

The Risk Models with Non-Local Poisson Processes



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Abstract

The Poisson process is the most commonly used point process in modelling counting phenomena [21]. Even if the counting process has non-stationary increments, it can be shown to converge to the Poisson process if observed sufficiently long after a transient period as long as it constitutes a renewal process [43]. As such, it is important to review the key characteristics of the Poisson process as it serves as the main building block of more complex models.

In the first part of this thesis, we propose two fractional risk models, where the classical risk process is time-changed by the mixture of tempered stable inverse subordinators. We characterise the risk processes by deriving the marginal distributions and establish the corresponding moments and covariance structure.

In the second part of this thesis, we study the main characteristics of these models such as ruin probability and time of ruin, and illustrate the results with Monte Carlo simulations. The data suggests that the time of ruin can be approximated by the inverse gaussian distribution and its generalisations.

Table of contents

List of figures	xi
List of tables	xiii
1 Introduction	1
2 Subordinator and Inverse subordinator	7
2.1 Subordinator and Inverse subordinator	7
2.2 α -stable subordinator and inverse α -stable subordinator	10
2.2.1 α -stable subordinator	10
2.2.2 Inverse α -stable subordinator	11
2.3 Tempered α -stable subordinator and inverse tempered α -stable subordinator	15
2.3.1 Tempered α -stable subordinator	15
2.3.2 Inverse tempered α -stable subordinator	15
2.4 Mixture of tempered α -stable subordinators and inverse mixture of tempered α -stable subordinators	16
2.4.1 Mixture of tempered α -stable subordinators	16
2.4.2 Inverse mixture of tempered α -stable subordinators	18

3	Count and risk processes	21
3.1	Poisson processes	21
3.1.1	Homogeneous Poisson process	21
3.1.2	Nonhomogeneous Poisson process	24
3.2	Fractional Poisson processes	26
3.2.1	Fractional homogeneous Poisson process	26
3.2.2	Fractional nonhomogeneous Poisson process	29
3.3	Mixed tempered fractional Poisson processes	30
3.3.1	Mixed tempered fractional homogeneous Poisson process	30
3.3.2	Mixed tempered fractional nonhomogeneous Poisson process	31
3.4	Fractional risk processes	32
3.4.1	Classical risk models	32
3.4.2	Mixed risk model	34
4	Simulations	37
4.1	Simulations methodologies	37
4.1.1	Simulation of processes	37
4.1.2	Simulation of ruin probability	41
4.2	Numerical investigation of the ruin probabilities and ruin times for the mixed risk model	44
4.2.1	Setups	44
5	Conclusions	51

Table of contents	ix
5.1 Summaries	51
5.2 Limitations	52
5.3 Future work	52
References	55
Appendix A Simulations' algorithms	61
Appendix B Full numerical results	65

List of figures

4.1	Simulations of the processes	40
4.2	Simulations of risk process	43
4.3	Distributions of the time of ruin for HPP, NPP, FHPP, and FNPP	45
4.4	Probability of ruin $\phi(t)$ for HPP, NPP, FHPP, and FNPP	47
4.5	Fitted distributions of simulations of time of ruin	47
4.6	Fitted distributions of simulations of time of ruin for Model 1 with different initial capital values	48
4.7	Fitted distributions of simulations of time of ruin for Model 2 with different initial capital values	48
4.8	Fitted distributions of simulations of time of ruin for Model 3 (MTFHPP) for different initial capital values	49
4.9	Fitted distributions of simulations of time of ruin for Model 4	49
4.10	Probability of ruin of Model 1 under different parameterisations	49
4.11	Probability of ruin of Model 2 under different parameterisations	49
4.12	Probability of ruin of model 3 under different parameterisations	49
4.13	Probability of ruin of Model 4 under different parameterisations	50

List of tables

- 4.1 Parameters of the fitted distributions using `scipy.stats.rv_continuous.fit` 46
- 4.2 Statistics for fitted distributions of the time of ruin 46
- 4.3 Finite-time ruin probabilities $\psi(u, t)$ 48

- B.1 Ruin probabilities with substituted α_1 and α_2 65
- B.2 Ruin probabilities with substituted λ_1 and λ_2 66
- B.3 Ruin probabilities with substituted c_1 and c_2 67
- B.4 Parameters of fitted distributions with default parameters (model 1) . . . 68
- B.5 Parameters of fitted distributions with default parameters (model 2) . . . 72

Chapter 1

Introduction

The homogenous Poisson process is used to model random occurrence of events. The modelling of events can be done in relation to time and space. However, the Poisson process can be applied only if events satisfy particular conditions, such as they are independent of each, there are no two events occurring at exactly the same time or located at exactly the same space and the probability of the event should increase with the increase in the interval of the time or space in which the event is expected. The rate of intensity of events is represented by a constant term which also represents the mean density of the events modelled in time or space. The homogeneous Poisson process assumes that the rate of occurrence is stationary [\[21\]](#).

[Proposition 3.1.1](#) shows that the distribution of arrivals depends only on the length of the interval and is independent of the location of the interval. In other words, the intensity of arrivals does not change over time. However, modelling real-life phenomena including risk probabilities often requires the rate of occurrence to vary with time as stationarity can only be assumed for a short period of time. A natural generalisation of the homogeneous Poisson process that would account for non-stationarity involves treating the rate λ as a function of t rather than a constant. As λ varies over time, it can be interpreted as instantaneous intensity of the arrival flow at time t [\[44\]](#). A constant intensity corresponds to the homogeneous Poisson process described in [Section 3.1.1](#).

Generally, a subordinated process is understood as a superposition of two independent stochastic processes [\[24\]](#). The main process is referred to as the parent process, or the outer process. The time of the process is replaced by an independent stochastic

process called the outer process, or the subordinator. This allows for altering certain characteristics of the parent process while still keeping some of its properties. Well-known subordinators covered in the present review include Poisson process, one-sided stable process with index $\alpha \in (0, 1)$ (α -stable subordinator), and tempered stable subordinators [24]. Subordinators arise naturally in risk theory. In a general perturbed risk process, the cumulative claim process has to be increasing. Assuming stationary independent increments, this implies that the cumulative claim process should be modelled using subordinators [19].

Studying subordinators involves describing the first-passage time of the process. The inverse subordinator is the process obtained by considering the first-passage time of a subordinator [51]. It is possible for a Lévy subordinator to be itself a first passage time of some other Lévy process. The use of inverse subordinators that is the most relevant to the present paper is achieving a process with sub-diffusion. Sub-diffusion, or anomalous diffusion, refers to a property of the process variance growing at a rate which varies non-linearly with time [51]. While the moments of a subordinator may be infinite, the moments of the inverses of specific types of subordinators are finite.

As described in Section 2.1, subordinators have infinite moments, but the first-hitting time process of the subordinator may have finite moments. For α -stable subordinators, all moments of order α or less are infinite while the first moment of its inverse grows as t^α [51]. In general, stable distributions possess heavy tails and are infinitely divisible which provides a rich class of Lévy processes [24]. The key limitation of α -stable processes is that generally they have non-finite second-order moments, which will be addressed by considering tempered stable subordinators in Section 2.3.

The inverse α -stable subordinator is defined using the Mittag-Leffler family of functions and fractional derivatives. The link between fractional calculus and probability is reflected in the interpretation of equations describing diffusion processes [38]. Specifically, in a Brownian motion, the probability of a jump exceeding a certain length x falls off as a power law $x^{-\alpha}$ for some $\alpha \in (0, 2)$. Then the random walk limit is an α -stable Lévy process with particle traces being fractals of dimension α [1]. The particle density is described by a diffusion equation which includes a fractional derivative of order α .

The link between subordinators and risk models is in the use of inverse subordinators instead of time in Poisson processes [31]. As will be shown in Section 3.2, the fractional Poisson process defined as a renewal process with Mittag-Leffler waiting times can be also obtained by time-changing via an inverse α -stable subordinator [37].

The Mittag-Leffler function can be regarded as an extension of the complex exponential function. [Section 2.2.2](#) introduces a one-parameter generalisation as an infinite series. This parametrisation corresponds to the commonly used two-parameter definition with $\beta = 1$. The Mittag-Leffler distribution is a heavy-tailed distribution. As such, when used to define waiting times for a Poisson-type renewal process it implies fewer arrivals on average for sufficiently large t compared to the classical Poisson process described in [Section 3.1](#) [9].

As noted in [Section 2.2](#), α -stable processes do not have finite second-order moments unless they correspond to the Gaussian case [27]. This issue is addressed by the process of tempering which in essence makes the tail probabilities decay faster. The present study follows established literature on risk models and considers exponential tempering of α -stable distributions [16]. This ensures that the resulting distributions are infinitely divisible, have finite moments of all orders, and have exponentially decaying tail probabilities [24]. Exponential tempering generates a class of continuous time Lévy processes that are no longer self-similar. In line with [Section 2.1](#), inverse tempered α -stable subordinators are defined as first-hitting times of a tempered α -stable subordinator [27]. Tempering allows the inverse subordinator to model power law waiting times while still having finite moments [1].

Mixtures of tempered α -stable subordinators generalise both tempered stable subordinators and α -stable subordinators [17]. As such, they can also be used instead of the process time in subordinated Poisson processes to allow for more flexibility in modelling deviations from Poisson processes [21].

[Sections 2.1–2.4](#) have introduced the notions of inverse subordinators and some of the major classes of these processes. These developments tie in with risk modelling by unifying fractional and time-fractal processes. In particular, the fractional Poisson process is defined as a renewal process with waiting times that are distributed according to the Mittag-Leffler distribution. Such a process can be shown to be equivalent to an ordinary Poisson process subordinated so that the process time is replaced by an inverse α -stable subordinator [37]. One of the key differences between an ordinary Poisson process and a fractional Poisson process is that the latter does not have stationary and independent increments [43]. It follows that fractional Poisson processes are not Lévy processes or Markov processes. However, this is not necessarily a shortcoming as the light-tailed distributed waiting times of Lévy processes fail to accurately describe phenomena with long memory [21].

The key advantage of fractional processes is that they allow for modelling anomalous diffusion described by space-time fractional diffusion equations for processes whose variances vary in time according to a power law [2]. More generally, the fractional Poisson process can also be viewed as the thinning limit of counting processes associated with renewal processes with inter-arrival times that are power-law distributed [43]. The power law describing how variance changes with time is reflected in the notion of long range dependence [34] which was proved for fractional Poisson processes by Biard and Saussereau [2]. Fractional Poisson processes are particularly well-suited for risk modelling as they allow for capturing heavy-tailed distributed interarrival times [10].

Section 3.2 considers fractional extensions for both homogeneous and non-homogeneous Poisson processes. Defining a fractional non-homogeneous Poisson process as a classical renewal process is not possible since it is an additive process with an intensity function that is deterministic and time-dependent [31]. This illustrates the usefulness of the subordinator construction developed in Sections 2.1–2.4 as a general renewal process can be considered instead. This is described in Section 3.4 which defines the fractional non-homogeneous Poisson process as an ordinary non-homogeneous Poisson process time-changed by the inverse α -stable subordinator.

Section 2.4 described mixtures of tempered α -stable subordinators while Section 3.2 introduced fractional Poisson processes. Section 3.3 illustrates further applications of notions developed in Section 2.4 to the Poisson process. This yields a generalisation of both homogeneous and non-homogeneous fractional Poisson processes by time-changing via a mixture of tempered stable subordinators [16]. The resulting process is more flexible compared to a Poisson process subordinated by a single inner process while still retaining the advantages of fractional processes and tempered subordinators that allow its application to real-life risk models.

The classical risk model describes the number of claims on a time interval as a Poisson process [40]. The total claims on the interval can therefore be retrieved as a sum of claim sizes, assumed to be non-negative and *i.i.d.*, over the Poisson-modelled number of claims. The risk, or surplus, process is then understood as the sum of the initial surplus and premiums less total claims at a specific time. This characterisation, while being relatively simplistic, allows for explicitly calculating the time of ruin, the infinite-horizon survival probabilities, and the infinite-horizon ruin probabilities as a function of the initial surplus. The finite-horizon ruin probability and finite-horizon survival probability can be retrieved as a function of time and the initial surplus. Both finite- and infinite-horizon survival

probabilities are non-decreasing functions of the initial surplus and a non-increasing function of time, but in general can be non-differentiable or even discontinuous [40]. The classical risk model is extended by considering a fractional Poisson process as the underlying process for the number of claims on a time interval.

The safety loading ρ is required to be strictly positive for the net profit condition to hold [25]. Non-positive values of the safety loading parameter correspond to companies that will necessarily go bankrupt. The time scale of the counting process $N_\alpha(t)$ and the payment process $Y_\alpha(t)$ should be coordinated as otherwise, the model would overestimate the number of claims or premiums and the model would violate the profit condition. Compared to a model based on the ordinary Poisson process, the fractional model implies a covariance structure that captures accumulation effects [25]. In particular, the premium associated with the simple compound Poisson process described in Definition 3.4.1 is strictly lower than the premium associated with the fractional compound Poisson process described in Definition 3.4.3.

Comparing the expressions for covariance between $R_1(t)$ and $R_2(t)$ shows that the main difference is in the safety loading factor ρ entering the expression for the covariance of $R_2(t)$. As such, the risk process of the first type exhibiting the long range dependence property implies that the risk process of the second type also exhibits the long range dependence property [20].

For fractional Poisson processes, simulation of trajectories is enabled by relating subordinators to α -stable distributions [25]. This illustrates why the description of fractional Poisson processes as a process time-changed by an inverse stable subordinator is particularly useful for applications and modelling. The present study relies on the acceptance-rejection method to generate increments from mixtures of subordinators. This is a general method for drawing from a random distribution for which the density is bounded by a density of another distribution scaled by some constant. The study fixes model parameters such as intensity rate for the homogeneous Poisson process, functional form of the intensity rate for the non-homogeneous Poisson process, tempering parameters, and subordinator mixture weights. This allows for focusing on the differences between process types given identical parameters which should help illustrate how more advanced models may be more suitable for describing real-life scenarios. The impact and choice of risk model parameters on ruin probabilities is discussed in later chapters of the study.

Chapter 2

Subordinator and Inverse subordinator

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2.1 Subordinator and Inverse subordinator

Definition 2.1.1 (Homogeneous Lévy process). The process $L := \{L(t)\}_{t \geq 0}$ is said to be a homogeneous Lévy process if all the following conditions hold:

- The process has independent and stationary increments,
- $L(0) = 0$ almost surely (a.s.),
- L is stochastically continuous, i.e. for any $\varepsilon > 0$ and $t > 0$,

$$\lim_{h \rightarrow 0} \mathbb{P}(|L(t+h) - L(t)| > \varepsilon) = 0,$$

- L is cadlag a.s., also called right continuous with left limits (RCLL),
- For $0 \leq s \leq t$, $L(t) - L(s) \stackrel{d}{=} L(t-s)$.

Definition 2.1.2 (Nonhomogeneous Lévy process). The process $L := \{L(t)\}_{t \geq 0}$ is said to be a nonhomogeneous Lévy process if all the following conditions hold:

- The process has independent increments,
- $L(0) = 0$ a.s.,
- L is stochastically continuous,
- L is cadlag a.s.

Examples of homogeneous Lévy processes include HPP and homogeneous Wiener process; and NPP is an example of nonhomogeneous Lévy process.

Definition 2.1.3 (Subordinators). A Lévy process $L := \{L(t)\}_{t \geq 0}$ is said to be a subordinator if it is non-decreasing a.s. [28]

Definition 2.1.4 (Inverse subordinators). An inverse subordinator is the first-hitting time process of the given subordinator, i.e. for a subordinator $L(t)$, the inverse subordinator is defined as [26]

$$Y(t) = \inf \{s \geq 0 : L(s) > t\}. \quad (2.1)$$

Definition 2.1.5 (Renewal function). The renewal function of a process $Y(t)$ is defined as

$$U(t) = \mathbb{E}[Y(t)]. \quad (2.2)$$

Properties

Definition 2.1.6 (Laplace exponent). For the following Laplace transform,

$$\mathbb{E} \left[e^{-sL(t)} \right] = e^{-t\phi(s)}, \quad s, t \geq 0, \quad (2.3)$$

the function ϕ is unique, and it is said to be the Laplace exponent.

From [50], we have Equations 2.4 and 2.5.

Let $L(t)$, $t \geq 0$ be a Lévy subordinator with Laplace exponent

$$\phi(s) = \mu s + \int_{(0, \infty)} (1 - e^{-sx}) \Pi(dx), \quad s \geq 0, \quad (2.4)$$

where $\mu \geq 0$ is the drift and the Lévy measure Π on $\mathbb{R}_+ \cup \{0\}$ satisfies

$$\int_0^\infty (1 \wedge x) \Pi(dx) < \infty. \quad (2.5)$$

The Lévy measure in Equation 2.4 is a measure on the σ -algebra of Borel sets of $(0, \infty)$, which satisfies the condition of Equation 2.5. However, as this is the only condition needs to be met for a Lévy measure, it is not necessarily to be finite, for example, the Lévy measure of an α -stable subordinator is not finite, as the integral of such a measure diverges at zero (see Equation 2.14).

The process $Y(t)$, $t \geq 0$, is non-decreasing and its sample paths are a.s. continuous if L is strictly increasing. Also $Y(t)$ is, in general, non-Markovian with non-stationary and non-independent increments.

We have

$$\{L(u_i) < t_i, i = 1, \dots, n\} = \{Y(t_i) > u_i, i = 1, \dots, n\}. \quad (2.6)$$

Let

$$H_u(t) = P \{Y(u) < t\}. \quad (2.7)$$

From Equation 2.4 we have

$$\int_0^\infty e^{-st} dH_u(t) = e^{-t\phi(s)}, \quad (2.8)$$

and

$$\tilde{U}(s) = \int_0^\infty U(t) e^{-st} dt = \frac{1}{s\phi(s)}, \quad (2.9)$$

where $\tilde{U}(s)$ denotes the Laplace transform of $U(s)$; thus, \tilde{U} characterizes the process $Y(t)$ (since ϕ characterizes L) [50].

And we have the covariance function [50, Equation (17)]

$$\text{Cov}(Y(t_1), Y(t_2)) = \int_0^{t_1 \wedge t_2} (U(t_1 - \tau) + U(t_2 - \tau)) dU(\tau) - U(t_1)U(t_2). \quad (2.10)$$

2.2 α -stable subordinator and inverse α -stable subordinator

In this section we present some results from [25].

2.2.1 α -stable subordinator

Let $L_\alpha(t)$, $t \geq 0$, be an α -stable subordinator with $\phi(s) = s^\alpha$, $0 < \alpha < 1$ (cadlag, continuous in probability, with independent and stationary increments), whose density $g(t, x)$ is such that $L_\alpha(1)$ has probability density function

$$\begin{aligned} g_\alpha(x) &= g(1, x) \\ &= \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma(\alpha k + 1)}{k!} \frac{1}{x^{\alpha k + 1}} \sin(\pi k \alpha) \\ &= \frac{1}{x} W_{-\alpha, 0}(-x^{-\alpha}), \quad x > 0, \end{aligned} \quad (2.11)$$

and with Laplace transform

$$\mathbb{E} \left[e^{-s L_\alpha(t)} \right] = \exp(-t s^\alpha), \quad s \geq 0. \quad (2.12)$$

Here we use the Wright's generalised Bessel function (see, e.g., [18])

$$W_{\gamma, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k) \Gamma(\beta + \gamma k)}, \quad z \in \mathbb{C},$$

where $\gamma > -1$, and $\beta \in \mathbb{C}$ [47], and for $\beta = 0, \gamma = -\alpha \in (-1, 0)$

$$W_{-\alpha, 0}(z) = \sum_{k=1}^{\infty} \frac{\sin(\pi k \alpha)}{\pi} \frac{z^k \Gamma(1 + \alpha k)}{k!} \quad (2.13)$$

by reciprocity relation for the Γ -function. Also

$$g_\alpha(x) \sim \frac{\left(\frac{\alpha}{x}\right)^{\frac{2-\alpha}{2(1-\alpha)}}}{\sqrt{2\pi\alpha(1-\alpha)}} \exp \left\{ -(1-\alpha) \left(\frac{x}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}} \right\}, \quad x \rightarrow 0, \quad (2.14)$$

and

$$g_\alpha(x) \sim \frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}, \quad x \rightarrow \infty. \quad (2.15)$$

Remark.

1. The α -stable subordinator has no finite moments.
2. The sample path of an α -stable subordinator is not continuous as it contains jumps at the points of events.

2.2.2 Inverse α -stable subordinator

Then we have the inverse α -stable subordinator

$$Y_\alpha(t) = \inf\{u \geq 0 : L_\alpha(u) > t\}, \quad (2.16)$$

which has density function of

$$f_\alpha(t, x) = \frac{t}{\alpha} x^{-1-\frac{1}{\alpha}} g_\alpha(tx^{-\frac{1}{\alpha}}), \quad x \geq 0, t > 0. \quad (2.17)$$

Definition 2.2.1 (Fractional Caputo-Djrbashian derivative). The fractional Caputo-Djrbashian derivative ${}^C D_t^\alpha$ is given by [5]

$${}^C D_t^\alpha = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{df(\tau)}{d\tau} \frac{d\tau}{(t-\tau)^\alpha}, \quad \alpha \in (0, 1). \quad (2.18)$$

Let

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, z \in \mathbb{C}$$

be the Mittag-Leffler function [13],[18].

Remark. It reduces to $E_\alpha(-z) = e^{-z}$ if $\alpha = 1$.

Asymptotic behaviour

From [36], we have the following asymptotic behaviours for $E_\alpha(-t^\alpha)$:

$$E_\alpha(-t^\alpha) \sim \exp\left(-\frac{t^\alpha}{\Gamma(1+\alpha)}\right), \quad t \rightarrow 0, \quad (2.19)$$

$$E_\alpha(-t^\alpha) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \rightarrow \infty. \quad (2.20)$$

Recall the following:

- (i) The Laplace transform of the Mittag-Leffler function is of the form

$$\int_0^\infty e^{-st} E_\alpha(-t^\alpha) dt = \frac{s^{\alpha-1}}{1+s^\alpha}, \quad 0 < \alpha < 1, \quad s \geq 0, \quad t \geq 0.$$

- (ii) The Mittag-Leffler function is a solution of the fractional equation with fractional Caputo-Djrbashian derivative ${}^C D_t^\alpha$

$${}^C D_t^\alpha E_\alpha(at^\alpha) = a E_\alpha(at^\alpha).$$

Proposition.

- (i)

$$\mathbb{E} \left[e^{-sY_\alpha(t)} \right] = \sum_{n=0}^{\infty} \frac{(-st^\alpha)^n}{\Gamma(\alpha n + 1)} = E_\alpha(-st^\alpha), \quad s \geq 0,$$

- (ii) both processes $L_\alpha(t), t \geq 0$ and $Y_\alpha(t)$ are self-similar

$$\frac{L_\alpha(at)}{a^{1/\alpha}} \stackrel{d}{=} L_\alpha(t), \quad \frac{Y_\alpha(at)}{a^\alpha} \stackrel{d}{=} Y_\alpha(t), \quad a > 0,$$

- (iii)

$$\frac{\partial \mathbb{E}(Y_\alpha(t_1) \cdots Y_\alpha(t_k))}{\partial t_1 \cdots \partial t_k} = \frac{1}{\Gamma^k(\alpha)} \frac{1}{[t_1(t_2 - t_1) \cdots (t_k - t_{k-1})]^{1-\alpha}}, \quad 0 < t_1 < \cdots < t_k.$$

In particular,

- (a) we have the first moment of $Y_\alpha(t)$,

$$\mathbb{E}[Y_\alpha(t)] = \frac{t^\alpha}{\Gamma(1+\alpha)};$$

(b) and we have the covariance function as

$$\begin{aligned} & \text{Cov}(Y_\alpha(s), Y_\alpha(t)) \\ &= \frac{1}{\Gamma(1+\alpha)\Gamma(\alpha)} \int_0^{\min(s,t)} ((s-\tau)^\alpha + (t-\tau)^\alpha) \tau^{\alpha-1} d\tau - \frac{(st)^\alpha}{\Gamma^2(1+\alpha)}. \end{aligned} \quad (2.21)$$

Comments.

1. Notice that this last property can be interpreted as long-range dependence.
2. There is a (complicated) form of all finite-dimensional distributions of $Y_\alpha(t), t \geq 0$, in the form of Laplace transforms, see [3].

Remark.

1. The inverse α -stable subordinator has finite moments.
2. Unlike α -stable subordinator, the sample path of an inverse α -stable subordinator is continuous.

Alternate form of α -stable and inverse α -stable subordinator

The density function of the α -stable subordinator, $g_\alpha(x, t)$, has the following integral form [23]

$$g_\alpha(x, t) = \frac{1}{\pi} \int_0^\infty e^{-ux - tu^\alpha \cos \alpha\pi} \sin(tu^\alpha \sin \alpha\pi) du, \quad x > 0. \quad (2.22)$$

The density function of the inverse α -stable subordinator, $f_\alpha(x, t)$, has the following integral form [23]

$$f_\alpha(x, t) = \frac{1}{\pi} \int_0^\infty u^{\alpha-1} e^{-tu - xu^\alpha \cos(\alpha\pi)} \sin(\alpha\pi - xu^\alpha \sin \alpha\pi) du, \quad x > 0. \quad (2.23)$$

The density function of the α -stable subordinator at $t = 1$, shorten as $g_\alpha(x)$, could also be expressed as followed (see [39]):

$$g_\alpha(x) = \frac{\alpha}{1-\alpha} \frac{1}{\pi x} \int_0^\pi u e^{-u} d\varphi, \quad 0 < \alpha < 1, \quad (2.24)$$

where

$$u = u(\varphi) = \frac{\sin(1-\alpha)\varphi}{\sin \varphi} \left(\frac{\sin \alpha\varphi}{x \sin \varphi} \right)^{\frac{\alpha}{1-\alpha}}.$$

The density function of the inverse α -stable subordinator at $t = 1$, $f_\alpha(x)$, could also be expressed as followed (see [45]):

$$f_\alpha(x) = \frac{1}{\alpha} x^{-(1+\frac{1}{\alpha})} g_\alpha\left(x^{-\frac{1}{\alpha}}\right), \quad x > 0. \quad (2.25)$$

Asymptotic behaviour

From [45], we have the following asymptotic behaviours for $f_\alpha(x)$:

$$f_\alpha(x) \rightarrow \frac{\sin(\alpha\pi)}{\alpha\pi} \Gamma(1+\alpha), \quad x \rightarrow 0^+, \quad (2.26)$$

$$f_\alpha(x) \sim \frac{K}{\alpha} x^{\frac{2\alpha-1}{2(1-\alpha)}} \exp\left(-Ax^{\frac{1}{1-\alpha}}\right), \quad x \rightarrow \infty, \quad (2.27)$$

where

$$A = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}},$$

$$K = \frac{\alpha^{\frac{1}{2(1-\alpha)}}}{\sqrt{2\pi(1-\alpha)}}.$$

From [39], we have the following asymptotic behaviors for $g_\alpha(x)$:

$$g_\alpha(x) \rightarrow K x^{-\frac{2-\alpha}{2(1-\alpha)}} \exp\left(-Ax^{-\frac{\alpha}{1-\alpha}}\right), \quad x \rightarrow 0^+, \quad (2.28)$$

$$g_\alpha(x) \sim \frac{\sin \alpha\pi}{\pi} \Gamma(1+\alpha) x^{-(1+\alpha)}, \quad x \rightarrow \infty. \quad (2.29)$$

Thus,

$$f_\alpha(x, t) \rightarrow \frac{1}{t^\alpha} \frac{\sin(\alpha\pi)}{\alpha\pi} \Gamma(1+\alpha), \quad x \rightarrow 0^+, \quad (2.30)$$

and

$$f_\alpha(x, t) \sim \frac{K t^{-\frac{\alpha}{2(1-\alpha)}}}{\alpha} x^{\frac{2\alpha-1}{2(1-\alpha)}} \exp\left(-\frac{A x^{\frac{1}{1-\alpha}}}{t^{\frac{\alpha}{1-\alpha}}}\right), \quad x \rightarrow \infty \quad (2.31)$$

for any fixed t .

2.3 Tempered α -stable subordinator and inverse tempered α -stable subordinator

2.3.1 Tempered α -stable subordinator

Definition 2.3.1 (Tempered α -stable subordinator). The tempered α -stable subordinator $S_{\alpha,\lambda}(t)$, with tempering parameter $\lambda > 0$ and stability index $\alpha \in (0, 1)$, is a Lévy process which has density function

$$g_{\alpha,\lambda}(x, t) = e^{-\lambda x + \lambda^\alpha t} g_\alpha(x, t), \quad (2.32)$$

where $g_\alpha(x, t)$ is the density function of the α -stable subordinator; and with Laplace transform

$$\mathbb{E} \left(e^{-s S_{\alpha,\lambda}(t)} \right) = e^{-t[(s+\lambda)^\alpha - \lambda^\alpha]}, \quad s > 0. \quad (2.33)$$

The first two moments and covariance of $S_{\alpha,\lambda}(t)$ are given by

$$\begin{aligned} \mathbb{E}(S_{\alpha,\lambda}(t)) &= \alpha \lambda^{\alpha-1} t \\ \mathbb{E}(S_{\alpha,\lambda}^2(t)) &= \alpha(1-\alpha) \lambda^{\alpha-2} t + (\alpha \lambda^{\alpha-1} t)^2 \\ \text{Cov}(S_{\alpha,\lambda}(s), S_{\alpha,\lambda}(t)) &= \alpha(1-\alpha) \lambda^{\alpha-2} \min(s, t), \quad s, t \geq 0 \end{aligned} \quad (2.34)$$

as shown in [17].

2.3.2 Inverse tempered α -stable subordinator

Definition 2.3.2. We have the inverse tempered α -stable subordinator $Y_{\alpha,\lambda}(t)$, defined by

$$Y_{\alpha,\lambda}(t) = \inf\{u \geq 0 : S_{\alpha,\lambda}(u) > t\}, \quad t \geq 0. \quad (2.35)$$

Let $f_{\alpha,\lambda}(x, t)$ denotes the density function of $Y_{\alpha,\lambda}(t)$. Then $f_{\alpha,\lambda}(x, t)$ has the Laplace transform given by

$$\mathcal{L}(f_{\alpha,\lambda}(x, t)) = \frac{1}{s} [(s+\lambda)^\alpha - \lambda^\alpha] e^{-x[(s+\lambda)^\alpha - \lambda^\alpha]}, \quad (2.36)$$

and

$$f_{\alpha,\lambda}(x,t) = \frac{1}{\pi} e^{\lambda^\alpha x - \lambda t} \int_0^\infty \frac{e^{-ty - xy^\beta \cos(\beta\pi)}}{y + \lambda} \left[\lambda^\beta \sin(xy^\beta \sin(\beta\pi)) + y^\beta \sin(\beta\pi - xy^\beta \sin(\beta\pi)) \right] dy, \quad x > 0, \lambda > 0, \beta \in (0, 1). \quad (2.37)$$

2.4 Mixture of tempered α -stable subordinators and inverse mixture of tempered α -stable subordinators

In this section we present some results from [17].

2.4.1 Mixture of tempered α -stable subordinators

In this section we present some results from [17]. We define a mixture of tempered stable subordinators (MTSS) as Lévy stochastic process $\{S_m(t); t \geq 0\}$ with the following Laplace exponent

$$\mathbb{E}[\exp\{-sS_m(t)\}] = e^{-t\phi(s)}, \quad s > 0, \quad (2.38)$$

where

$$\phi(s) = c_1 ((s + \lambda_1)^{\alpha_1} - \lambda_1^{\alpha_1}) + c_2 ((s + \lambda_2)^{\alpha_2} - \lambda_2^{\alpha_2}), \quad s > 0, \quad (2.39)$$

and $\alpha_i \in (0, 1)$, $\lambda_i \geq 0$, $c_i \geq 0$, $i = 1, 2$, $c_1 + c_2 = 1$.

Note that for $\alpha_1 = \alpha$, $c_1 = 1$, $c_2 = 0$, $\lambda_1 = 0$ the process $S_m(t)$ reduces to the α -stable subordinator $S_\alpha(t)$ defined in Equation 2.11.

An alternative definition of the MTSS can be given as a sum of the independent tempered stable Lévy processes $S_i(t)$, $i = 1, 2$ with Laplace exponents $\phi_i(s) = (s + \lambda_i)^{\alpha_i} - \lambda_i^{\alpha_i}$, $i = 1, 2$ with time scaling and the conditions $c_1 + c_2 = 1$ such that

$$S_m(t) = S_{\alpha_1, \lambda_1, \alpha_2, \lambda_2}(t) = S_1(c_1 t) + S_2(c_2 t), \quad t \geq 0. \quad (2.40)$$

The advantage of tempered stable subordinators over α -stable subordinators is that they have finite moments of all orders and their density is also infinitely divisible.

The first moment and covariance structure of the MTSS are given by

$$\mathbb{E}[S_m(t)] = t \left(c_1 \alpha_1 \lambda_1^{\alpha_1-1} + c_2 \alpha_2 \lambda_2^{\alpha_2-1} \right) \quad (2.41)$$

and

$$\text{Cov}(S_m(s), S_m(t)) = \min(t, s) \left[c_1 \alpha_1 (1 - \alpha_1) \lambda_1^{\alpha_1-2} + c_2 \alpha_2 (1 - \alpha_2) \lambda_2^{\alpha_2-2} \right]. \quad (2.42)$$

The pdf $g_m(x, t) = \frac{d}{dx} \mathbb{P}[S_m(t) \leq x]$ has the following representation for $\lambda_1 \neq \lambda_2$:

$$\begin{aligned} g_m(x, t) = & \frac{1}{\pi} \int_0^\infty \exp(-x\lambda_2) \exp(-wx) \exp[t(c_1 \lambda_1^{\alpha_1} + c_2 \lambda_2^{\alpha_2})] \\ & \times \exp \left[-t \left(c_1 (\lambda_1 - \lambda_2)^{\alpha_1} \sum_{k=0}^\infty \binom{\alpha_1}{k} \frac{w^k}{(\lambda_1 - \lambda_2)^k} \cos \pi k + c_2 w^{\alpha_2} \cos \pi \alpha_2 \right) \right] \\ & \times \sin \left(c_1 t (\lambda_1 - \lambda_2)^{\alpha_1} \sum_{k=0}^\infty \binom{\alpha_1}{k} \frac{w^k}{(\lambda_1 - \lambda_2)^k} \sin(\pi k) + c_2 t w^{\alpha_2} \sin(\pi \alpha_2) \right) dw \\ & + \frac{1}{\pi} \int_0^{\lambda_2 - \lambda_1} \exp(-x\lambda_1) \exp(-wx) \exp[t(c_1 \lambda_1^{\alpha_1} + c_2 \lambda_2^{\alpha_2})] \\ & \times \exp \left[-t \left(c_1 w^{\alpha_1} \cos(\pi \alpha_1) + c_2 (\lambda_1 - \lambda_2)^{\alpha_2} \sum_{k=0}^\infty \binom{\alpha_2}{k} \frac{w^k}{(\lambda_1 - \lambda_2)^k} \cos(\pi k) \right) \right] \\ & \times \sin \left(c_1 t w^{\alpha_1} \sin(\pi \alpha_1) + c_2 t (\lambda_1 - \lambda_2)^{\alpha_2} \sum_{k=0}^\infty \binom{\alpha_2}{k} \frac{w^k}{(\lambda_1 - \lambda_2)^k} \sin(\pi k) \right) dw, \end{aligned} \quad (2.43)$$

if $\lambda_1, \lambda_2 = \lambda$ and $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$, then

$$\begin{aligned} g_m(x, t) = & \frac{1}{\pi} \int_0^\infty \exp(-x\lambda) \exp(-wx) \exp[t(c_1 \lambda^{\alpha_1} + c_2 \lambda^{\alpha_2})] \\ & \times \exp[-t(c_1 w^{\alpha_1} \cos(\pi \alpha_1) + c_2 w^{\alpha_2} \cos(\pi \alpha_2))] \\ & \times \sin(t(c_1 w^{\alpha_1} \sin(\pi \alpha_1) + c_2 w^{\alpha_2} \sin(\pi \alpha_2))) dw, \end{aligned} \quad (2.44)$$

where $c_1 + c_2 = 1$ and $c_1, c_2 \geq 0$.

For the special case $\alpha_1 = \alpha_2 = \alpha$ and $\lambda_1 = \lambda_2 = 0$ with the condition $c_1 + c_2 = 1$ the formula (2.43) and (2.44) reduce to the known formulae for the α -stable subordinator:

$$g_m(x, t) = \frac{1}{\alpha t x^{1+\alpha}} M_\alpha \left(\frac{t}{x^\alpha} \right),$$

where the M-Wright function $M_\alpha(z)$ is defined as

$$M_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k! \Gamma(-\alpha k + (1 - \alpha))}, \quad z \in \mathbb{C}. \quad (2.45)$$

2.4.2 Inverse mixture of tempered α -stable subordinators

Next, we define the inverse MTSS (IMTSS) as the right continuous stochastic process

$$Y_m(t) = \inf\{u > 0 : S_{\alpha_1, \lambda_1, \alpha_2, \lambda_2}(u) > t\} = \inf\{u > 0 : S_m(u) > t\}, \quad t \geq 0. \quad (2.46)$$

Note that for $\alpha_1 = \alpha$, $c_1 = 1$, $c_2 = 0$, $\lambda_1 = 0$ the process $Y_m(t)$ reduces to the inverse α -stable subordinator $Y_\alpha(t)$ defined in [Equation 2.16](#).

The renewal function for the IMTSS is given by

$$U_m(t) = \mathbb{E}[Y_m(t)] \quad (2.47)$$

and its Laplace transform

$$\tilde{U}(s) = \frac{1}{\phi(s)}, \quad (2.48)$$

where $\phi(s)$ is given by [\(2.39\)](#). The asymptotic behavior of this renewal function is as follows:

$$U(t) \sim \begin{cases} \frac{t^{\alpha_1 + \alpha_2 - \min(\alpha_1, \alpha_2)}}{\Gamma(1 + \min(\alpha_1, \alpha_2)) (c_1 t^{\alpha_2 - \min(\alpha_1, \alpha_2)} + c_2 t^{\alpha_1 - \min(\alpha_1, \alpha_2)})}, & \text{as } t \rightarrow 0, \\ \frac{t}{(c_1 \alpha_1 \lambda_1^{\alpha_1 - 1} + c_2 \alpha_2 \lambda_2^{\alpha_2 - 1})}, \quad \lambda_1, \lambda_2 > 0, & \text{as } t \rightarrow \infty, \\ \frac{t^{\alpha_1 + \alpha_2 - \min(\alpha_1, \alpha_2)}}{\Gamma(1 + \min(\alpha_1, \alpha_2)) (c_1 t^{\alpha_2 - \min(\alpha_1, \alpha_2)} + c_2 t^{\alpha_1 - \min(\alpha_1, \alpha_2)})}, \quad \lambda_1 = \lambda_2 = 0, & \text{as } t \rightarrow \infty. \end{cases}$$

The cdf of IMTSS is given by $\mathbb{P}[Y_m(t) \leq x] = \int_t^\infty g_m(x, u) du$, and the pdf

$$h_m(x, t) = \frac{d}{dx} \mathbb{P}[Y_m(t) \leq x] = \frac{d}{dx} \left(1 - \int_t^\infty g_m(x, u) du \right). \quad (2.49)$$

The covariance structure is

$$Cov(Y_m(s), Y_m(t)) = \int_0^{s \wedge t} [U(t - \tau) - U(s - \tau)] d\tau - U(s)U(t), \quad s, t \geq 0, \quad (2.50)$$

where $U(t)$ is defined by (2.47).

Chapter 3

Count and risk processes

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3.1 Poisson processes

3.1.1 Homogeneous Poisson process

Definition 3.1.1 (Homogeneous Poisson process (HPP) (first definition)). The process $N^\mu = \{N^\mu(t), t \in [0, \infty)\}$ is said to be a HPP having rate μ , $\mu > 0$, if the following conditions are met.

- $N^\mu(0) = 0$.
- The process has independent and stationary increments.
- The number of arrivals P_m in any interval of length $\tau > 0$ is Poisson distributed with mean $(\mu\tau)$.

Thus, we have

$$\begin{aligned} P_m(t) &= P\{N^\mu(t+u) - N^\mu(u) = m\} \\ &= \frac{e^{-[\mu(t+u)-\mu u]} [\mu(t+u) - \mu u]^m}{m!}, \quad m = 0, 1, 2, \dots \end{aligned} \quad (3.1)$$

HPP could also be defined as follow.

Definition 3.1.2 (HPP (second definition)). The process $N^\mu = \{N^\mu(t), t \in [0, \infty)\}$ is said to be a HPP having rate $\mu, \mu > 0$, if the following conditions are met.

- $N^\mu(0) = 0$.
- The process has independent and stationary increments.
- For any $t \in [0, \infty)$, as $h \rightarrow 0$,

$$\begin{aligned} P(N^\mu(t+h) = m \mid N^\mu(t) = m) &= P\{N^\mu(t+h) - N^\mu(t) = 0\} \\ &= 1 - \mu th + o(h); \\ P(N^\mu(t+h) = m+1 \mid N^\mu(t) = m) &= P\{N^\mu(t+h) - N^\mu(t) = 1\} \\ &= \mu th + o(h); \\ P(N^\mu(t+h) = m+k \mid N^\mu(t) = m) &= P\{N^\mu(t+h) - N^\mu(t) = k\} \\ &= o(h), k \geq 2. \end{aligned}$$

Proposition 3.1.1. Let $N^\mu(t)$ be a HPP with rate μ . Let T_n be the time of n^{th} arrival, and Y_n be the inter-arrival time between the $(n-1)^{\text{th}}$ and the n^{th} arrival, i.e. $Y_n = T_n - T_{n-1}$. Then,

$$\begin{aligned} P(Y_1 > t) &= P(\text{no arrival in } (0, t]) \\ &= e^{-\mu t}; \end{aligned}$$

$$\begin{aligned} P(Y_2 > t \mid Y_1 = t_1) &= P(\text{no arrival in } (t_1, t_1 + t] \mid Y_1 = t_1) \\ &= P(\text{no arrival in } (t_1, t_1 + t]) \quad (\text{independent increments}) \\ &= e^{-\mu t}. \end{aligned}$$

$$\begin{aligned}
P(Y_{n+1} > t \mid Y_1 = t_1, \dots, Y_n = t_n) &= P(\text{no arrival in } (t_n, t_n + t] \mid Y_1 = t_1, \dots, Y_n = t_n) \\
&= P(\text{no arrival in } (t_n, t_n + t]) \\
&= e^{-\mu t}.
\end{aligned}$$

Then we have

$$F_{Y_n}^{\text{HPP}}(t) = \begin{cases} 1 - e^{-\mu t}, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}, \quad n \in \mathbb{Z}^+.$$

Thus, we have $Y_n \sim \text{Exp}(\mu)$.

Definition 3.1.3 (Erlang distribution). Let Y_1, Y_2, \dots, Y_n denotes n *i.i.d.* random variables, each of them has an exponential distribution with parameter μ . Then we have $\sum_{i=1}^n Y_i = T_n$, where T_n has a Erlang distribution with parameters n and μ , i.e. $T_n \sim \text{Erlang}(n, \mu)$.

An Erlang distribution has the following probability density function:

$$f_{T_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0, \lambda \geq 0, n \in \mathbb{Z}^+, \quad (3.2)$$

and the cumulative distribution function:

$$F_{T_n}(t) = 1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{1}{k!} (\lambda t)^k. \quad (3.3)$$

Remark. The Erlang distribution reduces to an exponential distribution when $n = 1$.

Proposition 3.1.2 (Arrival time). Recall that $T_n = Y_1 + \dots + Y_n$, since Y_1, \dots, Y_n are *i.i.d.* exponential random variable, thus, we have $T_n \sim \text{Erlang}(n, \mu)$.

Following the results from above, the Poisson process could also be defined as follow:

Definition 3.1.4 (HPP (third definition) (renewal representation)).

$$N^\mu(t) := \max \{n \geq 0 : Y_1 + \dots + Y_n \leq t\}, \quad (3.4)$$

where Y_1, \dots, Y_n are n *i.i.d* exponential random variables, and we have $N^\mu(t)_{t \geq 0}$ is a HPP with independent and stationary increments.

Covariance function

Let $X(t), t \geq 0$ be a HPP with rate $\mu > 0$, we have the covariance function:

$$\text{Cov}(X(s), X(t)) = \mu (\min(s, t)), \quad s \geq 0, t \geq 0. \quad (3.5)$$

3.1.2 Nonhomogeneous Poisson process

Definition 3.1.5 (Nonhomogeneous Poisson process). Let $\lambda(t) : [0, \infty) \rightarrow [0, \infty)$ be a pre-specified integrable function. The process $NN^{\Lambda(t)} = \{NN^{\Lambda(t)}(t), t \in [0, \infty)\}$ is said to be a nonhomogeneous Poisson process (NPP) having rate $\lambda(t)$, $\lambda(t) \geq 0$, if the following conditions are met.

- $NN^{\Lambda(t)}(0) = 0$.
- The process has independent increments.
- For any $t \in [0, \infty)$, as $h \rightarrow 0$,

$$\begin{aligned} P(NN^{\Lambda(t)}(t+h) = m \mid NN^{\Lambda(t)}(t) = m) &= P\{NN^{\Lambda(t)}(t+h) - NN^{\Lambda(t)}(t) = 0\} \\ &= 1 - \lambda(t)h + o(h); \\ P(NN^{\Lambda(t)}(t+h) = m+1 \mid NN^{\Lambda(t)}(t) = m) &= P\{NN^{\Lambda(t)}(t+h) - NN^{\Lambda(t)}(t) = 1\} \\ &= \lambda(t)h + o(h); \\ P(NN^{\Lambda(t)}(t+h) = m+k \mid NN^{\Lambda(t)}(t) = m) &= P\{NN^{\Lambda(t)}(t+h) - NN^{\Lambda(t)}(t) = k\} \\ &= o(h), k \geq 2. \end{aligned}$$

Let

$$\Lambda(t) = \int_0^t \lambda(u) du.$$

Then we have

$$\begin{aligned} P_m(t) &= P\{NN^{\Lambda(t)}(t) - NN^{\Lambda(t)}(0) = m\} \\ &= \frac{e^{-[\Lambda(t)-\Lambda(0)]} [\Lambda(t) - \Lambda(0)]^m}{m!}, \quad m = 0, 1, 2, \dots, \end{aligned} \quad (3.6)$$

and

$$F(t) = P(Y_1 \leq t) = 1 - \exp(-\Lambda(t)), \quad t \geq 0. \quad (3.7)$$

Remark. If $\lambda(t)$ is a positive constant at all t , then

$$\Lambda(t) = \lambda t, \Lambda(t+u) - \Lambda(u) = \lambda(t+u) - \lambda u = \lambda t.$$

i.e. it reduces to a HPP with rate λ .

Arrival time

T_n has the following distribution functions [32]:

$$F_{T_n}^{\text{NPP}}(t) = 1 - e^{-\Lambda(t)} \sum_{k=0}^{n-1} \frac{[\Lambda(t)]^k}{k!}, \quad (3.8)$$

and

$$f_{T_n}^{\text{NPP}}(t) = e^{-\Lambda(t)} \frac{\lambda(t) [\Lambda(t)]^{n-1}}{(n-1)!}. \quad (3.9)$$

Covariance function

Let $X^{\Lambda(t)}(t), t \geq 0$ be a NPP with rate $\lambda(t), \lambda(t) \geq 0$, we have the covariance function:

$$\text{Cov} \left(X^{\Lambda(t)}(s), X^{\Lambda(t)}(t) \right) = \Lambda(\min(s, t)), \quad s \geq 0, t \geq 0. \quad (3.10)$$

Similar to Definition 3.1.4, NPP could also be constructed as follow: [31]

Definition 3.1.6 (NPP (second definition) (renewal representation)). Let Y_1, Y_2, \dots be a sequence of *i.i.d.* non-negative random variables with distribution function

$$F(t) = P(Y_n \leq t) = 1 - e^{-\Lambda(t)}, \quad t \geq 0,$$

with

$$\Lambda(t) \rightarrow \infty, \quad t \rightarrow \infty.$$

Define

$$\begin{aligned} \zeta'_n &:= \max \{Y_1, \dots, Y_n\}, \quad n = 1, 2, \dots, \\ \chi_n &:= \inf \{k \in \mathbb{N} : \zeta'_k > \zeta'_{n-1}\}, \quad n = 2, 3, \dots \end{aligned}$$

with $\chi_1 = 1$, let

$$\zeta_n := \zeta'_{\chi_n},$$

and

$$NN^{\Lambda(t)}(t) := \sup \{k \in \mathbb{N} : \zeta_k \leq t\} = \sum_{n=0}^{\infty} n \mathbb{1}_{\{\zeta_n \leq t \leq \zeta_{n+1}\}}, \quad t \geq 0$$

where $\zeta_0 = 0$, and we have $NN^{\Lambda(t)}(t)_{t \geq 0}$ is a NPP with independent increments.

3.2 Fractional Poisson processes

3.2.1 Fractional homogeneous Poisson process

Definition 3.2.1 (Fractional homogeneous Poisson process (first definition)). The Fractional homogeneous Poisson process (FHPP) $N_\alpha(t)$, is defined by

$$N_\alpha(t) = N^1(Y_\alpha(t)), \quad t \geq 0, \quad \alpha \in (0, 1), \quad (3.11)$$

where $N^1(t)$ denotes a HPP as defined in [subsection 3.1.1](#) with parameter $\mu = 1$, independent of $Y_\alpha(t)$, which is the inverse stable subordinator as defined in [subsection 2.2.2](#).

Remark. It reduces to a HPP with rate μ if $\alpha = 1$ as shown in [\[29\]](#).

Definition 3.2.2 (FHPP (second definition) (renewal representation)). FHPP could also be defined with Mittag-Leffler distribution:

$$\begin{aligned} N_\alpha(t) &= \max(n : Y_1 + \cdots + Y_n \leq t) \\ &= \sum_{j=1}^{\infty} \mathbb{1}_{\{Y_1 + \cdots + Y_j \leq t\}} \\ &= \sum_{j=1}^{\infty} \mathbb{1}_{U_j \leq G_\alpha(t)}, \quad t \geq 0, \end{aligned} \quad (3.12)$$

where $\{Y_j\}, j = 1, 2, \dots$, denote the interarrival times of FHPP, which are *i.i.d.* random variables with strictly monotone Mittag-Leffler distribution, with probability function (see [\[25\]](#)):

$$F_\alpha(t) = P(Y_j \leq t) = 1 - E_\alpha(-\mu t^\alpha), \quad t \geq 0, \alpha \in (0, 1), j = 1, 2, \dots, \quad (3.13)$$

$$G_\alpha(t) = P(Y_1 + \cdots + Y_k \leq t) = \int_0^t h^{(k)}(x) dx,$$

where

$$\begin{aligned} h^{(k)}(x) &= \alpha \mu^k \frac{x^{k\alpha-1}}{(k-1)!} E_\alpha^{(k)}(-\mu x^\alpha) \\ &= \mu^k x^{\alpha k-1} E_{\alpha, \alpha k+1}^k(-\mu x^\alpha), \quad \alpha \in (0, 1), x > 0 \end{aligned}$$

Remark.

1. Mittag-Leffler distribution has no finite moments for $\alpha \in (0, 1)$. Also, it reduces to an exponential distribution if $\alpha = 1$ as shown in [subsection 2.2.2](#).
2. The interarrival time of the HPP $Y_j^{HPP} \stackrel{i.i.d.}{\sim} \text{Exp}(\mu)$, whereas the interarrival time of the FHPP are *i.i.d.* Mittag-Leffler distributed.

Note that, FHPP has the following probability mass function [\[25\]](#), [\[29\]](#):

$$\begin{aligned} P_\alpha(t, k) &= P(N_\alpha(t) = k) \\ &= \frac{(\mu t^\alpha)^k}{k!} \sum_{j=1}^{\infty} \frac{(j+k)!}{j!} \frac{(-\mu t^\alpha)^j}{\Gamma(\alpha(j+k)+1)} \\ &= \frac{(\mu t^\alpha)^k}{k!} E_\alpha^{(k)}(-\mu t^\alpha), \\ &= (\mu t^\alpha)^k E_{\alpha, \alpha k+1}^{k+1}(-\mu t^\alpha), \quad k = 0, 1, 2, \dots, t \geq 0, \alpha \in (0, 1) \end{aligned} \quad (3.14)$$

where $E_\alpha(z)$ is the Mittag-Leffler function, $E_\alpha^{(k)}(z)$ is the k -th derivative of $E_\alpha(z)$, and $E_{\alpha, \beta}^\gamma(z)$ is the three-parametric generalised Mittag-Leffler function defined as follows:

$$E_{\alpha, \beta}^\gamma(z) = \sum_{j=0}^{\infty} \frac{\gamma^{(j)} z^j}{j! \Gamma(\alpha j + \beta)}, \quad \alpha > 0, \beta > 0, \gamma > 0, z \in \mathbb{C}, \quad (3.15)$$

where $\gamma^{(j)}$ is the rising factorial, defined as:

$$\gamma^{(j)} := \begin{cases} 1, & \text{if } j = 0 \\ \frac{\Gamma(\gamma + j)}{\Gamma(\gamma)}, & \text{if } j = 1, 2, \dots \end{cases}$$

The probability function, $P_\alpha(t, k)$, follows the following fractional Kolmogorov equation:

$${}^C D_t^\alpha P(N_\alpha(t) = k) = \mu(P_\alpha(t, k-1) - P_\alpha(t, k)), \quad 0 < \alpha < 1, \quad (3.16)$$

with initial condition

$$P_\alpha(0, k) = \begin{cases} 0, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0, \end{cases} \quad \alpha \in (0, 1),$$

and we defined that $P_\alpha(k, -1) \equiv 0$.

Moments and covariance

We have

$$\mathbb{E}(N_\alpha(t)) = \frac{\mu t^\alpha}{\Gamma(1+\alpha)} \quad (3.17)$$

$$\text{Var}(N_\alpha(t)) = \frac{\mu^2 t^{2\alpha}}{\Gamma^2(1+\alpha)} \left[\frac{\alpha \Gamma(\alpha)}{\Gamma(2\alpha)} - 1 \right] + \frac{\mu t^\alpha}{\Gamma(1+\alpha)}, \quad t \geq 0, \quad (3.18)$$

$$\text{Cov}(N_\alpha(s), N_\alpha(t)) = \frac{\mu(\min(s, t))^\alpha}{\Gamma(1+\alpha)} + \mu^2 \text{Cov}(Y_\alpha(s), Y_\alpha(t)), \quad (3.19)$$

where $\text{Cov}(Y_\alpha(s), Y_\alpha(t))$ is given in [Equation 2.21](#) (see [\[25\]](#)).

Arrival time

The arrival time T_n of FHPP has the following distribution function [\[32\]](#):

$$\begin{aligned} F_{T_n}^{\text{FHPP}}(t) &= \int_0^\infty f_\alpha(t, u) F_{T_n}^{\text{HPP}}(u) du \\ &= 1 - \sum_{k=0}^{n-1} \frac{(\mu t^\alpha)^k}{k!} E_\alpha(-\mu t^\alpha), \end{aligned} \quad (3.20)$$

where f_α is given in [Equation 2.17](#), and $E_\alpha(z)$ is the one parameter Mittag-Leffler function.

3.2.2 Fractional nonhomogeneous Poisson process

Definition 3.2.3 (Fractional nonhomogeneous Poisson process (first definition)). The fractional nonhomogeneous Poisson process (FNPP) $NN_\alpha(t)$, is defined as [25]:

$$\begin{aligned} NN_\alpha(t) &:= N^1(\Lambda(Y_\alpha(t))), \quad t \geq 0, \quad \alpha \in (0, 1), \\ \Lambda(t) &:= \int_0^t \lambda(s) ds, \quad t \geq 0, \end{aligned} \quad (3.21)$$

with

$$\Lambda(t) \rightarrow \infty, t \rightarrow \infty,$$

where $N^1 = \{N^1(t), t \geq 0\}$ denotes a HPP with $\mu = 1$, and $Y_\alpha(t)$ denotes an inverse α -stable subordinator, which is independent of N^1 .

Following Definition 3.1.6, FNPP could also be constructed as follow: [31]

Definition 3.2.4 (FNPP (second definition) (renewal representation)).

$$NN_\alpha(t) = \sum_{n=0}^{\infty} n \mathbb{1}_{\{\zeta_n \leq Y_\alpha(t) \leq \zeta_{n+1}\}} \stackrel{a.s.}{=} \sum_{n=0}^{\infty} n \mathbb{1}_{\{L_\alpha(\zeta_n) \leq t \leq L_\alpha(\zeta_{n+1})\}},$$

where ζ_n is defined in Definition 3.1.6, and where $L_\alpha(Y_\alpha(t)) = t$ iff t is not a jump time of L_α .

Remark. The renewal representations of HPP and FHPP are based on the interarrival time Y_j , whereas the renewal representations of the nonhomogeneous counterpart are based on $\zeta_j = \zeta'_{\chi_j} = \max(Y_1, \dots, Y_{\chi_j})$, i.e. the maximum value of the interarrival time.

The probability mass function is given by [25]

$$P(NN_\alpha(t)) = \int_0^\infty e^{-\Lambda(u)} \frac{\Lambda(u)^k}{k!} f_\alpha(t, u) du, \quad k = 0, 1, 2, \dots, \quad (3.22)$$

where $f_\alpha(t, u)$ is given by Equation 2.17.

Arrival time

The arrival time T_n has the following distribution function [32]:

$$F_{T_n}^{\text{FNPP}}(t) = \int_0^\infty f_\alpha(t, u) F_{T_n}^{\text{NPP}}(u) du. \quad (3.23)$$

3.3 Mixed tempered fractional Poisson processes

3.3.1 Mixed tempered fractional homogeneous Poisson process

Definition 3.3.1 (Mixed tempered fractional homogeneous Poisson process). We introduce mixed tempered fractional homogeneous Poisson process (MTFHPP) as the counting process

$$\{Z_m(t) = N^1(Y_m(t)); t \geq 0\}, \quad (3.24)$$

where $N^1 = \{N^1(t); t \geq 0\}$ is a homogeneous Poisson process with intensity $\mu = 1$, independent of IMTSS $Y_m(t)$ defined by Equation 2.46.

The marginal distribution are as follows:

$$f_m(x) = \mathbb{P}[Z_m^h(t) = x] = \frac{1}{x!} \int_0^\infty e^{\mu u} [\mu u]^x h_m(t, u) du, \quad x \in \mathbb{N}_0, \quad (3.25)$$

where $h_m(t, u)$ is defined by Equation 2.49. It satisfies the governing equations with the same fractional operators (see ([17]) Proposition 9).

Employing the results of Leonenko et al. (2014) [30] Proposition 9, one can show that

$$\text{Cov}(Z_m(s), Z_m(t)) = \frac{\mu(\min(t, s))^\alpha}{\Gamma(1 - \alpha)} + \mu^2 \text{Cov}(Y_m(s), Y_m(t)), \quad 0 \leq s \leq t, \quad (3.26)$$

where $\text{Cov}(Y_m(s), Y_m(t))$ is given by Equation 2.50.

3.3.2 Mixed tempered fractional nonhomogeneous Poisson process

Definition 3.3.2 (Mixed tempered fractional non-homogeneous Poisson process). We introduce mixed tempered fractional nonhomogeneous Poisson process (MTFNPP) as the counting process

$$\{NZ_m(t) = N^1(\Lambda(Y_m(t))); t \geq 0\}, \quad (3.27)$$

where $N^1 = \{N^1(t); t \geq 0\}$ is a homogeneous Poisson process with intensity $\mu = 1$, independent of IMTSS $Y_m(t)$, and $\Lambda : [0, \infty) \rightarrow [0, \infty)$ is a right continuous, non-decreasing function such that

$$\begin{aligned} \Lambda(0) = 0, \quad \Lambda(t) - \Lambda(t-) \leq 1, \quad \Lambda(\infty) = \infty, \quad t > 0 \\ \Lambda(t) = \int_0^t \lambda(u) du < \infty, \end{aligned}$$

where $\lambda(u)$, $u \geq 0$ is an intensity function.

The marginal distribution are as follows:

$$f_m(x) = \mathbb{P}[NZ_m(t) = x] = \frac{1}{x!} \int_0^\infty e^{\Lambda(u)} [\Lambda(u)]^x h_m(t, u) du, \quad x \in \mathbb{N}_0, \quad (3.28)$$

where $h_m(t, u)$ is defined by [Equation 2.49](#). Equations [3.25](#) and [3.28](#) satisfy the governing equations with the same fractional operators (see ([\[17\]](#)) Proposition 9). This fact justifies our terminology.

Again, by employing the results of Leonenko et al. (2014) [\[30\]](#) Proposition 9, we can show that

$$\text{Cov}(Z_m(s), Z_m(t)) = \frac{\Lambda((\min(t, s))^\alpha)}{\Gamma(1 - \alpha)} + \mu^2 \text{Cov}(Y_m(s), Y_m(t)), \quad 0 \leq s \leq t, \quad (3.29)$$

where $\text{Cov}(Y_m(s), Y_m(t))$ is given by [Equation 2.50](#).

Following Leonenko et al. (2019) [31] one can approximate the distributions of the stochastic process $Z_m^h(t)$ as follows:

$$\frac{[Z_m^h(t) - \mu Y_m(t)]}{\sqrt{\mu}} \xrightarrow{J_1} B(Y_m(t)), \quad (3.30)$$

where $\{B(t); t \geq 0\}$ is a standard Brownian Motion independent of $Y_m(t)$ and convergence holds in J_1 Skorokhod topology. For other approximations see again Leonenko et al. (2019) [31].

3.4 Fractional risk processes

3.4.1 Classical risk models

Definition 3.4.1 (Compound Poisson process). Suppose that $N^\mu(t)$ is a HPP, and $X_1, X_2, \dots = X$ are *i.i.d.* random variables which independent of $N^\mu(t)$. Then the sum of $X_1, X_2, \dots, X_{N^\mu(t)}$, is said to be a compound Poisson process, i.e.

$$S(t) = \sum_{k=1}^{N^\mu(t)} X_k \quad (3.31)$$

is said to be a compound Poisson process [11], [12], [49].

We have

$$\mathbb{E}(S(t)) = \mu t \mathbb{E}(X) \quad (3.32)$$

$$\text{Var}(S(t)) = \mu t \mathbb{E}(X^2) \quad (3.33)$$

$$\text{Cov}(S(s), S(t)) = \mu \min(s, t) \mathbb{E}(X^2) \quad (3.34)$$

Definition 3.4.2 (Classical risk process). In classical Cramer-Lundberg risk process (also known as classical compound Poisson risk model), the risk process $R(t), t \geq 0$, is defined by:

$$R(t) := u + ct - \sum_{k=1}^{N^\mu(t)} X_k, \quad t \geq 0, \quad (3.35)$$

where u is the initial surplus, c is the gross risk premium rate, $X_k : k \geq 1$ are *i.i.d.* random variables denoting the claim sizes, with $X_k > 0 : k \geq 1$, and $N^\mu(t)$ denotes the number of claims during the interval $(0, t]$ [48].

We consider a generalisation of the classical risk model, by replacing $N^\mu(t)$ with desired processes.

Definition 3.4.3 (Fractional compound Poisson process). Suppose that $N_\alpha(t)$ is a FHPP, and $X_1, X_2, \dots = X$ are *i.i.d.* random variables which independent of $N_\alpha(t)$. Then the sum of $X_1, X_2, \dots, X_{N_\alpha(t)}$, is said to be a fractional compound Poisson process, i.e.

$$S_\alpha(t) = \sum_{k=1}^{N_\alpha(t)} X_k \quad (3.36)$$

is said to be a fractional compound Poisson process, other processes could also be computed similarly by replacing $N_\alpha(t)$.

Definition 3.4.4 (Fractional homogeneous risk process). The classical risk process was extended by Kumar, et al. (see [25]), consider

$$R(t) := u + m\mu(1 + \rho)Y_\alpha(t) - \sum_{k=1}^{N_\alpha(t)} X_k, \quad t \geq 0, \quad (3.37)$$

where u is the initial surplus, $X_k : k \geq 1$ are *i.i.d.* random variables denoting the claim sizes, with $X_k > 0 : k \geq 1$ with mean $m > 0$, ρ is the safety loading, $Y_\alpha(t)$ is an inverse α -stable subordinator, and $N_\alpha(t)$ is a FHPP.

For nonhomogeneous version, the process is defined as

Definition 3.4.5 (Fractional nonhomogeneous risk process).

$$R(t) := u + m(1 + \rho)\Lambda(Y_\alpha(t)) - \sum_{k=1}^{NN_\alpha(t)} X_k, \quad t \geq 0, \quad (3.38)$$

where u is the initial surplus, $X_k : k \geq 1$ are *i.i.d.* random variables denoting the claim sizes, with $X_k > 0 : k \geq 1$, ρ is the safety loading, $Y_\alpha(t)$ is an inverse α -stable subordinator, and $NN_\alpha(t)$ is a FNPP.

3.4.2 Mixed risk model

To extend the usage of classical Cramer-Lundberg risk model for mixed risk model, we consider a generalisation of it, namely we consider the risk process of the first type as

$$R_1(t) = u + ct - \sum_{i=1}^{NZ_m(t)} X_i, \quad (3.39)$$

where u is the initial capital of the insurance company, c is the constant premium rate, and $\{X_i\}_{i \geq 1}$ are positive *i.i.d.* random variables independent of the fractional counting process $Z_m(t)$ defined by (Equation 3.24). We interpret $\{X_i\}_{i \geq 1}$ as the claim sizes of the counting process $NZ_m(t)$ with inter-arrival times $\{W_i\}_{i \geq 1}$. The net profit condition $c\mathbb{E}[W_i] > \mathbb{E}[X_i]$ is imposed to ensure that ruin does not happen with certainty. We also assume that random variables X_i have two moments and employ the following notation: $m = \mathbb{E}[X_i]$.

Note that for $\lambda_1 = \lambda_2 = \lambda$ the process (Equation 3.39) was introduced and studied in Kataria and Khandakar (2021) [22]. Observe that for the homogeneous Poisson process (i.e. $\Lambda(t) = \mu t$) we get

$$\mathbb{E}[R_1(t)] = u + m\rho\mu U(t),$$

$$\text{Cov}(R_1(s), R_1(t)) = \mathbb{E}[X_1^2]\mathbb{E}[NZ_m(s)] + \mu^2(\mathbb{E}[X_1])^2 \text{Cov}(NZ_m(s), NZ_m(t)), \quad 0 < s \leq t,$$

where $U(t)$ is given by (Equation 3.26).

Following Kumar et. al (2019) [25], we also introduce the risk process of the second type

$$R_2(t) = u + m(1 + \rho)\Lambda(Y_m(t)) - \sum_{i=1}^{NZ_m(t)} X_i, \quad (3.40)$$

where ρ is the safety loading factor defined as $\rho = \frac{c\mathbb{E}[W_1] - \mathbb{E}[X_1]}{\mathbb{E}[X_1]}$. Note that

$$\mathbb{E}[R_2(t)] = u + m\rho\mu U(t)$$

and

$$\text{Cov}(R_2(s), R_2(t)) = \mathbb{E}[X_1^2]\mathbb{E}[NZ_m(s)] + m^2\mu^2\rho^2 \text{Cov}(NZ_m(s), NZ_m(t)),$$

where $\text{Cov}(NZ_m(s), NZ_m(t))$ is given by (Equation 3.26).

Observe that the homogeneous versions of models (Equation 3.39), (Equation 3.40) are

obtained by setting the intensity function $\lambda(t) = \mu t$. This yields

$$R_3(t) = u + ct - \sum_{i=1}^{Z_m(t)} X_i, \quad (3.41)$$

and

$$R_4(t) = u + m(1 + \rho)Y_m(t) - \sum_{i=1}^{Z_m(t)} X_i. \quad (3.42)$$

For all risk processes $R_i(t), i = 1, 2, 3, 4$ we introduce the ruin probabilities with finite horizon (finite-time ruin probabilities) [49]

$$\psi_i(u, T) = P[\inf_{0 \leq t \leq T} R_i(t) < 0 \mid R_i(0) = u] \quad i = 1, 2, \quad T > 0 \quad (3.43)$$

and the time of ruin

$$T_i = \inf\{t > 0 : R_i(t) < 0\}, \quad i = 1, 2. \quad (3.44)$$

These characteristics will be investigated numerically in [Section 4.2](#).

Chapter 4

Simulations

This chapter is based on the ‘Risk process with mixture of tempered stable inverse subordinators: analysis and synthesis’ coauthor with Tetyana Kadankova accepted on 28th October, 2021 by ‘Random Operators and Stochastic Equations’ and has been reproduced here with the permission of the copyright holder.

4.1 Simulations methodologies

The python code for simulations could be found on <https://github.com/VinventN/Processes>.

4.1.1 Simulation of processes

In this section we provide simulations of the sample paths of the two traditional types of Poisson processes: homogeneous Poisson process $N(t)$ (HPP) defined in [Section 3.1.1](#), and non-homogeneous Poisson process $NN(t)$ (NPP) defined in [Section 3.1.2](#); along with four types of fractional Poisson processes: fractional homogeneous Poisson process $N_\alpha(t)$ (FHPP) defined by [Section 3.2.1](#), and its non-homogeneous version $NN_\alpha(t)$ (FNPP) defined by [Section 3.2.2](#), the mixed tempered stable fractional homogeneous Poisson process $Z_m(t)$ (MTFHPP) defined by [Definition 3.3.1](#), and its non-homogeneous version $NZ_m(t)$ (MTFNPP) defined by [Definition 3.3.2](#). These processes serve as counting

processes of the claims in the models defined in [Equations \(3.39\)–\(3.42\)](#). We use the following base parameterisation:

- the homogeneous Poisson process $N^\mu(t)$ has the intensity $\mu = 4$;
- the intensity function of the non-homogeneous Poisson process is

$$\lambda(t) = \frac{\gamma}{\beta} \left(\frac{t}{\beta} \right)^{\gamma-1}, \quad \gamma = 0.9, \beta = 0.2;$$

- the parameters of the mixture tempered stable inverse process are set as follows:

$$\alpha_1 = 0.9, \alpha_2 = 0.5, \lambda_1 = 0.3, \lambda_2 = 0.7, c_1 = 0.75, c_2 = 0.25.$$

and the results of the simulations could be found in [Figure 4.1](#).

Homogeneous Poisson process

From [Proposition 3.1.1](#), it was shown that the waiting time of HPP follows an exponential distribution with parameter λ , and a HPP could be simulated using [Algorithm 1](#). The results of the simulation could be found in [Figure 4.1a](#).

Nonhomogeneous Poisson process

Since NPP has a time dependent rate $\lambda(t)$, [Algorithm 1](#) is not suitable for such simulation. However, the thinning algorithm could be used for NPP simulation (see [8], [33], [41]), a proof of that could be found in [8, Theorem 4.2]. An NPP could be simulated using [Algorithm 2](#), and the results could be found in [Figure 4.1b](#).

α -stable subordinator

α -stable subordinator $S_\alpha(t)$ could be simulated using $S_\alpha(t)$ using $S_\alpha(t + dt) - S_\alpha(t) \stackrel{d}{=} S_\alpha(dt) \stackrel{d}{=} (dt)^{1/\alpha}$ [17], where

$$S_\alpha(1) \stackrel{d}{=} \frac{\sin(\alpha\pi U_1) [\sin(1-\alpha)\pi U_1]^{1/\alpha-1}}{[\sin(\pi U_1)]^{1/\alpha} |\ln U_2|^{1/\alpha-1}};$$

Inverse α -stable subordinator

An inverse α -stable subordinator could be simulated by swapping the axes of an α -stable subordinator due to the nature of an inverse function.

Fractional homogeneous Poisson process

Simulating of the paths of the fractional homogeneous Poisson process (FHPP) is based on the following fact: the inter-arrival times $\{T_i; i \geq 1\}$ of FHPP have the Mittag-Leffler distribution. We employ the approach proposed in [4], where the authors used the following representation for $\{T_i; i \geq 1\}$:

$$T_i \stackrel{d}{=} T = \frac{|\ln U_1|^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} L_{\alpha}(1), \quad (4.1)$$

where $U_1 \stackrel{i.i.d.}{\sim} U(0, 1)$, and $L_{\alpha}(1)$ is an α -stable subordinator evaluated at $t = 1$. The following result from [6] provides an algorithm for simulation of the inter-arrival times of FHPP:

$$L_{\alpha}(1) \stackrel{d}{=} \frac{\sin(\alpha\pi U_2) [\sin((1 - \alpha)\pi U_2)]^{\frac{1}{\alpha}-1}}{[\sin(\pi U_2)] |\ln U_3|^{\frac{1}{\alpha}-1}}, \quad (4.2)$$

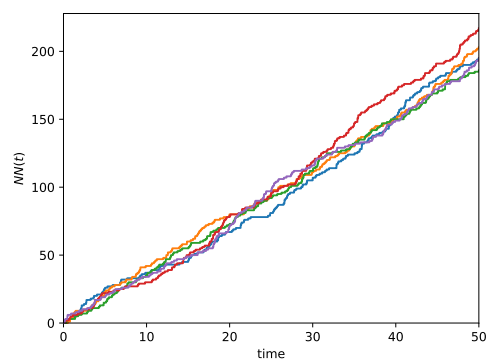
where $U_2, U_3 \stackrel{i.i.d.}{\sim} U(0, 1)$.

Fractional nonhomogeneous Poisson process

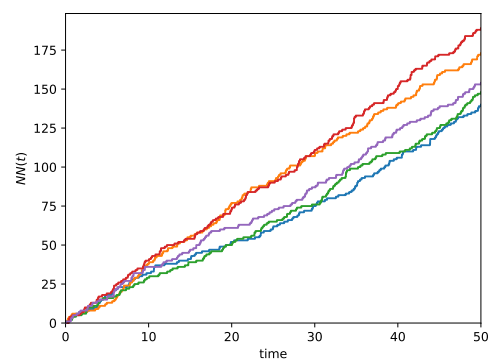
The simulation of FNPP could be done by using the same thinning algorithm to a FHPP, as shown in Algorithm 4, the results is shown in Figure 4.1d.

Mixed tempered fractional Poisson process

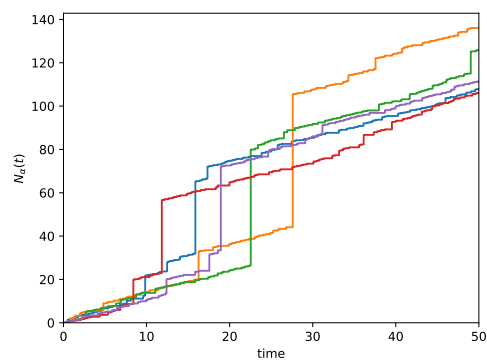
The simulations of the mixed tempered processes (MTFHPP and MTFNPP) could be done with the ‘acceptance-rejection method’ as shown in [17], and the algorithms could be found in Algorithms 5 and 6.



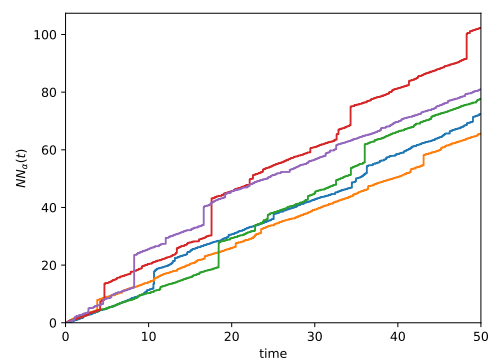
(a) HPP



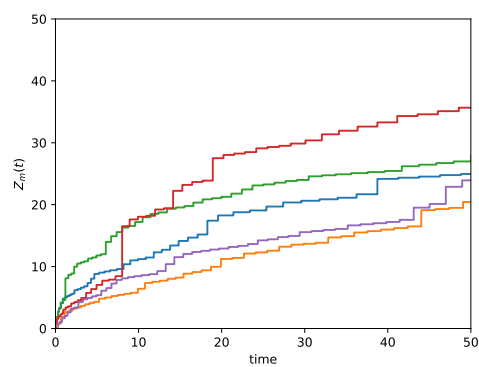
(b) NPP



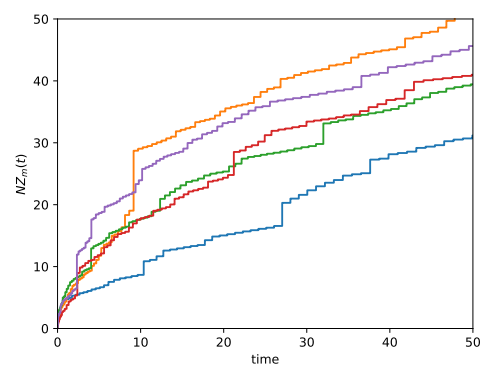
(c) FHPP



(d) FNPP



(e) MTFHPP



(f) MTFNPP

Fig. 4.1 Simulations of the processes

4.1.2 Simulation of ruin probability

Definition 4.1.1 (Inverse Gaussian distribution). The probability density function of an inverse Gaussian (IG) distribution is given by (see [46])

$$f(x; \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp \left[-\frac{\lambda}{2\mu^2} \frac{(x - \mu^2)^2}{x} \right], \quad x > 0, \mu > 0, \lambda > 0, \quad (4.3)$$

with cumulative distribution function:

$$F(x; \mu, \lambda) = \Phi \left(\sqrt{\frac{\lambda}{x}} \left(\frac{x}{\mu} - 1 \right) \right) + \exp \left(\frac{2\lambda}{\mu} \right) \Phi \left(-\frac{\lambda}{x} \left(\frac{x}{\mu} + 1 \right) \right). \quad (4.4)$$

The Laplace transform of an inverse Gaussian distribution is given by (see [46])

$$\mathbb{E} [e^{-sX}] = \exp \left\{ \frac{\lambda}{\mu} \left[1 - \left(1 + \frac{2\mu^2 s}{\lambda} \right)^{\frac{1}{2}} \right] \right\}. \quad (4.5)$$

The first two moments of an inverse Gaussian distribution is given by

$$\begin{aligned} \mathbb{E} [X] &= \mu \\ \mathbb{E} [X^2] &= \frac{\mu^2(\lambda + \mu)}{\lambda} \end{aligned}$$

Definition 4.1.2 (Generalised Inverse Gaussian distribution). The probability density function of a generalised inverse Gaussian (GIG) distribution is given by (see [14])

$$f(x; a, b, p) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{ax+b/x}{2}}, \quad x > 0, a > 0, b > 0, p \in \mathbb{R}, \quad (4.6)$$

where K_p is the modified Bessel function of the second kind, which is given by (see [52])

$$K_p(x) = \int_0^\infty e^{-x \cosh u} \cosh(pu) du, \quad \Re(x) > 0.$$

with cumulative distribution function:

$$F(x; a, b, p) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} \left(\frac{2}{a} \right)^p \gamma \left(p, \frac{ax}{2}, \frac{ab}{4} \right) \quad (4.7)$$

where $\gamma(\alpha, x, b)$ is the generalised lower incomplete gamma function, given by (see [7])

$$\gamma(\alpha, x, b) = \int_0^x t^{\alpha-1} e^{-t-\frac{b}{t}} dt.$$

The first two moments of a generalised inverse Gaussian distribution is given by

$$\begin{aligned}\mathbb{E}[X] &= \frac{\sqrt{b}K_{p+1}(\sqrt{ab})}{\sqrt{a}K_p(\sqrt{ab})} \\ \mathbb{E}[X^2] &= \frac{bK_{p+2}(\sqrt{ab})}{aK_p(\sqrt{ab})}.\end{aligned}$$

Definition 4.1.3 (Exponentially modified Gaussian distribution). The probability density function of a exponentially modified Gaussian distribution (exGaussian) (EMG) distribution is given by (see [15])

$$\begin{aligned}f(x; \mu, \sigma, \lambda) &= \frac{\lambda}{2} \exp(2\mu + \lambda\sigma^2 - 2x) \operatorname{erfc}\left(\frac{\mu + \lambda\sigma^2 - x}{\sqrt{2}\sigma}\right), \\ x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0, \lambda > 0,\end{aligned}\tag{4.8}$$

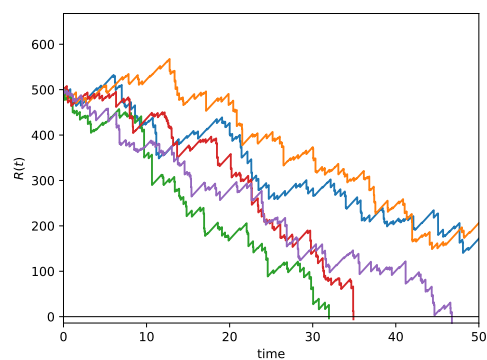
where $\operatorname{erf}(x)$ is the error function defined as $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, and $\operatorname{erfc}(x)$ is the complementary error function defined as $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$; with cumulative distribution function:

$$F(x; \mu, \sigma, \lambda) = \Phi(u, 0, v) - \exp\left(-\frac{u + v^2}{2 + \log(\Phi(u, v^2, v))}\right)\tag{4.9}$$

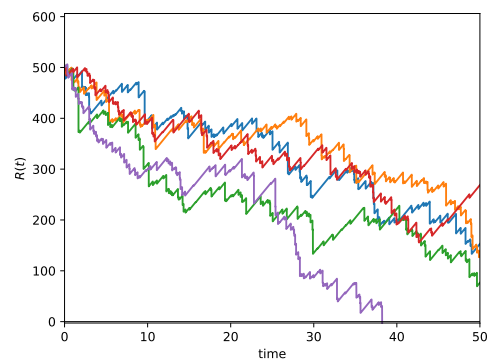
where $u = \lambda(x - \mu)$, $v = \lambda\sigma$.

The first two moments of a exponentially modified Gaussian distribution is given by

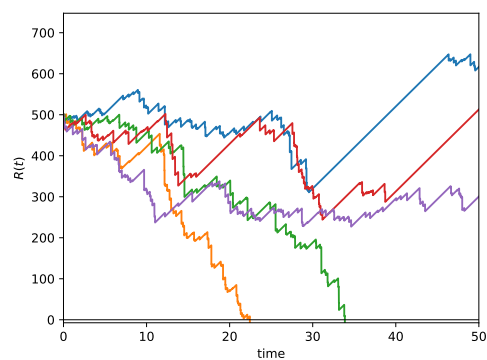
$$\begin{aligned}\mathbb{E}[X] &= \mu + \frac{1}{\lambda} \\ \mathbb{E}[X^2] &= \sigma^2 + \mu^2 + \frac{2(1 + \lambda\mu)}{\lambda^2}.\end{aligned}$$



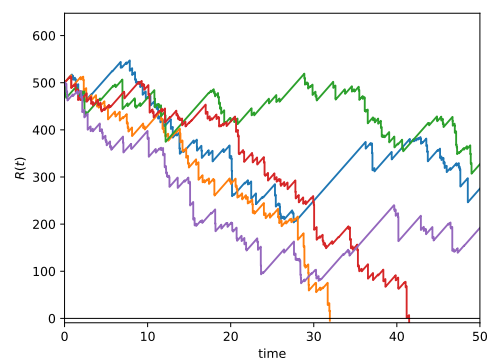
(a) HPP



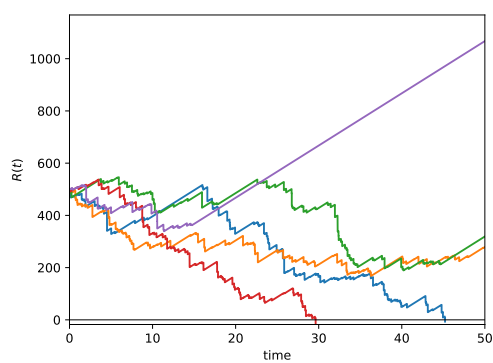
(b) NPP



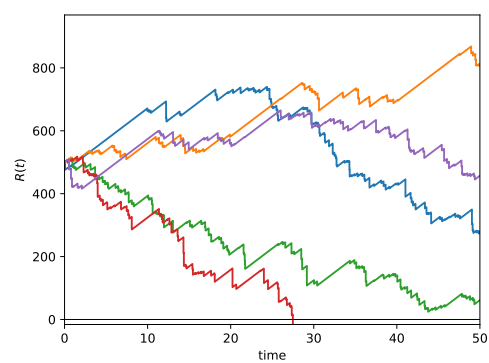
(c) FHPP



(d) FNPP



(e) MTFHPP



(f) MTFNPP

Fig. 4.2 Simulations of risk process

4.2 Numerical investigation of the ruin probabilities and ruin times for the mixed risk model

4.2.1 Setups

In this section we adhere to the following assumptions (unless specified otherwise):

- the initial capital of the insurance company is set to $u = 500$;
- claim sizes $\{X_i\}_{i \geq 0}$ are *i.i.d.* exponentially distributed, with expected claim size equal to 10;
- the gross risk premium rate is set to $c = 20$, the safety loading factor is $\rho = 0.1$;
- the claims arrive according to the homogeneous Poisson process with intensity $\mu = 4$;
- for the non-homogeneous Poisson process we assume that the intensity function is

$$\lambda(t) = \frac{\gamma}{\beta} \left(\frac{t}{\beta} \right)^{\gamma-1} \quad \gamma = 0.9, \beta = 0.2;$$

- the parameters of the mixture tempered stable inverse process are set as

$$\alpha_1 = 0.9, \alpha_2 = 0.5, \lambda_1 = 0.3, \lambda_2 = 0.7, c_1 = 0.75, c_2 = 0.25.$$

For the sake of simplicity we adhere to the following notation: the risk models (3.39) - (3.42) will be denoted by Model 1, Model 2, Model 3 and by Model 4 respectively. We first simulated 200 000 realisations of sample paths of each risk process $R_i(t)$, $i = 1, \dots, 4$. Based on these simulations we created samples of ruin times and then we fitted different distributions to these samples. All available distributions in `scipy` package from python were used for distribution fitting using `scipy.stats.rv_continuous.fit`. The best fit is produced by (i) IG; (ii) GIG and (iii) the EMG with the following parameterisations:

(i) for the IG:

$$f(x; \delta, \eta) = \sqrt{\frac{\eta}{2\pi x^3}} \exp \left[-\frac{\eta}{2\delta^2} \frac{(x - \delta^2)}{x} \right], \quad x > 0, \delta > 0, \eta > 0; \quad (4.10)$$

(ii) GIG has the density:

$$f(x; a, b, p) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{ax+b/x}{2}}, \quad x > 0, a > 0, b > 0, p \in \mathbb{R}, \quad (4.11)$$

(iii) and for the EMG:

$$f(x; \kappa, \sigma, q_0) = \frac{q_0}{2} \exp(2\kappa + q_0\sigma^2 - 2x) \operatorname{erfc}\left(\frac{\kappa + q_0\sigma^2 - x}{\sqrt{2}\sigma}\right), \quad (4.12)$$

$$x \in \mathbb{R}, \kappa \in \mathbb{R}, \sigma > 0, q_0 > 0,$$

where $\operatorname{erf}(x)$ is the error function defined as $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, and $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$.

Table 4.1 contains the maximum likelihood estimators of the fitted distributions. Table 4.2 presents the errors and the goodness of fit measures such as the sum of squared error (SSE), Akaike information criterion (AIC), Bayesian information criterion (BIC), and the log-likelihood (LLH). As shown in Table 4.2, the GIG and IG have the lowest value of SSE, therefore, they will be used for further analyses in this paper.

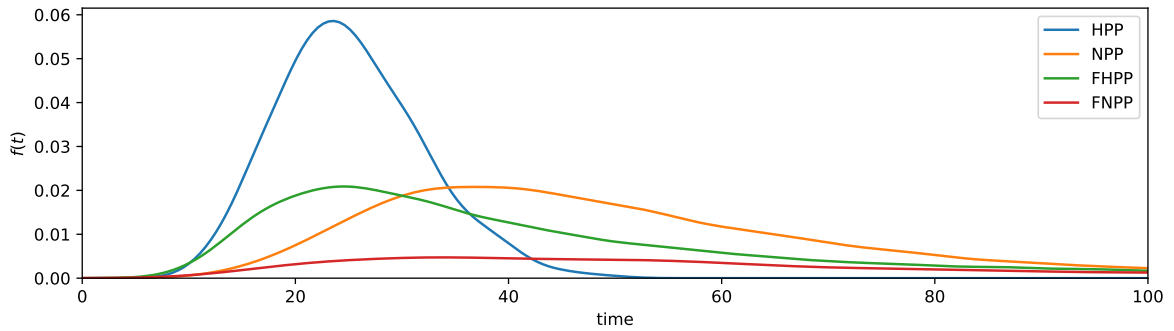


Fig. 4.3 Distributions of the time of ruin for HPP, NPP, FHPP, and FNPP

Further, we compared the behaviour of the finite-time ruin probabilities $\psi(u, t)$ as a function of some model parameters. To do this, we simulated the finite-time ruin probabilities obtained from Models 1-4.

Table 4.3 shows that under Model 2 and Model 4 the ruin probabilities are remarkably higher than those under Model 1 and Model 3. Simulations suggest that the ruin probabilities drop more rapidly as the initial capital u increases for Model 1 and Model

Table 4.1 Parameters of the fitted distributions using
`scipy.stats.rv_continuous.fit`

Distribution	Parameters
Model 1	
IG	$\delta = 94.752, \eta = 164.627$
GIG	$a = 151.685, b = 0.020, p = -0.294$
EMG	$\kappa = 22.543, \sigma = 7.268, q_0 = 0.003$
Model 2	
IG	$\delta = 26.391, \eta = 38.051$
GIG	$a = 99.070, b = 2.960 \times 10^{-8}, p = -3.179$
EMG	$\kappa = 10.171, \sigma = 0.004, q_0 = 0.073$
Model 3	
IG	$\delta = 93.504, \eta = 158.924$
GIG	$a = 135.072, b = 0.022, p = -0.108$
EMG	$\kappa = 23.644, \sigma = 8.051, q_0 = 0.014$
Model 4	
IG	$\delta = 26.216, \eta = 38.680$
GIG	$a = 99.493, b = 3.888 \times 10^{-5}, p = -3.183$
EMG	$\kappa = 7.587, \sigma = 1.192, q_0 = 0.054$

Table 4.2 Statistics for fitted distributions of the time of ruin

Distribution	SSE	AIC	BIC
Model 1			
IG	3.592×10^{-6}	1888.216	-103635.007
GIG	$3.086 \times 10^{-6*}$	1867.221*	-104374.570*
EMG	4.972×10^{-6}	1942.557	-102033.463
Model 2			
IG	$3.80 \times 10^{-5*}$	5600.448	$-8.987 \times 10^5*$
GIG	3.75×10^{-4}	2890.382*	-7.995×10^5
EMG	5.10×10^{-5}	10403.993	-8.852×10^5
Model 3			
IG	2.033×10^{-6}	1833.934*	-105738.071
GIG	$1.980 \times 10^{-6*}$	1840.089	-105859.845*
EMG	6.098×10^{-6}	1870.819	-100360.953
Model 4			
IG	$3.60 \times 10^{-5*}$	5546.795	$-8.904 \times 10^5*$
GIG	3.40×10^{-4}	2887.251*	-7.953×10^5
EMG	4.70×10^{-5}	10502.987	-8.800×10^5

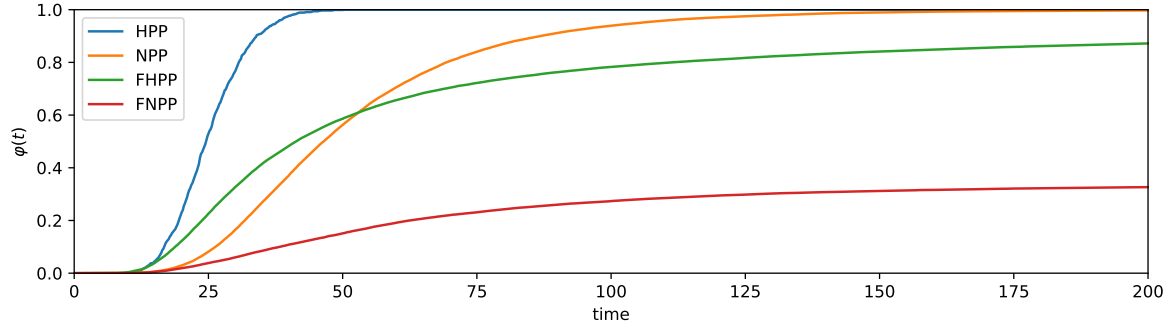


Fig. 4.4 Probability of ruin $\phi(t)$ for HPP, NPP, FHPP, and FNPP

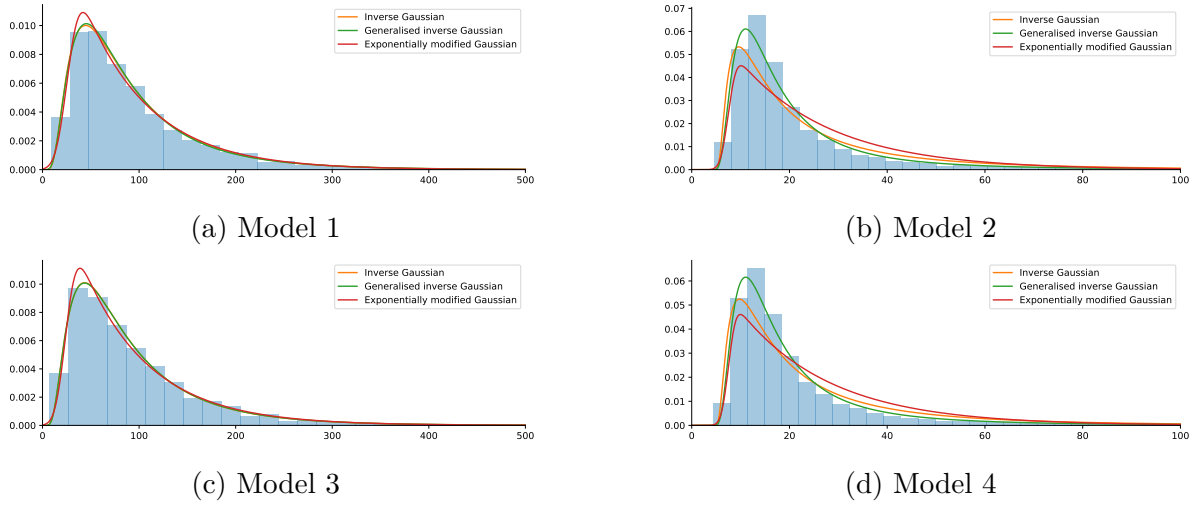


Fig. 4.5 Fitted distributions of simulations of time of ruin

3. By comparing $\psi(u, 2000)$ and $\psi(u, 100)$, we established that Model 1 and Model 3 have lower proportion of ruins occurred before time $t = 100$, which indicates that their distributions have heavier tails compared to those of Model 2 and Model 4.

Figures 4.3 and 4.4 show the time of ruin and probabilities of ruin of HPP, NPP, FHPP, FNPP. Figure 4.5 shows the probability density functions of the ruin probability with their fitted distributions. Figures 4.6–4.9 contain the ruin probabilities of Model 1 (MTFNPP) and Model 3 (MTFHPP) with different initial capitals u . Figures 4.10–4.13 show the ruin probabilities of MTFHPP and MTFNPP with different values of α_1 and u under Models 1 - 4.

From Figures 4.10b, 4.11b, 4.12b and 4.13b, we observe that the following behavior as u increases:

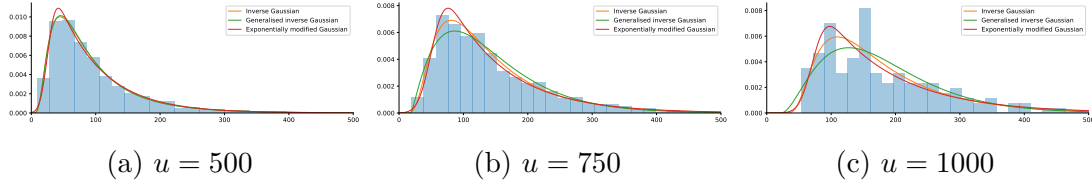


Fig. 4.6 Fitted distributions of simulations of time of ruin for Model 1 with different initial capital values

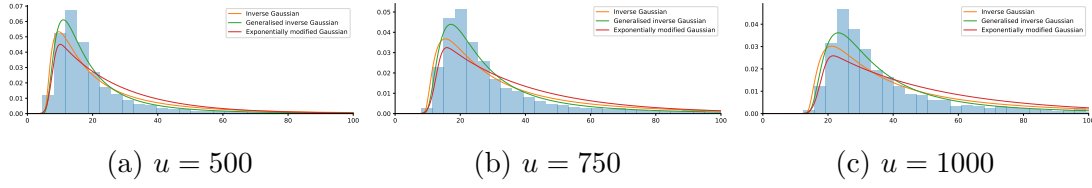


Fig. 4.7 Fitted distributions of simulations of time of ruin for Model 2 with different initial capital values

- the probabilities of ruin reduce;
- the distributions of time of ruin are less left skewed;
- the distributions of time of ruin have heavier tails.

Complete numerical results can be found in [Appendix B](#).

Table 4.3 Finite-time ruin probabilities $\psi(u, t)$

	Model 1	Model 2	Model 3	Model 4
$\psi(500, 2000)$	0.024635	0.0448	0.02448	0.044195
$\psi(500, 100)$	0.016425	0.04362	0.01626	0.04301
$\psi(750, 2000)$	0.00422	0.02429	0.00411	0.02441
$\psi(750, 100)$	0.00166	0.023135	0.001585	0.0232
$\psi(1000, 2000)$	0.00072	0.01591	0.00066	0.01649
$\psi(1000, 100)$	0.000165	0.01472	0.00013	0.015245

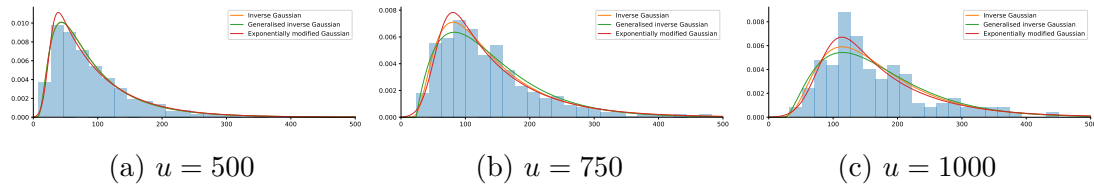


Fig. 4.8 Fitted distributions of simulations of time of ruin for Model 3 (MTFHPP) for different initial capital values

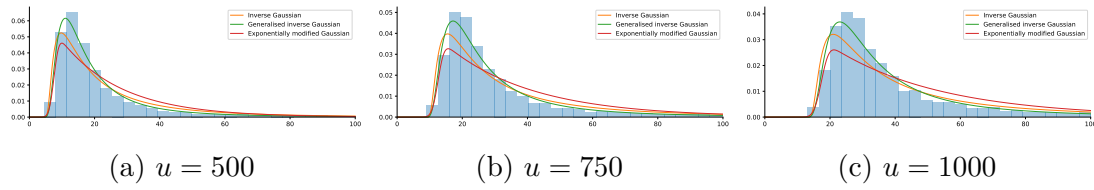


Fig. 4.9 Fitted distributions of simulations of time of ruin for Model 4

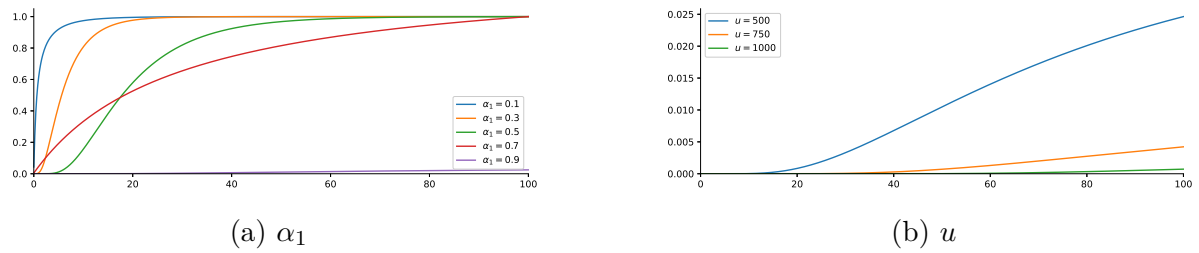


Fig. 4.10 Probability of ruin of Model 1 under different parameterisations

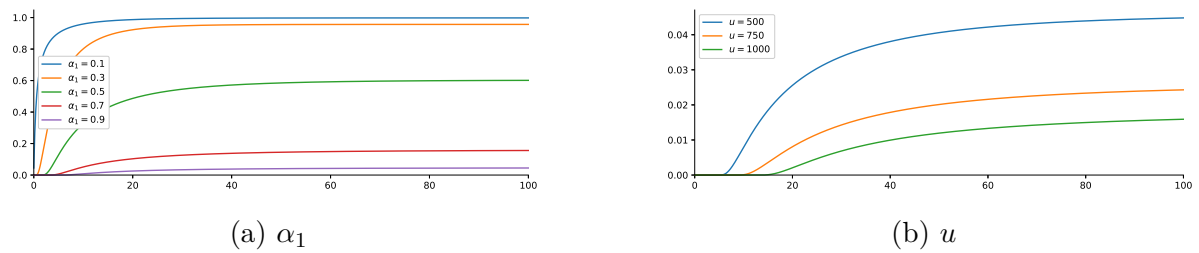


Fig. 4.11 Probability of ruin of Model 2 under different parameterisations

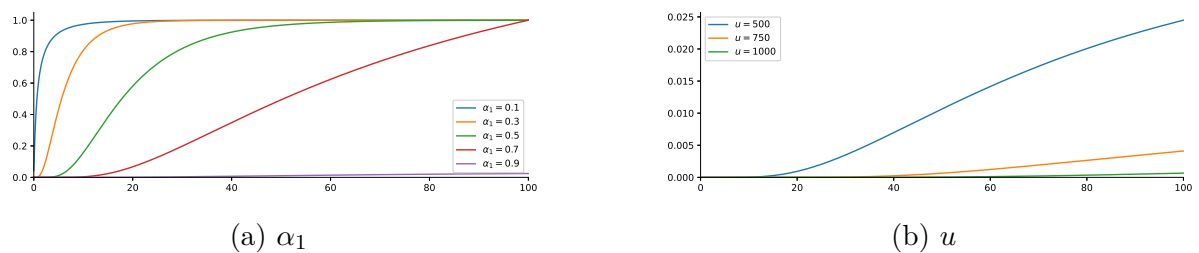


Fig. 4.12 Probability of ruin of model 3 under different parameterisations

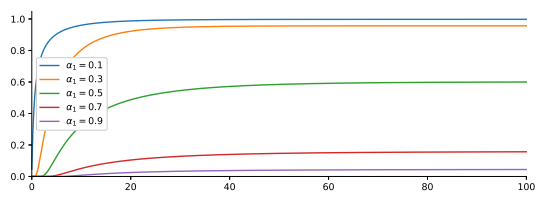
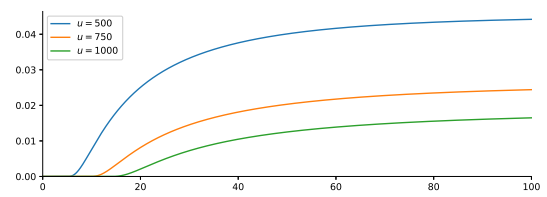
(a) α_1 (b) u

Fig. 4.13 Probability of ruin of Model 4 under different parameterisations

Chapter 5

Conclusions

In this thesis, we introduce the mixed tempered fractional Poisson process, which would be more flexible in real-life modelling due to the extra parameters.

This thesis had the aim of exploring the relationship between ruin probabilities and the newly introduced mixed tempered fractional risk models.

5.1 Summaries

The results in [Section 4.2](#) suggest that the inverse Gaussian distribution fits well for the risk models under mixed tempered fractional Poisson process, especially model 1 and 3, but also the other processes as they are special cases within MTFPP.

Furthermore, there are noteworthy drops in ruin probabilities of model 1 and 3 as the initial capital increase in comparison to the other models.

Although the exact formula of the risk process remains a complex open problem, we propose a simple but practical method for time of ruin and finite-time ruin probability approximation in [Chapter 4](#), with the use of the inverse Gaussian and generalised inverse Gaussian distribution, it is far less complex and time consuming than full Monte Carlo simulation.

5.2 Limitations

Insurance companies are often not received the premium payments continuously and uniformly throughout the year in real life, e.g. travel insurance tends to have higher number of claims and more policies are bought during summer, i.e. more claims and higher premium rate. Also, the use of exponential distribution for claim sizes X_i in [Section 4.1.2](#) might not be suitable for some insurance providers.

However, for simplicity, and also to reduce the need of computational power, the Monte Carlo simulations in [Section 4.2](#) are done in the assumption that the premium payments are received continuously and uniformly with exponentially distributed claim sizes.

5.3 Future work

We believe that there are still many open problems for further research. We suggest that further research may be carried out in the following areas:

1. For insurance providers which have higher proportion of seasonal policies, we could extend the model that was discussed in [Equation \(3.39\)](#),

$$R(t) = u + c(t) - \sum_{i=1}^{NZ_m(t)} X_i, \quad (5.1)$$

where $c(t)$ is the premium rate function over time. Also, a periodic function could be useful for $\lambda(t)$ as the distribution of claim sizes and the number of claims over each calendar year are usually very similar every year.

2. For insurance companies which have high proportion of lower claim sizes, lighter-tailed distributions would be more suitable, e.g. generalised rectified Gaussian distribution [\[42\]](#).
3. The obtained results in this paper do not include the effects of reinsurance or copayment which are common in some types of policies.
4. Further research may be carried out in the relationship between the collective risk model used in this paper, and the individual risk model [\[53\]](#).

-
5. Mixture of more than 2 MTSS could be introduced in further studies.
 6. The theory for ruin probability and time of ruin for the risk models with some particular non-local Poisson processes.
 7. Translated gamma approximation discussed in [\[12\]](#) could also be further studied for non-local processes.

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Appendix A

Simulations' algorithms

Algorithm 1: Simulation of Homogeneous Poisson process with rate μ , on $[0, T]$ [8]

Input: μ, T

Step 1: generate increments of $Y_n \sim \text{Exp}(\mu)$ (see [Propositions 3.1.1](#) and [3.1.2](#));

Step 2: sum up the increments to give $N^\mu(t)$;

Output: $N^\mu(t)$

Algorithm 2: Simulation of Nonhomogeneous Poisson process with time dependent rate $\lambda(t)$, on $[0, T]$ [8]

Input: $\lambda(t), T$

Step 1: generate increments of $Y_n \sim \text{Exp}(\mu)$;

Step 2: generate $NN^\mu(t)$ using the following ‘thinning algorithm’:

(i) generate Y_n ;

(ii) generate $U \stackrel{i.i.d.}{\sim} U(0, 1)$;

(iii) if $U \leq \frac{\lambda(t)}{\sup_{0 \leq t \leq T} \lambda(t)}$, accept Y_n ;

(iv) goto (i);

Step 3: sum up the increments to give $NN^\mu(t)$;

Output: $NN^\mu(t)$

Algorithm 3: Simulation of Fractional Homogeneous Poisson process with rate λ , and fractional index α , on $[0, T]$ [4], [35]

Input: μ, T, α

Step 1: generate $U_1, U_2, U_3, \overset{i.i.d.}{\sim} U(0, 1)$;

Step 2: generate increments of Y_n , where

$$Y_n = \frac{|\ln U_1|^{\frac{1}{\alpha}} \sin(\alpha\pi U_2) [\sin((1-\alpha)\pi U_2)]^{\frac{1}{\alpha}-1}}{\lambda^{\frac{1}{\alpha}} [\sin(\pi U_2)] |\ln U_3|^{\frac{1}{\alpha}-1}};$$

Step 3: sum up the increments to give $N_\alpha(t)$;

Output: $N_\alpha(t)$

Algorithm 4: Simulation of Fractional Nonhomogeneous Poisson process with time dependent rate $\lambda(t)$, and fractional index α , on $[0, T]$

Input: $\lambda(t), T, \alpha$

Step 1: generate $U_1, U_2, U_3, \overset{i.i.d.}{\sim} U(0, 1)$;

Step 2: generate increments of Y_n , where

$$Y_n = \frac{|\ln U_1|^{\frac{1}{\alpha}} \sin(\alpha\pi U_2) [\sin((1-\alpha)\pi U_2)]^{\frac{1}{\alpha}-1}}{\lambda^{\frac{1}{\alpha}} [\sin(\pi U_2)] |\ln U_3|^{\frac{1}{\alpha}-1}};$$

Step 3: apply the thinning algorithm (see [Algorithm 2](#));

Step 4: sum up the increments to give $N_\alpha(t)$;

Output: $N_\alpha(t)$

Algorithm 5: Simulation of mixture tempered fractional Poisson processes with rate μ or $\Lambda(t)$, and fractional indices α_1, α_2 , tempered indices λ_1, λ_2 , and mixture indices c_1, c_2 , on $[0, T]$ [17]

Input: $\mu, T, \alpha_1, \alpha_2, \lambda_1, \lambda_2, c_1, c_2$

Step 1: generate $U_1, U_2 \stackrel{i.i.d.}{\sim} U(0, 1)$;

Step 2: generate increments of the α -stable subordinator $S_\alpha(t)$ using

$S_\alpha(t + dt) - S_\alpha(t) \stackrel{d}{=} S_\alpha(dt) \stackrel{d}{=} (dt)^{1/\alpha}$, where

$$S_\alpha(1) \stackrel{d}{=} \frac{\sin(\alpha\pi U_1) [\sin(1 - \alpha)\pi U_1]^{1/\alpha-1}}{[\sin(\pi U_1)]^{1/\alpha} |\ln U_2|^{1/\alpha-1}};$$

Step 3: generate $S_1(c_1t)$ and $S_2(c_2t)$ using the following ‘accept-reject algorithm’:

(i) generate $S_\alpha(dt)$;

(ii) generate $U_3 \stackrel{i.i.d.}{\sim} U(0, 1)$, if $U_3 \leq e^{-\lambda S_\alpha(dt)}$, accept $S_\alpha(dt)$; otherwise, go back to (i);

(iii) sum up the increments gives $S_1(c_1t)$ and repeat to get $S_2(c_2t)$;

Step 4: generate MTSS $S_m(t)$ by summing up $S_1(c_1t)$ and $S_2(c_2t)$ (see

[Equation 2.40](#));

Step 5: generate IMTSS $Y_m(t)$ by inverting the axes;

Step 6: generate MTFHPP $NZ_m(t)$ using [Equation 3.24](#);

Output: $Z_m(t)$

Algorithm 6: Simulation of mixture tempered fractional Poisson processes with rate μ or $\Lambda(t)$, and fractional indices α_1, α_2 , tempered indices λ_1, λ_2 , and mixture indices c_1, c_2 , on $[0, T]$ [17]

Input: $\lambda(t), T, \alpha_1, \alpha_2, \lambda_1, \lambda_2, c_1, c_2$

Step 1: generate $U_1, U_2 \stackrel{i.i.d.}{\sim} U(0, 1)$;

Step 2: generate increments of the α -stable subordinator $S_\alpha(t)$ using

$S_\alpha(t + dt) - S_\alpha(t) \stackrel{d}{=} S_\alpha(dt) \stackrel{d}{=} (dt)^{1/\alpha}$, where

$$S_\alpha(1) \stackrel{d}{=} \frac{\sin(\alpha\pi U_1) [\sin(1 - \alpha)\pi U_1]^{1/\alpha-1}}{[\sin(\pi U_1)]^{1/\alpha} |\ln U_2|^{1/\alpha-1}};$$

Step 3: apply the thinning algorithm (see Algorithm 2);

Step 4: generate $S_1(c_1 t)$ and $S_2(c_2 t)$ using the following ‘accept-reject algorithm’:

(i) generate $S_\alpha(dt)$;

(ii) generate $U_3 \stackrel{i.i.d.}{\sim} U(0, 1)$, if $U_3 \leq e^{-\lambda S_\alpha(dt)}$, accept $S_\alpha(dt)$; otherwise, go back to (i);

(iii) sum up the increments gives $S_1(c_1 t)$ and repeat to get $S_2(c_2 t)$;

Step 5: generate MTSS $S_m(t)$ by summing up $S_1(c_1 t)$ and $S_2(c_2 t)$ (see Equation 2.40);

Step 6: generate IMTSS $Y_m(t)$ by inverting the axes;

Step 7: generate MTFNPP $NZ_m(t)$ using Equation 3.27;

Output: $NZ_m(t)$

Appendix B

Full numerical results

Table B.1 Ruin probabilities with substituted α_1 and α_2

	α_1					α_2				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
model 1										
$\psi(500, 2000)$	1	1	1	1	0.024635	0.024515	0.02424	0.024635	0.025235	0.02528
$\psi(500, 100)$	1	1	0.999245	0.62831	0.016425	0.016135	0.015885	0.016425	0.016585	0.016975
$\psi(750, 2000)$	1	1	1	0.99999	0.00422	0.00423	0.004125	0.00422	0.00408	0.004265
$\psi(750, 100)$	1	1	0.997115	0.38287	0.00166	0.00175	0.001595	0.00166	0.001545	0.0016
$\psi(1000, 2000)$	1	1	1	0.99999	0.00072	0.000875	0.00072	0.00072	0.000645	0.00077
$\psi(1000, 100)$	1	1	0.991185	0.189275	0.000165	0.00013	0.000125	0.000165	0.0001	0.00013
model 2										
$\psi(500, 2000)$	0.9979	0.95665	0.60129	0.15593	0.0448	0.044855	0.044575	0.0448	0.04438	0.044
$\psi(500, 100)$	0.9969	0.953405	0.594195	0.152305	0.04362	0.0435	0.04339	0.04362	0.043075	0.04281
$\psi(750, 2000)$	0.997685	0.93539	0.429315	0.083135	0.02429	0.02464	0.024705	0.02429	0.02444	0.024175
$\psi(750, 100)$	0.996535	0.930115	0.41913	0.079155	0.023135	0.02331	0.02349	0.023135	0.0231	0.02291
$\psi(1000, 2000)$	0.997485	0.91031	0.30683	0.055145	0.01591	0.016625	0.016445	0.01591	0.01667	0.016335
$\psi(1000, 100)$	0.99621	0.902835	0.294325	0.05099	0.01472	0.015285	0.01522	0.01472	0.01531	0.015055
model 3										
$\psi(500, 2000)$	1	1	1	1	0.02448	0.02521	0.02502	0.02448	0.02522	0.024315
$\psi(500, 100)$	1	1	0.99925	0.628715	0.01626	0.01672	0.016395	0.01626	0.01651	0.01593
$\psi(750, 2000)$	1	1	1	1	0.00411	0.00429	0.004245	0.00411	0.004105	0.00394
$\psi(750, 100)$	1	1	0.99733	0.38232	0.001585	0.00173	0.00174	0.001585	0.001585	0.001545
$\psi(1000, 2000)$	1	1	1	1	0.00066	0.000685	0.000845	0.00066	0.00068	0.00064
$\psi(1000, 100)$	1	1	0.991465	0.18908	0.00013	0.000125	0.00016	0.00013	0.000145	0.00009
model 4										
$\psi(500, 2000)$	0.997895	0.95672	0.59985	0.156675	0.044195	0.04464	0.044095	0.044195	0.043765	0.044225
$\psi(500, 100)$	0.996875	0.953735	0.59346	0.153315	0.04301	0.04343	0.04294	0.04301	0.0425	0.043015
$\psi(750, 2000)$	0.99763	0.93537	0.42681	0.08326	0.02441	0.02466	0.02465	0.02441	0.02403	0.024275
$\psi(750, 100)$	0.99649	0.93065	0.417205	0.07968	0.0232	0.023405	0.02346	0.0232	0.02275	0.02302
$\psi(1000, 2000)$	0.99744	0.910655	0.306885	0.05536	0.01649	0.016655	0.016605	0.01649	0.016035	0.01594
$\psi(1000, 100)$	0.996145	0.90372	0.295105	0.051675	0.015245	0.01539	0.015385	0.015245	0.01473	0.014625

Table B.2 Ruin probabilities with substituted λ_1 and λ_2

	α_1					α_2				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
model 1										
$\psi(500, 2000)$	0.00187	0.024635	0.218995	0.90633	0.99998	0.003025	0.00672	0.01302	0.024635	0.04448
$\psi(500, 100)$	0.001755	0.016425	0.074645	0.21114	0.42669	0.002695	0.00545	0.00987	0.016425	0.025825
$\psi(750, 2000)$	0.00009	0.00422	0.10491	0.848245	0.99995	0.000185	0.000555	0.00158	0.00422	0.00968
$\psi(750, 100)$	0.00007	0.00166	0.01323	0.06067	0.17547	0.00014	0.000305	0.000765	0.00166	0.003035
$\psi(1000, 2000)$	0.00001	0.00072	0.049985	0.7837	0.99986	0.000005	0.000035	0.000215	0.00072	0.002115
$\psi(1000, 100)$	0.000005	0.000165	0.001725	0.011755	0.052075	0	0.000005	0.000045	0.000165	0.000255
model 2										
$\psi(500, 2000)$	0.05172	0.0448	0.04338	0.045505	0.04938	0.04145	0.040745	0.04289	0.0448	0.04666
$\psi(500, 100)$	0.04914	0.04362	0.042545	0.04477	0.048795	0.04014	0.03965	0.04176	0.04362	0.045475
$\psi(750, 2000)$	0.031885	0.02429	0.02247	0.02245	0.02301	0.022725	0.022355	0.023845	0.02429	0.02575
$\psi(750, 100)$	0.02923	0.023135	0.02163	0.02171	0.02235	0.0214	0.02127	0.022695	0.023135	0.024535
$\psi(1000, 2000)$	0.022705	0.01591	0.0145	0.01403	0.01428	0.01539	0.01477	0.015815	0.01591	0.01734
$\psi(1000, 100)$	0.020015	0.01472	0.01366	0.013285	0.01361	0.014025	0.013585	0.014565	0.01472	0.016075
model 3										
$\psi(500, 2000)$	0.00203	0.02448	0.219865	0.905655	0.99994	0.002905	0.00709	0.013225	0.02448	0.04468
$\psi(500, 100)$	0.001885	0.01626	0.07502	0.21098	0.426665	0.00267	0.00583	0.00995	0.01626	0.02597
$\psi(750, 2000)$	0.00011	0.00411	0.104535	0.849115	0.999885	0.000165	0.000585	0.00163	0.00411	0.00992
$\psi(750, 100)$	0.000075	0.001585	0.013425	0.060395	0.176805	0.000135	0.000385	0.000765	0.001585	0.003105
$\psi(1000, 2000)$	0.00001	0.00066	0.04992	0.784365	0.999815	0.000005	0.0004	0.0002	0.00066	0.00225
$\psi(1000, 100)$	0.000005	0.00013	0.001625	0.012005	0.0516	0	0.000015	0.000035	0.00013	0.00026
model 4										
$\psi(500, 2000)$	0.053195	0.044195	0.04431	0.04629	0.049345	0.040565	0.04081	0.04348	0.044195	0.046595
$\psi(500, 100)$	0.05054	0.04301	0.043355	0.04556	0.048705	0.03937	0.039655	0.042325	0.04301	0.04545
$\psi(750, 2000)$	0.032735	0.02441	0.022945	0.02244	0.02319	0.02214	0.022745	0.02417	0.02441	0.025885
$\psi(750, 100)$	0.03007	0.0232	0.021975	0.02169	0.02254	0.020945	0.02156	0.02298	0.0232	0.024695
$\psi(1000, 2000)$	0.023515	0.01649	0.014675	0.014005	0.01427	0.01503	0.01517	0.01584	0.01649	0.017565
$\psi(1000, 100)$	0.02072	0.015245	0.013725	0.01325	0.0136	0.0138	0.013875	0.01467	0.015245	0.01637

Table B.3 Ruin probabilities with substituted c_1 and c_2

	$c_1=0.25$	$c_1=0.5$	$c_1=0.75$
model 1			
$\psi(500, 2000)$	0.373705	0.095175	0.024635
$\psi(500, 100)$	0.103525	0.044965	0.016425
$\psi(750, 2000)$	0.22971	0.03029	0.00422
$\psi(750, 100)$	0.021495	0.00635	0.00166
$\psi(1000, 2000)$	0.139765	0.009415	0.00072
$\psi(1000, 100)$	0.00311	0.000675	0.000165
model 2			
$\psi(500, 2000)$	0.042295	0.02448	0.0448
$\psi(500, 100)$	0.041625	0.02397	0.04362
$\psi(750, 2000)$	0.02097	0.013115	0.02429
$\psi(750, 100)$	0.020265	0.01258	0.023135
$\psi(1000, 2000)$	0.013345	0.00876	0.01591
$\psi(1000, 100)$	0.01261	0.008215	0.01472
model 3			
$\psi(500, 2000)$	0.373185	0.095175	0.02448
$\psi(500, 100)$	0.10449	0.04349	0.01626
$\psi(750, 2000)$	0.22966	0.03084	0.00411
$\psi(750, 100)$	0.02149	0.006105	0.001585
$\psi(1000, 2000)$	0.14103	0.01006	0.00066
$\psi(1000, 100)$	0.003365	0.00066	0.00013
model 4			
$\psi(500, 2000)$	0.04241	0.02421	0.044195
$\psi(500, 100)$	0.041645	0.023715	0.04301
$\psi(750, 2000)$	0.02069	0.01325	0.02441
$\psi(750, 100)$	0.019905	0.01277	0.0232
$\psi(1000, 2000)$	0.013355	0.00852	0.01649
$\psi(1000, 100)$	0.01252	0.00805	0.015245

Table B.4 Parameters of fitted distributions with default parameters (model 1)

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
MTFHPP ($u = 500$)		
$\alpha_1=0.1$	1.736544	0.161875
$\alpha_1=0.3$	6.931490	12.907127
$\alpha_1=0.5$	20.863197	56.630527
$\alpha_1=0.7$	108.422033	119.023242
$\alpha_1=0.9$	93.504016	158.924275
$\alpha_2=0.1$	93.743205	160.274760
$\alpha_2=0.3$	95.504299	163.632697
$\alpha_2=0.5$	93.504016	158.924275
$\alpha_2=0.7$	94.601584	168.793540
$\alpha_2=0.9$	94.984285	156.547996
$\lambda_1=0.1$	49.585078	141.541559
$\lambda_1=0.3$	93.504016	158.924275
$\lambda_1=0.5$	241.039832	176.352383
$\lambda_1=0.7$	401.690447	208.811837
$\lambda_1=0.9$	164.058703	184.016630
$\lambda_2=0.1$	55.934370	170.507499
$\lambda_2=0.3$	67.717129	152.813741
$\lambda_2=0.5$	79.435996	162.824444
$\lambda_2=0.7$	93.504016	158.924275
$\lambda_2=0.9$	111.593350	167.446835
$c_1=0.25$	322.703787	189.122655
$c_1=0.5$	154.198241	171.124131
$c_1=0.75$	93.504016	158.924275
MTFHPP ($u = 750$)		
$\alpha_1=0.1$	2.570388	0.624654
$\alpha_1=0.3$	10.224518	28.897045
$\alpha_1=0.5$	30.852025	124.012570
$\alpha_1=0.7$	160.407647	262.474519
$\alpha_1=0.9$	139.559746	371.002633
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Table B.4 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$\alpha_2=0.1$	137.459643	358.439063
$\alpha_2=0.3$	137.008760	327.395095
$\alpha_2=0.5$	139.559746	371.002633
$\alpha_2=0.7$	138.697805	347.513061
$\alpha_2=0.9$	139.231662	386.475193
$\lambda_1=0.1$	87.289769	473.240187
$\lambda_1=0.3$	139.559746	371.002633
$\lambda_1=0.5$	353.492403	395.139156
$\lambda_1=0.7$	564.995102	489.344836
$\lambda_1=0.9$	243.063599	409.232935
$\lambda_2=0.1$	76.616405	359.947592
$\lambda_2=0.3$	97.898509	364.438426
$\lambda_2=0.5$	120.447093	362.036079
$\lambda_2=0.7$	139.559746	371.002633
$\lambda_2=0.9$	164.149611	366.846586
$c_1=0.25$	466.120151	435.295913
$c_1=0.5$	228.767093	382.226798
$c_1=0.75$	139.559746	371.002633
MTFHPP ($u = 1000$)		
$\alpha_1=0.1$	3.376175	1.378184
$\alpha_1=0.3$	13.467178	50.379486
$\alpha_1=0.5$	40.767683	216.125647
$\alpha_1=0.7$	212.245295	458.294992
$\alpha_1=0.9$	169.613602	560.698531
$\alpha_2=0.1$	189.397460	560.073063
$\alpha_2=0.3$	177.511707	603.975238
$\alpha_2=0.5$	169.613602	560.698531
$\alpha_2=0.7$	178.709830	586.213447
$\alpha_2=0.9$	188.181705	806.270969
$\lambda_1=0.1$	103.545402	806.642560
$\lambda_1=0.3$	169.613602	560.698531
Continued on next page		

Table B.4 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$\lambda_1=0.5$	468.260981	722.099034
$\lambda_1=0.7$	709.528981	925.983689
$\lambda_1=0.9$	322.052542	724.029683
$\lambda_2=0.1$	unknown	
$\lambda_2=0.3$	134.181191	613.916658
$\lambda_2=0.5$	166.415997	918.439923
$\lambda_2=0.7$	169.613602	560.698531
$\lambda_2=0.9$	215.795323	691.987799
$c_1=0.25$	597.753722	798.033270
$c_1=0.5$	302.456751	666.933403
$c_1=0.75$	169.613602	560.698531
MTFNPP ($u = 500$)		
$\alpha_1=0.1$	1.729972	0.163872
$\alpha_1=0.3$	6.948907	12.860763
$\alpha_1=0.5$	20.908181	56.789506
$\alpha_1=0.7$	107.757513	119.490737
$\alpha_1=0.9$	94.751570	164.626525
$\alpha_2=0.1$	96.186092	165.567190
$\alpha_2=0.3$	94.831148	159.865012
$\alpha_2=0.5$	94.751570	164.626525
$\alpha_2=0.7$	95.839577	157.859758
$\alpha_2=0.9$	93.979272	159.765785
$\lambda_1=0.1$	50.867421	177.220949
$\lambda_1=0.3$	94.751570	164.626525
$\lambda_1=0.5$	238.836382	178.097994
$\lambda_1=0.7$	401.791533	207.650535
$\lambda_1=0.9$	164.386916	183.951069
$\lambda_2=0.1$	56.438122	152.149749
$\lambda_2=0.3$	68.948354	147.452805
$\lambda_2=0.5$	79.918747	162.167089
$\lambda_2=0.7$	94.751570	164.626525
Continued on next page		

Table B.4 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$\lambda_2=0.9$	113.424835	165.079768
$c_1=0.25$	325.011451	188.617942
$c_1=0.5$	151.226695	170.527887
$c_1=0.75$	94.751570	164.626525
MTFNPP ($u = 750$)		
$\alpha_1=0.1$	2.555041	0.626731
$\alpha_1=0.3$	10.234961	28.882778
$\alpha_1=0.5$	30.934851	124.515495
$\alpha_1=0.7$	159.780635	263.694484
$\alpha_1=0.9$	141.474429	360.521813
$\alpha_2=0.1$	141.098453	359.667019
$\alpha_2=0.3$	138.528194	335.361465
$\alpha_2=0.5$	141.474429	360.521813
$\alpha_2=0.7$	139.645979	364.252405
$\alpha_2=0.9$	141.006311	341.317003
$\lambda_1=0.1$	73.460819	383.213809
$\lambda_1=0.3$	141.474429	360.521813
$\lambda_1=0.5$	351.620543	399.055337
$\lambda_1=0.7$	563.954678	487.120537
$\lambda_1=0.9$	243.765766	409.853490
$\lambda_2=0.1$	85.764071	419.487770
$\lambda_2=0.3$	104.737569	386.064342
$\lambda_2=0.5$	120.719256	376.950985
$\lambda_2=0.7$	141.474429	360.521813
$\lambda_2=0.9$	167.716522	361.646135
$c_1=0.25$	467.946740	430.665416
$c_1=0.5$	225.595726	376.663208
$c_1=0.75$	141.474429	360.521813
MTFNPP ($u = 1000$)		
$\alpha_1=0.1$	3.366629	1.378773
$\alpha_1=0.3$	13.466059	50.584546
Continued on next page		

Table B.4 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$\alpha_1=0.5$	40.814873	216.095932
$\alpha_1=0.7$	211.638625	460.340812
$\alpha_1=0.9$	180.856516	600.071880
$\alpha_2=0.1$	179.325641	650.471084
$\alpha_2=0.3$	189.367231	631.319072
$\alpha_2=0.5$	180.856516	600.071880
$\alpha_2=0.7$	189.052168	658.643196
$\alpha_2=0.9$	193.197130	594.478922
$\lambda_1=0.1$	87.580711	3433.771089
$\lambda_1=0.3$	180.856516	600.071880
$\lambda_1=0.5$	460.306563	712.559314
$\lambda_1=0.7$	710.746326	924.525727
$\lambda_1=0.9$	322.437436	722.582631
$\lambda_2=0.1$	unknown	
$\lambda_2=0.3$	143.349094	1904.818292
$\lambda_2=0.5$	153.228757	835.820697
$\lambda_2=0.7$	180.856516	600.071880
$\lambda_2=0.9$	225.601339	605.589345
$c_1=0.25$	597.427959	798.859329
$c_1=0.5$	297.381151	647.705621
$c_1=0.75$	180.856516	600.071880

Table B.5 Parameters of fitted distributions with default parameters (model 2)

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
MTFHPP ($u = 500$)		
$\alpha_1=0.1$	1.944541	0.146251
$\alpha_1=0.3$	6.133941	6.918411
$\alpha_1=0.5$	13.740798	15.685142
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Table B.5 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$\alpha_1=0.7$	20.856469	25.904670
$\alpha_1=0.9$	26.215863	38.679535
$\alpha_2=0.1$	26.111083	39.258574
$\alpha_2=0.3$	26.352391	37.955265
$\alpha_2=0.5$	26.215863	38.679535
$\alpha_2=0.7$	28.287866	35.505485
$\alpha_2=0.9$	26.916467	36.809816
$\lambda_1=0.1$	35.309804	39.931642
$\lambda_1=0.3$	26.215863	38.679535
$\lambda_1=0.5$	23.632937	35.058761
$\lambda_1=0.7$	20.542332	35.101576
$\lambda_1=0.9$	18.006967	37.356901
$\lambda_2=0.1$	28.459905	39.901391
$\lambda_2=0.3$	27.771790	39.490623
$\lambda_2=0.5$	26.232487	40.310410
$\lambda_2=0.7$	26.215863	38.679535
$\lambda_2=0.9$	25.222393	38.945344
$c_1=0.25$	21.422327	36.541342
$c_1=0.5$	22.949237	39.657906
$c_1=0.75$	26.215863	38.679535
MTFHPP ($u = 750$)		
$\alpha_1=0.1$	2.721104	0.561131
$\alpha_1=0.3$	9.351144	12.734376
$\alpha_1=0.5$	22.528616	25.196908
$\alpha_1=0.7$	33.453887	43.636532
$\alpha_1=0.9$	39.336248	63.806251
$\alpha_2=0.1$	38.726392	65.461947
$\alpha_2=0.3$	39.322319	61.442361
$\alpha_2=0.5$	39.336248	63.806251
$\alpha_2=0.7$	42.828310	57.437312
$\alpha_2=0.9$	40.539829	59.525870
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Table B.5 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$\lambda_1=0.1$	50.050510	67.815398
$\lambda_1=0.3$	39.336248	63.806251
$\lambda_1=0.5$	36.319014	56.589791
$\lambda_1=0.7$	32.056950	55.632292
$\lambda_1=0.9$	27.857572	60.514024
$\lambda_2=0.1$	42.499186	66.146091
$\lambda_2=0.3$	41.214940	64.978841
$\lambda_2=0.5$	38.938678	67.359193
$\lambda_2=0.7$	39.336248	63.806251
$\lambda_2=0.9$	37.363169	64.452708
$c_1=0.25$	33.537896	57.131860
$c_1=0.5$	33.760255	66.421843
$c_1=0.75$	39.336248	63.806251
MTFHPP ($u = 1000$)		
$\alpha_1=0.1$	3.467921	1.245402
$\alpha_1=0.3$	12.739262	18.612081
$\alpha_1=0.5$	32.172029	34.896776
$\alpha_1=0.7$	45.065155	61.314441
$\alpha_1=0.9$	51.010255	88.889889
$\alpha_2=0.1$	50.120573	91.626150
$\alpha_2=0.3$	51.538577	84.753615
$\alpha_2=0.5$	51.010255	88.889889
$\alpha_2=0.7$	56.274942	77.453973
$\alpha_2=0.9$	53.073602	80.582050
$\lambda_1=0.1$	63.822064	94.957389
$\lambda_1=0.3$	51.010255	88.889889
$\lambda_1=0.5$	48.441919	75.612427
$\lambda_1=0.7$	42.904147	75.474593
$\lambda_1=0.9$	37.171731	82.916679
$\lambda_2=0.1$	55.077654	90.996280
$\lambda_2=0.3$	54.552877	87.805984
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Table B.5 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$\lambda_2=0.5$	50.445184	94.294598
$\lambda_2=0.7$	51.010255	88.889889
$\lambda_2=0.9$	48.315660	88.532556
$c_1=0.25$	45.023931	77.565666
$c_1=0.5$	44.091545	90.966499
$c_1=0.75$	51.010255	88.889889
MTFNPP ($u = 500$)		
$\alpha_1=0.1$	1.978713	0.145692
$\alpha_1=0.3$	6.155867	6.869668
$\alpha_1=0.5$	13.901867	15.415544
$\alpha_1=0.7$	21.761345	24.802214
$\alpha_1=0.9$	26.390501	38.051303
$\alpha_2=0.1$	27.677103	35.185516
$\alpha_2=0.3$	26.425973	38.546059
$\alpha_2=0.5$	26.390501	38.051303
$\alpha_2=0.7$	26.838277	37.254953
$\alpha_2=0.9$	26.315400	38.194973
$\lambda_1=0.1$	36.105480	39.119173
$\lambda_1=0.3$	26.390501	38.051303
$\lambda_1=0.5$	22.030985	38.554697
$\lambda_1=0.7$	20.230347	36.222607
$\lambda_1=0.9$	18.416626	35.598887
$\lambda_2=0.1$	28.552110	40.264927
$\lambda_2=0.3$	26.528826	41.710645
$\lambda_2=0.5$	26.468013	39.962175
$\lambda_2=0.7$	26.390501	38.051303
$\lambda_2=0.9$	26.104646	36.623078
$c_1=0.25$	21.184601	37.175090
$c_1=0.5$	23.076361	38.783114
$c_1=0.75$	26.390501	38.051303
MTFNPP ($u = 750$)		
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Table B.5 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$\alpha_1=0.1$	2.765155	0.563039
$\alpha_1=0.3$	9.443699	12.538340
$\alpha_1=0.5$	22.688377	24.977842
$\alpha_1=0.7$	35.393897	41.164240
$\alpha_1=0.9$	38.984329	62.818654
$\alpha_2=0.1$	41.886873	57.418830
$\alpha_2=0.3$	39.313055	62.882382
$\alpha_2=0.5$	38.984329	62.818654
$\alpha_2=0.7$	40.316835	61.512244
$\alpha_2=0.9$	39.496764	63.473851
$\lambda_1=0.1$	51.955311	65.473105
$\lambda_1=0.3$	38.984329	62.818654
$\lambda_1=0.5$	33.495066	64.645329
$\lambda_1=0.7$	31.481985	59.652981
$\lambda_1=0.9$	29.439696	55.865323
$\lambda_2=0.1$	42.790942	66.681207
$\lambda_2=0.3$	39.328498	69.104184
$\lambda_2=0.5$	39.266339	66.539661
$\lambda_2=0.7$	38.984329	62.818654
$\lambda_2=0.9$	39.505285	59.313581
$c_1=0.25$	32.973702	60.764216
$c_1=0.5$	34.741395	65.117889
$c_1=0.75$	38.984329	62.818654
MTFNPP ($u = 1000$)		
$\alpha_1=0.1$	3.505921	1.228717
$\alpha_1=0.3$	12.764805	18.618484
$\alpha_1=0.5$	32.791577	34.315365
$\alpha_1=0.7$	47.985459	57.920695
$\alpha_1=0.9$	51.342390	87.608275
$\alpha_2=0.1$	54.698405	77.982701
$\alpha_2=0.3$	51.369249	85.674501
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Table B.5 – continued from previous page

	Parameters for fitted Inverse Gaussian distribution	
	δ	η
$\alpha_2=0.5$	51.342390	87.608275
$\alpha_2=0.7$	52.533037	85.193807
$\alpha_2=0.9$	51.024759	88.726060
$\lambda_1=0.1$	66.537056	89.540016
$\lambda_1=0.3$	51.342390	87.608275
$\lambda_1=0.5$	43.687266	90.030634
$\lambda_1=0.7$	41.959928	81.237019
$\lambda_1=0.9$	39.751174	76.091345
$\lambda_2=0.1$	56.054164	92.048740
$\lambda_2=0.3$	51.704456	94.146151
$\lambda_2=0.5$	51.593545	90.694492
$\lambda_2=0.7$	51.342390	87.608275
$\lambda_2=0.9$	51.322267	81.175297
$c_1=0.25$	44.087969	82.388442
$c_1=0.5$	45.467919	91.174670
$c_1=0.75$	51.342390	87.608275