birthsforecast$SSE
[1] 281.8759
```

Holt-Winters Method: All Components

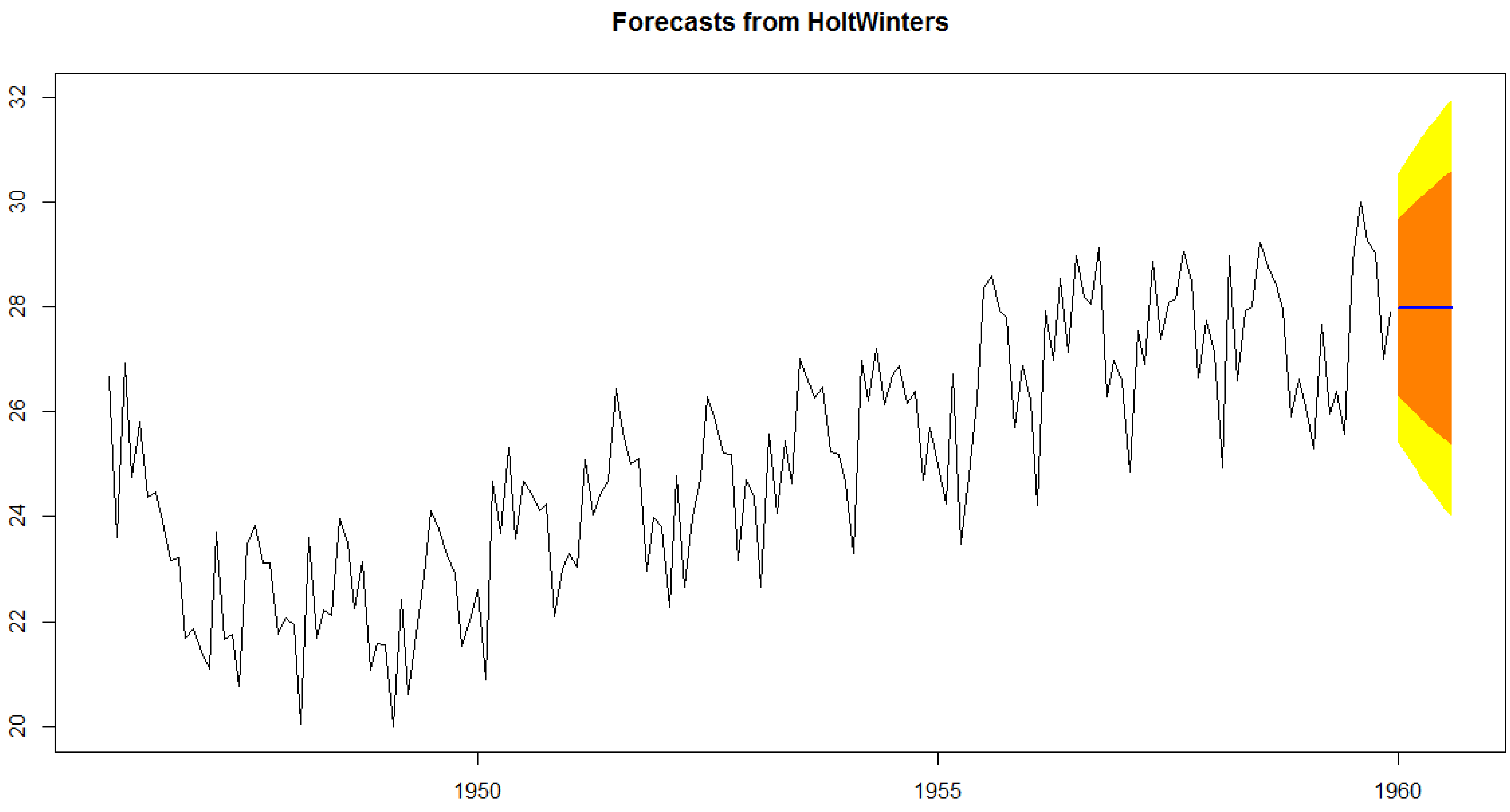



```
> birthsforecast$SSE
[1] 90.94058
```

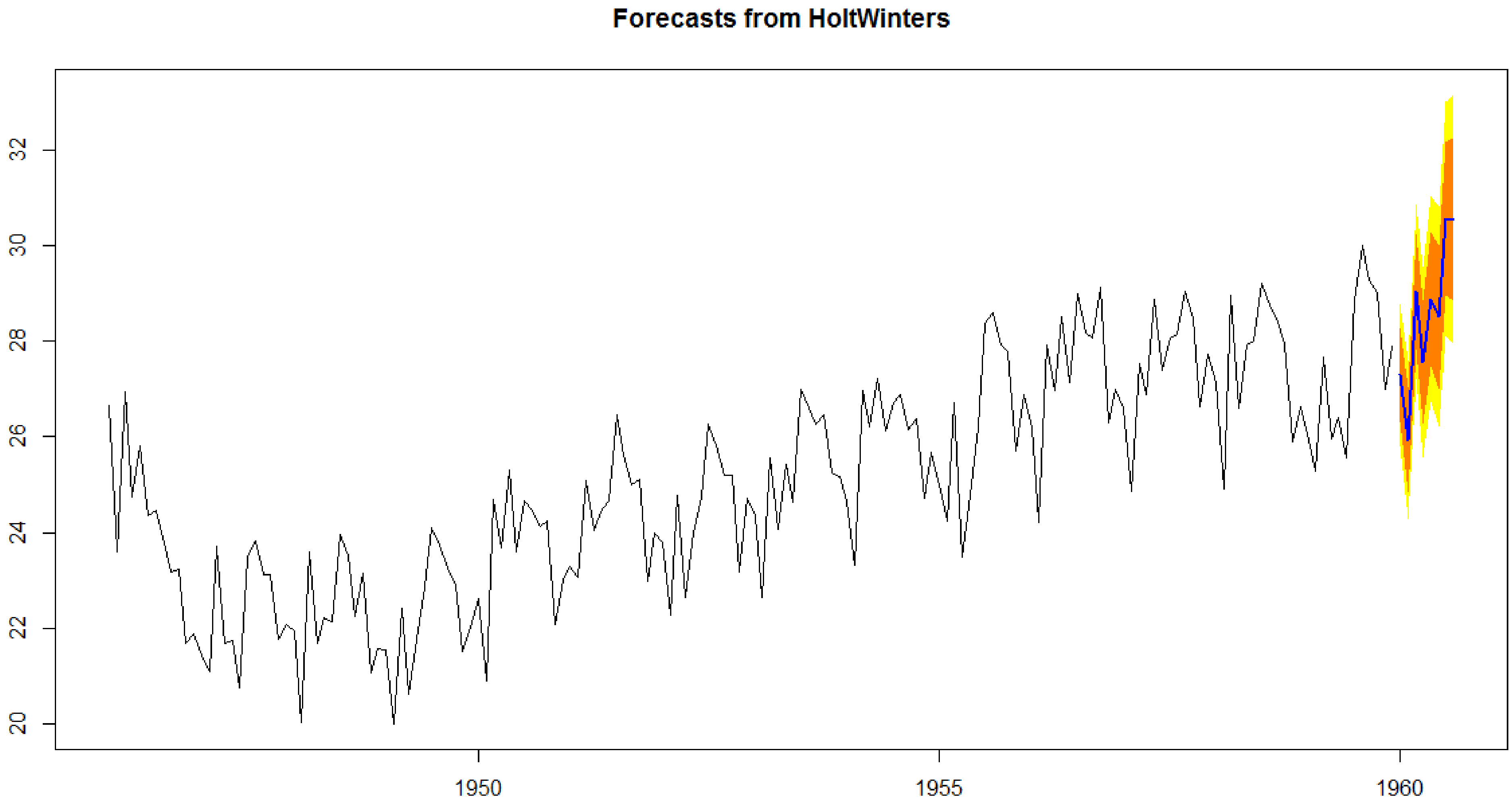
Holt-Winters Method: All Components

```
Holt-Winters exponential smoothing with trend and additive seasonal component.

Call:
HoltWinters(x = birthstimeseries)

Smoothing parameters:
 alpha: 0.4823655
 beta : 0.02988495
 gamma: 0.563186

Coefficients:
           [,1]
a   28.04366357
b    0.04199921
s1  -0.78546221
s2  -2.19944507
s3   0.87813012
s4  -0.65164728
s5   0.63427267
s6   0.21182821
s7   2.23177191
s8   2.17167733
s9   1.52077678
s10  1.16900861
s11 -0.97500043
s12 -0.18636055
> birthsforecast$fitted
             xhat    level         trend      season
Jan 1947 23.13579 23.81055 -0.1567618007 -0.51798958
```

Holt-Winters Method: Forecasting with No Trend and Seasonality

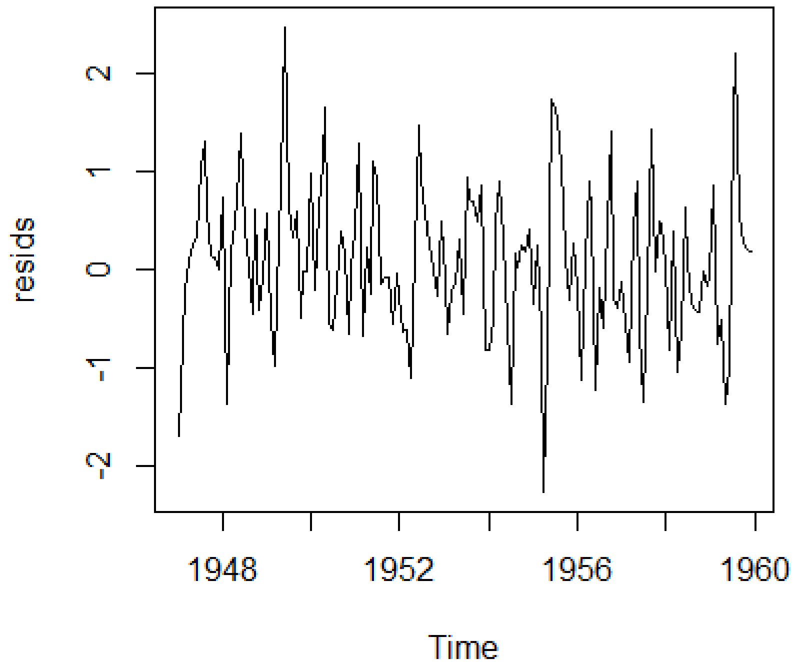


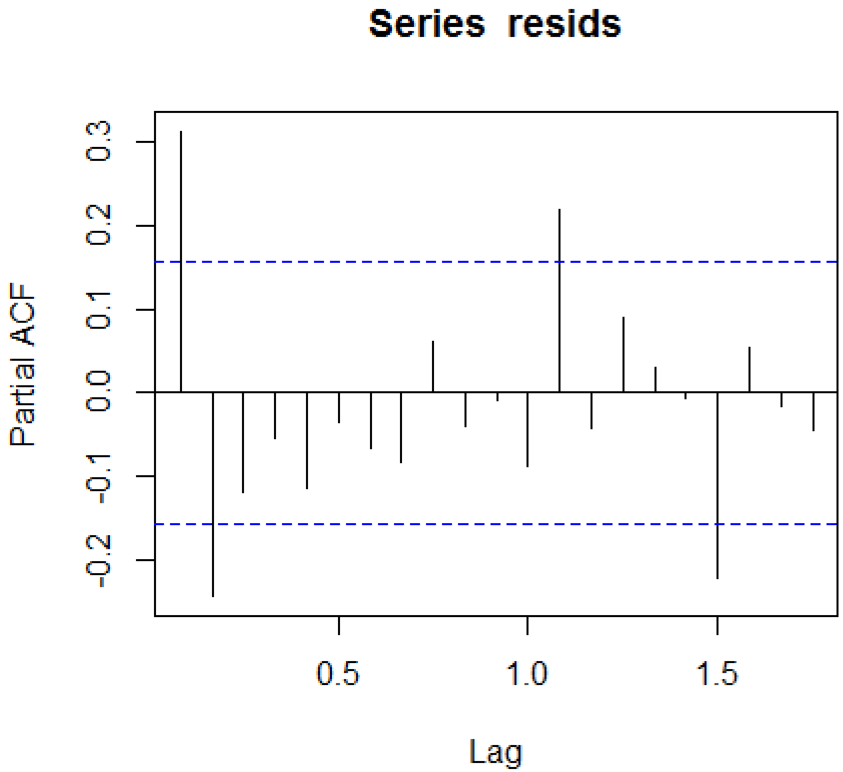
Holt-Winters Method: Forecasting



Residual Plots are useful

```
> resids <- birthstimeseries- birthsforecast$fitted[,1]
> plot(resids)
> pacf(resids)
> |
```





If the residuals are random and have no auto-correlation, then it means all useful information has been extracted. If the residuals show auto-correlation, it means more extractable information exists in the residual.

AR, MA AND ARIMA MODELS

CSE 7202c

AR(p) models

- Auto-regressive model of order p

$$\hat{y}_t = \alpha + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \cdots + \beta_p y_{t-p}$$

- We find the best value of parameters (β_1, β_2,…) that minimize the errors in forecast of $\hat{y}_t$.

- The order of the model p is determined based on the number beyond which PACF terms are zero.

Moving Average or MA(q) models

- Model attempts to predict future values using past error in predictions $\varepsilon_1 = \hat{y}_1 - y_1$
- So MA(2) model is

$$\hat{y}_t = \mu + \phi_1 \varepsilon_{t-1} + \phi_2 \varepsilon_{t-2}$$

Where, μ is the average value of the time series

- Again, the parameters (ϕ_1 , ϕ_2) are determined so that prediction error is minimized.
- The number of terms, *q*, is determined from the ACF plot. Its the maximum lag beyond which the ACF is 0

ARMA(p,q) model

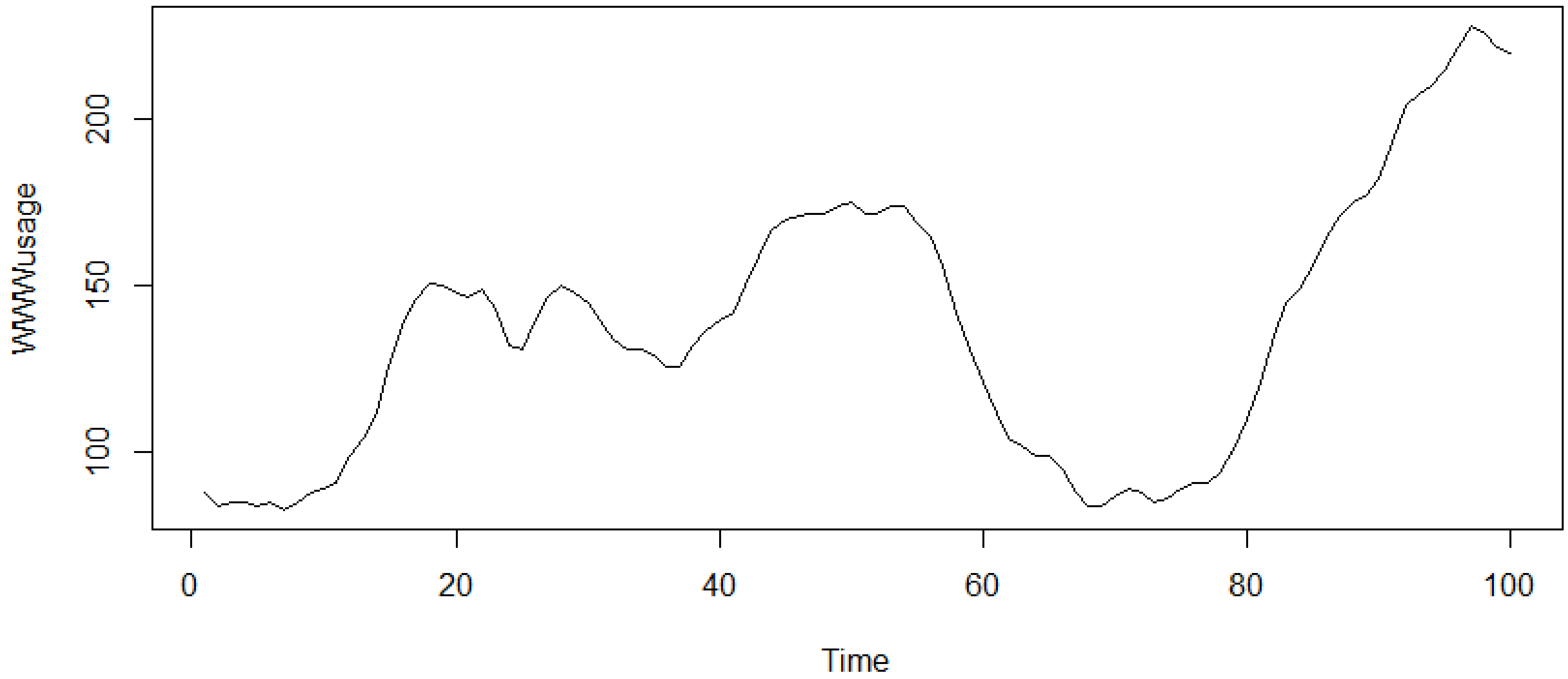
- Combines both AR(p) and MA(q) models
- For eg: a ARMA(2,1) model is

$$\hat{y}_t = \alpha + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \phi_1 \varepsilon_{t-1}$$

ARIMA(p,d,q) Model

- *p* is the number of autoregressive terms (a linear regression of the current value of the series against one or more prior values of the series)
  - Maximum lag beyond which PACF is 0
- *d* is the number of non-seasonal differences (order of the differencing) used to make the time series stationary
- *q* is the number of past prediction error terms used for the future forecasts.

Using ARIMA to forecast



A time series of the numbers of users connected to the Internet through a server every minute.

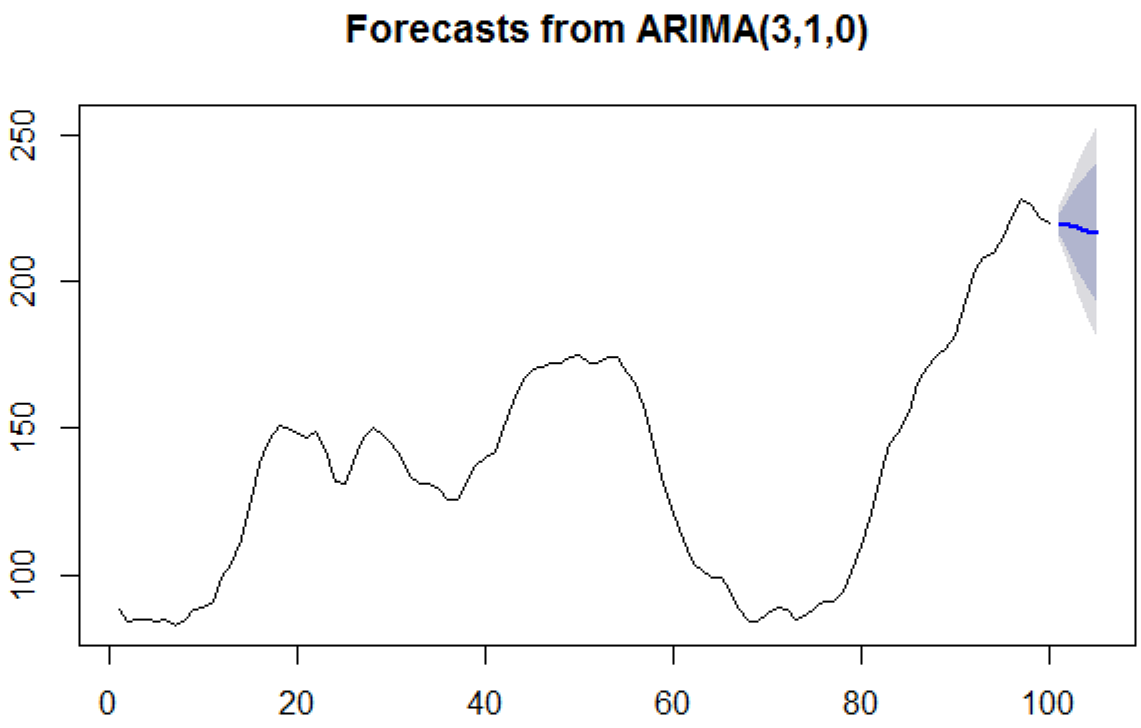
Using ARIMA to forecast

Let us see what the forecast looks like if we use Arima(3,1,0) model

```
>
> fit <- Arima(WWWusage,c(3,1,0))
> plot(forecast(fit,h=5))
>
```

Fit using Arima(3,1,0) model

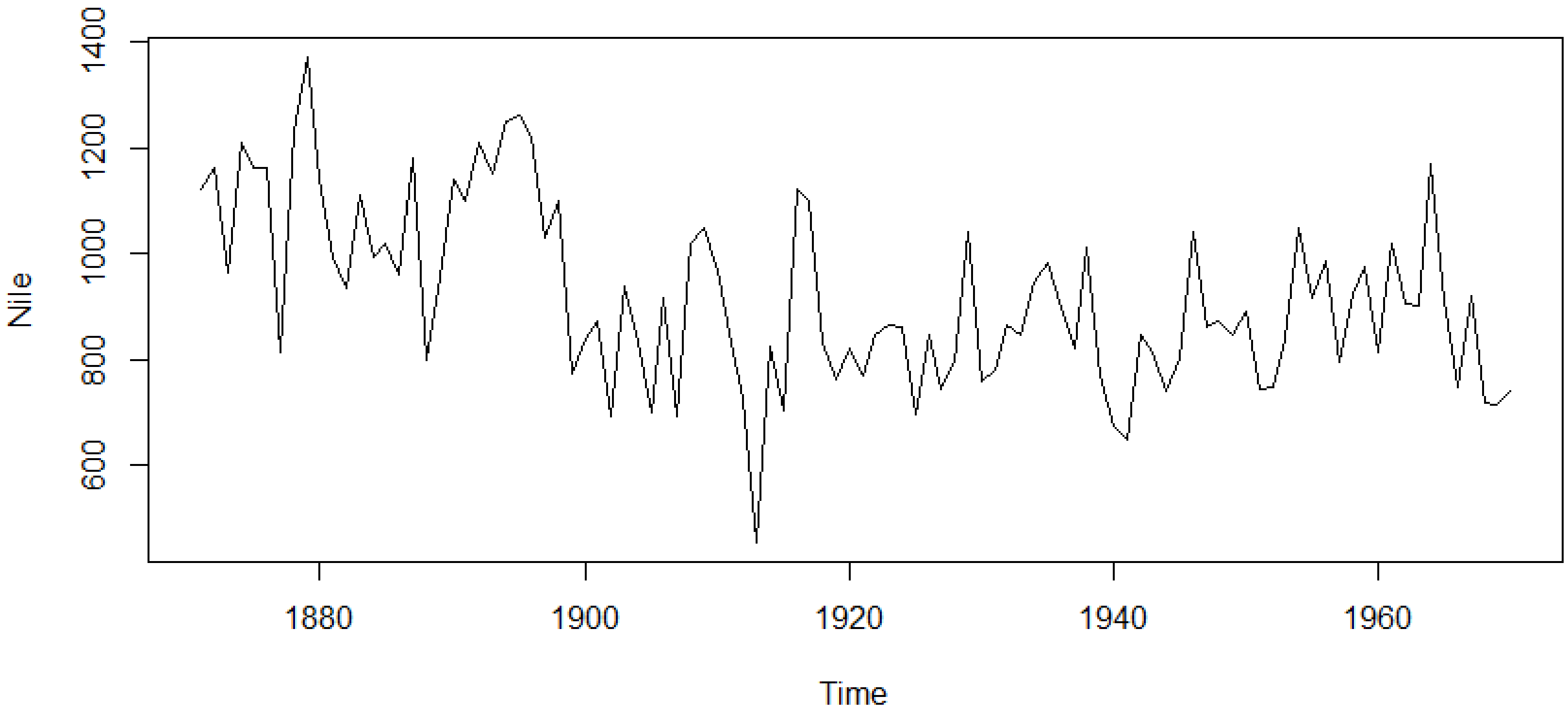
Forecast for 5 periods ahead



Model Selection

- The number of parameters (p,d,q) needed to fit, depends on the dataset
- There are techniques that automate model selection
- *auto.Arima* command in R picks the best p,d & q parameters for ARIMA(p,d,q)

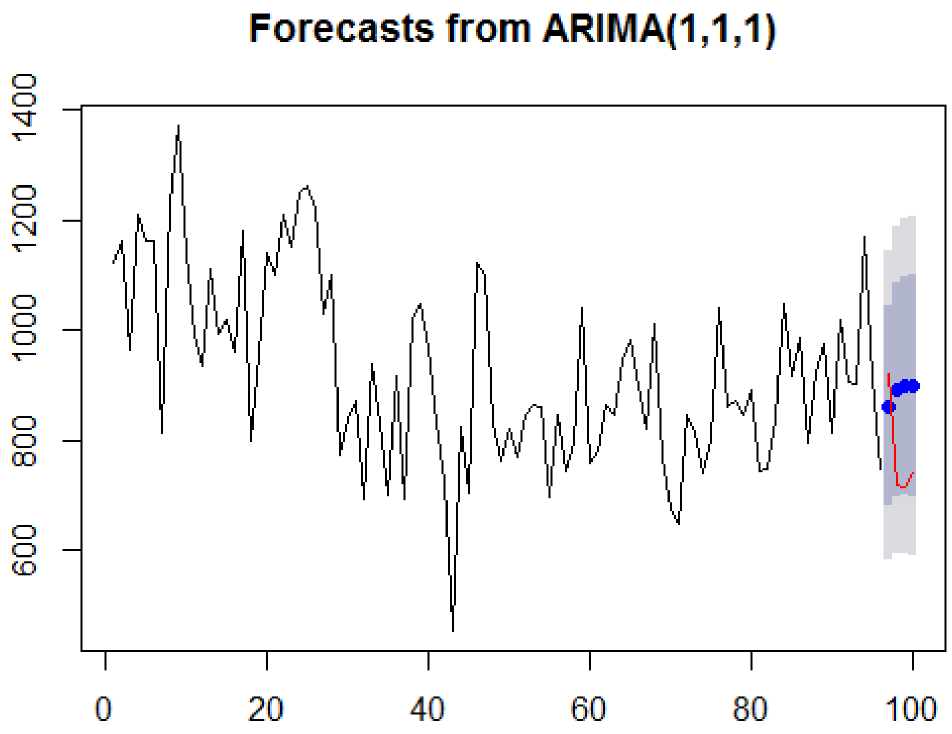
Auto.Arima: Annual Flow in River Nile




```
>
> plot(Nile)
>
```

Auto.Arima: Annual Flow in River Nile

```
> # Fit auto.arima to the first 96 points
> fitNile <- auto.arima(Nile[1:96])
> fitNile
Series: Nile[1:96]
ARIMA(1,1,1)

Coefficients:
         ar1      ma1
      0.2389  -0.8711
s.e.  0.1214   0.0585

sigma^2 estimated as 20372:  log likelihood=-605.59
AIC=1217.17   AICc=1217.44   BIC=1224.84
>
> #Now we predict last 4 points using the fit
> plot(forecast(fitNile,h=4))
> lines(97:100,Nile[97:100],col="red")
```



Model Identification

- Before Automated functions were available, one used to use ACF plots to determine the best value of (p,d,q) for a given dataset
- Box–Jenkins Methodology: Model identification and model selection
  - Make sure variables are stationary. Difference as necessary to get a constant mean and transformations to get constant variance.
  - Check for seasonality: Decays and spikes at regular intervals in ACF plots.
- Parameter estimation
  - Compute coefficients that best fit the selected model.
- Model checking
  - Check if residuals are independent of each other and constant in mean and variance over time (white noise).

http://www.ncss.com/wp-content/themes/ncss/pdf/Procedures/NCSS/The_Box-Jenkins_Method.pdf

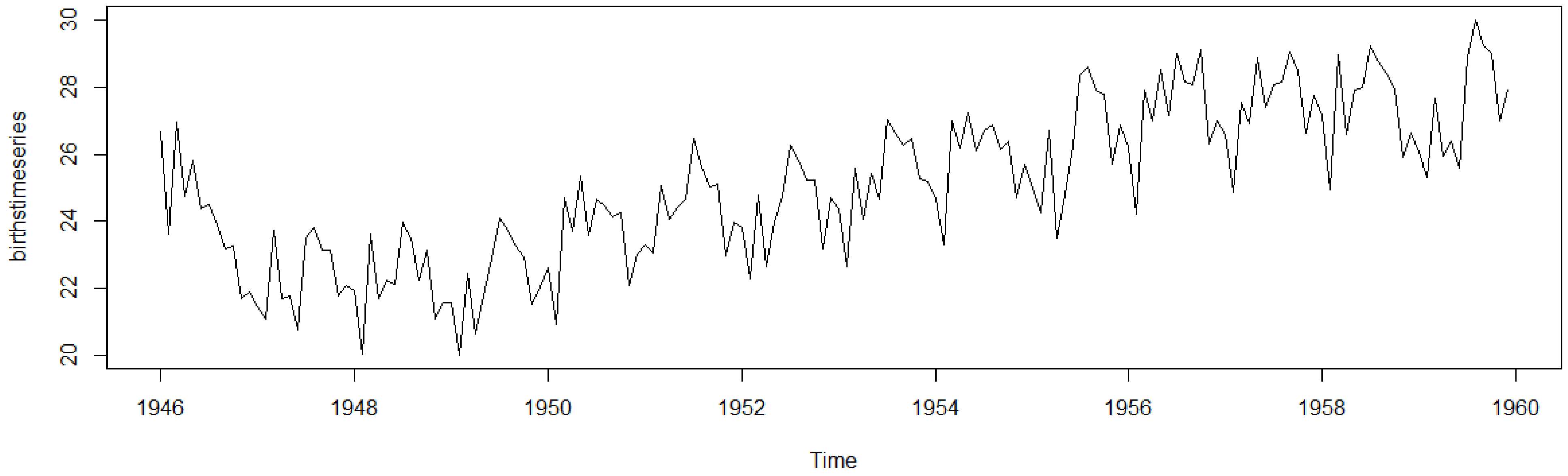
Model Selection

- Check ACF, PACF
- Identify important lag periods
- Create a data frame (table) with these past lag values as independent variables and value to be predicted as dependent variable
- Perform autoregression (AR models)
- To incorporate randomness, use MA

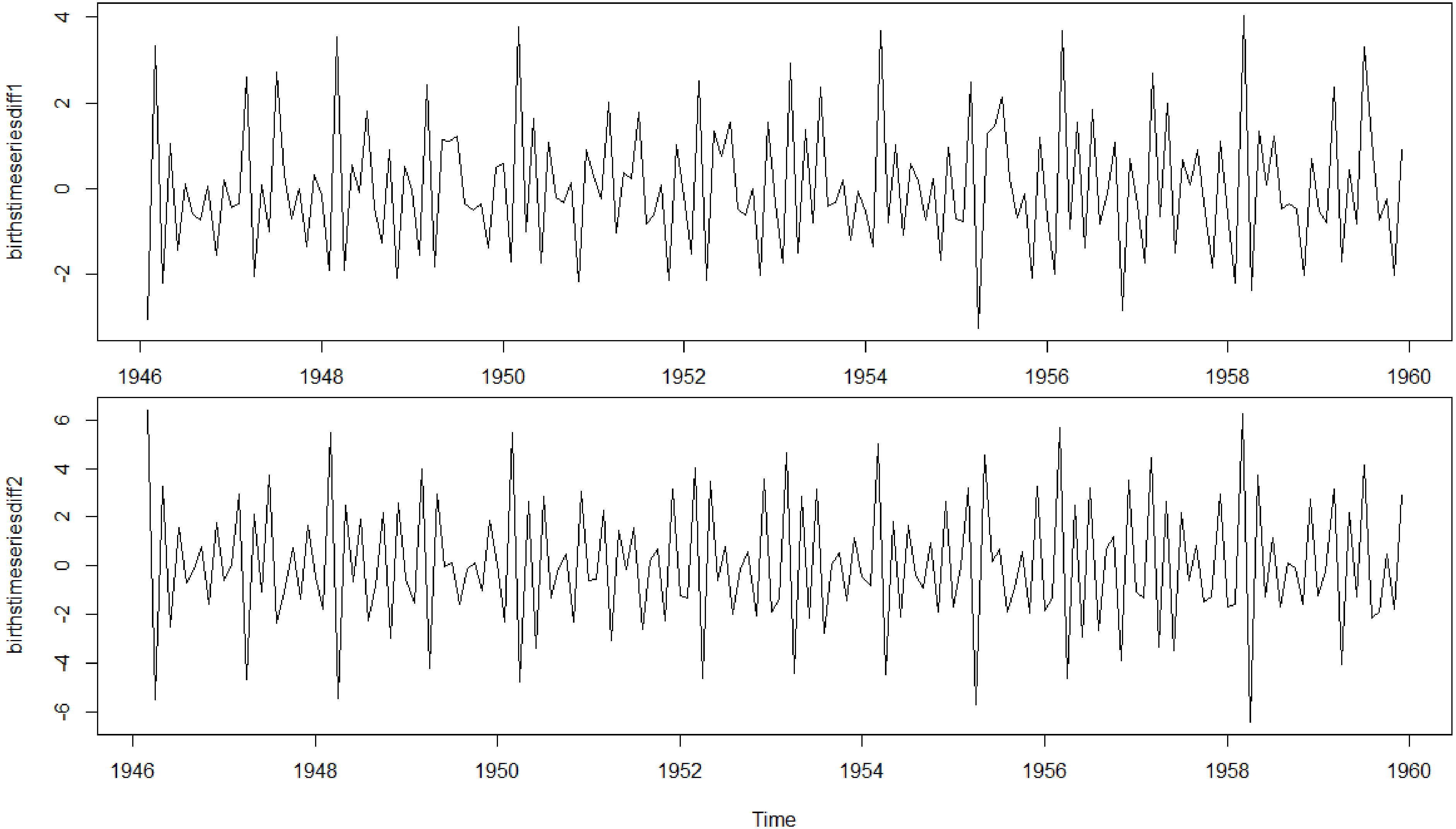
SHAPE	INDICATED MODEL
Exponential, decaying to zero	Autoregressive model. Use the partial autocorrelation plot to identify the order of the autoregressive model.
Alternating positive and negative, decaying to zero	Autoregressive model. Use the partial autocorrelation plot to help identify the order.
One or more spikes, rest are essentially zero	Moving average model, order identified by where plot becomes zero.
Decay, starting after a few lags	Mixed autoregressive and moving average model.
All zero or close to zero	Data is essentially random.
High values at fixed intervals	Include seasonal autoregressive term.
No decay to zero	Series is not stationary.

- Non-seasonal ARIMA models are denoted ARIMA(p,d,q)
- Seasonal ARIMA (SARIMA) models are denoted ARIMA(p,d,q)(P,D,Q)$_m$, where m refers to the number of periods in each season and (P,D,Q) refer to the autoregressive, differencing and moving average terms of the seasonal part of the ARIMA model.

Birth Timeseries: Stationary?



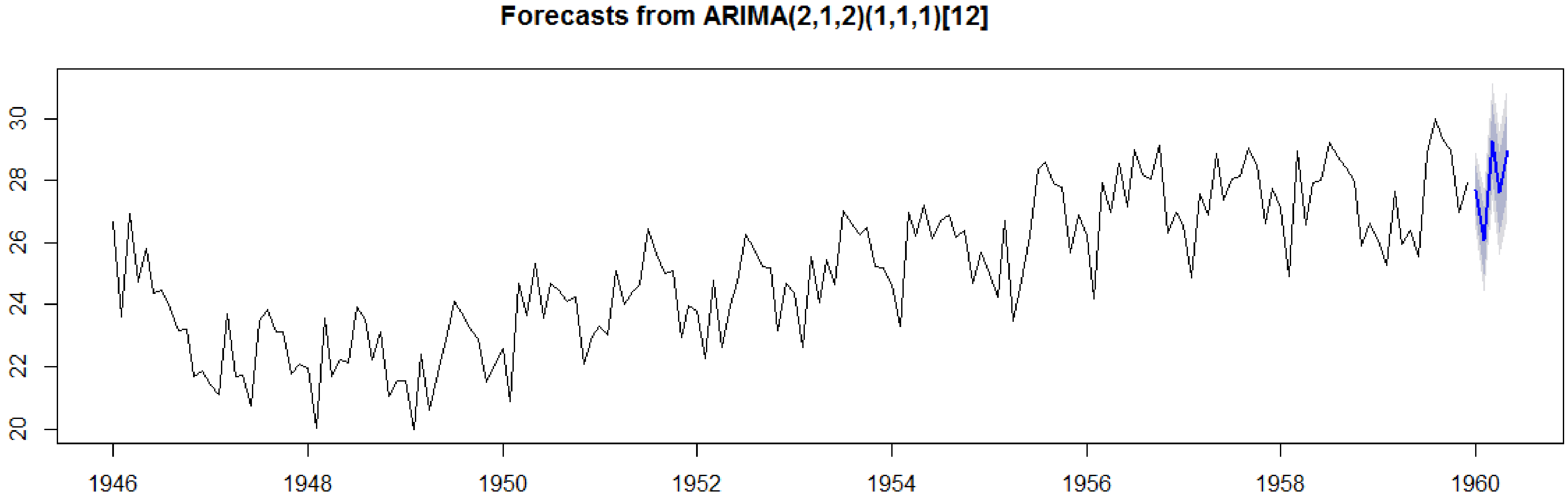
Differencing once vs twice



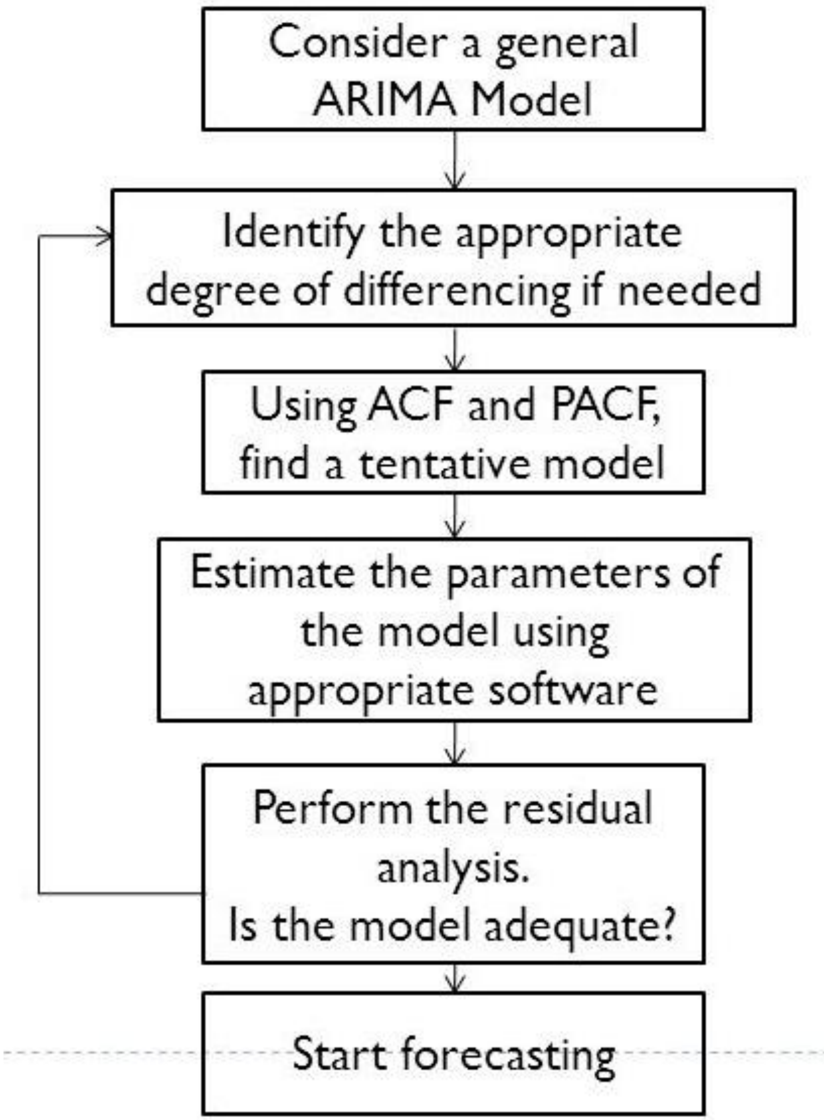
Seasonal ARIMA Model

```
Series: birthstimeseries
ARIMA(2,1,2)(1,1,1)[12]

Coefficients:
         ar1      ar2      ma1     ma2     sar1     sma1
      0.6539  -0.4540  -0.7255  0.2532  -0.2427  -0.8451
s.e.  0.3004   0.2429   0.3228  0.2879   0.0985   0.0995

sigma^2 estimated as 0.3918:  log likelihood=-157.45
AIC=328.91   AICc=329.67   BIC=350.21
>
```

Seasonal ARIMA Model - Forecast



Time Series Model Building Using ARIMA

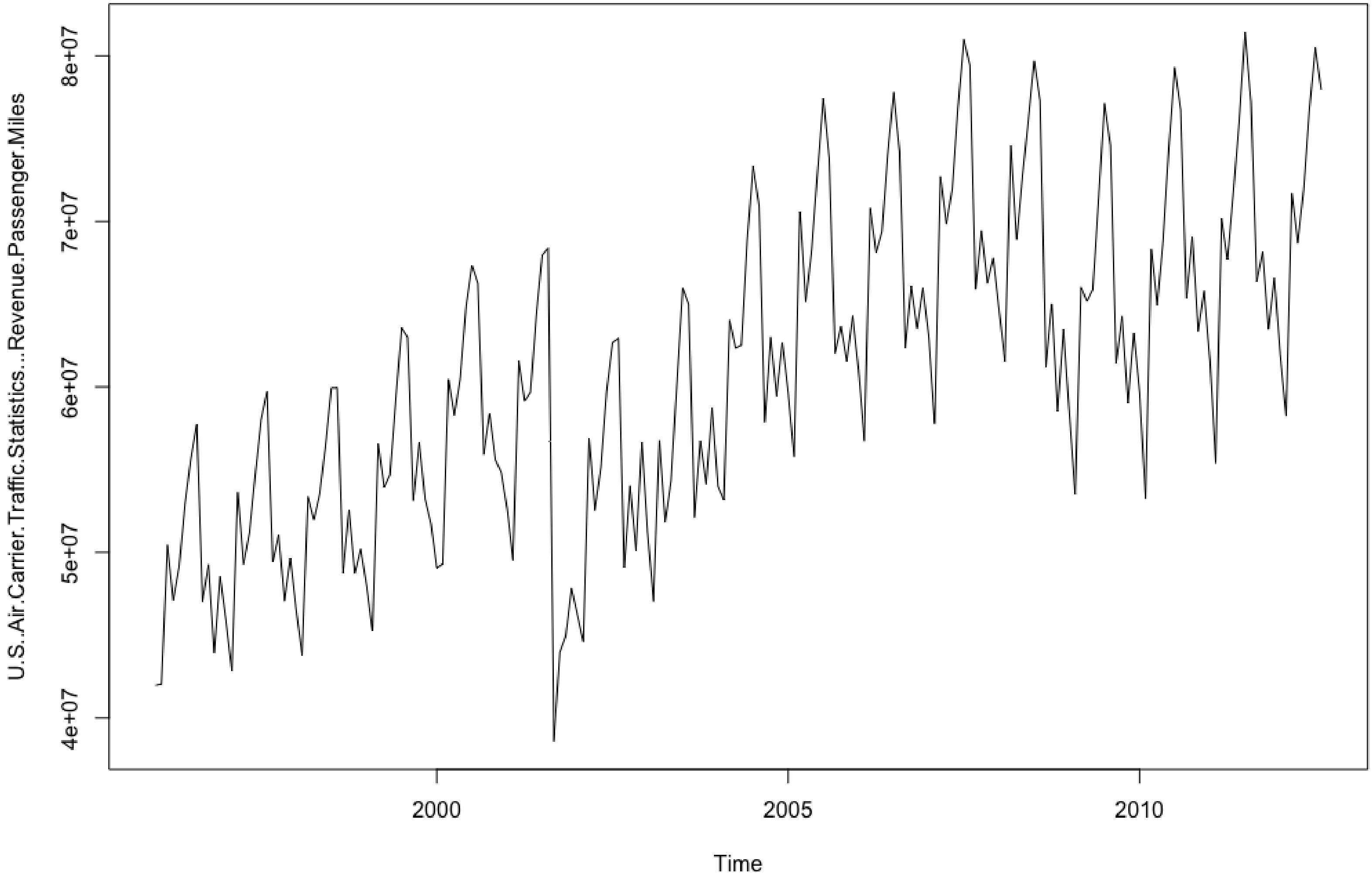


Time Series Model Building Using ARIMA

A nice summary of rules for identifying ARIMA models

http://people.duke.edu/~rnau/arimrule.htm

Time Series Model Building Using ARIMA - RPM



Time Series Model Building Using ARIMA - RPM

Auto ARIMA

```
Series: milestimeseries
ARIMA(1,0,1)(0,1,1)[12] with drift

Coefficients:
         ar1      ma1     sma1      drift
      0.9078  -0.2093  -0.7266  110280.44
s.e.  0.0364   0.0885   0.0682   31856.26

sigma^2 estimated as 3.901e+12:  log likelihood=-2994.93
AIC=5999.86   AICc=6000.19   BIC=6016.04
```

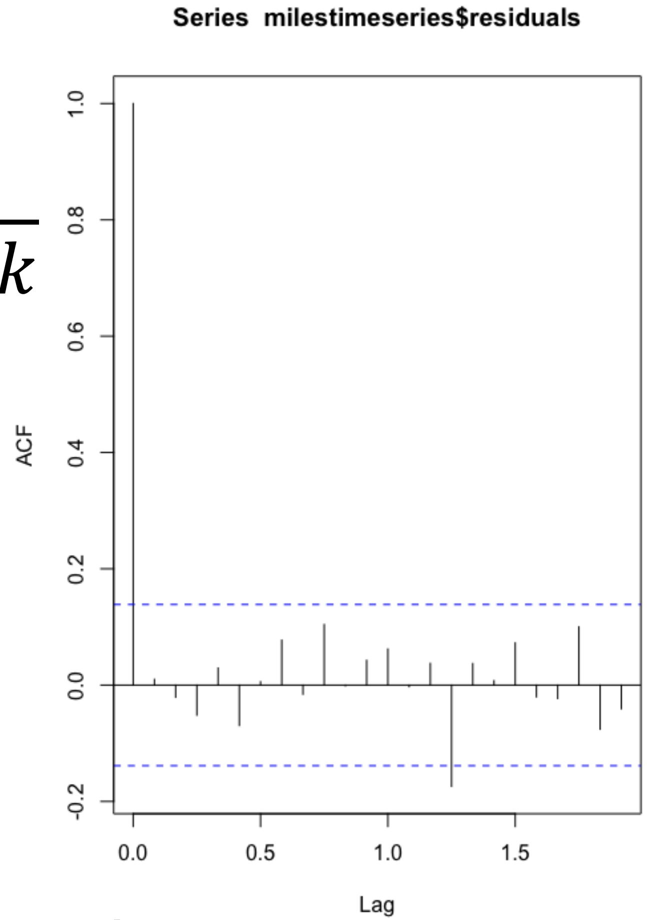
Time Series Model Building Using ARIMA - RPM

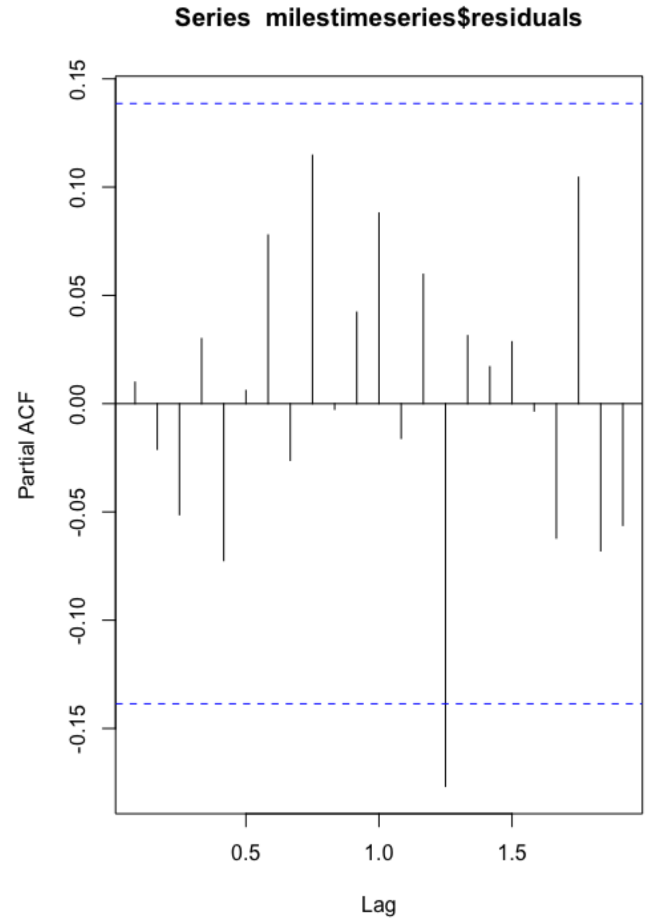
Residuals

$$Q^* = n(n+2)\sum_{k=1}^{h}\frac{{r_k}^2}{n-k}$$

h is the maximum lag being considered
n is the # of observations
r_k is the autocorrelation

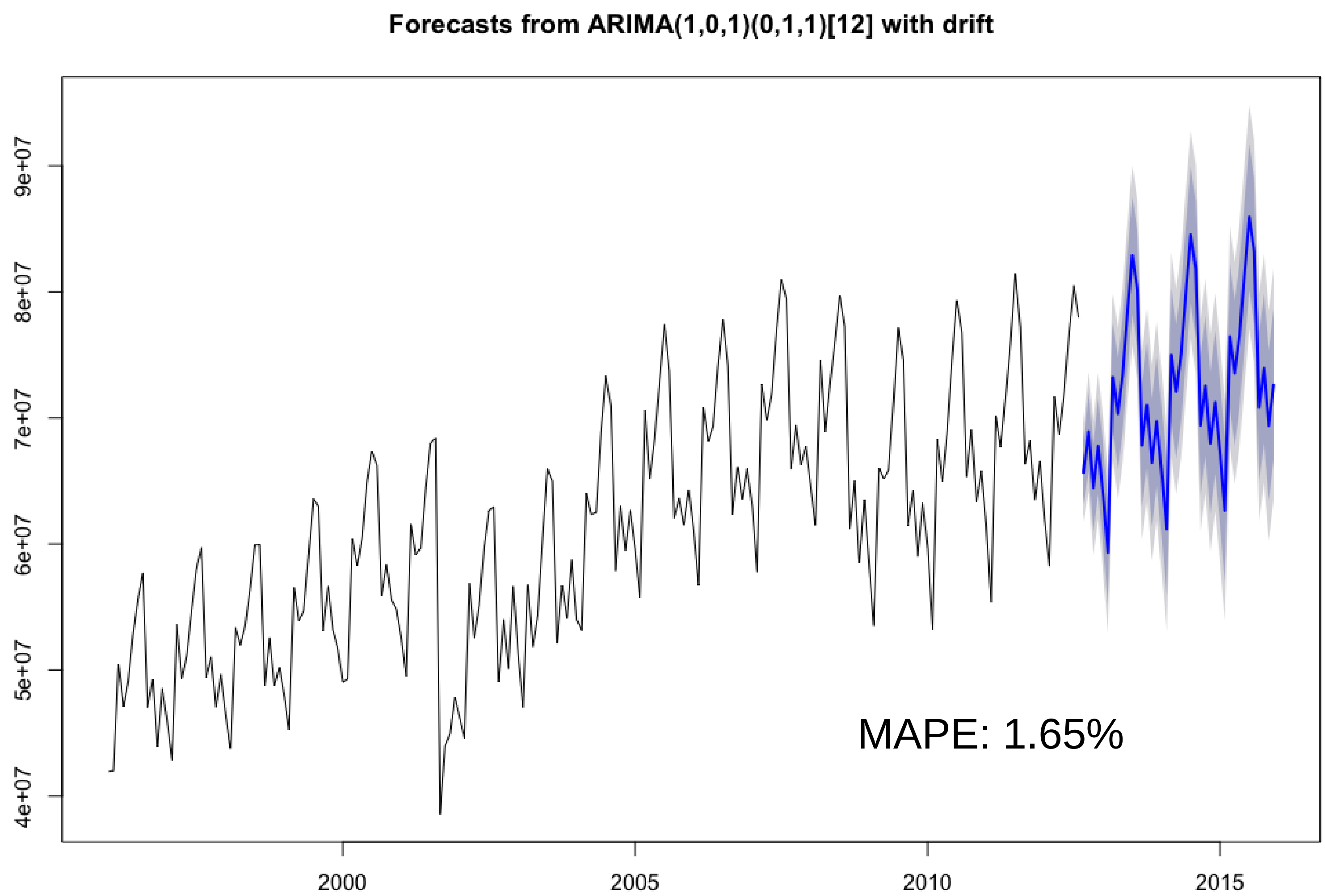
If residuals are white noise (purely random), then Q^* has a χ^2 distribution



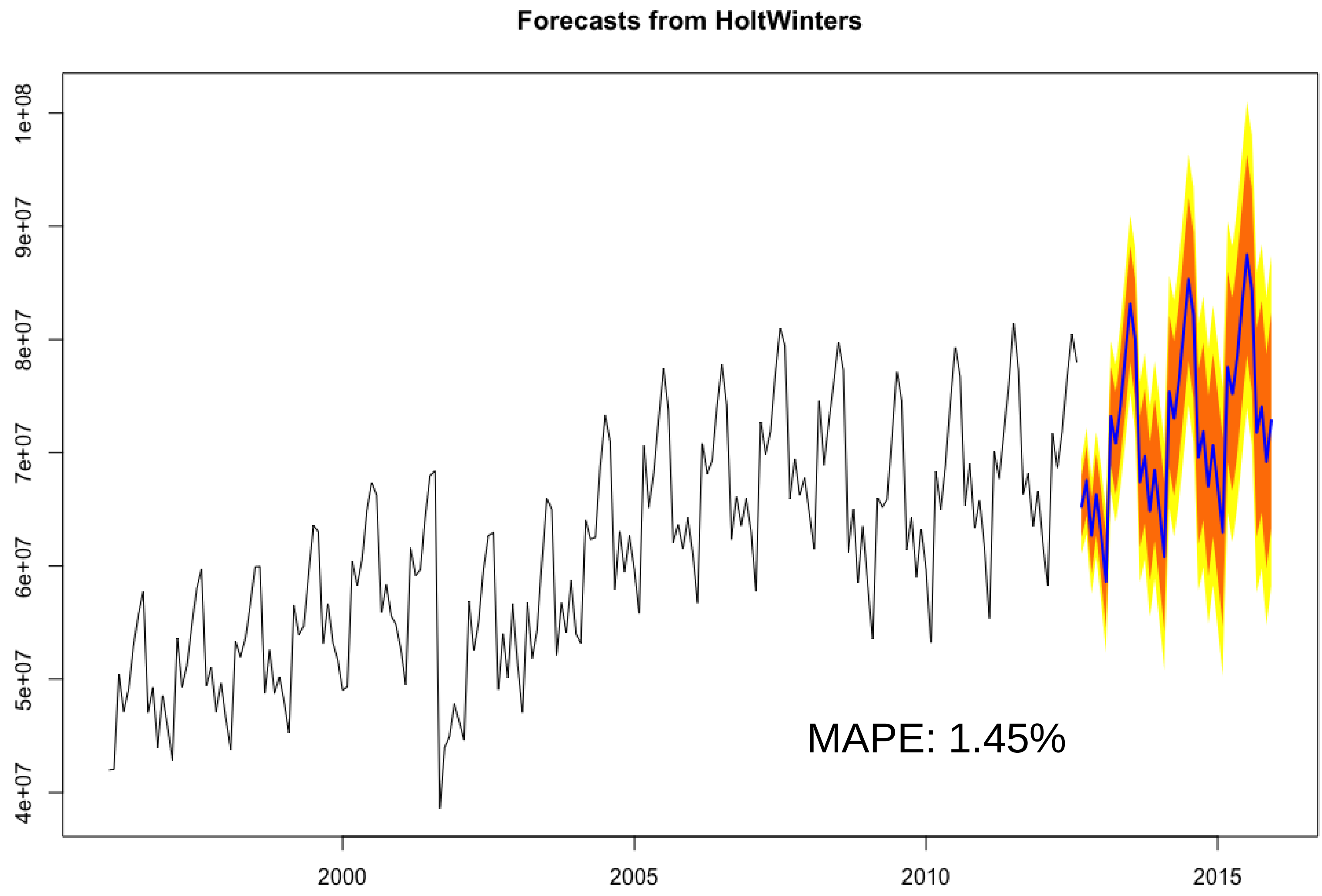



```
        Box-Ljung test

data:  milestimeseries$residuals
X-squared = 15.288, df = 20, p-value = 0.7597
```

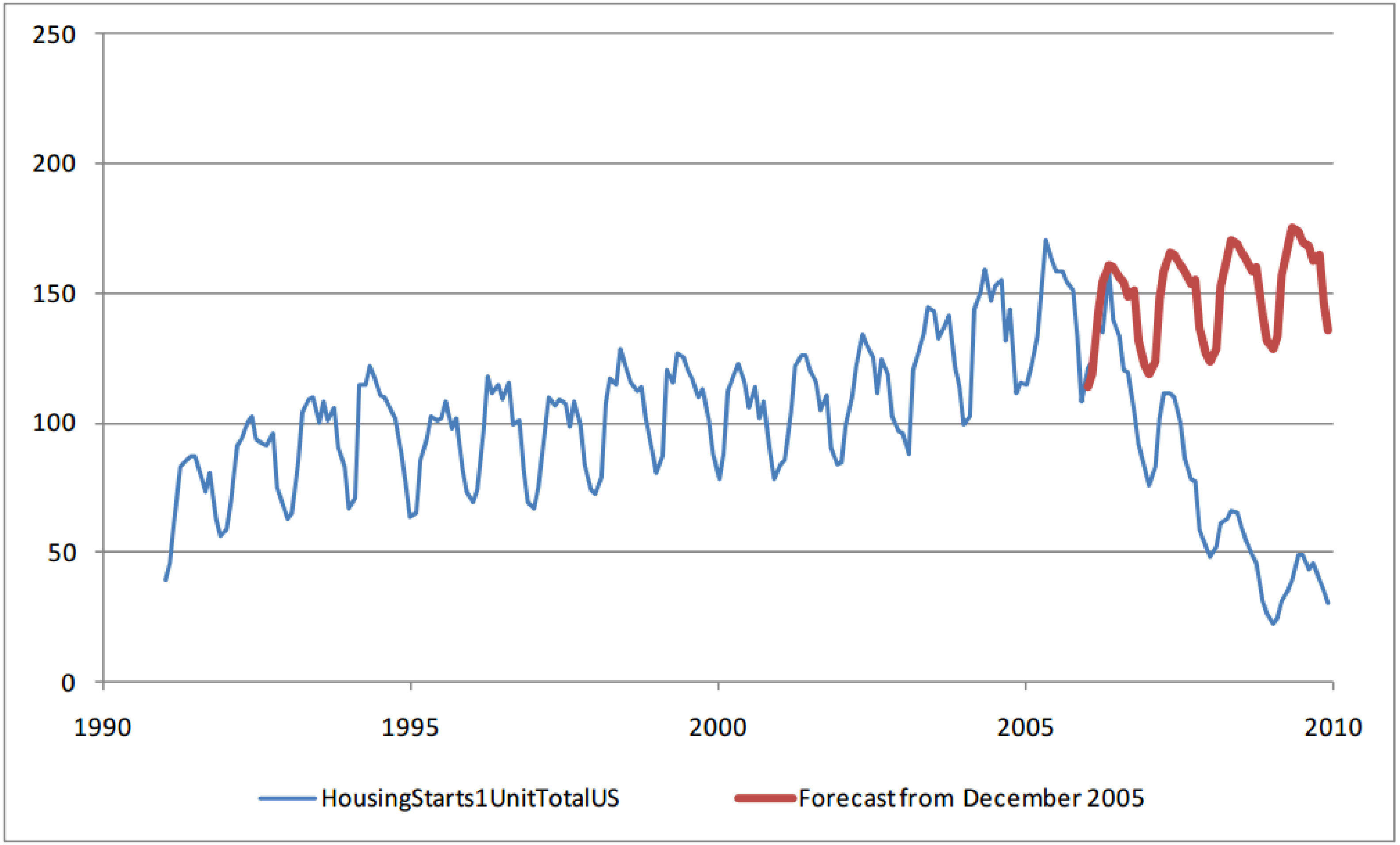
Time Series Model Building Using ARIMA - RPM Forecast



Forecast using Holt-Winters - RPM



Caution: Forecasting is Risky!



"Prediction is very difficult, especially if it's about the future."
--Niels Bohr, Nobel laureate in Physics

Resources

- https://www.otexts.org/fpp

An good open online book on Forecasting methods and practices

- http://a-little-book-of-r-for-time-series.readthedocs.io/en/latest/src/timeseries.html A short condensed summary on time-series
- https://www.analyticsvidhya.com/blog/2015/12/complete-tutorial-time-series-modeling/ A short tutorial on using ARIMA models