Rotation 10

By

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Angular impulse and Impulse momentum theorems

I: moment of inertia.

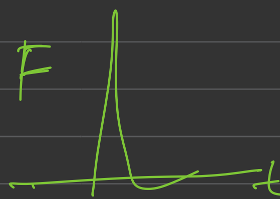
$\mathcal{J} = \int F\,dt$

Linear impulse

During collisions, extremely large magnitude forces act for extremely small durations.

$F \to 10^{10}$

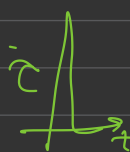
$\Delta t \to 10^{-10}$. $\left(\begin{matrix} F \to \infty \\ \Delta t \to 0 \end{matrix} \right)$



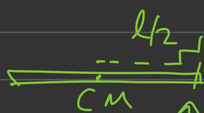
Angular Impulse $= \int \tau\, dt$ is created by impulsive forces creating impulsive torques.

$$J = \int F r_\perp\, dt = r_\perp \int F\,dt$$

$\tau = F r_\perp$.



$$\boxed{\begin{matrix} J = \mathcal{J}\, r_\perp \\ \text{like } \tau = F r_\perp \end{matrix}}$$



sudden hit

$r_\perp =$ finite

$\tau = F r_\perp$.

Direction of angular impulse about A will be imagined by swinging $\mathcal{J}$ vector around A.

We calculate J by looking at Impulse diagram

$$\mathcal{J} = \int F\,dt = \int \frac{dp}{dt}\,dt = \int dp = \Delta p \Rightarrow \boxed{\bar{\mathcal{J}} = \Delta \bar{p}} \checkmark$$

$$J = \int \tau\, dt = \int \frac{dL}{dt}\,dt = \int dL = \Delta L \Rightarrow \boxed{\bar{J} = \Delta \bar{L}} \checkmark$$

Impulse–Momentum theorems.

$\bar{J}_P = \Delta \bar{L}_P$ is applied about a point P which remains stationary before/during/after impact.

$\vec{J}_{CM} = \Delta \vec{L}_{CM}$ about CM.

$$\boxed{J_P = I_P \omega_f - I_P \omega_i \quad \text{OR} \quad \vec{J}_{CM} = I_{CM}\omega_f - I_{CM}\omega_i}$$

$\tau_P = I_P \alpha$ OR $\tau_{CM} = I_{CM}\alpha$.

$L_P = I_P \omega$ About st. pt. P.

$L_{CM} = I_{CM}\omega$ About CM

$L_A = I_{CM}\omega + M V_{CM} r_\perp$ General.

RIGID BODY

$\vec{P} = M \vec{V}_{CM}$

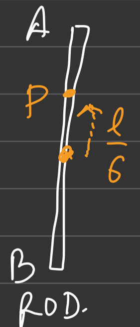
particle

$\vec{p} = m\vec{v}$

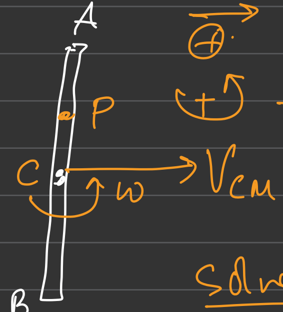
Illustration - 26 *A uniform rod AB of mass m and length l is at rest on a smooth horizontal surface. An impulse I is applied to the end B perpendicular to the rod in horizontal direction. Speed of particle P at a distance l/6* from the centre towards *A of the rod at time* $t = \frac{\pi m l}{12 I}$ *is :*

(A) $2\frac{I}{m}$ (B) $\frac{I}{\sqrt{2}m}$ (C) $\frac{I}{m}$ (D) $\sqrt{2}\frac{I}{m}$

SOLUTION :



before hit



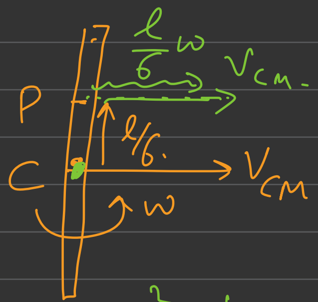
After hit.

$\overrightarrow{\oplus}$: $+J = +MV_{CM} - 0$.

$\circlearrowleft(+)$: $+J\frac{l}{2} = \frac{Ml^2}{12}\omega - 0$. (about CM)

Solve: $V_{CM} = \frac{J}{M}$

$\omega = \frac{6J}{Ml}$.



just after impact.

V of P $= V_{CM} + \frac{l}{6}\omega$

$= \frac{J}{M} + \frac{l}{6}\frac{6J}{Ml}$

$= 2J/M$.

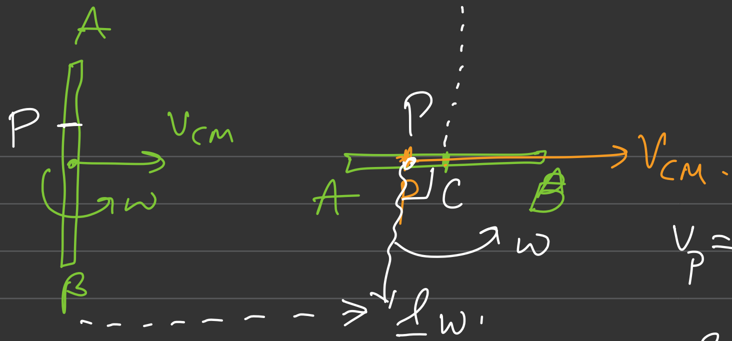
$K_i = 0$ $\quad K_f = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$

At $t = \frac{\pi Ml}{12J}$.

$t = 0$ to $t = \frac{\pi Ml}{12J}$

$S_{CM} = V_{CM}t$

$\Delta\theta = \omega\Delta t = \frac{6\cancel{J}}{\cancel{Ml}} \cdot \frac{\pi \cancel{Ml}}{12\cancel{J}} = \frac{\pi}{2}$



$$v_p = \sqrt{v_{cm}^2 + \frac{l^2\omega^2}{36}}$$

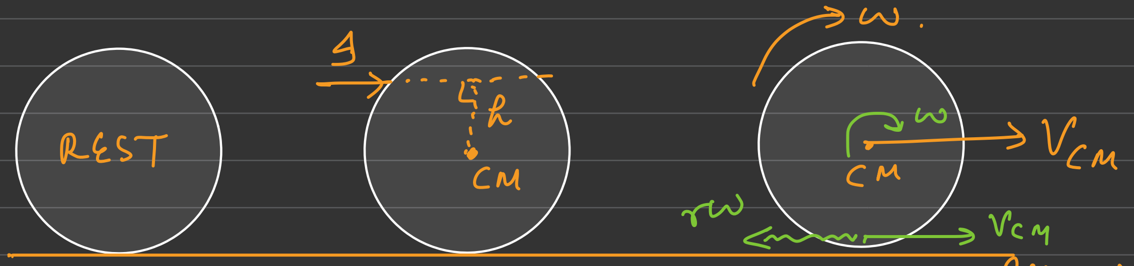
solve for v_p.

$$S_{CM} = v_{cm}t = \frac{g}{M} \cdot \frac{\pi M l}{12 g} = \frac{\pi l}{12}$$

$$\bar{S}_p = \frac{l}{12}(\pi - 2)\hat{i} - \frac{l}{6}\hat{j}.$$

$$\left.\begin{aligned} S_x &= \frac{\pi l}{12} - \frac{l}{6} \\ S_y &= -l/6. \end{aligned}\right]$$

Illustration - 25 *A billiard ball, initially at rest, is given a sharp impulse by a rod. The rod is held horizontally at a height h above the centre of the ball. The ball immediately begins to roll without slipping after the impact. Calculate the height h in terms of the radius of the ball.*



Pure rolling begins just after hit.

$+J = mV_{CM} - 0$

$+Jh = \frac{2}{5}Mr^2\omega - \frac{2}{5}Mr^2(0)$

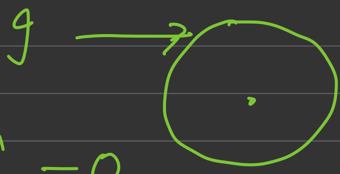
About CM:

$V_{CM} = r\omega$

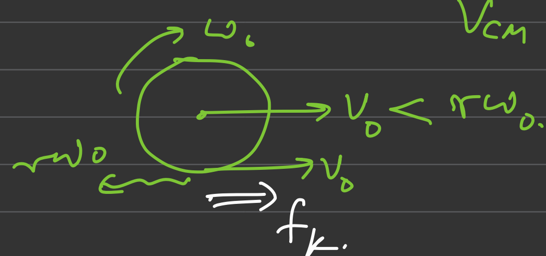
Solve to get $h = 2r/5$.

in general $h = (k^2/r^2)\, r$ to cause pure rolling immediately after hit.

For $h > 2r/5$ (for solid ball).



rω will be more than V_{CM}

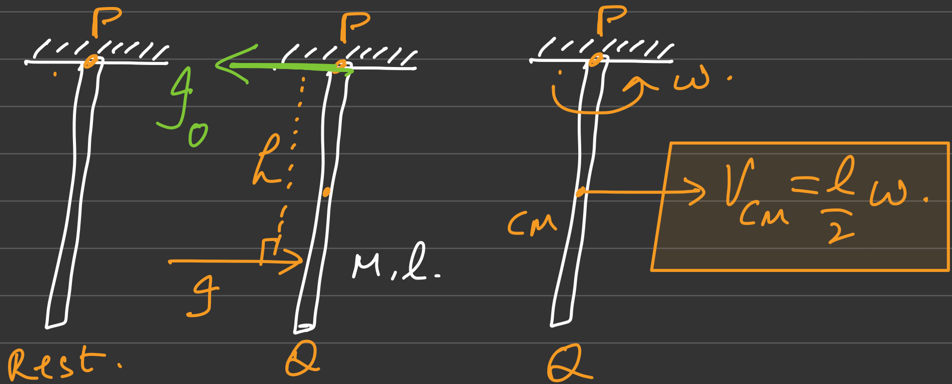


$J = mV_{CM} - 0$

$Jh = I_{cm}\omega$

increased.

A thin rod of mass M and length L is hanging from a fixed support and is hit by an impulse J acting at a point distant h from fixed end perpendicular to its length. Calculate the impulse acting on rod at the fixed end and angular velocity of rod after the hit.



Due to hit, upper end P tries to move horizontally but pivot/support hinge exerts horizontal impulse at fixed end to keep P at rest.

Let us now apply Impulse Momentum theorem.

$\xrightarrow{}$ (+):

$$\boxed{+J - J_0 = MV_{CM} - M(0).}$$

(+) About P.

$$\boxed{Jh = \frac{Ml^2}{3}\omega - \frac{Ml^2}{3}(0)}$$

$(L_P = I_P\omega)$

Solve to get:

$$J_0 = J\left(1 - \frac{3h}{2l}\right)$$

If we hit at $h = \frac{2l}{3}$ then $J_0 = 0$ ..

Conservation of angular momentum in collisions

$$J = \Delta L.$$

If $J_A = 0$ about A in the body, then $\Delta L_A = 0$

$\Rightarrow L_A$ remains conserved.

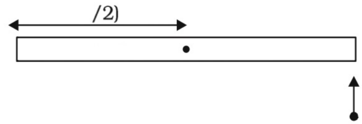
Angular Momentum of a body/system is conserved about a point A if angular impulse about A is zero.

For conservation of L we look for a point A about which $J_A = 0$.

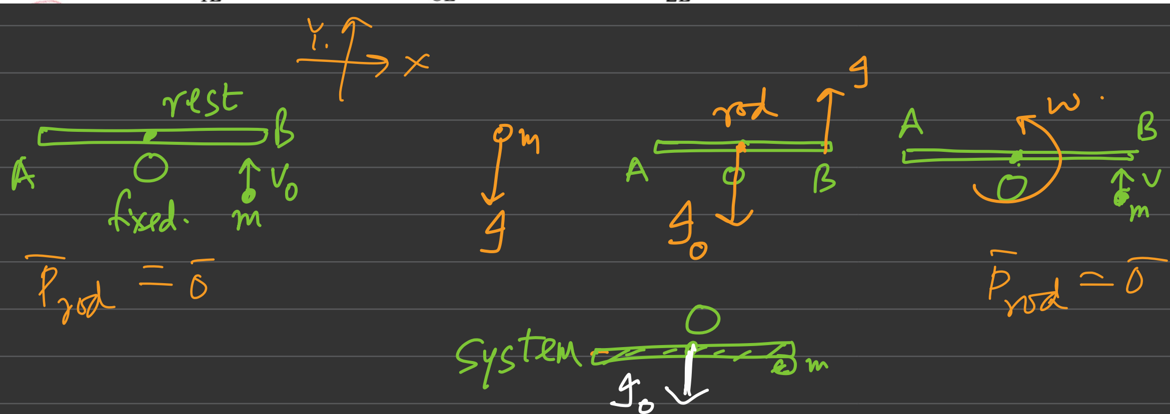
$\left[L_A \right.$ can be done only if all impulses act at one point only at A.

L can be conserved about any point if there is NO impulse.

55. A uniform rod of mass M and length L, which is free to rotate about a fixed vertical axis through O, is lying on a frictionless horizontal table. A particle of equal mass strikes the rod with a velocity V_0 and sticks to it. The angular velocity of the combination immediately after the collision is:



(A) $\frac{3V_0}{4L}$ **(B)** $\frac{3V_0}{8L}$ **(C)** $\frac{3V_0}{2L}$ **(D)** None of these



1. Conservation of $\vec{p}$ along an axis where $J = 0$. component.
2. Conservation of $\bar{L}$

Conservation theorems

for Collisions

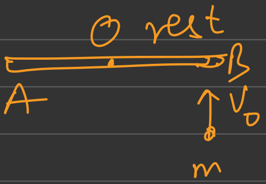
3. $V_{sep} = e\, V_{app}$ (along CN)

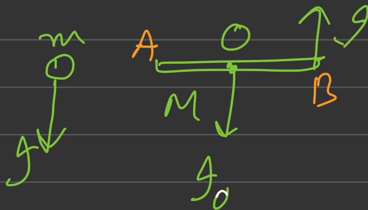
4. $\bar{J} = \Delta\bar{p}$

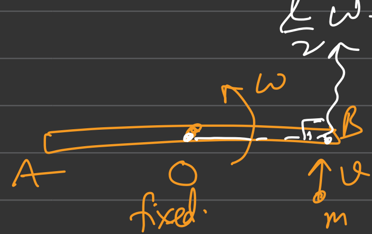
5. $\bar{J} = \Delta\bar{L}$ → $\bar{J}_{CM} = \Delta\bar{L}_{CM}$; → $\bar{J}_p = \Delta\bar{L}_p$ (P: st. pt.)

Impulse-Momentum theorems.

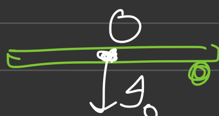
Here $\bar{p}$ is not conserved along Y axis for particle & system.
(For rod $\bar{p} = 0$ all the time)







System.



(+↺) About O.

$$\boxed{m v_0 \frac{l}{2} + 0 = m v \frac{l}{2} + \frac{M l^2}{12}\omega}$$

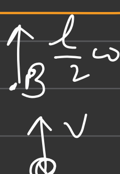
$\left(\begin{array}{l} L_{cm} = I_{cm}\omega \\ L = m v r_{\perp} \end{array} \right)$

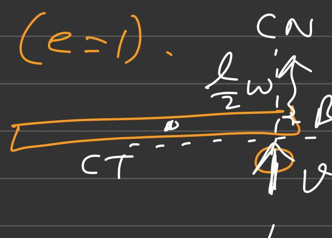
$\sum_{\text{about } O} P_{\text{system}}$ = conserved because $\tau_{\text{about } O} = 0$ for system.

As particle sticks to rod at B. $\Rightarrow$ $v_{\text{particle}} = v_{\text{of } B}$

$$\boxed{v = \frac{l}{2}\omega}$$

FOR ELASTIC IMPACT $(e=1)$.





$V_{sep} = e\, V_{app}$

$$\boxed{\frac{l}{2}\omega - v = e\, v_0}$$

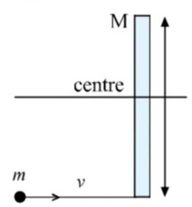
($e = 1$ for elastic).

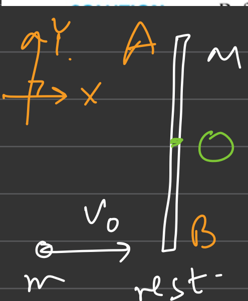
(Put $e=0$ if particle sticks) (put $e=1$ if elastic)

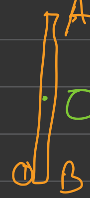
Example - 18 *A meter stick lies on a frictionless horizontal table. It has a mass M and is free to move in any way on the table. A hockey puck m, moving as shown with speed v collide elastically with the stick.*

(a) *What is the velocity of the puck after impact ?*

(b) *What is the velocity of the CM and the angular velocity of the stick after impact ?*

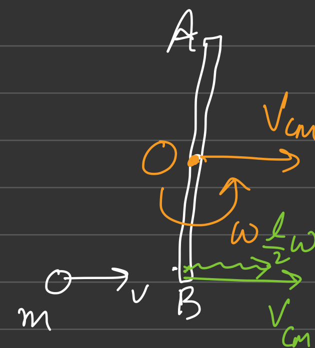






system.

No impulse on system.



three unknowns v, V_{cm}, ω.

$\sum P_{system}$ = Conserved

as there is NO ext. impulse on system.

$$\boxed{mv_0 + 0 = mv + MV_{cm}} \quad (1)$$

(+)

L of system can be conserved about any pt.

Let us conserve L of system about mid point of rod O.

$\sum_{System} L_O$ = Conserved.

$$\boxed{mv_0\frac{l}{2} + 0 = mv\frac{l}{2} + \frac{Ml^2}{12}\omega} \quad (2)$$

$(L_O = I_{cm}\omega)$ → CM.

$v_{sep} = e\,v_{app.} \Rightarrow$

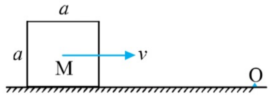
$$\boxed{\left(V_{cm} + \frac{l}{2}\omega\right) - v = e\,v_0}$$

for elastic impact, $e = 1$.

(for m sticking to rod, $e = 0$)

(L of rod about B can be conserved)

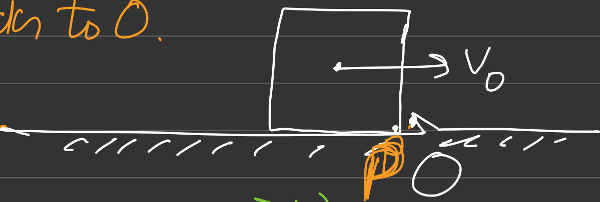
6. *A cubical block of side a is moving with velocity v on a horizontal smooth plane as shown in figure. It hits a ridge at point O. The angular speed of the block just after it hits O is :*

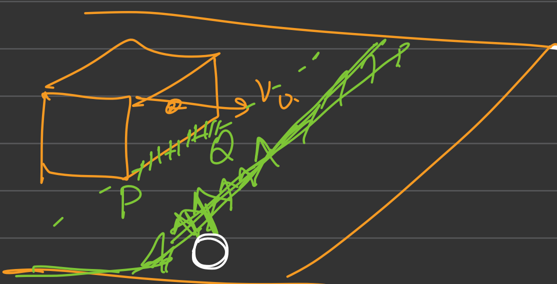


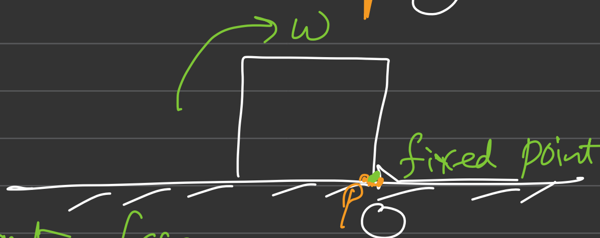
(A) $\frac{3v}{(4a)}$ (B) $\frac{3v}{(2a)}$

(C) $\sqrt{3}v(\sqrt{2}a)$ (D) Zero

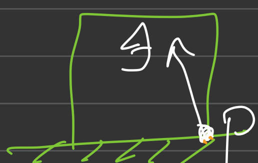
point P of cube sticks to O.







lower edge PQ of front face of cube gets stuck (clamped) (trapped) in the obstacle/ridge/nails



As J acts at P, $J_{about\ P} = 0$.

L_P = conserved.

∃ unknown impulse on lower edge of front face of cube by ridge/clamp. $L_A = I_{cm}\omega + M V_{cm} r_\perp$.

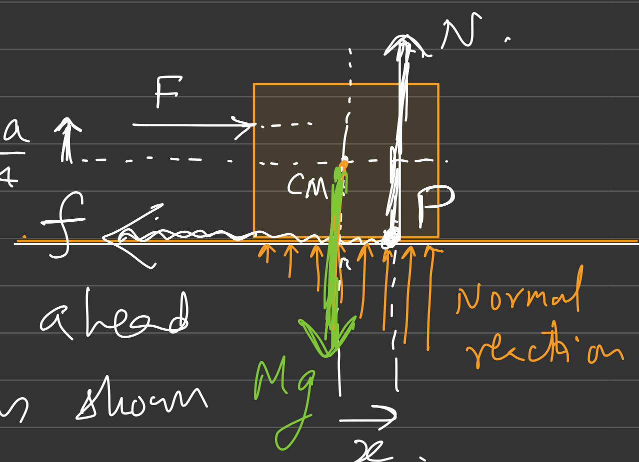
$I_{cm}(0) + M v_0 \frac{a}{2} = I_P \omega$ as P has become fixed point after hit.

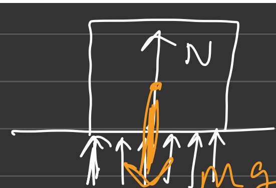
$I_P = I_{cm} + M\left(\frac{a}{\sqrt{2}}\right)^2$

$I_P = \frac{Ma^2}{6} + \frac{Ma^2}{2}$

61. A uniform cube of side a and mass m rests on a rough horizontal surface. A horizontal force F is applied normal to one face at a point that is directly above the centre of the face at a height $\frac{a}{4}$ above the centre. The minimum value of F for which the cube begins to topple above an edge without sliding is:

(A) $\frac{1}{4}mg$ (B) $2mg$ (C) $\frac{1}{2}mg$ (D) $\frac{2}{3}mg$





uniformly distributed N

$N = Mg.$

N shifts ahead of CM as shown by x.

N and f act at P.

$\tau_{P\circlearrowright} = \tau_{P\circlearrowleft}$ balance torques about P.

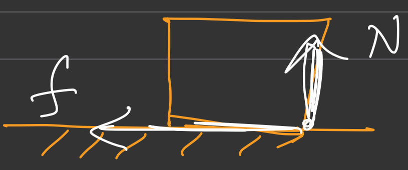
$F\left(\frac{a}{2}+\frac{a}{4}\right) = Mgx$

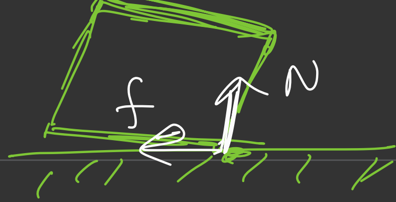
$N = Mg.$
$f = F$

$$x = \frac{3Fa}{4Mg}.$$

If F goes on increasing then x increases.
N shifts towards front corner.

CRITICAL POINT when N shifts & reaches front corner. Now cube is about to be toppled over.





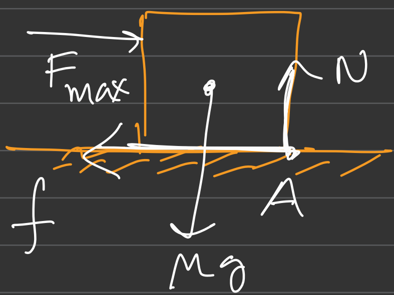
toppling point.

All points are about to lift off the ground.

At toppling point $x = \frac{a}{2} = \frac{3F_{max}a}{4Mg}$.

$F_{max} = \frac{2Mg}{3}$ to avoid toppling.

We balance torques at toppling point.



$\tau_A \circlearrowleft = \tau_A \circlearrowright$

$F_{max} \frac{3}{4} a = Mg \frac{a}{2}$

$F_{max} = 2Mg/3$.

F_{max} to avoid slipping $= \mu_s Mg$.

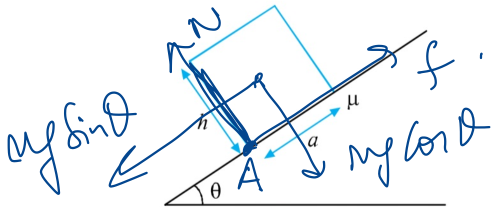
*7. *A block with a square base measuring a × a, and height h, is placed on an inclined plane. The coefficient of friction is μ. The angle of inclination (θ) of the plane is gradually increased. The block will :*

(A) topple before sliding if $\mu > \frac{a}{h}$

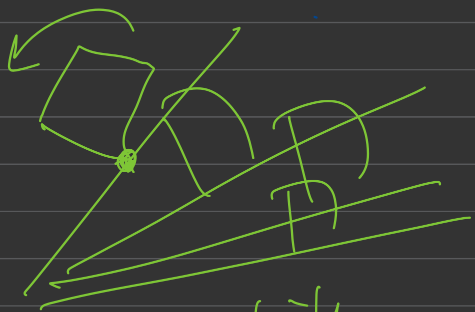
(B) topple before sliding if $\mu < \frac{a}{h}$

(C) slide before toppling if $\mu > \frac{a}{h}$

(D) slide before toppling if $\mu < \frac{a}{h}$



find θ_{max} to avoid toppling.



$\tau_A \circlearrowright = \tau_A \circlearrowleft$

$\theta_{max} = \tan^{-1}\mu_s$ to avoid slipping.

$Mg\cos\theta\frac{a}{2} = Mg\sin\theta\frac{h}{2}$.

$\theta_{max} = \tan^{-1}\frac{a}{h}$, to avoid toppling