# Fast Signature-Based Generative Models for Time Series

## 1 Background: Path Signatures

Let $X : [0,T] \to \mathbb{R}^d$ be a continuous path of finite variation. The (*truncated*) signature of $X$ over $[0,T]$ to level $m$, denoted $S^{(m)}(X)_{0,T} \in \bigoplus_{k=0}^{m}(\mathbb{R}^d)^{\otimes k}$, collects the iterated integrals

$$S^{(k)}(X)_{0,T}^{i_1,\ldots,i_k} = \int_{0<t_1<\cdots<t_k<T} dX_{t_k}^{i_k} \cdots dX_{t_1}^{i_1}, \qquad k = 1,2,\ldots,m.$$

In practice, for a one-dimensional series $S_t$ we embed time and value to form a two-dimensional path $Z_t = (t, S_t)$ before computing the signature. Key properties used here: (i) Chen's identity for concatenation, (ii) faithfulness/uniqueness (up to tree-like equivalence), and (iii) *expected signatures* characterize laws, which motivates the linear signature MMD used for evaluation.

**Notation.** We use sliding windows of length $L$ (lookback) and forward segment length $F$. For a window ending at index $t$, write $W_t = Z_{[t-L,\,t]}$ and $s_t = S^{(m)}(W_t) \in \mathbb{R}^{d(m)}$ for the truncated signature feature.

## 2 Model 1: Path-wise Signature Bootstrap

### 2.1 Idea

Build an empirical library of *cause/effect* pairs

$$\mathcal{D} = \{(s_t,\, p_t)\}, \quad s_t = S^{(m)}(Z_{[t-L,\,t]}), \quad p_t = S_{t+1:t+F} - S_t,$$

i.e., store the signature of each lookback window and the corresponding future *relative* segment to ensure continuity when stitching. At generation time, compute the current lookback signature $s_{\text{gen}}$, find its $k$ nearest neighbors in $\mathcal{D}$, and *sample* one of the stored future segments to append.

## 2.2 Pseudo-code

**Algorithm 1** Path-wise Signature Bootstrap

**Input:** historical paths $\mathcal{R}$, lookback $L$, forward $F$, level $m$, $k$
**Library creation:**
$\mathcal{D} \leftarrow \emptyset$
**for** $R \in \mathcal{R}$ **do**
  **for** $t = L, \ldots, |R| - F$ **do**
    $s_t \leftarrow S^{(m)}\big((\tau, R_\tau)_{\tau=t-L}^{t}\big)$
    $p_t \leftarrow (R_{t+1}, \ldots, R_{t+F}) - R_t$
    $\mathcal{D} \leftarrow \mathcal{D} \cup \{(s_t, p_t)\}$
  **end for**
**end for**
**Generation:** given seed path $R^{\text{seed}}_{0:L}$
$R^{\text{gen}} \leftarrow R^{\text{seed}}$
**while** length$(R^{\text{gen}})$ < target **do**
  $s_{\text{gen}} \leftarrow S^{(m)}$ of last $L+1$ points
  $N_k \leftarrow k$-NN of $s_{\text{gen}}$ in $\mathcal{D}$
  Sample $(s_j, p_j) \in N_k$ uniformly (or softmax by distance)
  Append $p_j$ shifted by current level: $R^{\text{gen}} \leftarrow R^{\text{gen}} \cup \big(R^{\text{gen}}_{-1} + p_j\big)$
**end while**

**Remarks.** Using time-normalized windows (time reparameterized to $[0, 1]$) stabilizes the signature features; log-signatures may further de-correlate components. Soft neighbor sampling (e.g., softmax on distances) reduces jumps.

# 3 Model 2: Hybrid Drift + Signature Residual

## 3.1 Idea

Decompose one-step dynamics into a simple parametric drift plus a nonparametric innovation sampled from a signature-conditioned library. With $X_t = \log S_t$ and $\Delta X_t = X_{t+1} - X_t$ (step $\Delta t$), learn a mean-reverting drift $\mu_\theta(x) = c\,x$ by least squares on residuals, then build a library of residuals conditioned on window signatures.

## 3.2 Training

$$\text{Given } \{X_t\}, \quad \text{solve} \quad \min_c \sum_t \Big(\Delta X_t - \mu_c(X_t)\,\Delta t\Big)^2 + \lambda c^2, \qquad \mu_c(x) = c\,x.$$

Then, for each $t \geq L$, define the residual $r_t := \Delta X_t - \mu_c(X_t)\,\Delta t$ and store pairs $(s_t, r_t)$ with $s_t = S^{(m)}(W_t)$ in a residual library $\mathcal{R}$.

## 3.3 Generation (Euler step with resampled residual)

$$X_{t+1} \;=\; X_t + \mu_c(X_t)\,\Delta t + \widetilde{r}_t, \qquad \widetilde{r}_t \sim \text{Empirical}\big(\{r_j : (s_j, r_j) \in N_k(s_t)\}\big).$$

Pseudocode mirrors Model 1, except the future segment is a scalar residual added per step, not a multi-step block. Soft $k$-NN sampling is recommended.

### Model 2 Pseudocode

**Algorithm 2** Hybrid Drift + Signature Residual: Training

**Input:** detrended log-paths $\{\{X_t^{(n)}\}_t\}_{n=1}^N$, lookback $L$, step $\Delta t$, signature level $m$, ridge $\lambda$
**Build one-step dataset**
$\mathcal{T} \leftarrow \emptyset$
**for** $n = 1$ to $N$ **do**
  **for** $t = L$ to $T_n - 1$ **do**
    $W_t \leftarrow (X_{t-L}^{(n)}, \dots, X_t^{(n)})$
    $s_t \leftarrow S^{(m)}\big((\tau, W_t(\tau))_{\tau\in[0,1]}\big)$ ▷ time normalized to $[0, 1]$
    $y_t \leftarrow X_t^{(n)}, \quad \Delta X_t \leftarrow X_{t+1}^{(n)} - X_t^{(n)}$
    $\mathcal{T} \leftarrow \mathcal{T} \cup \{(s_t, y_t, \Delta X_t)\}$
  **end for**
**end for**
**Fit linear mean-reversion drift** $\mu_c(x) = c\,x$:

$$c^\star \in \arg\min_c \sum_{(s_t,y_t,\Delta X_t)\in\mathcal{T}} \big(\Delta X_t - c\,y_t\,\Delta t\big)^2 + \lambda c^2$$

**Build residual library** $\mathcal{R}$:

$$r_t \leftarrow \Delta X_t - c^\star\,y_t\,\Delta t, \qquad \mathcal{R} \leftarrow \mathcal{R} \cup \{(s_t, r_t)\}$$

**Output:** drift coefficient $c^\star$, residual library $\mathcal{R} = \{(s_t, r_t)\}$

**Algorithm 3** Hybrid Drift + Signature Residual: Generation

**Input:** seed log-path $(X_0, \dots, X_L)$, horizon $T$, step $\Delta t$, level $m$, $k$-NN, temperature $\tau > 0$, drift $c^\star$, residual library $\mathcal{R}$
**for** $t = L$ to $T - 1$ **do**
  $W_t \leftarrow (X_{t-L}, \dots, X_t)$
  $s_t \leftarrow S^{(m)}\big((\tau, W_t(\tau))_{\tau\in[0,1]}\big)$
  Find $k$ nearest neighbors $N_k(s_t) \subset \mathcal{R}$ by Euclidean distance in signature space
  Compute weights $w_j \propto \exp\big(-\operatorname{dist}(s_t, s_j)/\tau\big)$ for $(s_j, r_j) \in N_k(s_t)$
  Sample residual $\widetilde{r}_t$ from $N_k(s_t)$ with probabilities $\{w_j\}$
  $X_{t+1} \leftarrow X_t + c^\star X_t \Delta t + \widetilde{r}_t$
**end for**
**Output:** generated path $(X_0, \dots, X_T)$ (convert to levels if needed: $S_t = e^{X_t}$)

# 4 Model 3: Kernel Ridge Regression (KRR) in Signature Space

## 4.1 Feature construction and targets

Work on $X_t = \log S_t$. For each window $W_t$ (length $L$) compute $s_t = S^{(m)}(W_t) \in \mathbb{R}^{d(m)}$ and define two targets:

$$y_t^{(\mu)} = \frac{X_{t+1} - X_t}{\Delta t}, \qquad y_t^{(\log\sigma)} = \log\Big(\frac{\operatorname{std}(\{\Delta X \text{ in } W_t\})}{\sqrt{\Delta t}} + \varepsilon\Big).$$

Stacking rows gives the signature design matrix $S \in \mathbb{R}^{N\times d(m)}$.

### 4.2 Training (linear kernel in signature space)

Form the Gram matrix $K = SS^\top \in \mathbb{R}^{N\times N}$ and solve two ridge systems:

$$\boldsymbol{\alpha}_\mu = (K+\lambda I_N)^{-1}\boldsymbol{y}_\mu, \qquad \boldsymbol{\alpha}_{\log\sigma} = (K+\lambda I_N)^{-1}\boldsymbol{y}_{\log\sigma}.$$

For a new window with signature $s_{\text{new}}$, predict via the kernel trick

$$\widehat{\mu} = (Ss_{\text{new}})^\top\boldsymbol{\alpha}_\mu, \qquad \widehat{\log\sigma} = (Ss_{\text{new}})^\top\boldsymbol{\alpha}_{\log\sigma}, \quad \widehat{\sigma} = e^{\widehat{\log\sigma}}.$$

### 4.3 Generation (Euler–Maruyama)

$$X_{t+1} \;=\; X_t + \widehat{\mu}\,\Delta t + \widehat{\sigma}\,\Delta W_t, \quad \Delta W_t \sim \mathcal{N}(0,\Delta t).$$

Repeat with a rolling window to refresh $s_{\text{new}}$ at each step.

### Model 3 Pseudocode

**Algorithm 4** KRR in Signature Space: Training (drift and log-vol)

**Input:** training log-paths $\{\{X_t^{(n)}\}_t\}_{n=1}^N$, lookback $L$, step $\Delta t$, level $m$, ridge $\lambda$, floor $\varepsilon > 0$
**Build windowed signature dataset**
$S \leftarrow [\,]$ ▷ design matrix rows
$\boldsymbol{y}_\mu \leftarrow [\,]$, $\boldsymbol{y}_{\log\sigma} \leftarrow [\,]$
**for** $n = 1$ to $N$ **do**
  **for** $t = L$ to $T_n - 1$ **do**
    $W_t \leftarrow (X_{t-L}^{(n)}, \dots, X_t^{(n)})$
    $s_t \leftarrow S^{(m)}\big((\tau, W_t(\tau))_{\tau\in[0,1]}\big)$
    $\Delta X_t \leftarrow X_{t+1}^{(n)} - X_t^{(n)}$
    $y_t^{(\mu)} \leftarrow \Delta X_t/\Delta t$
    $\sigma_t \leftarrow \max\big(\text{std}(\{\Delta X \text{ inside } W_t\})/\sqrt{\Delta t},\ \varepsilon\big)$
    $y_t^{(\log\sigma)} \leftarrow \log\sigma_t$
    Append row $s_t$ to $S$; append $y_t^{(\mu)}$ to $\boldsymbol{y}_\mu$; append $y_t^{(\log\sigma)}$ to $\boldsymbol{y}_{\log\sigma}$
  **end for**
**end for**
**(Optional) feature scaling:** center/scale columns of $S$ to get $Z$
**Kernel (linear in signature space):** $K \leftarrow ZZ^\top$
Solve

$$\boldsymbol{\alpha}_\mu = (K+\lambda I)^{-1}\boldsymbol{y}_\mu, \qquad \boldsymbol{\alpha}_{\log\sigma} = (K+\lambda I)^{-1}\boldsymbol{y}_{\log\sigma}.$$

**Output:** $(\boldsymbol{\alpha}_\mu, \boldsymbol{\alpha}_{\log\sigma}, Z)$ and scaling stats for signatures

**Algorithm 5** KRR in Signature Space: Generation (Euler–Maruyama)

**Input:** seed log-path $(X_0, \ldots, X_L)$, horizon $T$, step $\Delta t$, level $m$, training data $(\boldsymbol{\alpha}_\mu, \boldsymbol{\alpha}_{\log\sigma}, Z)$, signature scaling stats
**for** $t = L$ to $T-1$ **do**
  $W_t \leftarrow (X_{t-L}, \ldots, X_t)$
  $s_t \leftarrow S^{(m)}\big((\tau, W_t(\tau))_{\tau\in[0,1]}\big)$
  Standardize $s_t$ with saved stats to get $z_t$;  $k_t \leftarrow Z z_t$  ▷ $k_t$ is kernel vector
  $\widehat{\mu}_t \leftarrow k_t^\top \boldsymbol{\alpha}_\mu$
  $\widehat{\log\sigma}_t \leftarrow k_t^\top \boldsymbol{\alpha}_{\log\sigma}$;  $\widehat{\sigma}_t \leftarrow \exp(\widehat{\log\sigma}_t)$
  Sample $\Delta W_t \sim \mathcal{N}(0, \Delta t)$
  $X_{t+1} \leftarrow X_t + \widehat{\mu}_t\,\Delta t + \widehat{\sigma}_t\,\Delta W_t$
**end for**
**Output:** generated path $(X_0, \ldots, X_T)$ (levels via $S_t = e^{X_t}$ if desired)

# 5 Evaluation Protocol

We benchmark generative quality along four complementary axes: (1) geometry via signature MMD, (2) marginal distributions (KS/Wasserstein and moments), (3) temporal dependence (ACF and volatility clustering), and (4) downstream ML utility.

Throughout, let $\mathcal{P}_{\text{real}}$ be the set of real paths and $\mathcal{P}_{\text{gen}}$ the set of generated paths. For a path $S = (S_0, \ldots, S_T)$ define log-returns $R_t = \log S_t - \log S_{t-1}$.

## 5.1 Signature MMD (linear kernel)

Fix a window length $L$ and signature level $m$. For each path, slide a window of length $L$ and form a two-channel path $Z(\tau) = (\tau, S(\tau))$ with the time channel normalized to $\tau \in [0,1]$. Let $s \in \mathbb{R}^{d(m)}$ denote the truncated signature (or log-signature) of the window.

Let $\{s_i^{(\text{real})}\}_{i=1}^{n_r}$ be the collection of window-signatures from $\mathcal{P}_{\text{real}}$ and $\{s_j^{(\text{gen})}\}_{j=1}^{n_g}$ from $\mathcal{P}_{\text{gen}}$. With the linear kernel $k(u,v) = u^\top v$, the MMD reduces to the distance between mean signatures:

$$\mathrm{MMD}^2_{\text{sig}} \;=\; \|\, \bar{s}_{\text{real}} - \bar{s}_{\text{gen}} \,\|_2^2, \qquad \bar{s}_{\text{real}} = \frac{1}{n_r}\sum_{i=1}^{n_r} s_i^{(\text{real})}, \quad \bar{s}_{\text{gen}} = \frac{1}{n_g}\sum_{j=1}^{n_g} s_j^{(\text{gen})}.$$

**Variants.** (i) *log-signature* features in place of signatures; (ii) *multi-scale* aggregation over window lengths $\mathcal{L}$ and levels $\mathcal{M}$:

$$\mathrm{MMD}^2_{\text{multi}} \;=\; \sum_{L\in\mathcal{L}} \sum_{m\in\mathcal{M}} w_{L,m} \left\| \bar{s}^{(L,m)}_{\text{real}} - \bar{s}^{(L,m)}_{\text{gen}} \right\|_2^2, \quad w_{L,m} \geq 0, \; \sum w_{L,m} = 1.$$

**Notes.** Match the number of windows (or reweight) to mitigate small-sample bias.

## 5.2 Distributional congruence (KS/Wasserstein and moments)

We compare (a) terminal levels $S_T$, (b) terminal log-returns $G = \log(S_T/S_0)$, and (c) pooled per-step log-returns $\{R_t\}$.

**Two-sample KS.** Let $F_n$ and $G_m$ be empirical CDFs of samples $x_{1:n}$ and $y_{1:m}$. The KS statistic is

$$D_{n,m} \;=\; \sup_x \left| F_n(x) - G_m(x) \right|.$$

Lower $D_{n,m}$ (higher p-value) indicates closer marginals.

**Wasserstein-1.** The one-dimensional $W_1$ distance admits a quantile representation:

$$W_1(F,G) \;=\; \int_0^1 \left|F^{-1}(u) - G^{-1}(u)\right| du \;\approx\; \frac{1}{N}\sum_{i=1}^{N} \left|x_{(i)} - y_{(i)}\right|,$$

where $x_{(i)}$ and $y_{(i)}$ are order statistics (with interpolation if $n \neq m$). Smaller $W_1$ indicates closer distributions.

**Moment diagnostics.** Report (robust) moments for $X \in \{S_T, G, R_t\}$:

$$\text{mean} = \mathbb{E}[X], \quad \text{stdev} = \sqrt{\text{Var}(X)}, \quad \text{skew} = \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3}, \quad \text{kurt} = \frac{\mathbb{E}[(X-\mu)^4]}{\sigma^4}.$$

Optionally include robust analogues (median, MAD, trimmed moments).

### 5.3 Temporal dependence: ACF, squared-ACF, Ljung–Box

For a series $x_1, \ldots, x_n$ with mean $\bar{x}$, the sample autocorrelation at lag $k$ is

$$\hat{\rho}_k \;=\; \frac{\sum_{t=1}^{n-k}(x_t - \bar{x})(x_{t+k} - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2}, \qquad k = 1, 2, \ldots.$$

We compute mean ACF curves across paths for $x_t = R_t$ (linear dependence) and for $x_t = R_t^2$ (volatility clustering). Compare curves by an $\ell_2$ gap:

$$\text{ACFGap}(h) \;=\; \left(\frac{1}{h}\sum_{k=1}^{h}\left(\hat{\rho}_k^{\text{real}} - \hat{\rho}_k^{\text{gen}}\right)^2\right)^{1/2},$$

and analogously for squared returns.

**Ljung–Box test (iid check).** For a chosen horizon $h$, define the Q-statistic on $x_t$ as

$$Q(h) \;=\; n(n+2)\sum_{k=1}^{h}\frac{\hat{\rho}_k^2}{n-k},$$

which is asymptotically $\chi_h^2$ under the iid null. We report $(Q, p)$ on the concatenated *returns* and on *squared* returns to probe linear and second-order dependence, respectively. Higher p-values imply closer-to-iid behavior.

### 5.4 Downstream ML utility (task-based evaluation)

We quantify whether synthetic data help or harm learning under realistic splits.

**Tasks.** (1) One-step *regression* of log-return $R_{t+1}$ (report MSE/RMSE, MAE), (2) *classification* of return sign $\mathbb{1}\{R_{t+1} > 0\}$ (report AUC/accuracy), and (3) *volatility* forecasting (regress $|R_{t+1}|$ or $R_{t+1}^2$; report MSE).

**Regimes.** Let $\mathcal{D}_{\text{real}}^{\text{train}}$ and $\mathcal{D}_{\text{real}}^{\text{test}}$ be disjoint real splits, and $\mathcal{D}_{\text{gen}}$ be generated samples matched in horizon and sampling. We compare:

$$\begin{aligned}
&\mathbf{R}: \quad \text{train on } \mathcal{D}_{\text{real}}^{\text{train}}, \ \text{test on } \mathcal{D}_{\text{real}}^{\text{test}}, \\
&\mathbf{G}: \quad \text{train on } \mathcal{D}_{\text{gen}}, \ \text{test on } \mathcal{D}_{\text{real}}^{\text{test}}, \\
&\mathbf{R{+}G}: \quad \text{train on } \mathcal{D}_{\text{real}}^{\text{train}} \cup \mathcal{D}_{\text{gen}}, \ \text{test on } \mathcal{D}_{\text{real}}^{\text{test}}.
\end{aligned}$$

For a loss functional $\mathcal{L}$ (e.g., MSE or cross-entropy), define

$$\Delta_{\text{aug}} \ = \ \mathcal{L}(\mathbf{R{+}G}) \ - \ \mathcal{L}(\mathbf{R}),$$

with $\Delta_{\text{aug}} < 0$ indicating that synthetic data *improve* generalization. We also report $\mathcal{L}(\mathbf{G})$ to gauge domain shift and realism.

**Reporting.** For each metric, provide mean $\pm$ standard error over $B$ bootstrap resamples of the test set. When comparing methods, include paired confidence intervals for deltas (e.g., $\Delta_{\text{aug}}$).

### 5.5 Summary score (optional)

To summarize across views, combine standardized distances:

$$\mathcal{S} \ = \ \alpha\,\widetilde{\text{MMD}}_{\text{multi}} \ + \ \beta\,\widetilde{W}_1(\text{returns}) \ + \ \gamma\,\text{ACFGap}(h) \ + \ \eta\,\max\{0,\ \mathcal{L}(\mathbf{R{+}G}) - \mathcal{L}(\mathbf{R})\},$$

where tildes denote z-scored metrics across models to make scales comparable, and $(\alpha, \beta, \gamma, \eta)$ are user-chosen weights. Lower $\mathcal{S}$ is better.

**Practical defaults.** Use $L \in \{10, 15, 20\}$ and $m \in \{3, 4\}$ for signatures; compare terminal values, terminal log-returns, and pooled per-step log-returns; set $h \in [10, 20]$ for ACF/Ljung–Box; in ML tasks, hold out the last 20% of each series for testing.

## 6 Practical Notes (matching the code)

- **Path representation.** If you store *levels* or *log-levels*, use relative segments $p_t = S_{t+1:t+F} - S_t$; if you store *returns*, do not re-center segments.
- **Time normalization.** Normalize the time channel within each window to $[0, 1]$ for stable signatures.
- **k-NN sampling.** Prefer soft sampling (e.g., softmax on distances) to reduce discontinuities.
- **KRR solver.** Use Cholesky with jitter on $K + \lambda I$; center/scale signature features columnwise.

## 7 Experiment: Path-wise Signature Bootstrap on S&P 500 (2010–2024)

### 7.1 Data and Preprocessing

We source daily close prices via `yfinance` for S&P 500 (`^GSPC`), DJIA (`^DJI`), and Nasdaq (`^IXIC`) over `2010-01-01` to `2025-01-01`. For this experiment we use only S&P 500 for generation/evaluation; the others are reserved for future multi-asset tests.

We consider two training datasets:

1. **Yearly paths (*multi*)**: for each year $y \in \{2010, \ldots, 2024\}$ we extract the first 250 trading days of `^GSPC` as one path $S^{(y)}$, normalize by $S_0^{(y)}$ to obtain a price index $I_t^{(y)} = S_t^{(y)}/S_0^{(y)}$, then remove a linear trend on the fixed grid $t = 0, \ldots, 249$ via least squares: $I_t^{(y)} = \hat{a}^{(y)}t + \hat{b}^{(y)} + R_t^{(y)}$, keep residuals $R^{(y)}$, and store the trend line $\hat{T}_t^{(y)} = \hat{a}^{(y)}t + \hat{b}^{(y)}$ for re-adding after generation.

2. **Whole path (*one*)**: we take the entire `^GSPC` series up to 2024 end as one long path $S$, form the index $I_t = S_t/S_0$, detrend it linearly on its native grid to get residuals $R_t$ and trend $\hat{T}_t$.

We compute per-window signatures on the 2D path $(\tau, X_\tau)$ where $\tau$ is reparameterized to $[0, 1]$ within each window and $X$ is the detrended index level (residual). Generation is performed in residual space and the saved linear trend is added back to produce final levels for evaluation.

## 7.2 Model and Training Setup

We use the *path-wise signature bootstrap* library: for each lookback window of length $L = 20$ and forward window $F = 5$, we record $(s_t, p_t)$ where $s_t$ is the truncated signature at level $m = 3$ (and in a second variant, the truncated *log-signature*), and $p_t = (X_{t+1}, \ldots, X_{t+F}) - X_t$ is the relative future segment. At generation time, we roll a size-$L$+1 window on the evolving path, compute its signature, find $k = 10$ nearest neighbors in the library (Euclidean distance), sample one neighbor (uniform), and append the stored segment. For the multi dataset we generate 50 paths (seed = 1234); for the one dataset we generate 10 paths. MMD window = 15. All signatures are computed on time-normalized windows ($\tau \in [0, 1]$). Generation is performed in residual space; linear trends are re-added for evaluation.

## 7.3 Evaluation Metric

We report the *linear signature MMD* (window size 15, level $m = 3$), i.e., the squared $\ell_2$ distance between mean (log-)signatures of sliding windows from real vs generated sets:

$$\mathrm{MMD}^2_{\mathrm{sig}} \; = \; \| \, \bar{s}_{\mathrm{real}} - \bar{s}_{\mathrm{gen}} \, \|_2^2 \, .$$

Lower is better.

## 7.4 Results

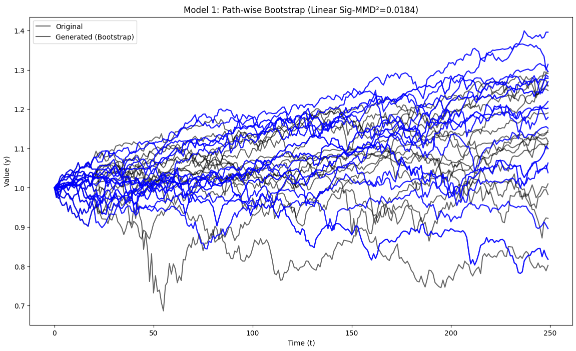
### 7.4.1 Model 1: Pathwise bootstrap

**Configuration constants.** Lookback $L = 20$, forward $F = 5$, level $m = 3$, neighbors $k = 10$, window for MMD = 15. Number of generated paths: multi = 50, one = 10.

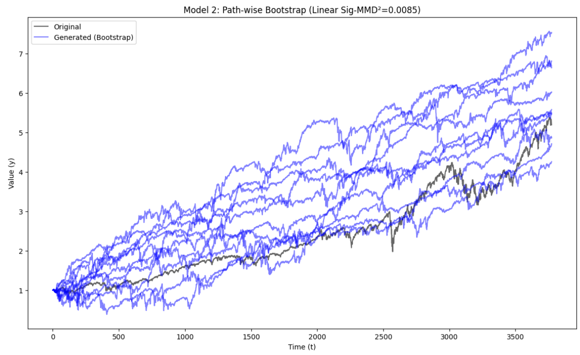
Table 1: Linear Signature MMD$^2$ across datasets and feature types (lower is better).

| **Dataset & Feature** | **Sig-MMD$^2$** | **Notes** |
|---|---|---|
| Yearly paths (*multi*) + Signature | `<0.018412>` | $N_{\mathrm{gen}} = 50$, seed = 1234 |
| Whole path (*one*) + Signature | `<0.008518>` | $N_{\mathrm{gen}} = 10$, seed unset |
| Yearly paths (*multi*) + Log-signature | `<0.028589>` | $N_{\mathrm{gen}} = 50$, seed = 1234 |
| Whole path (*one*) + Log-signature | `<0.116951>` | $N_{\mathrm{gen}} = 10$, seed unset |

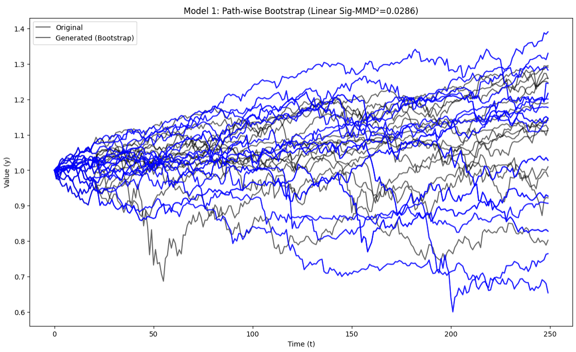
**Qualitative samples.** Figure 2 shows representative generated samples for each setting (residual paths with trend re-added to produce index levels).



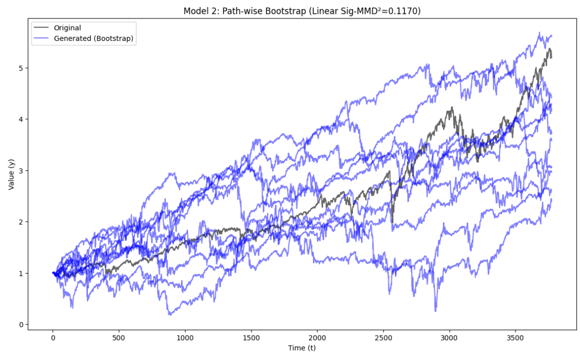

(a) Yearly paths + Signature (multi).




(b) Whole path + Signature (one).




(c) Yearly paths + Log-signature (multi).




(d) Whole path + Log-signature (one).

Figure 1: Path-wise bootstrap samples under four settings (trend added back).

### 7.4.2 Model 2: Hybrid bootstrap

Table 2: Linear Signature $\text{MMD}^2$ across datasets and feature types.

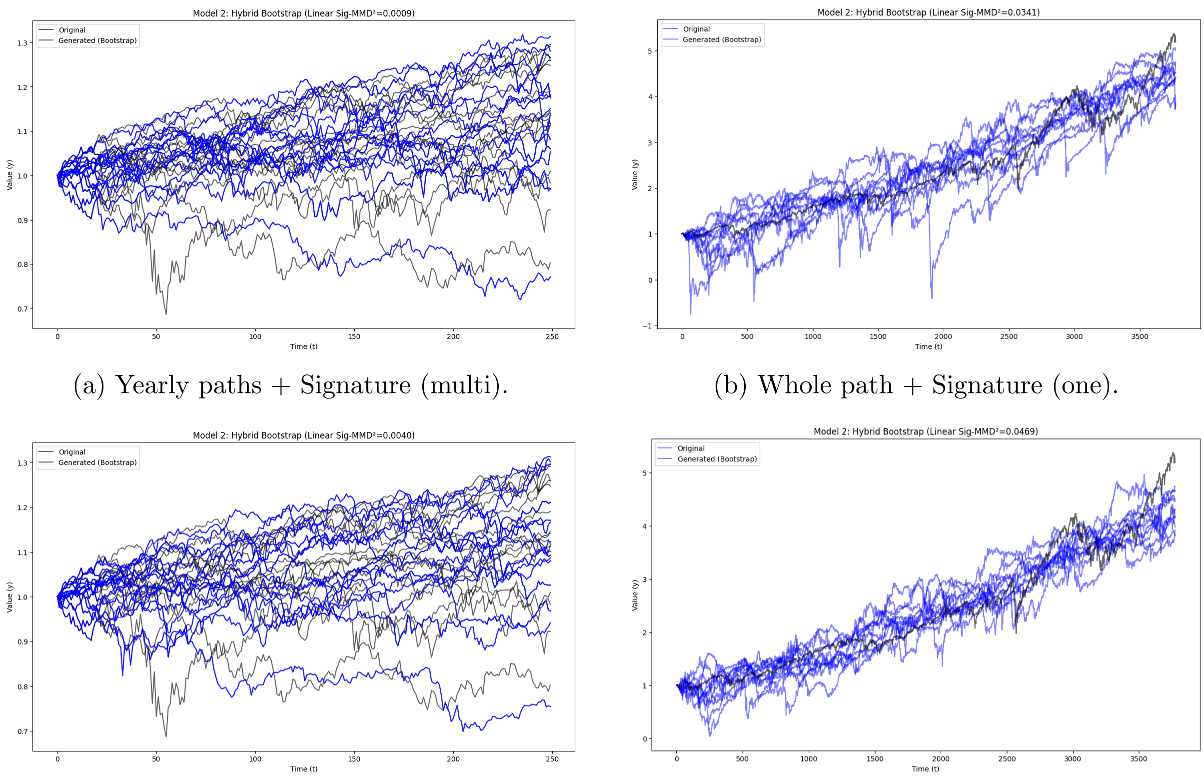
| **Dataset & Feature** | **Sig-MMD$^2$** | **Notes** |
|---|---|---|
| Yearly paths (*multi*) + Signature | <0.000867> | $N_{\text{gen}} = 50$, seed $= 1234$ |
| Whole path (*one*) + Signature | <0.034129> | $N_{\text{gen}} = 10$, seed unset |
| Yearly paths (*multi*) + Log-signature | <0.004026> | $N_{\text{gen}} = 50$, seed $= 1234$ |
| Whole path (*one*) + Log-signature | <0.046949> | $N_{\text{gen}} = 10$, seed unset |

**Qualitative samples.** Figure 2 shows representative generated samples for each setting (residual paths with trend re-added to produce index levels).

### 7.4.3 Model 3: Kernel Ridge Regression

**Configuration constants.** Lookback $L = 10$, signature level $m = 4$, linear kernel in signature space, ridge $\lambda = 2.0$, Euler–Maruyama with $\Delta W_t \sim \mathcal{N}(0, \Delta t)$, MMD window $= 15$. Number of generated paths: *multi* $= 50$, *one* $= 15$. (For the "one" + logsig run, $\widehat{\log \sigma}$ is clipped to the $[1, 99]$th percentiles as in code.)

**Qualitative samples.** Figure 3 shows representative generated log-price paths under the four settings.




(a) Yearly paths + Signature (multi).

(b) Whole path + Signature (one).

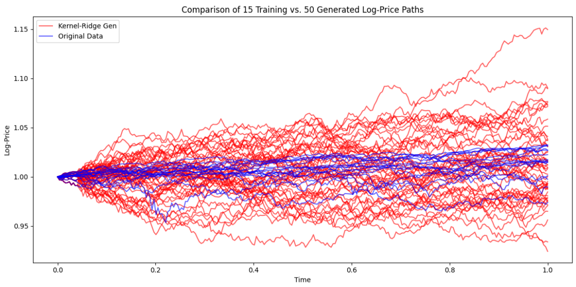
(c) Yearly paths + Log-signature (multi).

(d) Whole path + Log-signature (one).

Figure 2: Hybrid bootstrap samples under four settings (trend added back).

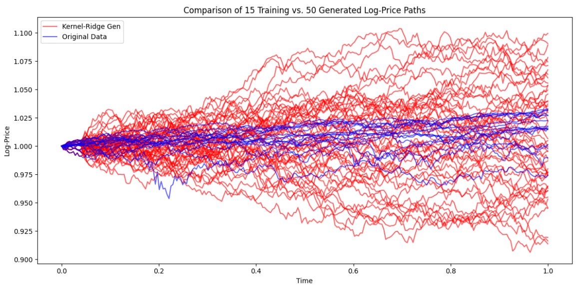
Table 3: Model 3 (KRR) — Linear Signature MMD$^2$ across datasets and feature types (lower is better).

| **Dataset & Feature** | **Sig-MMD$^2$** | **Notes** |
|---|---|---|
| Yearly paths (*multi*) + Signature | `<2.0218544503313323e-09>` | $N_{\text{gen}} = 50$, seed if set |
| Whole path (*one*) + Signature | `<6.119611899692986e-07>` | $N_{\text{gen}} = 15$, seed unset |
| Yearly paths (*multi*) + Log-signature | `<1.3504824216178214e-07>` | $N_{\text{gen}} = 50$, seed if set |
| Whole path (*one*) + Log-signature | `<4.066745963097663e-07>` | $N_{\text{gen}} = 15$, $\widehat{\log \sigma}$ clipped |

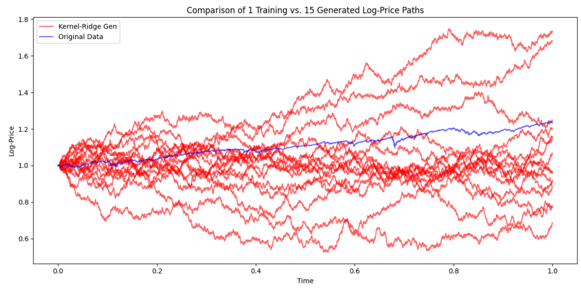



(a) Yearly paths + Signature (multi).


(b) Whole path + Signature (one).




(c) Yearly paths + Log-signature (multi).




(d) Whole path + Log-signature (one).

Figure 3: Kernel Ridge Regression (signature–SDE) samples under four settings.