Крайова задача для рівняння теплопровідності

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = a^2 \frac{\partial^2 u(t,x)}{\partial x^2} + g(t,x), & 0<x<l \quad 0<t\leq T \\ u(0,x) = u_0(x), & 0\leq x\leq l, \\ \left.\begin{array}{l} u(t,0) = \psi_1(t) \\ u(t,l) = \psi_2(t) \end{array}\right\} & 0\leq t\leq T \end{cases}$$

$$g(t,x) = 0$$

$$u_0(x) = \begin{cases} 0, & x\in[0, x_1] \\ h_1 \frac{x-x_1}{x_2-x_1}, & x\in[x_1, x_2] \\ h_1+(h_2-h_1)\frac{x-x_2}{x_3-x_2}, & x\in[x_2, x_3] \\ h_2 \frac{x_4-x}{x_4-x_3}, & x\in[x_3, x_4] \\ 0, & x\in[x_4, l] \end{cases}$$

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = 9 \frac{\partial^2 u(t,x)}{\partial x^2}, & 0<x<6 \quad 0<t\leq 8 \\ u(0,x) = \begin{cases} 0, & x\in[0,1] \\ -2x+2 & x\in[1,2] \\ x-2 & x\in[2,3] \\ \frac{5-x}{2} & x\in[3;5] \\ 0 & x\in[5;6] \end{cases} & 0\leq x\leq 6 \\ \left.\begin{array}{l} u(t,0) = \frac{t}{8} \\ u(t,l) = \sin\left(\frac{3\pi t}{8}\right) \end{array}\right\} & 0\leq t\leq 8 \end{cases}$$

$$q(t,x) = \frac{6t - xt + 8x\sin\left(\frac{3\pi t}{8}\right)}{48} \qquad [5.5]$$

$$u(t,x) = \frac{6t-xt+8x\sin\left(\frac{3\pi t}{8}\right)}{48} + v(t,x) + w(t,x) \qquad [5.4]$$

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial v}{\partial t} + \dfrac{\partial w}{\partial t} + \dfrac{6+3\pi x\cos\left(\frac{3\pi t}{8}\right) - x}{48} \\[2ex]
\dfrac{\partial^2 u(t,x)}{\partial x^2} = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 w}{\partial x^2} \\[2ex]
u(0,x) = u_0(x) = v(0,x) + w(0,x)
\end{cases}
$$

[5.34]
[5.7]

$$
\begin{cases}
\dfrac{\partial v}{\partial t} + \dfrac{\partial w}{\partial t} = 9\dfrac{\partial^2 v}{\partial x^2} + 9\dfrac{\partial^2 w}{\partial x^2} - \dfrac{6+3\pi x\cos\left(\frac{3\pi t}{8}\right) - x}{48} & 0<x<6 \quad 0<t\leq T \\[2ex]
v(0,x) + w(0,x) = u_0(x) & 0\leq x\leq 6 \\[2ex]
\left.\begin{aligned} v(t,0) + w(t,0) = 0 \\ v(t,6) + w(t,6) = 0 \end{aligned}\right\} & 0\leq t\leq 8
\end{cases}
$$

[5.8]

$$
\begin{cases}
\dfrac{\partial v}{\partial t} = 9\dfrac{\partial^2 v}{\partial x^2} & 0<x<6 \quad 0<t\leq T \\[2ex]
v(0,x) = v_0(x) & 0\leq x\leq 6 \\[2ex]
\left.\begin{aligned} v(t,0) = 0 \\ v(t,6) = 0 \end{aligned}\right\} & 0\leq t\leq 8
\end{cases}
$$

[5.9]

$$
\begin{cases}
\dfrac{\partial w(t,x)}{\partial t} = 9\dfrac{\partial^2 w}{\partial x^2} - \dfrac{6+3\pi x\cos\left(\frac{3\pi t}{8}\right) - x}{48} & 0<x<6 \quad 0<t\leq 8 \\[2ex]
w(0,x) = 0 & 0\leq x\leq 6 \\[2ex]
\left.\begin{aligned} w(t,0) = 0 \\ w(t,6) = 0 \end{aligned}\right\} & 0\leq t\leq 8
\end{cases}
$$

$$v_{o\mu} = \frac{2}{\ell}\int_0^\ell v_o(x)\sin\frac{\mu\pi x}{\ell}\,dx$$

$$v(t,x) = \sum_{\mu=1}^{\infty} v_{o\mu} e^{-\left(\frac{\mu\pi 3}{6}\right)^2 t}\sin\frac{\mu\pi x}{6}$$

$$b_{\mu_1} = \frac{2}{6}\left(\int_1^2 (2-2x)\sin\left(\frac{\mu\pi x}{6}\right)dx + \int_2^3 (x-2)\sin\left(\frac{\mu\pi x}{6}\right)dx + \right.$$

$$\left. + \int_3^5\left(\frac{5-x}{2}\right)\sin\left(\frac{\mu\pi x}{6}\right)dx\right) = \frac{1}{3}\left(I_1 + I_2 + I_3\right)$$

$$I_1 = \int_1^2 (2-2x)\sin\left(\frac{\mu\pi x}{6}\right)dx = 2\int_1^2 (1-x)\sin\left(\frac{\mu\pi x}{6}\right)dx =$$

$$= \left\{\begin{array}{ll} u = 1-x & du = dx \\ dv = \sin\left(\frac{\mu\pi x}{6}\right)dx & v = -\cos\left(\frac{\mu\pi x}{6}\right)\frac{6}{\mu\pi}\end{array}\right\} =$$

$$= uv - \int v\,du = \frac{2\cdot 6(x-1)}{\mu\pi}\cos\left(\frac{\mu\pi x}{6}\right)\Bigg|_1^2 + 2\int_1^2 \frac{6\cos\left(\frac{\mu\pi x}{6}\right)}{\mu\pi}dx =$$

$$= 2\frac{6}{\mu\pi}\cos\left(\frac{\mu\pi}{3}\right) + 2\left(\frac{6}{\mu\pi}\right)^2\sin\left(\frac{\mu\pi x}{6}\right)\Bigg|_1^2 = \frac{2\cdot 6}{\mu\pi}\cos\left(\frac{\mu\pi}{3}\right) + 2\left(\frac{6}{\mu\pi}\right)^2\left(\sin\left(\frac{\mu\pi}{3}\right) - \sin\left(\frac{\mu\pi}{6}\right)\right)$$

$$I_2 = \int_2^3 (x-2)\sin\left(\frac{\mu\pi x}{6}\right)dx = \left\{\begin{array}{ll} u = x-2 & du = dx \\ dv = \sin\left(\frac{\mu\pi x}{6}\right)dx & v = -\frac{6}{\mu\pi}\cos\left(\frac{\mu\pi x}{6}\right)\end{array}\right\}$$

$$= \frac{(2-x)\,6}{\mu\pi}\cos\left(\frac{\mu\pi x}{6}\right)\Bigg|_2^3 + \int_2^3 \frac{6}{\mu\pi}\cos\left(\frac{\mu\pi x}{6}\right)dx =$$

$$-\frac{6}{\mu\pi}\cos\left(\frac{\mu\pi}{2}\right) + \left(\frac{6}{\mu\pi}\right)^2\sin\left(\frac{\mu\pi x}{6}\right)\Bigg|_2^3 = -\frac{6}{\mu\pi}\cos\left(\frac{\mu\pi}{2}\right) + \left(\frac{6}{\mu\pi}\right)^2\left(\sin\left(\frac{\mu\pi}{2}\right) - \sin\left(\frac{\mu\pi}{3}\right)\right)$$

$$I_3 = \int_3^5 \left(\frac{5-x}{2}\right) \sin\left(\frac{\mu\pi x}{6}\right) dx = \frac{1}{2}\int_3^5 (5-x)\sin\left(\frac{\mu\pi x}{6}\right) dx =$$

$$= \left\{ \begin{array}{ll} u = 5-x & du = dx \\ dv = \sin\left(\frac{\mu\pi x}{6}\right) dx & V = \frac{-6}{\mu\pi}\cos\left(\frac{\mu\pi x}{6}\right) \end{array} \right\} =$$

$$= \frac{1}{2}\left(\frac{(x-5)6}{\mu\pi}\cos\left(\frac{\mu\pi x}{6}\right)\Bigg|_3^5 + \int_3^5 \frac{6}{\mu\pi}\cos\left(\frac{\mu\pi x}{6}\right) dx \right) =$$

$$= \frac{1}{2}\left(\frac{12\cos\left(\frac{\mu\pi}{2}\right)}{\mu\pi} + \left(\frac{6}{\mu\pi}\right)^2 \left(\sin\left(\frac{\mu\pi 5}{6}\right) - \sin\left(\frac{\mu\pi}{2}\right)\right)\right) =$$

$$= \frac{6\cos\left(\frac{\mu\pi}{2}\right)}{\mu\pi} + \frac{1}{2}\left(\frac{6}{\mu\pi}\right)^2\left(\sin\left(\frac{\mu\pi 5}{6}\right) - \sin\left(\frac{\mu\pi}{2}\right)\right)$$

$$b_\mu = \frac{1}{3}\frac{12}{\mu\pi}\cos\left(\frac{\mu\pi}{3}\right) + \frac{2}{3}\left(\frac{6}{\mu\pi}\right)^2\left(\sin\left(\frac{\mu\pi}{3}\right) - \sin\left(\frac{\mu\pi}{6}\right)\right) +$$

$$+ \frac{1}{3}\left(\frac{6}{\mu\pi}\right)^2\left(\sin\left(\frac{\mu\pi}{2}\right) - \sin\left(\frac{\mu\pi}{3}\right)\right) - \frac{6}{3\mu\pi}\cos\left(\frac{\mu\pi}{2}\right) +$$

$$+ \frac{1}{3}\cdot\frac{1}{2}\left(\frac{6}{\mu\pi}\right)^2\left(\sin\left(\frac{\mu\pi 5}{6}\right) - \sin\left(\frac{\mu\pi}{2}\right)\right) + \frac{6}{3\mu\pi}\cos\left(\frac{\mu\pi}{2}\right) =$$

$$\cancel{= \frac{4}{\mu\pi}\cos\left(\frac{\mu\pi}{3}\right) + \frac{2}{3}\left(\frac{6}{\mu\pi}\right)^2\sin\left(\frac{\mu\pi}{3}\right) - \frac{2}{3}\left(\frac{6}{\mu\pi}\right)^2\sin\left(\frac{\mu\pi}{6}\right) +}$$

$$\cancel{+ \frac{1}{3}\left(\frac{6}{\mu\pi}\right)^2\sin\left(\frac{\mu\pi}{2}\right) - \frac{1}{3}\left(\frac{6}{\mu\pi}\right)^2\sin\left(\frac{\mu\pi}{3}\right) + }$$

$$= \frac{4}{\mu\pi}\cos\left(\frac{\mu\pi}{3}\right) + \left(\frac{6}{\mu\pi}\right)^2\left(\frac{2}{3}\sin\left(\frac{\mu\pi}{3}\right) - \frac{2}{3}\sin\left(\frac{\mu\pi}{6}\right) + \frac{1}{3}\sin\left(\frac{\mu\pi}{2}\right) - \frac{1}{3}\sin\left(\frac{\mu\pi}{3}\right) +\right.$$

$$\left. + \frac{1}{6}\sin\left(\frac{\mu\pi 5}{6}\right) - \frac{1}{6}\sin\left(\frac{\mu\pi}{2}\right)\right) =$$

$$= \frac{4}{\mu\pi}\cos\left(\frac{\mu\pi}{3}\right) + \left(\frac{6}{\mu\pi}\right)^2\left(\frac{1}{6}\sin\left(\frac{\mu\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{\mu\pi}{3}\right) - \frac{2}{3}\sin\left(\frac{\mu\pi}{6}\right) + \frac{1}{6}\sin\left(\frac{\mu\pi 5}{6}\right)\right)$$

$$v(t,x) = \sum_{\mu=1}^{\infty} b_\mu e^{-\left(\frac{3\mu\pi}{6}\right)^2 t} \cdot \sin\left(\frac{\mu\pi x}{6}\right)$$

$$w(t,x) = \sum_{\mu=1}^{\infty} w_\mu(t) \sin\left(\frac{\mu\pi x}{6}\right)$$

$$w_\mu(t) = \int_0^t e^{-\left(\frac{\mu\pi 3}{6}\right)^2 (t-\tau)} \cdot h_\mu(\tau)\, d\tau$$

$$h_\mu(t) = \frac{2}{6}\int_0^6 h(\tau,x)\sin\left(\frac{\mu\pi x}{6}\right)dx$$

$$h_\mu(t) = \frac{1}{3}\int_0^6 -\frac{6+x\cos\left(\frac{3\pi t}{8}\right)\cdot 3\pi - x}{48}\sin\left(\frac{\mu\pi x}{6}\right)dx =$$

$$\cancel{h(t,x) = \frac{x-6-x\cos\left(\frac{3\pi t}{8}\right)}{48} = \sum_{\mu=1}^{\infty} h_\mu(t)\sin\left(\frac{\mu\pi x}{6}\right)}$$

$$\cancel{h_\mu(t) = \frac{2}{6}\int_0^6 h(t,x)\sin\left(\frac{\mu\pi x}{6}\right)dx =}$$

$$= -\frac{1}{3}\int_0^6 +\frac{6}{48}\sin\left(\frac{\mu\pi x}{6}\right)dx - \frac{1}{3}\int_0^6 \frac{x\cos\left(\frac{3\pi t}{8}\right)\cdot 3\pi\cdot\sin\left(\frac{\mu\pi x}{6}\right)}{48}dx +$$

$$+\frac{1}{3}\int_0^6 \frac{x\sin\left(\frac{\mu\pi x}{6}\right)}{48}dx =$$

$$= +\frac{1}{3}\cdot\frac{6}{48}\cdot\frac{6}{\mu\pi}\left(\cos\left(\frac{\mu\pi 6}{6}\right) - \cos 0\right) - \frac{1}{3}\frac{\cos\left(\frac{3\pi t}{8}\right)}{48}\cdot 3\pi\int_0^6 x\sin\left(\frac{\mu\pi x}{6}\right)dx +$$

$$+\frac{1}{3\cdot 48}\int_0^6 x\sin\left(\frac{\mu\pi x}{6}\right)dx =$$

$$= \frac{1}{4\mu\pi}\left(\cos\left(\mu\pi\right)-1\right) - \frac{\cos\left(\frac{3\pi t}{8}\right)\pi}{48}\int_0^6 x\sin\left(\frac{\mu\pi x}{6}\right)dx + \frac{1}{3\cdot 48}\int_0^6 x\sin\left(\frac{\mu\pi x}{6}\right)dx$$

$$= \frac{1}{4\mu\pi}\left(\cos(\mu\pi)-1\right) + \int_0^6 x\sin\left(\frac{\mu\pi x}{6}\right)dx\left(\frac{1}{3\cdot 48} - \frac{\pi\cos\left(\frac{3\pi t}{8}\right)}{48}\right)$$

$$\left\{\begin{array}{ll} u = x & du = dx \\ dv = \sin\left(\frac{\mu\pi x}{6}\right)dx & v = -\frac{6}{\mu\pi}\cos\left(\frac{\mu\pi x}{6}\right) \end{array}\right\}$$

$$= \frac{1}{4\mu\pi}\left(\cos(\mu\pi)-1\right) + \left(\left(\frac{6}{\mu\pi}\right)^2\left(\sin(\mu\pi)\right) - \frac{36}{\mu\pi}\cos\mu\pi\right)\left(\frac{1}{3\cdot 48} - \frac{\pi\cos\left(\frac{3\pi t}{8}\right)}{48}\right) =$$

$$= \frac{3\pi}{4}\cos\left(\frac{3\pi t}{8}\right) - \frac{1}{4\mu\pi}$$

$$h_\mu(\tau) = \frac{3\pi}{4}\cos\left(\frac{3\pi\tau}{8}\right) - \frac{1}{4\mu\pi} = \frac{4\mu\pi \cdot \frac{3\pi}{4}\cos\left(\frac{3\pi\tau}{8}\right) - 1}{4\mu\pi} = \frac{3\mu\pi^2\cos\left(\frac{3\pi\tau}{8}\right) - 1}{4\mu\pi}$$

$$w_\mu = \int_0^t e^{-\left(\frac{\mu\pi 3}{6}\right)^2(t-\tau)} h_\mu(\tau)\,d\tau = e^{-\left(\frac{\mu\pi}{2}\right)^2 t}\int_0^t e^{\left(\frac{\mu\pi}{2}\right)^2\tau} h_\mu(\tau)\,d\tau$$

$$I_1 = \int_0^t \frac{3\pi}{4}\cos\left(\frac{3\pi\tau}{8}\right)\cdot e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\,d\tau =$$

$$= \frac{\frac{9\pi^2}{32} e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\sin\left(\frac{3\pi\tau}{8}\right) + \frac{3\pi}{4}\left(\frac{\mu\pi}{2}\right)^2 e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\cos\left(\frac{3\pi\tau}{8}\right)}{\left(\frac{3\pi}{8}\right)^2 - 3\pi\left(\frac{\mu\pi}{2}\right)^4}$$

$$I_2 = \int_0^t e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\cdot\frac{1}{4\mu\pi}\,d\tau = \frac{1}{4\mu\pi}\int_0^t e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\,d\tau$$

$$h_\mu(\tau) = \frac{3\pi}{4}\cos\left(\frac{3\pi\tau}{8}\right) - \frac{1}{4\mu\pi}$$

$$w_\mu = e^{-\left(\frac{\mu\pi}{2}\right)^2 t}\int_0^t e^{\left(\frac{\mu\pi}{2}\right)^2\tau} h_\mu(\tau)\,d\tau = e^{-\left(\frac{\mu\pi}{2}\right)^2 t}\left(I_1 - I_2\right)$$

$$I_1 = \int_0^t e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\frac{3\pi}{4}\cos\left(\frac{3\pi\tau}{8}\right)d\tau =$$

$$= \left.\frac{\frac{9\pi}{8}^2 e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\cdot\sin\left(\frac{3\pi}{8}\tau\right) + 3\pi\left(\frac{\mu\pi}{2}\right)^2 e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\cos\left(\frac{3\pi}{8}\tau\right)}{\left(\frac{3\pi}{8}\right)^2 - 3\pi\left(\frac{\mu^2\pi^2}{4}\right)^2}\right|_0^t =$$

$$= \left.\frac{e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\left(\frac{9\pi}{8}^2\sin\left(\frac{3\pi}{8}\tau\right) + 3\pi\left(\frac{\mu\pi}{2}\right)^2\cos\left(\frac{3\pi}{8}\tau\right)\right)}{\left(\frac{3\pi}{8}\right)^2 - 3\pi\left(\frac{\mu^2\pi^2}{4}\right)^2}\right|_0^t =$$

$$= \frac{e^{\left(\frac{\mu\pi}{2}\right)^2 t}\left(\frac{9\pi}{8}^2\sin\left(\frac{3\pi}{8}t\right) + 3\pi\left(\frac{\mu\pi}{2}\right)^2\cos\left(\frac{3\pi}{8}t\right)\right)}{\left(\frac{3\pi}{8}\right)^2 - 3\pi\left(\frac{\mu^2\pi^2}{4}\right)^2} - \frac{3\pi\left(\frac{\mu\pi}{2}\right)^2}{\left(\frac{3\pi}{8}\right)^2 - 3\pi\left(\frac{\mu^2\pi^2}{4}\right)^2}$$

$$I_2 = \int_0^t e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\frac{1}{4\mu\pi}\,d\tau = \left.\frac{1}{4\mu\pi}e^{\left(\frac{\mu\pi}{2}\right)^2\tau}\cdot\frac{4}{(\mu\pi)^2}\right|_0^t =$$

$$= \frac{1}{(\mu\pi)^3}\left(e^{\left(\frac{\mu\pi}{2}\right)^2 t} - 1\right) = \frac{e^{\left(\frac{\mu\pi}{2}\right)^2 t} - 1}{(\mu\pi)^3}$$

$$w_\mu = e^{-\left(\frac{\mu\pi}{2}\right)^2 t}\left(\frac{e^{\left(\frac{\mu\pi}{2}\right)^2 t}\left(\frac{9\pi}{8}^2\sin\left(\frac{3\pi t}{8}\right) + 3\pi\left(\frac{\mu\pi}{2}\right)^2\cos\left(\frac{3\pi}{8}t\right)\right) - 3\pi\left(\frac{\mu\pi}{2}\right)^2}{\left(\frac{3\pi}{8}\right)^2 - 3\pi\left(\frac{\mu^2\pi^2}{4}\right)^2} - \frac{e^{\left(\frac{\mu\pi}{2}\right)^2 t} - 1}{(\mu\pi)^3}\right)$$

$$\boxed{w(t,x) = \sum_{\mu=1}^{\infty} w_\mu\cdot\sin\frac{\mu\pi x}{6}}$$