Contoh Soal Bab 5

1. X_1, X_2, X_3, X_4 berdistribusi independen identik dengan $\mu = 5$ dan $\sigma = 3$.

$$Y = X_1 + 2X_2 + X_3 - X_4$$

a. $E(Y) = E(X_1 + 2X_2 + X_3 - X_4)$

$$E(Y) = E(X_1) + 2.E(X_2) + E(X_3) - E(X_4)$$

$$E(Y) = 5 + 2 \times 5 + 5 - 5$$

$$E(Y) = 15$$

b. Bukti $Var(X - Y)$?

$$Var(X - Y) = E(X - Y)^2 - \left(E(X - Y)\right)^2$$

$$Var(X - Y) = E(X^2 - 2XY + Y^2) - [E(X) - E(Y)]^2$$

$$Var(X - Y) = E(X^2) - 2E(XY) + E(Y^2) - \left[\left(E(X)\right)^2 - 2E(X).E(Y) + \left(E(Y)\right)^2\right]$$

$$Var(X - Y)$$
$$= \left[E(X^2) - \left(E(X)\right)^2\right] + \left[E(Y^2) - \left(E(Y)\right)^2\right] - 2E(XY)$$
$$+ 2E(X).E(Y)$$

$$Var(X - Y) = Var(X) + Var(Y) - 2[E(XY) \overset{-}{\cancel{+}} E(X).E(Y)]$$

$$Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y)$$

4. $f(x,y) = \begin{cases} \frac{4}{(5xy)} & , x = 1,2 \ , y = 2,3 \\ 0 & , \textit{otherwise} \end{cases}$

a. $E(X) = \sum x . f(x)$

$$f_x(x) = \sum_y f(x,y) = \frac{4}{5x.2} + \frac{4}{5x.3} = \frac{4}{10x} + \frac{4}{15x} = \frac{12+8}{30x} = \frac{2}{3x}$$

$$E(X) = \sum_x x . f(x)$$

$$E(X) = 1 . \frac{2}{3.1} + 2 . \frac{2}{3.2} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

b. $E(Y) = \sum y . f(y)$

$$f_y(y) = \sum_x f(x,y) = \frac{4}{5.y} + \frac{4}{5.2y} = \frac{4}{5y} + \frac{4}{10y} = \frac{8+4}{10y} = \frac{6}{5y}$$

$$E(Y) = \sum_y y . f(y)$$

$$E(Y) = 2 . \frac{6}{5.2} + 3 . \frac{6}{5.3} = \frac{12}{5}$$

c. $E(XY)$

$$E(XY) = \sum_y \sum_x xy\, f(x, y)$$

$$E(XY) = \sum_y \sum_x xy\, \frac{4}{5xy}$$

$$E(XY) = \sum_y \left(1.y.\frac{4}{5y} + 2.y.\frac{4}{5.2y}\right)$$

$$E(XY) = \sum_y \left(\frac{4y}{5y} + \frac{8y}{10y}\right)$$

$$E(XY) = \sum_{y=2}^{3} \frac{16y}{10y} = \frac{16.2}{10.2} + \frac{16.3}{10.3} = \frac{32}{10} = \frac{16}{5}$$

d. $Cov(x, y)$

$$Cov(x, y) = E(XY) - E(X).E(Y)$$

$$Cov(x, y) = \frac{16}{5} - \frac{4}{3}.\frac{12}{5}$$

$$Cov(x, y) = \frac{16}{5} - \frac{48}{15}$$

$$Cov(x, y) = \frac{48 - 48}{15} = 0$$

5. $f(x.y) = \begin{cases} x + y, 0 < x < 1,\ 0 < y < 1 \\ 0 \qquad\qquad , otherwise \end{cases}$

a. $E(X)$

$$E(X) = \int_0^1 \int_0^1 x(x+y)\, dy\, dx$$
$$E(X) = \int_0^1 \int_0^1 (x^2 + xy)\, dy\, dx$$
$$E(X) = \int_0^1 \left[x^2 y + \frac{1}{2}xy^2\right]_0^1 dx$$

$$E(X) = \int_0^1 \left(x^2 + \frac{1}{2}x\right) dx$$
$$E(X) = \left[\frac{1}{3}x^3 + \frac{1}{4}x^2\right]_0^1 = \frac{1}{2} + \frac{1}{4} = \frac{7}{12}$$

b. $E(X+Y)$

$$E(X+Y) = E(X) + E(Y)$$
$$E(X+Y) = \frac{7}{12} + \int_0^1 \int_0^1 y(x+y)\, dxdy$$

$$E(X+Y) = \frac{7}{12} + \frac{7}{12} = \frac{14}{12} = \frac{7}{6}$$

c. $E(XY)$

$$E(XY) = \int_0^1 \int_0^1 xy(x+y)\, dy\, dx$$

$$E(XY) = \int_0^1 \int_0^1 (x^2y + xy^2)\, dy\, dx$$

$$E(XY) = \int_0^1 \left[\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]_0^1 dx$$

$$E(XY) = \int_0^1 \left(\frac{1}{2}x^2 + \frac{1}{3}x\right) dx$$

$$E(XY) = \left[\frac{1}{6}x^3 + \frac{1}{6}x^2\right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

d. $Cov(2X, 3Y) = 2\,.\,3\,.\,Cov(X, Y)$

$$= 6\,[E(XY) - E(X).E(Y)]$$

$$= 6\,\left[\frac{2}{6} - \frac{7}{12}.\frac{7}{12}\right]$$

$$= 6\,\left[\frac{2}{6} - \frac{49}{144}\right]$$

$$= -\frac{1}{24}$$

e. $E(Y|x) = \int_0^1 y f(y|x) dy$

$$f(y|x) = \frac{f(x,y)}{fx} = \frac{x+y}{\int_0^1 (x+y)dy} = \frac{x+y}{x+\frac{1}{2}}$$

$$E(Y|x) = \int_0^1 y\left(\frac{x+y}{x+\frac{1}{2}}\right)dy$$

$$= \frac{1}{x+\frac{1}{2}}\int_0^1 (xy+y^2)dy$$

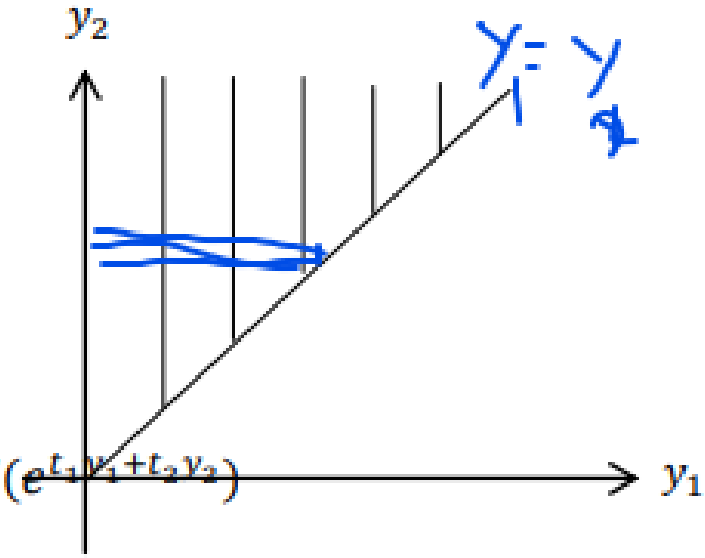
$$= \frac{1}{x+\frac{1}{2}}\left[\frac{1}{2}xy^2+\frac{1}{3}y^3\right]_0^1$$

$$= \frac{\frac{1}{2}x+\frac{1}{3}}{x+\frac{1}{2}}$$

$$= \frac{3x+2}{6x+3}$$

21. y_1 dan y_2 kontinu

$$f(y_1, y_2) = \begin{cases} 2e^{-y_1-y_2} & , 0 < y_1 < y_2 < \infty \\ 0 & , y_1, y_2 \; yang \; lain \end{cases}$$



$$M_{y_1 y_2}(t_1, t_2) = E(e^{t_1 y_1 + t_2 y_2})$$

$$= \int_0^\infty \int_0^{y_2} e^{t_1 y_1 + t_2 y_2} 2e^{-(y_1+y_2)} dy_1 dy_2$$

$$= 2 \int_0^\infty e^{t_2 y_2} e^{-y_2} \int_0^{y_2} e^{t_1 y_1} e^{-y_1} \, dy_1 dy_2$$

$$= 2 \int_0^\infty e^{-y_2(1-t_2)} \int_0^{y_2} e^{-y_1(1-t_1)} \, dy_1 dy_2$$

$$= 2 \int_0^\infty e^{-y_2(1-t_2)} \left[\frac{1}{t_1 - 1} e^{-y_1(1-t_1)} \right]_0^{y_2} dy_2$$

$$= \frac{2}{t_1-1}\int_0^\infty e^{-y_2(1-t_2)}\left(e^{-y_2(1-t_1)}-1\right)dy_2$$

$$= \frac{2}{t_1-1}\int_0^\infty e^{-y_2(1-t_2)-y_2(1-t_1)} - e^{-y_2(1-t_2)}dy_2$$

$$= \frac{2}{t_1-1}\left[-\frac{1}{2-t_2-t_1}e^{-y_2(2-t_2-t_1)}+\frac{1}{1-t_2}e^{-y_2(1-t_2)}\right]_0^\infty$$

$$= \frac{-2}{1-t_1}\left(0+\frac{1}{2-t_2-t_1}+0-\frac{1}{1-t_2}\right)$$

$$= \frac{-2}{(1-t_1)(2-t_2-t_1)}+\frac{2}{(1-t_1)(1-t_2)}$$

$$= \frac{-2(1-t_2)+2(2-t_2-t_1)}{(1-t_1)(1-t_2)(2-t_2-t_1)}$$

$$= \frac{-2+2t_2+4-2t_2-2t_1}{(1-t_1)(1-t_2)(2-t_2-t_1)}$$

$$= \frac{2-2t_1}{(1-t_1)(1-t_2)(2-t_2-t_1)}$$

$$= \frac{2(1-t_1)}{(1-t_1)(1-t_2)(2-t_2-t_1)}$$

$$= \frac{2}{(1-t_2)(2-t_2-t_1)}$$