Coxeter groups, graphs and Ricci curvature

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Discrete Ricci curvature

Definition Let $G = (\mathcal{V}(G), \mathcal{E}(G))$ be a simple, undirected, locally finite graph. Let $f : \mathcal{V}(G) \to \mathbb{R}$, we define the following operators:

- $L(f)(x) = \sum_{y \in B(1,x)} (f(x) - f(y))$;
- $\Gamma(f, g)(x) = \sum_{y \in B(1,x)} (f(x) - f(y))(g(x) - g(y))$;
- $\Gamma_2(f, f)(x) = \frac{1}{2} L\Gamma(f, f)(x) - \Gamma(f, L(f))(x)$.

Where $B(i, x) = \{v \in \mathcal{V}(G) | \text{distance}(x, v) = i\}$.
We define, following [2]:
-Local Ricci curvature of G at $x \in \mathcal{V}(G)$

$$Ric(G)_x = inf_f \frac{\Gamma_2(f, f)(x)}{\Gamma(f, f)(x)}$$

-Global Ricci curvature of G:

$$Ric(G) = inf_x Ric(G)_x$$

Remark: The local Ricci curvature only depends on a neighbourhood of x

Applications

From Section 3 and 4 of [3]. Let G a finite graph

Definition Let Δ the Laplacian operator, this acts linearly on the real functions on $\mathcal{V}(G)$. We can see it as a matrix (laplacian matrix), we call spectral gap its minimal non-zero eigenvalue.

Theorem Let G be a graph with $Ric(G) > K > 0$ and λ_G its spectral gap, then $\lambda_G \geq K$.

Theorem [Isoperimetric inequality] Suppose G has $Ric(G) \geq K$, for some $K \in \mathbb{R}$. Then, for any subset $A \subset \mathcal{V}(G)$,

$$|\partial A| \geq \frac{1}{2} min\{\sqrt{\lambda_G}, \frac{\lambda_G}{\sqrt{2|K|}}\} |A| \left(1 - \frac{|A|}{|\mathcal{V}(G)|}\right).$$

Here, by ∂A, we mean the collection of all edges connecting A to its complement.

Curvature of Bruhat graphs

Theorem If W is any finite Coxeter group and $B(W)$ is its Bruhat graph, then

$$Ric(B(W)) = 2.$$

Proof:

- all the distance 2 neighbours are isomorphic, it is sufficient to compute $Ric_e(B(W))$;
- the distance 2 neighbours can be seen as union of $B(I_2(m))$;
- $Ric(B(I_2(m))) = 2$.

Corollary Let W be a finite Coxeter group and $B(W)$ be its Bruhat graph. Then $\lambda_{B(W)} \geq 2$.

Corollary Let $B(W)$ be the Bruhat graph associated to any finite Coxeter group W, let A be a subset of W, then

$$|\partial A| \geq \frac{1}{2} |A| \left(1 - \frac{|A|}{|W|}\right).$$

References

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Goal

We want to compute the Ricci curvature for graphs associated to Coxeter groups. Given a Coxeter system (W, S) are interested in the following families:

- Bruhat graphs: $\mathcal{V}(G) = W$, $(u, v) \in \mathcal{E}(G)$ iff $u = tv$ where t is a reflections;
- Weak order graphs: $\mathcal{V}(G) = W$, $(u, v) \in \mathcal{E}(G)$ iff $u = sv$, $s \in S$;
- Hasse diagram of Bruhat order: $\mathcal{V}(G) = W$ and $(u, v) \in \mathcal{E}(G)$ iff u covers v according to the Bruhat order.

Ricci Curvature as Matrix eigenvalue

Theorem [Matrix Theorem]: Let $x \in V(G)$; we enumerate the elements connected to x: $B(1, x) = \{v_1, \ldots, v_{d(x)}\}$, then

$$Ric(G)_x = \min Eig(A_x)$$

Where $A(x)$ is a matrix $A \in \mathbb{R}^{d(x) \times d(x)}$ such that

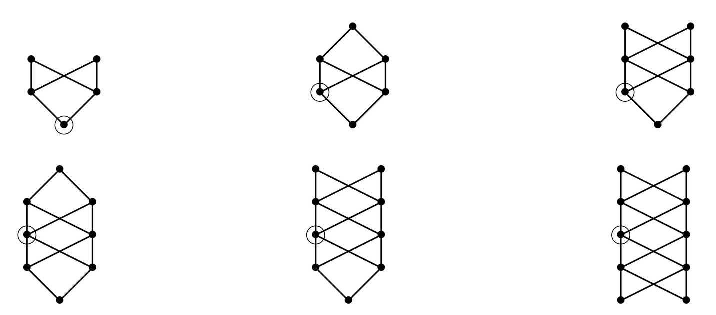
$$A_{ij}(x) = \begin{cases} \sum_{u \in \mathcal{U}_{v_i}} \frac{2(n_u - 1)}{n_u} + 1 + \frac{4 - d(x) - d(v_i)}{2} + \frac{3}{2} t_{v_i} & \text{if } i = j \\ \sum_{\mathcal{U}_{v_i} \cap \mathcal{U}_{v_j}} -\frac{2}{n_u} + 1 + 2T(v_i, v_j) & \text{if } i \neq j \end{cases}.$$

where

- n_u is the cardinality of $B(1, u) \cap B(1, x)$, $u \in B(2, x)$;
- $\mathcal{U}_v := B(2, x) \cap B(1, v)$, $v \in B(1, x)$;
- t_v number of triangles containing x and v;

$$T(v, v') = \begin{cases} 1 & \text{if there is a triangle with vertices } x, v, v' \\ 0 & \text{otherwise} \end{cases}.$$

Hasse diagram of Bruhat order



Distance-2 neighbours in dihedral case

Dihedral case: There is only a finite number of distance two neighbours that occur in the Hasse diagram of a dihedral group.

- $Ric(H(I_2(3))) = \frac{21 - \sqrt{33}}{12}$,
- $Ric(H(I_2(4))) = \frac{1}{2}$,
- $Ric(H(I_2(5))) = \frac{5 - \sqrt{17}}{2}$,
- $Ric(H(I_2(m))) = 0$ for any $m > 5$.

Cases A, B **and** D. As a corollary of [matrix Theorem] we have that:
Given G a triangle free graph, then

$$Ric(G) \geq 4 - 2d_{max}.$$

Where d_{max} is the maximal degree of the vertices in G.

- For $G = H(A_{n-1})$, $d_{max} = \lfloor \frac{n^2}{4} \rfloor + n - 2$. see [1]
- For $G = H(B_n)$

$$d_{max} = \begin{cases} 4(n-1) & n < 5, \\ \lfloor \frac{n^2}{2} \rfloor + n - 1 & n \geq 5. \end{cases}$$

- For $G = H(D_n)$

$$d_{max} = \lfloor \frac{n^2}{2} \rfloor + n - 1 \quad n \geq 3.$$

Weak order graphs

Theorem Let W a finite Coxeter group, $V(W)$ its weak order graph, then:

- $Ric(V(W)) = -2cos(\pi/n)$ if $W = A_n, B_n, H_3, H_4, I_2(m), F_4$;
- $-4 \leq Ric(V(W)) \leq -1$ for $W = D_n$;
- $Ric(V(W)) \approx -2.3$ for E_6, E_7, E_8.
- If $W = W_1 \times \ldots \times W_n$, with W_i irreducible Coxeter groups, $Ric(V(W)) = min_k Ric(V(W_k))$.

Proof The curvature is computed as a matrix eigenvalue:

- If $W = A_n, B_n, H_3, H_4, I_2(m), F_4$ we end up working with Toeplitz matrices, whose eigenvalues are described through trigonometric functions;
- If $W = D_n$ we obtain a matrix whose eigenvalues can be bounded (Gershgorin Theorem);
- Cases E_6, E_7, E_8 are a direct computation of the eigenvalues of three matrices.

Corollary Let (W, S) be a finite Coxeter system and $A \subset W$. Then the following isoperimetric inequalities hold for $V(W)$:

- If in the irreducible decomposition of W dihedrals don't appear:

$$|\partial A| \geq \frac{1}{2} \frac{|W|}{|S|^{|T|}|T|\sqrt{2|Ric(V(W))|}} |A| \left(1 - \frac{|A|}{|W|}\right)$$

- If at least one dihedral group appears in the irreducible decomposition of W:

$$|\partial A| \geq \frac{1}{2} \sqrt{\frac{|W|}{|S|^{|T|}|T|}} |A| (1 - \frac{|A|}{|W|})$$

where T denotes the set of reflections.