Math 286
Lab Project

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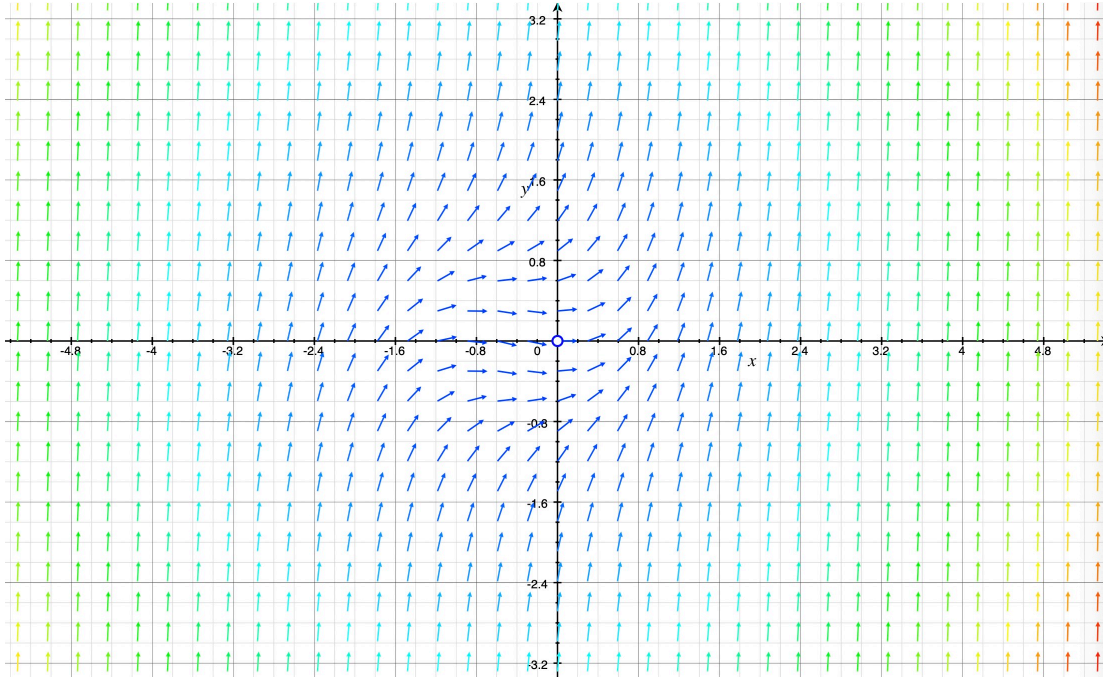
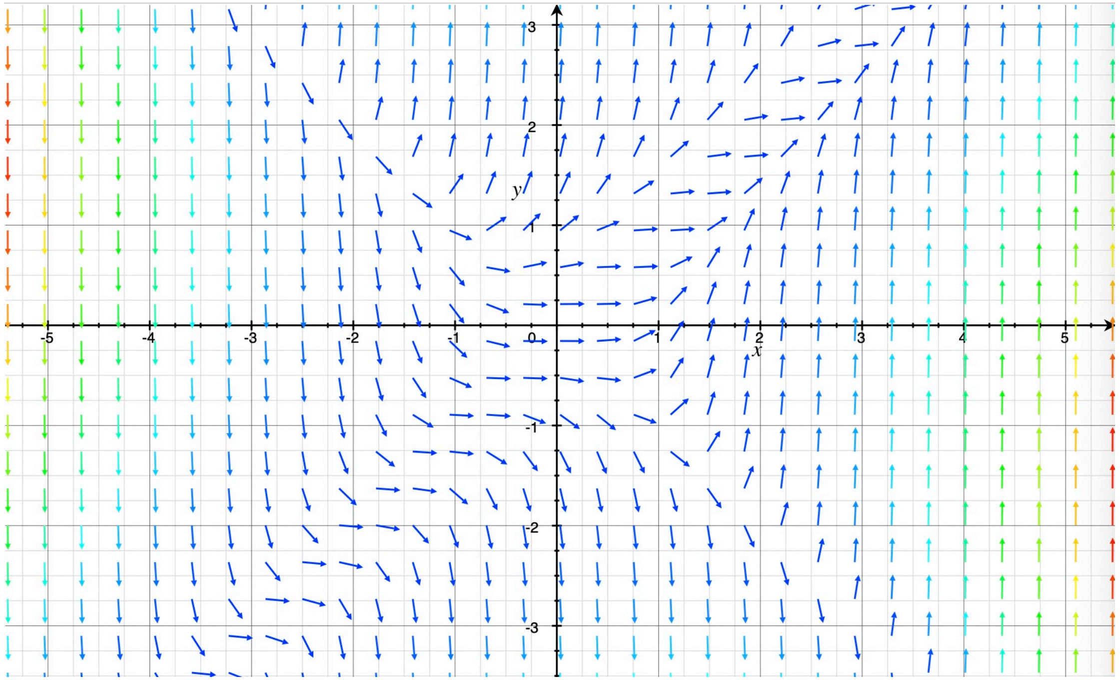
Outline

- Introduction
- Numerical Methods
- Asymptote analysis
- Power series representations
- Riccati transformation

Introduction

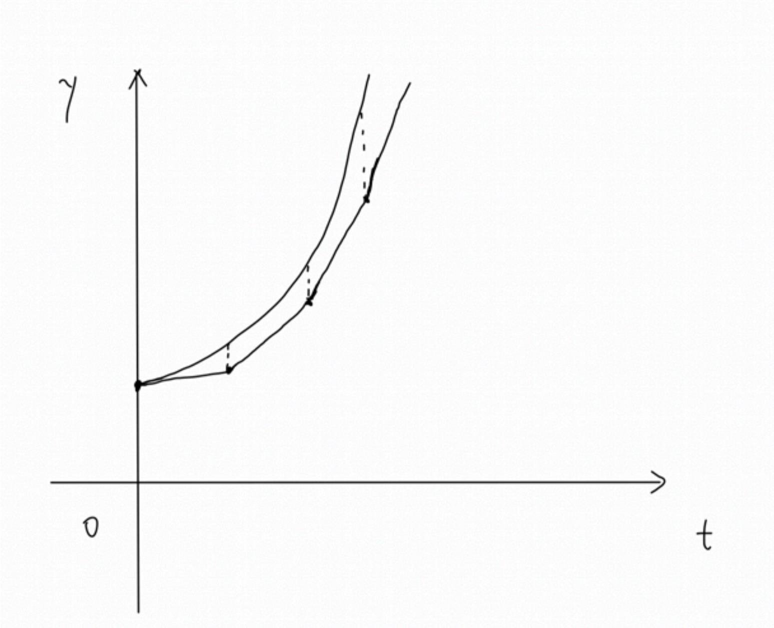
- $y' = y^2 + t^2 + t \qquad y(0) = 1 \quad \text{(IVP1)}$
- $y' = (y - t)(y^2 - t^2) \qquad y(0) = 1 \quad \text{(IVP2)}$





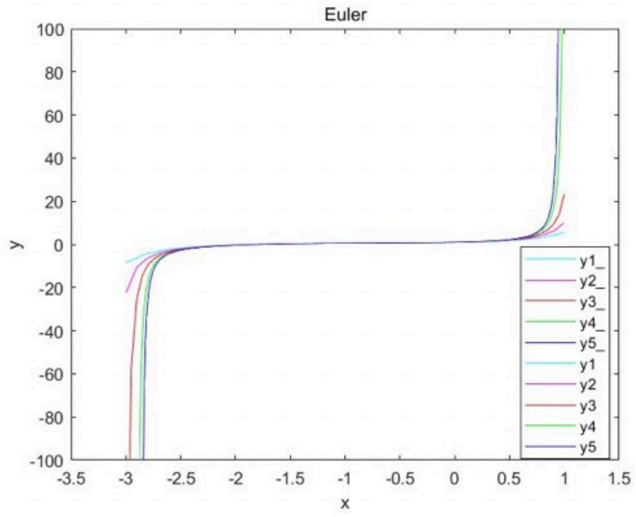
Numerical Methods: Euler

- Euler method is probably the simplest way to solve a differential equation numerically. For the IVP
  - $y' = f(t, y),\ y(t_0) = y_0$ (1)
- Euler method simply takes
  - $y(t+h) = y(t) + f(t, y(t)) * h$ (2)
- where h is the step size, positive or negative.
- Error: $O(h)$.

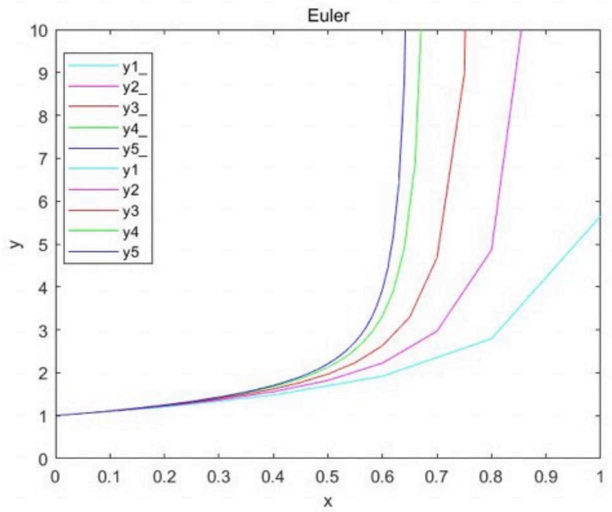


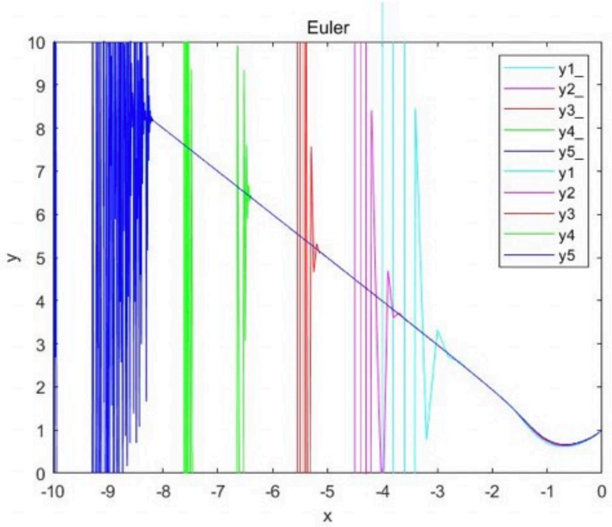
Result of Euler

IVP1:



IVP2:



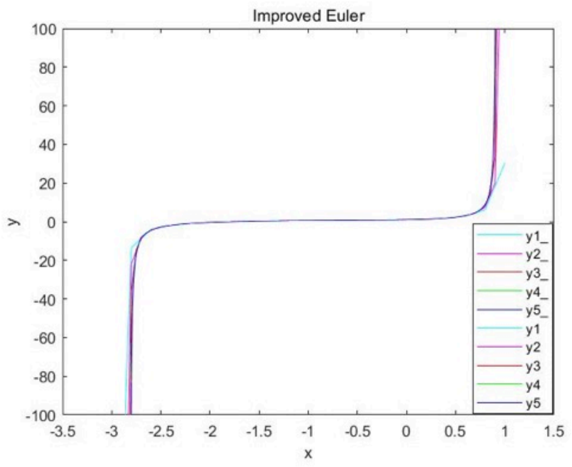


Numerical Methods: Improved Euler

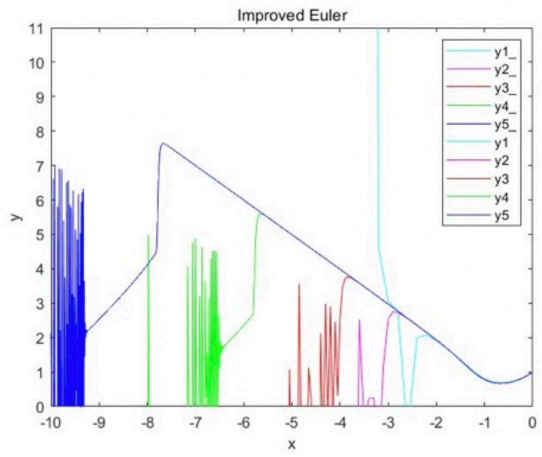
- The improved Euler method is based on Euler method. For the IVP (1), improved Euler method takes
  - $y\_predict(t+h) = y(t) + f(t, y(t)) * h$ (4)
  - $y(t+h) = y(t) + \frac{h}{2}(f(t, y(t)) + f(t+h, y\_predict(t+h))$ (5)
- where h is the step size, positive or negative.
- Error: $O(h^2)$.

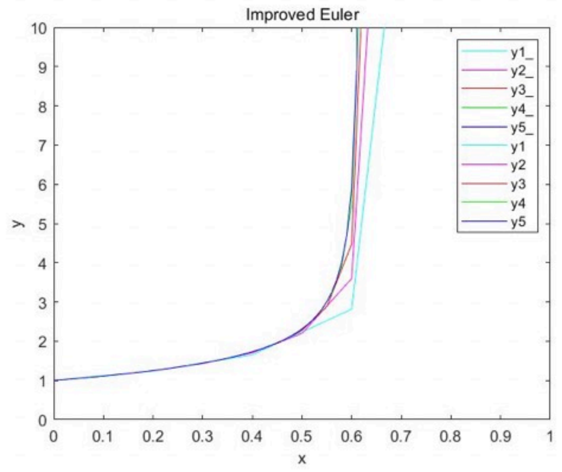
Result of improved Euler

IVP1:



IVP2:





Numerical Methods: Runge-Kutta

- We know that for the IVP, we can use:
- $y_{n+1}(x) = y_n(x) + \int_{x_n}^{x_{n+1}} f\big(x, y(x)\big)dx$
- By the Mean Value Theorems, there must exist a ξ that $x_n \le \xi \le x_{n+1}$, we can substitute $\int_{x_n}^{x_{n+1}} f\big(x, y(x)\big)dx$ by $f\big(\xi, y(\xi)\big)h$, where h is the step size.
- If we choose m points between x_n and x_{n+1} : $x_n \le t_1 \le t_2 \le \cdots \le t_m \le x_{n+1}$
- Let $K_i = f\big(t_i, y(t_i)\big)$
- We can use these points to calculate the area of $\int_{x_n}^{x_{n+1}} f\big(x, y(x)\big)dx$, here we construct a m-order Runge-Kutta.

Numerical Methods: Runge-Kutta

- Now we get $y_{n+1} = y_n + h\sum_i^m c_i K_i$, where c_i is the weighted combination coefficient.
- In order to calculate K_i , we choose $t_1 = x_n$ and $K_1 = f\big(x_n, y(x_n)\big)$, then we can use weighted combination to calculate other K_i.
- Let $t_i = t_1 + a_i h = x_n + a_i h$, we get the general form of Runge-Kutta:

- $K_1 = f(x_n + y_n)$
- $K_2 = f(x_n + a_2 h, y_n + h b_{21} K_1)$
- $K_3 = f\big(x_n + a_3 h, y_n + h(b_{31} K_1 + b_{32} K_2)\big)$
- $\cdots$
- $k_m = f\left(x_n + a_m h, y_n + h\big(b_{m_1} k_1 + b_{m_2} k_2 + \cdots + b_{mm-1} k_{m-1}\big)\right)$

- Where $\sum_{j=1}^{i-1} b_{ij} = a_i$ and $\sum_{i=1}^m c_i = 1$.
- By increment function of Taylor expansion method, we can determine those a, b and c.

Numerical Methods: Runge-Kutta

- The third order Runge-Kutta method:
  - (a)
  - $K_1 = f(x_n + y_n)$
  - $K_2 = f\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hK_1\right)$
  - $K_3 = f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hK_2\right)$
  - $y_{n+1} = y_n + \frac{h}{4}(K_1 + 3K_3)$
  - (b)
  - $K_1 = f(x_n + y_n)$
  - $K_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_1\right)$
  - $K_3 = f(x_n + h, y_n - hK_1 + 2hK_2)$
  - $y_{n+1} = y_n + \frac{h}{6}(K_1 + 4K_2 + K_3)$

Numerical Methods: Runge-Kutta

- The fourth order Runge-Kutta method:
  - (a)
  - $K_1 = f(x_n + y_n)$
  - $K_2 = f\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hK_1\right)$
  - $K_3 = f\left(x_n + \frac{2}{3}h, y_n - \frac{1}{3}hK_1 + hK_2\right)$
  - $K_4 = f(x_n + h, y_n + hK_1 - hK_2 + hK_3)$
  - $y_{n+1} = y_n + \frac{h}{8}(K_1 + 3K_2 + 3K_3 + K_4)$

  - (b)
  - $K_1 = f(x_n + y_n)$
  - $K_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_1\right)$
  - $K_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_2\right)$
  - $K_4 = f(x_n + h, y_n + hK_3)$
  - $y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

Numerical Methods: Runge-Kutta

- We use the second one of the fourth order Runge-Kutta to solve (IVP1)
  - $y' = y^2 + t^2 + t \quad y(0) = 1$

Firstly, we take h = 0.3 from t = 0 to t = 0.9

- Step 0 t_0 = 0, y_0 = 1
- Step 1 t_1 = 0.3 $y_1 = y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$= 1.498836865170347 1.4992
- Step 2 t_2 = 0.6 $y_2 = y_1 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$= 3.092616161249836 3.1068
- Step 3 t_3 = 0.9 $y_3 = y_2 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$= 23.914533197543687 99.8573

Numerical Methods: Runge-Kutta

Then, we take h = 0.1 from t = 0 to t = 0.9

- Step 0 t_0 = 0, y_0 = 1
- Step 1 t_1 = 0.1 y_1= 1.116844289929774 1.1168
- Step 2 t_2 = 0.2 y_2= 1.276584681611626 1.2766
- Step 3 t_3 = 0.3 y_3= 1.499185425629429 1.4992
- Step 4 t_3 = 0.4 y_4= 1.819293075299801 1.8193
- Step 5 t_3 = 0.5 y_5= 2.304206211779880 2.3043
- Step 6 t_3 = 0.6 y_6= 3.106554937431314 3.1068
- Step 7 t_3 = 0.7 y_7= 4.666647714910317 4.6688
- Step 8 t_3 = 0.8 y_8= 8.990372045980527 9.0342
- Step 9 t_3 = 0.9 y_9= 53.696283140577293 99.8573

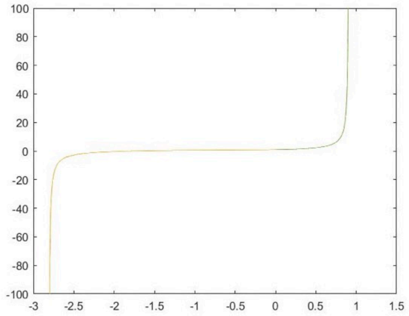
t_i	Power series solution $y(t_i)$	Numerical solution y_i	Error $\lvert y(t_i) - y_i \rvert$
0	1	1	0
0.3	1.4992	1.498836865170347	0.00036313
0.6	3.1068	3.092616161249836	0.01418384
0.9	99.8573	23.914533197543687	75.9427668

t_i	Power series solution $y(t_i)$	Numerical solution y_i	Error $\lvert y(t_i) - y_i \rvert$
0	1	1	0
0.1	1.1168	1.116844289929774	4.42899E-05
0.2	1.2766	1.276584681611626	1.53184E-05
0.3	1.4992	1.499185425629429	1.45744E-05
0.4	1.8193	1.819293075299801	6.9247E-06
0.5	2.3043	2.304206211779880	9.37882E-05
0.6	3.1068	3.106554937431314	0.000245063
0.7	4.6688	4.666647714910317	0.002152285
0.8	9.0342	8.990372045980527	0.043827954
0.9	99.8573	53.696283140577293	46.16101686

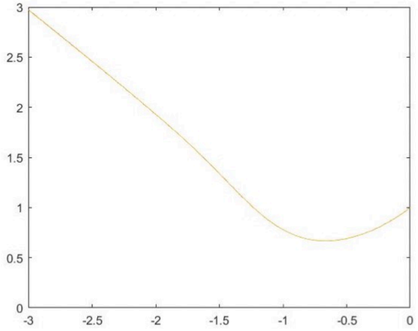
Numerical Methods: Linear multistep methods

- Adams-Bashforth-Moulton method
- $y'_{k+1} = y_k + \int_{t_k}^{t_{k+1}} f(t,y) = y_k + \frac{h}{24}(-9f_{k-3} + 37f_{k-2} - 59f_{k-1} + 55f_k)$
- $y_{k+1} = y_k + \int_{t_k}^{t_{k+1}} f(t,y) = y_k + \frac{h}{24}(f_{k-2} - 5f_{k-1} + 19f_k + 9f_{k+1})$
- Milne-Simpson method
- $y'_{k+1} = y_{k-3} + \int_{t_{k-3}}^{t_{k+1}} f(t,y) = y_{k-3} + \frac{4h}{3}(2f_{k-2} - f_{k-1} + 2f_k)$
- $y_{k+1} = y_{k-1} + \int_{t_{k-1}}^{t_{k+1}} f(t,y) = y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_k + f_{k+1})$.
- Hamming method
- $y'_{k+1} = y_{k-3} + \frac{4h}{3}(2f_{k-2} - f_{k-1} + 2f_k)$
- $y_{k+1} = \frac{-1}{8}y_{k-2} + \frac{9}{8}y_k + \frac{3h}{8}(-f_{k-1} + 2f_k + f_{k+1})$

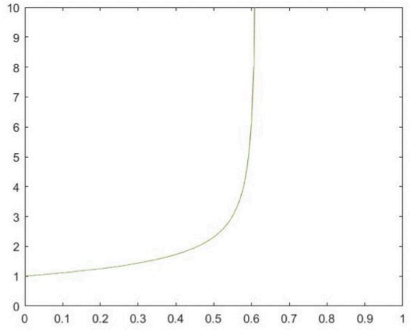
Result for IVP 1:



Result for IVP 2:

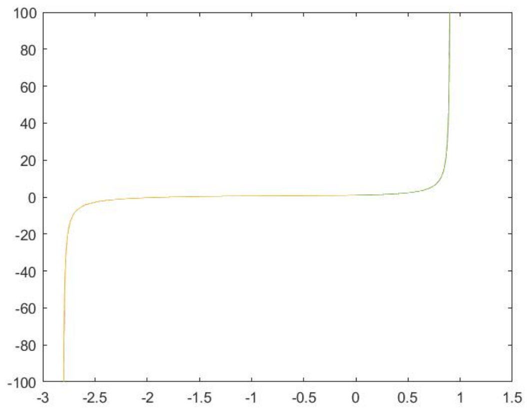


Negative part



Positive part

Asymptote analysis: Vertical asymptote

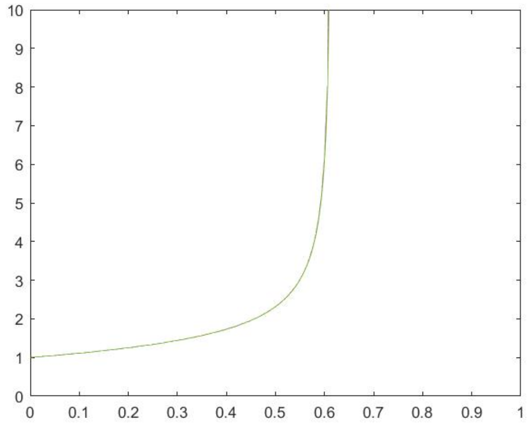


- **when y is a lot larger than t, they can be viewed as:**
- $\boldsymbol{y' = y^2}$ **(IVP1)**
- with solutions $\boldsymbol{y = \frac{1}{a-t}}$, with a to be determined by point (t, y).

- IVP1:
- We use RK-4 to compute two precise points (0.9100, 7685.915313082049579) and (-2.810, -12055038)

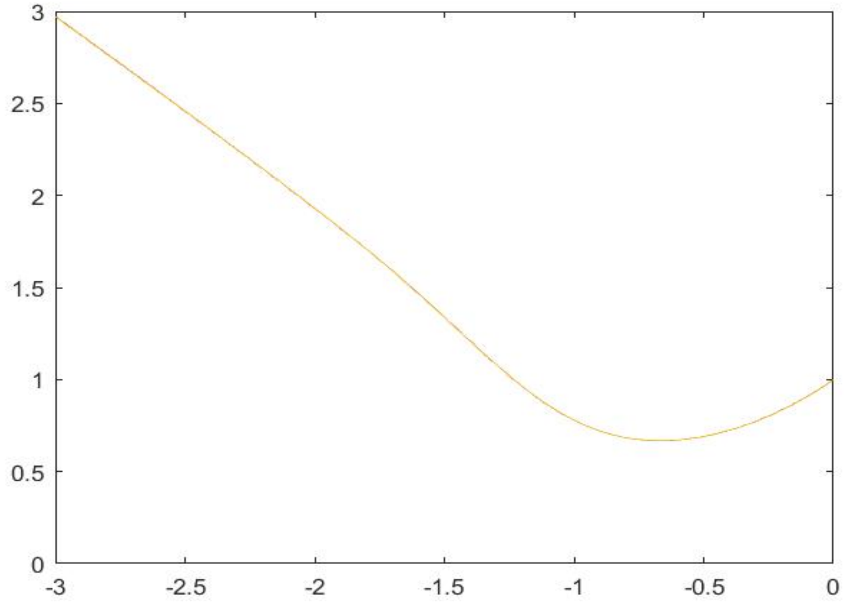
  the asymptote: $x = 0.910103$ and $x = -2.8100001$
- IVP2:
- We use RK-4 to compute the precise point(0.6142, 561.885927476)

  the asymptote $x = 0.61597972$



Asymptote analysis: Asymptote for IVP2's negative part

- From the plot we guess it's y = -t.
- "Semi-Homogeneous"!
- $y' = (y-t)(y^2-t^2) \Leftrightarrow \frac{y'}{t^3} = \frac{y^3}{t^3} - \frac{y^2}{t^2} - \frac{y}{t} + 1$
- For $t \to -\infty, \frac{y'}{t^3} = 0$, assume $\frac{y}{t} = \lambda$, then
- $0 = \lambda^3 - \lambda^2 - \lambda + 1 = (\lambda+1)(\lambda-1)^2$, from the plot $\lambda = -1$.
- therefore, the asymptote for the IVP2's negative part is $y = -t$.



Power series representations

Because $f(t, y) = y^2 + t^2 + t$ is analytic, the solution must be analytic as well. i.e., we can assume $y(t) = \sum_{n=0}^{\infty} a_n t^n$, $y(0) = a_0 = 1$. And by the Cauchy's multiplication formula, we have $y(t)^2 = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k\, a_{n-k}) t^n$.

Substitute $y(t)$, $y(t)^2$ into the ODE and equate coefficients. $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$, $y^2 + t^2 + t = t^2 + t + \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k\, a_{n-k}) t^n$

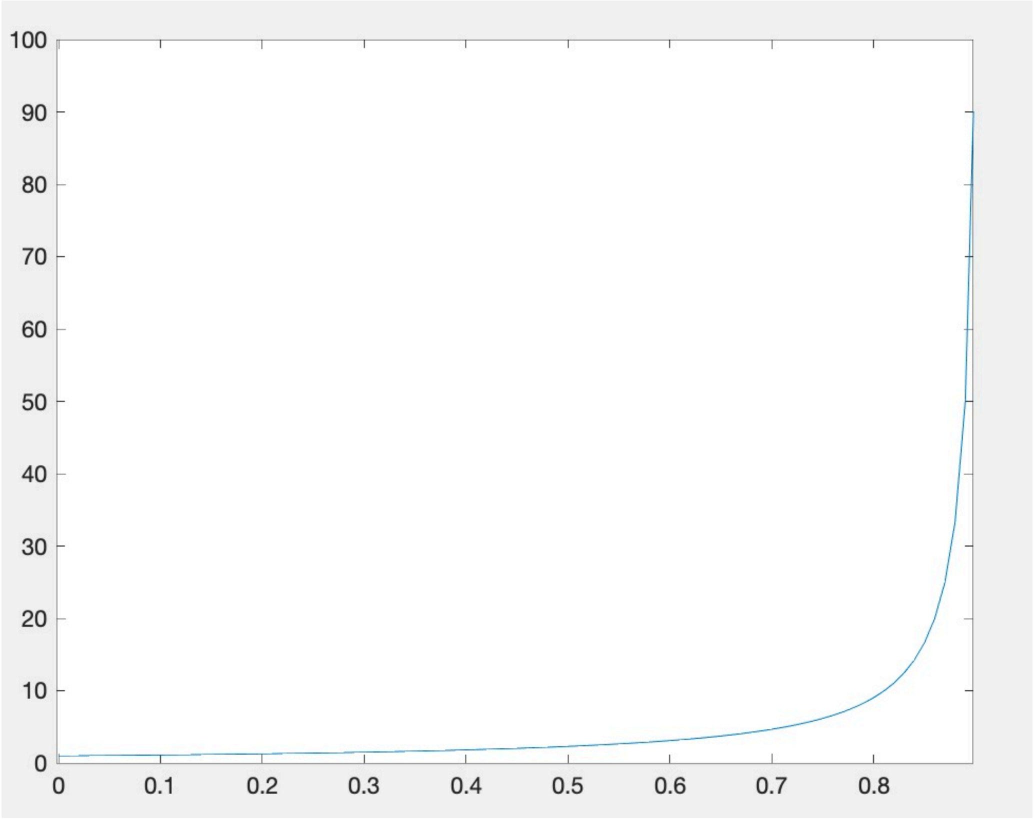
$$\Rightarrow (n+1) a_{n+1} = \begin{cases} 1 + \sum_{k=0}^{n} a_k\, a_{n-k}, n = 1,2 \\ \sum_{k=0}^{n} a_k\, a_{n-k}, n \neq 1,2 \end{cases}$$

For $n \geq 1$ this determines a_n from a_k , $k < n$, and thus together with the initial value $y(0) = a_0 = 1$ provides a recursion formula for a_n, which can easily be programmed:

$$y(t) = 1 + t + \frac{3t^2}{2} + \frac{5t^3}{3} + \frac{19t^4}{12} + 1.75t^5 + 1.94444t^6 + 2.13095t^7 + 2.33482t^8 + \cdots$$

Result of power series method

precise to t^{1000}



Power series representations: Radius of convergence

For every power series $y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$, the radius of convergence $\rho = \frac{1}{L}$, where

$L = \lim_{n\to\infty} sup \sqrt[n]{a_n}$. And if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists, $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$.

Use $n = 1000$, for the first IVP, $\rho \approx 0.91$; for the second IVP, $\rho \approx 0.625$

Riccati transformation

Riccati equations: $y' + p(t)y + q(t)y^2 + r(t) = 0$

Use substitution $y(t) = \frac{G(t)}{F(t)}$, then$[G' + p(t)G + r(t)F]F - [F' - q(t)G]G = 0$

Obviously, the choice of F, G is not unique, so we can choose some additional conditions that are convenient to solve. For example, we can make $[F' - q(t)G] = 0$.

Then we need to solve $[G' + p(t)G + r(t)F]F = 0$. Here, $G = \frac{F\prime}{q(t)}$, then $y(t) = \frac{G(t)}{F(t)} = \frac{F\prime}{q(t)F} = \frac{(\ln F)\prime}{q(t)}$, which is called Riccati transformation.

Then by solving $F'' + \left(p - \frac{q'}{q}\right)F' + qrF = 0$, we can get F, and hence y.

Riccati transformation

In the first IVP, $p(t) = 0, q(t) = -1, r(t) = -(t + t^2)$. So, we use substitution $y(t) = -\frac{F\prime(t)}{F(t)}$. And $y(0) = -\frac{F'(0)}{F(0)} = 1$, $F(0) + F'(0) = 1$

$\Rightarrow F'' + (t + t^2)F = 0$. Solve it by power series, assume $F(t) = \sum_{n=0}^{\infty} b_n t^n$, $b_0 = 1$, then we get $b_1 = -1$, $b_2 = 0$, $b_3 = \frac{1}{6}$, and when $n \geq 4, n(n-1)b_n + b_{n-4} + b_{n-3} = 0$

And let $-F'(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_n = -(n+1)b_{n+1}$. Let $y(t) = \sum_{n=0}^{\infty} c_n t^n$
We have $-F'(t) = y(t)F(t)$

$\Rightarrow a_n = \sum_{k=0}^{n} c_k\, b_{n-k} \Rightarrow c_n = \frac{a_n - \sum_{k=0}^{n-1} c_k b_{n-k}}{b_0}, (c_0 = \frac{a_0}{b_0})$.(From Lecture 30-34 Page 29/108)

Based on the recursion formulas of a_n, b_n, c_n, we can get $y(t) = 1 + t + \frac{3t^2}{2} + \frac{5t^3}{3} + \frac{19t^4}{12} + \cdots$

THANKS