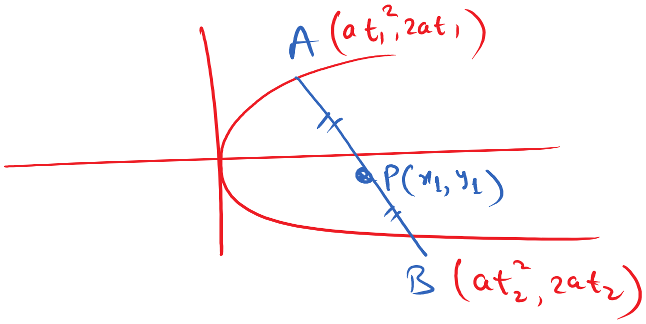
2023 GEN 1 AND 2 PARABOLA 3 AND PAIR OF STRAIGHT LINES

Locus of mid-point of Chord

Type- 2



$2x_1 = a(t_1^2 + t_2^2)$

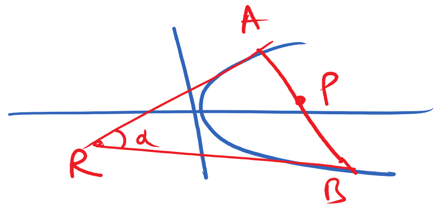
$\cancel{2}y_1 = \cancel{2}a(t_1 + t_2)$

$\Rightarrow \boxed{t_1 + t_2 = \dfrac{y_1}{a}}$ — (1)

$2x_1 = a\left[(t_1+t_2)^2 - 2t_1t_2\right]$

$\Rightarrow \boxed{t_1t_2 = \dfrac{y_1^2}{2a^2} - \dfrac{x_1}{a}}$ — (2)

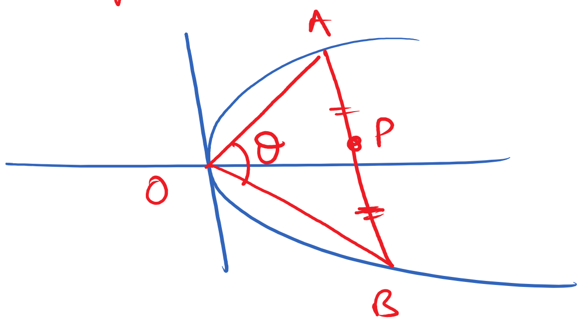
1. Find locus of P s.t. tangents drawn at end-points of chord include angle α



$$\tan\alpha = \frac{|t_2 - t_1|}{|1 + t_1t_2|}$$

$$\tan^2\alpha = \frac{(t_1+t_2)^2 - 4t_1t_2}{(1+t_1t_2)^2} \quad -(3)$$

2. Find locus of 'P' s.t. the chord subtends angle θ at vertex



$m_{OA} = \dfrac{2}{t_1}$

$m_{OB} = \dfrac{2}{t_2}$

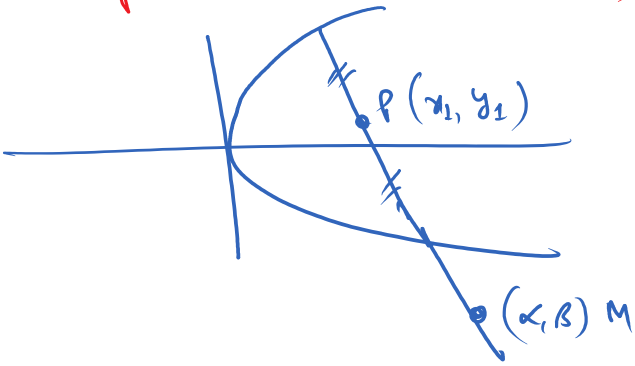
$$\tan\theta = \frac{\left|\dfrac{2}{t_1} - \dfrac{2}{t_2}\right|}{\left|1 + \dfrac{2}{t_1}\cdot\dfrac{2}{t_2}\right|}$$

— (3)

3. Find locus of P s.t. the chord passes through a fixed point $M \equiv (\alpha, \beta)$

$T =$

a fixed point $M \equiv (\alpha, \beta)$



$$T = S_1$$

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

(α, β)

$$\beta y_1 - 2a(\alpha + x_1) = y_1^2 - 4ax_1$$

4. Find locus of 'P' s.t. the chord touches $x^2+y^2=c^2$

Chord: $T = S_1$

$$yy_1 - 2a(x+x_1) = y_1^2 - 4ax_1$$

Cond^n. of tangency

$$\frac{|-2ax_1 + 4a^2 - y_1^2|}{\sqrt{y_1^2+(2a)^2}} = |c|$$

5. Locus of point 'P' s.t. the chord is || to $y = mx + c$

$$m = \frac{2a}{y_1} \Rightarrow y = \frac{2a}{m}$$

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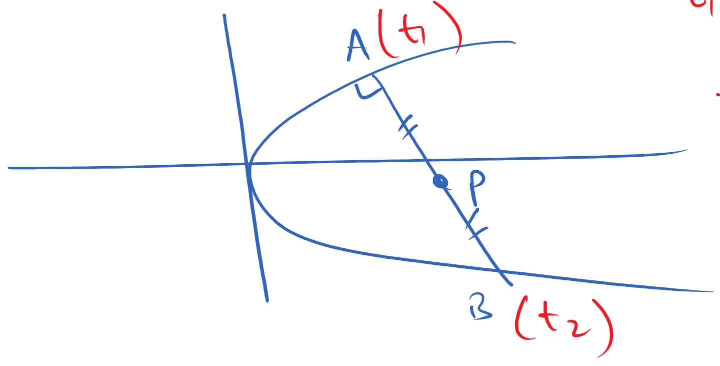
6. Find locus of P s.t. the chord is itself a normal chord.



$t_1 + t_2 = \frac{y_1}{a}$ — (1)

$t_1 t_2 = \frac{y_1^2}{2a^2} - \frac{x_1}{a}$ — (2)

$t_2 = -t_1 - \frac{2}{t_1}$ — (3)

7. Find locus of P s.t. the length of chord is 'l'.

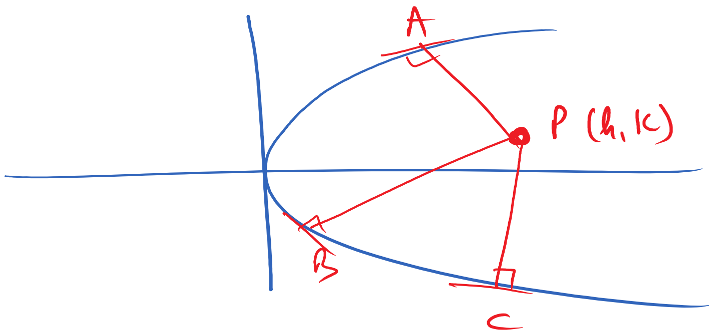
— (1)

— (2)

$l = a|t_2 - t_1| \sqrt{(t_1+t_2)^2+4}$

$l^2 = a^2\left[(t_1+t_2)^2 - 4t_1t_2\right]\left[(t_1+t_2)^2+4\right]$ — (3)

Co-Normal points



For normal,
$m = -t$
or $t = -m$

$A \equiv (at_1^2, 2at_1)$

$A \equiv (am_1^2, -2am_1)$

$B \equiv (am_2^2, -2am_2)$

$C \equiv (am_3^2, -2am_3)$

Eq^{n}. of normal:- $y = mx - 2am - am^3$

$k = mh - 2am - am^3$

$\Rightarrow am^3 + 0m^2 + m(2a-h) + k = 0$

Cubic in m, so max 3 Normals from a point

(1) — $m_1 + m_2 + m_3 = 0$ $\Rightarrow -2am_1 - 2am_2 - 2am_3 = 0$

$\Rightarrow y_A + y_B + y_C = 0$.

Learn,

(2) — $m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a-h}{a}$

(3) — $m_1m_2m_3 = \dfrac{-k}{a}$

Illustration - 19 *The locus of a point, such that two of the three normals drawn from it to the parabola* $y^2 = 4ax$ *are perpendicular is :*

(A) $y^2 = a(x - 3a)$ **(B)** $y^2 = a(x + 3a)$ **(C)** $x^2 = a(y - 3a)$ **(D)** $x^2 = a(y + 3a)$

$m_1 + m_2 + m_3 = 0$ — (1)

$m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a-h}{a}$ — (2)

$m_1m_2m_3 = -\dfrac{k}{a}$ — (3)

(extra) $m_1m_2 = -1$ — (4)

Use (4) in (3) $m_3 = \dfrac{k}{a}$ — (5)

(1) $\Rightarrow m_1 + m_2 = -m_3$

(2) $\Rightarrow (m_1 + m_2)m_3 + (-1) = \underline{2a-h}$

$$\Rightarrow \quad -m_3^2 - 1 = \frac{2a-h}{a}$$

$$\Rightarrow \quad -\frac{k^2}{a^2} - 1 = \frac{2a-h}{a}$$

$$\Rightarrow \quad \frac{-y^2}{a^2} - 1 = \frac{2a-x}{a}$$

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Illustration - 20 *Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point $(h, 0)$. then :*

$a = 1$

(A) $h < 2$ (B) $h > 2$ (C) $h < 3$ (D) $h > 3$

$$m_1 + m_2 + m_3 = 0$$

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{2-h}{1}$$

$$m_1 m_2 m_3 = \frac{-k}{1}$$

$$(m_1 + m_2 + m_3)^2 = \Sigma m_1^2 + 2\Sigma m_1 m_2$$

$$\Rightarrow 0 = \Sigma m_1^2 + 2(2-h)$$

$$\Rightarrow 0 > 2(2-h)$$

$$\Rightarrow 2-h < 0 \Rightarrow h > 2$$

$\boxed{\text{generalising } \quad h > 2a}$

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Pair of straight lines

$$f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

→ If $\Delta = 0$

$h^2 > ab$ pair of distinct straight lines

$h^2 = ab$ — — coincident lines

$h^2 < ab$ single point

→ If $\Delta \neq 0$

$h^2 = ab$ parabola

$h^2 < ab$ ellipse

$h^2 > ab$ hyperbola

→ Centre of conic $\left(\dfrac{bg - hf}{h^2 - ab}, \dfrac{af - gh}{h^2 - ab}\right)$

OR take partial derivatives w.r.t. x & y and solve

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Pair of lines passing through origin

M-1 $\quad ax^2 + 2hxy + by^2 = 0 \qquad$ Divide by x^2

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0 \qquad \frac{y}{x} = m$$

($\because$ lines pass through origin)

$$bm^2 + 2hm + a = 0$$

$$m_1, m_2 = \frac{-2h \pm 2\sqrt{h^2 - ab}}{2b}$$

$$m_1 = \frac{-h + \sqrt{h^2 - ab}}{b} \qquad m_2 = \frac{-h - \sqrt{h^2 - ab}}{b}$$

$h^2 > ab \Rightarrow$ 2 different lines

$h^2 = ab \Rightarrow$ 2 same lines

$h^2 < ab \Rightarrow$? ?

$$\boxed{m_1 + m_2 = \frac{-2h}{b}} \qquad \boxed{m_1 m_2 = \frac{a}{b}}$$

Angle b/w lines $\quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \Rightarrow \tan^2\theta = \frac{(m_1+m_2)^2 - 4m_1m_2}{(1+m_1m_2)^2}$

$\perp$ lines $\Rightarrow m_1 m_2 = -1 \Rightarrow \frac{a}{b} = -1 \Rightarrow a + b = 0$

$\Rightarrow$ coeff. of x^2 + coeff. of $y^2 = 0$.

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a+b|} \qquad \theta \text{ is acute}$$

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M-2

$$ax^2 + 2hxy + by^2 = 0 \quad — (1) \qquad (1) \equiv (2)$$

$$(y - m_1 x)(y - m_2 x) = 0 \quad — (2)$$

$$\Rightarrow y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0 \quad — (2)$$

$$\frac{1}{b} = \frac{-(m_1 + m_2)}{2h} = \frac{m_1 m_2}{a}$$

$$\Rightarrow m_1 + m_2 = -\frac{2h}{b} \qquad m_1 m_2 = \frac{a}{b}$$

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1.4 Angle bisector of lines

$ax^2 + 2hxy + by^2 = 0.$

Let m_1, m_2 be the slopes of lines $ax^2 + 2hxy + by^2 = 0$.

$\Rightarrow$ lines are $y - m_1 x = 0$ and $y - m_2 x = 0$ $\quad$ $[m_1 + m_2 = -2h/b$ and $m_1 m_2 = a/b]$

$\Rightarrow$ bisectors of angles are :

$$\frac{y - m_1 x}{\sqrt{1 + m_1^2}} = \pm\left(\frac{y - m_2 x}{\sqrt{1 + m_2^2}}\right) \Rightarrow \left(1 + m_2^2\right)(y - m_1 x)^2 - \left(1 + m_1^2\right)(y - m_2 x)^2 = 0$$

On simplification we get :

$$-y^2 (m_1 + m_2) + x^2 (m_1 + m_2) - 2xy (1 - m_1 m_2) = 0$$

$$\Rightarrow \quad x^2 - y^2 = \frac{2xy\,(1 - m_1 m_2)}{m_1 + m_2} \quad \Rightarrow \quad x^2 - y^2 = \frac{2xy\,(1 - a/b)}{-2h/b}$$

$\Rightarrow$ The equation of Bisectors is $\boxed{\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}}$.

1.5 Pair of lines perpendicular to the lines

Let L_1 and L_2 be the lines $ax^2 + 2hxy + by^2 = 0$.

Let P_1 be the line Perpendicular to L_1 and P_2 be the line perpendicular to L_2.

We have to find equation of $P_1 P_2$.

Let L_1 be $y - m_1 x = 0$ and L_2 be $y - m_2 x = 0$

$\Rightarrow$ P_1 is $m_1 y + x = 0$ and P_2 is $m_2 y + x = 0$

$\Rightarrow$ Pair $P_1 P_2$ is $(m_1 y + x) \cdot (m_2 y + x) = 0$

$$\Rightarrow \quad m_1 m_2 y^2 + xy\,(m_1 + m_2) + x^2 = 0$$

$$\Rightarrow \quad \frac{a}{b} y^2 + xy\left(-\frac{2h}{b}\right) + x^2 = 0$$

$\Rightarrow$ $bx^2 - 2hxy + ay^2 = 0$ is the equation of the pair of lines perpendicular to the pairs of lines $ax^2 + 2hxy + by^2 = 0$.

Note : By interchanging the coefficients of x^2 and y^2 and reversing the sign of the xy term, we can get equation of $P_1 P_2$ from $L_1 L_2$.

Pair of lines NOT passing through origin

Illustration - 4 *If the equation* $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$ *represents a pair of lines. Then the equations of each line.*

(A) $4x + 3y - 7 = 0$ (B) $5x - 3y - 2 = 0$ (C) $3x + 4y - 7 = 0$ (D) $2x - 3y + 2 = 0$

First consider homogenous part

$$6x^2 - xy - 12y^2 = 0$$

Factorise using middle term splitting or assume quadratic in x

$$6x^2 - 9xy + 8xy - 12y^2 = 0 \Rightarrow 3x(2x-3y) + 4y(2x-3y) = 0$$

$$\Rightarrow (3x+4y)(2x-3y) = 0$$

Now, $(3x+4y+c_1)(2x-3y+c_2) = 6x^2 - xy - 12y^2 - 8x + 29y - 14$

Compare coeff. of x & y to c_1 & c_2

and verify the const. to confirm it is pair of lines.

coeff. of x: $3c_2 + 2c_1 = -8$

" " y: $4c_2 - 3c_1 = 29$

const: $c_1 c_2 = -14$ (for verificn.)

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Illustration - 8 *If the equation of conic $2x^2 + xy + 3y^2 - 3x + 5y + \lambda = 0$ represents a single point, then find the value of λ.*

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

centre of conic | intersection of imaginary lines

If $h^2 < ab$

only centre is visible in case of $\Delta = 0$

$\Delta = 0$ $h = \frac{1}{2}$ $h^2 = \frac{1}{4}$ $a = 2$ $b = 3$

$h^2 < ab$ ✓

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2 & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & 3 & \frac{5}{2} \\ -\frac{3}{2} & \frac{5}{2} & \lambda \end{vmatrix} = 0$$

If we forget formula of centre of conic, how to take partial derivative

$f(x,y) = 2x^2 + xy + 3y^2 - 3x + 5y + \lambda$

$\frac{\partial f}{\partial x} = 4x + y - 3 = 0$

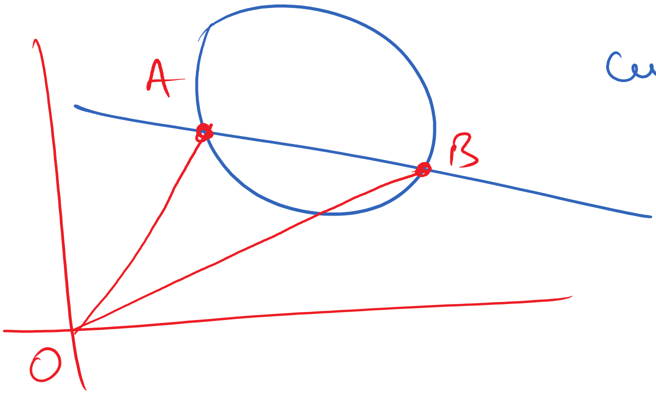
$\frac{\partial f}{\partial y} = 0 + x + 6y + 5 = 0$

$\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$

Homogenisn

→ How to homogenise? (Look for a suitable '1')

→ what does it represent? (pair of lines passing through origin)



Curve and a line passing through it

↳ $x^2+y^2+2gx+2fy+c=0$

↳ $lx+my+n=0$

$lx+my=-n$

$\dfrac{lx+my}{-n}=1$

$$x^2+y^2+2gx\cdot 1+2fy\cdot 1+c\cdot 1^2=0$$

$$(1) \; - \; x^2+y^2+2gx\cdot\left(\frac{lx+my}{-n}\right)+2fy\cdot\left(\frac{lx+my}{-n}\right)+c\cdot\frac{(lx+my)^2}{n^2}=0$$

For A & B $\dfrac{lx+my}{-n}=1$ and so the above eqn. is satisfied.

$(0,0)$ also satisfies above eqn.

Also comparing with $ax^2+2hxy+by^2+2gx+2fy+c=0$

$f=0 \quad g=0 \quad c=0$

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ \cancel{g}_{0} & \cancel{f}_{0} & \cancel{c}_{0} \end{vmatrix}=0 \Rightarrow \text{Eqn. (1) is pair of st. lines.}$$

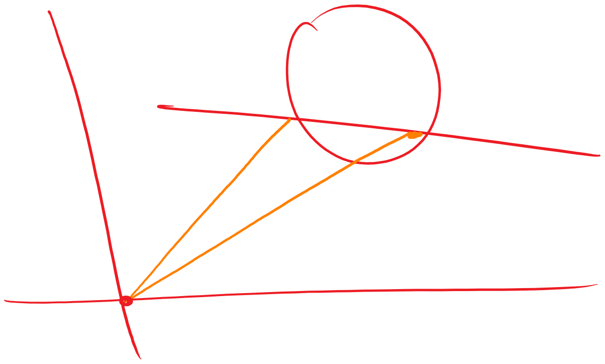
Homogenised eqn. represents pair of st. lines connecting origin with points of intn. of curve & line

Illustrating the concept :

Find the equation of the lines joining the origin to the points of intersection of the line $4x - 3y = 10$ *with the circle* $x^2 + y^2 + 3x - 6y - 20 = 0$ *and show that they are perpendicular.*

$C \equiv \left(-\frac{3}{2}, 3\right) \qquad r = \sqrt{\frac{9}{4} + 9 + 20} = \frac{5\sqrt{5}}{2}$

$$CP = \frac{\left|4\left(-\frac{3}{2}\right) - 3 \times 3 - 10\right|}{5} = 5 < \frac{5\sqrt{5}}{2}$$



$$\frac{4x - 3y}{10} = 1$$

$$x^2 + y^2 + 3x \cdot 1 - 6y \cdot 1 - 20 \cdot 1^2 = 0$$