Math 300

First order linear $y' = a(t)y + b(t)$

$$y_p(t) = e^{A(t)} \int_{t_0}^{t} b(s) e^{-A(s)}\, ds$$

$$y = y(t_0)e^{A(t)} + y_p(t) \qquad A(t) = \int_{t_0}^{t} a(s)\,ds$$

Logistic $y' = ay - by^2$

$$y(t) = \frac{a}{de^{-at} + b} \qquad d \in \mathbb{R}$$

Exact 1st order: $M\,dx + N\,dy = 0$

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\,dx\right) \text{ if } \frac{M_y - N_x}{N} = g(x)$$

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M}\,dy\right) \text{ if } \frac{M_y - N_x}{M} = g(y)$$

$$\mu(xy) = \exp\left(\int \frac{M_y - N_x}{N\cdot y - M\cdot x}\,d(xy)\right) \text{ if } \frac{M_y - N_x}{N\cdot y - M\cdot x} = g(xy)$$

$$\mu\left(\frac{y}{x}\right) = \exp\left(-\frac{x^2\cdot(M_y - N_x)}{N\cdot y + M\cdot x}\right)$$

if $\dfrac{x^2\cdot(M_y - N_x)}{N\cdot y + M\cdot x} = g(x\,y)$

(f_n) converges point-wise (on I) if for every $x \in I$ the sequence $(f_n(x))$, an ordinary sequence of real numbers, converges. If this is the case then $f(x) = \lim_{n\to\infty} f_n(x)$ defines a function $f: I \to \mathbb{R}$, called "limit function" or "point-wise limit" of the sequence (f_n).

(f_n) converges uniformly (on I) if it converges point-wise and the limit function $f: I \to \mathbb{R}$ has the following property: For every $\varepsilon > 0$ there is a "uniform" response $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \varepsilon$ for all $n > N$ and all $x \in I$.

If all functions f_n are continuous at $x_0 \in I$ and (f_n) converges uniformly on I then $f(x) = \lim f_n(x)$ $x \in I$ is continuous at x_0.

If all functions f_n are C^1-functions, (f_n) converges point-wise on I, and (f_n') converges uniformly on I, then $f(x) = \lim_{n\to\infty} f_n(x)$ $x \in I$ is a C^1-function and satisfies $f'(x) = \lim_{n\to\infty} f_n'(x)$

If I is a bounded interval, all functions f_n are (Lebesgue) integrable over I and (f_n) converges uniformly on I then the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ is integrable as well, and $\int_I f(x)\,dx = \lim_{n\to\infty} \int_I f_n(x)\,dx$.

Weierstrass's Criterion:

Suppose $f_n: D \to \mathbb{R}$ $(n = 0,1,2)$ are functions with common domain D and there exist "uniform" bounds $M_n \in \mathbb{R}$ such that $|f_n(x)| \le M_n$ for all $n \in \mathbb{R}$ and $x \in D$, If the series $\sum_{n=0}^{\infty} M_n$ converges in $\mathbb{R}$ then the function series $\sum_{n=0}^{\infty} f_n$ converges uniformly.

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\ln\left(2\sin\frac{x}{2}\right), \quad \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2} \quad (0 < x < 2\pi)$$

Picard-Lindelöf: $y' = f(t,y) \wedge y(t_0) = y_0$

$$\phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))\,ds$$

Metric Space: set M, map $d: M\times M \to \mathbb{R}$

① $d(x,y) \ge 0$; $d(x,y) = 0 \Leftrightarrow x = y$ non-negativity

② $d(x,y) = d(y,x)$ symmetry

③ $d(x,y) \le d(x,z) + d(z,y)$ triangle inequality

BANACH: contraction T + complete metric space $\Rightarrow$ 存在 $x^* \in M$, $T(x^*) = x^*$. fixed point.

Norm 范数: $\|A\| = \max\left\{\frac{|Ax|}{|x|}; x \in \mathbb{R}^n \setminus \{0\}\right\}$

$= \max\{|Ax|; x \in \mathbb{R}^n, |x| = 1\}$.

$\|A+B\| \le \|A\| + \|B\|$; $\|AB\| \le \|A\|\|B\|$

$\|A\|_F = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}$ Frobenius norm. $\|A\| \le \|A\|_F$.

Lipschitz condition: $\exists L \ge 0$,

$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$.

或 y' 对 y 偏导有界. 可以左导≠右导.

高阶 ODE: $2y^{(5)} - y''' + y'' = 0 \to 2\lambda^5 - \lambda^3 + \lambda^2 = 0$

二阶: i) $r_1 \ne r_2$ $\quad y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

ii) $r_1 = r_2$. $\quad y = (c_1 + c_2 t)\cdot e^{r_1 t}$

iii) $\Delta < 0$. $r_1 = \alpha + \beta i$ $\quad r_2 = \alpha - \beta i$

$y = e^{\alpha x}(c_1\cos(\beta t) + c_2 \sin(\beta t))$

n阶: i) 单根 $c \cdot e^{rt}$

ii) 一对单复根: $c_1 e^{\alpha t}\cos\beta t$, $c_2 e^{\alpha t}\sin\beta t$

iii) k重实根: $e^{rx}(c_1 + c_2 x + \dots + c_k x^{k-1})$

iv) 一对k重复根: 对应 $e^{\alpha x}(c_1 + c_2 x + \dots + c_k x^{k-1})\cos\beta x$

$e^{\alpha x}(D_1 + D_2 x + \dots + D_k x^{k-1})\cdot\sin\beta x$

$a(D)y = h\cdot e^{\lambda t}$

非齐次 ↓ : λ 是 k 重根.

特解 $y = \dfrac{t^k h \cdot e^{\lambda t}}{a^{(k)}(\lambda)}$.

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k = I_n + A + \frac{1}{2}A^2 + \cdots$$

$$W(t) = \begin{pmatrix} y_1(t) & y_2(t) & \cdots \\ y_1'(t) & y_2'(t) & \\ \vdots & \vdots & \end{pmatrix}$$ y_n 是 basis of solution space of $y' = Ay$. $e^{\lambda t}$

$e^{At} = W(t)\cdot W(0)^{-1}$

$y' = A(t)y + b(t)$ $\quad c(t) = \int_{t_0}^{t} \phi(s)^{-1} b(s)\,ds$

$y_p(t) = \phi(t)\cdot c(t)$ $\quad \phi' = A\phi$. $\phi = (v_1|v_2|v_3)$

$y(t) = \phi(t)\cdot c(t) + \phi(t)\cdot C_0$ $\quad C_0 \in \mathbb{C}^n$.

Euler equations: $t^2 y'' + \alpha t y' + \beta y = 0$.

$r^2 + (\alpha - 1)r + \beta = 0$ $\quad r_1 > r_2$.

i) $\Delta > 0$. $\phi_1(t) = t^{r_1}$ $\quad \phi_2(t) = t^{r_2}$; t^{r_1} for $t > 0$, $(-t)^{r_1}$ $t < 0$

ii) $\Delta = 0$. $\phi_1(t) = t^{r_1} = t^{\frac{1-\alpha}{2}}$. $\phi_2(t) = \ln t \cdot t^{r_1}$

iii) $\Delta < 0$. $r_1 = \lambda + \mu i$ $\quad r_2 = \lambda - \mu i$

$y_1(t) = t^{\lambda}\cos(\mu \ln t)$ $\quad y_2(t) = t^{\lambda}\sin(\mu\ln t)$

Bessel: $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+\nu)!}\left(\frac{x}{2}\right)^{\nu+2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{\nu+2m}$$

Power series: $P(x)y'' + Q(x)y' + R(x)y = 0$.

$y'' + p(x)y' + q(x)y = 0$. $\quad p(x) = \frac{Q(x)}{P(x)}$ $\quad q(x) = \frac{R(x)}{P(x)}$

$P(x_0) = 0 \Rightarrow$ singular otherwise ordinary point

regular singular point: $\lim_{x\to x_0}(x - x_0)p(x)$ (p_0)

(q_0) $\lim_{x\to x_0}(x - x_0)^2 q(x)$ exists.

$F(r) = r^2 + (p_0 - 1)r + q_0$ $\quad r_1 > r_2$.

$$a_n(r_1) = -\frac{1}{F(r_1 + n)}\sum_{k=0}^{n-1}\left[(r_1 + k)p_{n-k} + q_{n-k}\right]a_k(r_1)$$

i) $r_1 - r_2 \notin \mathbb{Z}$.

ii) $r_1 - r_2 = 0$ $\quad y_2 = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n$.

$b_n = a_n'(r_1)$

iii) $r_1 - r_2 \in \mathbb{Z}_+$. $\quad a = \lim_{r\to r_2}(r - r_2)a_N(r)$. $\quad N = r_1 - r_2$.

$$y_2 = a y_1(x)\ln x + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right)$$

$$c_n(r_2) = \frac{d}{dr}\left[(r - r_2)a_n(r)\right]\Big|_{r=r_2}.$$

For $f(x) = (x - \alpha_1)^{\beta_1}(x - \alpha_2)^{\beta_2}\cdots(x - \alpha_n)^{\beta_n}$

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x - \alpha_1} + \frac{\beta_2}{x - \alpha_2} + \cdots + \frac{\beta_n}{x - \alpha_n}.$$

在 $(-\infty, 0)$. 所有的 $(x - x_0)^r \to |x - x_0|^r$.

Laplace: $F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$

存在: set Δ of discontinuities is discrete continuous on each connected components one side limits exists. $|f(t)| \le k\cdot e^{at}$.

$1 \leftrightarrow \frac{1}{s}$ $\quad t \leftrightarrow \frac{1}{s^2}$ $\quad \frac{t^n}{n!} \leftrightarrow \frac{1}{s^{n+1}}$ $\quad t^n \leftrightarrow \frac{n!}{s^{n+1}}$ $(n > 1)$

$e^{-at} \leftrightarrow \frac{1}{s+a}$ $\quad t\cdot e^{-at} \leftrightarrow \frac{1}{(s+a)^2}$

$t^n e^{at} \leftrightarrow \frac{n!}{(s-a)^{n+1}}$

$\sin(wt) \leftrightarrow \frac{w}{s^2 + w^2}$ $\quad \cos(wt) \leftrightarrow \frac{s}{s^2 + w^2}$

$t\sin wt \leftrightarrow \frac{2sw}{(s^2 + w^2)^2}$ $\quad t\cos wt \leftrightarrow \frac{s^2 - w^2}{(s^2 + w^2)^2}$

$e^{\lambda t}\sin(wt) \leftrightarrow \frac{w}{(s-\lambda)^2 + w^2}$

$e^{\lambda t}\cos(wt) \leftrightarrow \frac{s - \lambda}{(s-\lambda)^2 + w^2}$

$t e^{\lambda t}\sin wt \leftrightarrow \frac{2w(s-\lambda)}{[(s-\lambda)^2 + w^2]^2}$

$t e^{\lambda t}\cos(wt) \leftrightarrow \frac{(s-\lambda)^2 - w^2}{[(s-\lambda)^2 + w^2]^2}$.

$a^{\frac{t}{T}} \leftrightarrow \frac{1}{s - \frac{1}{T}\ln a}$.

$\cos^2 t \leftrightarrow \frac{s^2 + 2}{s(s^2 + 4)}$ $\quad \sin^2 t \leftrightarrow \frac{2}{s(s^2 + 4)}$.

$$H_c(t) = H(t - c) = \begin{cases} 0 & t < c \\ 1 & t \ge c. \end{cases}$$

$\mathcal{L}\{H_c(t)f(t-c)\} = e^{-cs}F(s)$

$F(s - c) = \mathcal{L}\{e^{ct}f(t)\}$

$H_c(t) \leftrightarrow \frac{e^{-cs}}{s}$ $\quad \delta(t) \leftrightarrow 1$ $\quad \delta'(t) \leftrightarrow s$

$F(s)=\mathcal{L}\{f(t)\}$

$F'(s)=\mathcal{L}\{-t f(t)\}.$

$\mathcal{L}\{f^{(n)}(t)\}=s^n F(s)-s^{n-1}f(0)-s^{n-2}f'(0)$
$\quad -s^{n-3}f''(0)-\cdots-s^0 f^{(n-1)}(0).$

$\mathcal{L}\{\int_0^t f(\tau)d\tau\}=\frac{F(s)}{s}.$

$\mathcal{L}\{\frac{f(t)}{t}\}=\int_s^{\infty}F(\sigma)d\sigma.$

$kf(t)\leftrightarrow kF(s).$ linearity. ECE 210

$f(at)\leftrightarrow \frac{1}{a}F(\frac{s}{a}).$

$f(t-t_0)\leftrightarrow F(s)\cdot e^{-st_0}$

$f(t)e^{s_0t}\leftrightarrow F(s-s_0)$

$f(t)*g(t)\leftrightarrow F(t)\cdot g(t)$

$(f*g)t=\int_0^t f(\tau)g(t-\tau)d\tau.$

$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$ $\quad \frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$

$\sin x=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

$\cos x=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$

$\tan^{-1}x=\arctan x=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{2n+1}$

$\ln(1+x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$

$(1+x)^k=\sum_{n=0}^{\infty}\binom{n}{k}x^n=1+kx+\frac{k(k-1)}{2!}\cdot x^2+\cdots$

Taylor $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)(x-a)^n}{n!}$

$\int\tan x\,dx=\ln|\sec x|+C$

$\int\cot x\,dx=\ln|\sin x|+C$

$\int\sec x\,dx=\ln|\sec x+\tan x|+C$

$\int\csc x\,dx=\ln|\csc x-\cot x|+C$

$(\arctan x)'=\frac{1}{1+x^2}$

$(\arcsin x)'=\frac{1}{\sqrt{1-x^2}}$

$(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}.$

微分方程课:

1. 基解: fundamental system of solution

特征根. 特征向量: $Ax=\lambda x$

复根: $\lambda\to v$ $\bar{\lambda}\to\bar{v}$. $y_1=e^{\lambda t}v$ $y_2=e^{\bar{\lambda}t}\bar{v}$

real solution: $z_1=Re\{y_1\}$ $z_2=Im\{y_1\}$.

复杂情况: 广义特征向量:

Suppose $\chi_A(x)=(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_r)^{m_r}$

① $\mathbb{C}^n$ has basis $B=\{v_1,\cdots,v_n\}$ consisting of generalized eigenvectors of A.

② if $v_j\in B$ is associated to the eigenvalue λ_i of A and $y_j:\mathbb{R}\to\mathbb{C}^n$ is:

$$y_j(t)=\sum_{k=0}^{m_i-1}\frac{1}{k!}\cdot t^k\cdot e^{\lambda_i t}(A-\lambda_i I_n)^k v_j.$$

then $y_1,y_2,\cdots,y_n$ form a fundamental system of solutions of $y'=Ay$.

③ The matrix exponential function is:

$$t\to e^{At}=(y_1(t)|y_2(t)|\cdots|y_n(t))\cdot(v_1|\cdots|v_n)^{-1}.$$

$m=1$, $y_j(t)=e^{\lambda_i t}\cdot v_j.$

特征向量通过算 $(A-\lambda I)v=0$ 得到

广义 $(A-\lambda I)^{\alpha}v=0.$

伴随阵求逆:

$A^{-1}=\frac{1}{|A|}A^*.$ $A_{ij}=(-1)^{i+j}M_{ji}$

例: $A=\begin{pmatrix}1&2&3\\2&2&1\\3&4&3\end{pmatrix}$. $M=\begin{pmatrix}2&3&2\\-6&-6&-2\\-4&-5&-2\end{pmatrix}$

$M^T=\begin{pmatrix}2&-6&-4\\3&-6&-5\\2&-2&-2\end{pmatrix}$ $|A|=2$

$A^*=\begin{pmatrix}2&6&-4\\-3&-6&5\\2&2&-2\end{pmatrix}$ $A^{-1}=\begin{pmatrix}1&3&-2\\-\frac{3}{2}&-3&\frac{5}{2}\\1&1&-1\end{pmatrix}$

$\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc$, $\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$

可逆 $\Leftrightarrow\lambda_i\neq0$. $\overset{\text{迹}}{\text{Trace}}=\sum\lambda_i$ 相似矩阵迹相等.

$e^{At}=\phi(t)\cdot\phi^{-1}(0)$. $\phi(t)=(y_1|y_2|y_3)$

如果 $\chi_A(x)=(x-\lambda)^n$,

$$e^{At}=e^{\lambda t}[I+t(A-\lambda I)+\frac{t^2}{2!}(A-\lambda I)^2+\cdots+\frac{t^{n-1}}{(n-1)!}(A-\lambda I)^{n-1}].$$

0点 unstable $\Leftrightarrow$ 有一个λ s.t. $Re(\lambda)>0$.