CS215 Assignment-3

210050115 Patil Vipul Sudhir

210050119 Hari Prakash Reddy

1 Maximum Likelihood Estimate

Let D be the dataset of size N generated from x having Uniform Distribution in $[0,\theta]$

Uniform distribution can be defined as:

$$\text{Uniform}(x|\theta) = \frac{1}{\theta} \times I(x \in [0,\theta])$$

$$I(\text{true}) = 1 \text{ and } I(\text{false}) = 0$$

The likelihood estimation:

Likelihood function $P(D|\theta) = \frac{1}{\theta^N} \times I(\theta \geqslant \max\{x_1,\ldots,x_N\})$

Maximum likelihood estimation:

$$\hat{\theta}^{MLE} = \min(\theta) \text{ for } \theta \geqslant \max\{x_1,\ldots,x_N\}$$

$$= \max\{x_1,\ldots,x_N\}$$

$$\therefore \underline{\hat{\theta}^{MLE} = \max\{x_1,\ldots,x_N\}}$$

Maximum Likelihood $= \underline{\dfrac{1}{(\hat{\theta}^{MLE})^N}}$

2 Maximum-a-posteriori estimate

2.1 Posterior distribution

Given prior : Pareto distribution

$$P(\theta) = \begin{cases} K\left(\frac{\theta_m}{\theta}\right)^{\alpha} & \text{for } \theta \geq \theta_m \\ 0 & \text{otherwise} \end{cases}$$

K = Normalising constant

$$\int_{-\infty}^{\infty} P(\theta)\,d\theta = 1$$

$$\int_{\theta_m}^{\infty} K\left(\frac{\theta_m}{\theta}\right)^{\alpha} = 1$$

$$K = \frac{\alpha-1}{\theta_m}$$

$$P(\theta) = \begin{cases} (\alpha-1)\dfrac{\theta_m^{\alpha-1}}{\theta^{\alpha}} & , \theta \geq \theta_m \\ 0 & , \text{otherwise} \end{cases}$$

The posterior distribution, is the likelihood times the prior, normalized :

$$P(\theta|D) = \frac{P(\theta)\,P(D|\theta)}{\int P(\theta)\,P(D|\theta)\,d\theta}$$

$$= \frac{(\alpha-1)\dfrac{\theta_m^{\alpha-1}}{\theta^{\alpha}} \times I(\theta \geq \theta_m) \times \dfrac{1}{\theta^N} \times I(\theta \geq \max\{x_1, x_2 \dots x_N\})}{\int_{\theta \in R} \dfrac{\theta_m^{\alpha-1}}{\theta^{\alpha}}(\alpha-1)\; I(\theta \geq \theta_m) \times \dfrac{1}{\theta^N} \times I(\theta \geq \max\{x_1, \dots x_N\})\,d\theta}$$

$$= \frac{\dfrac{1}{\theta^{\alpha+N}} \times I(\theta \geq \theta_m) \times I(\theta \geq \max\{x_1, \dots, x_N\})}{K}$$

$$\text{where } K = \int_{\theta \in R} \frac{1}{\theta^{\alpha+N}} \times I(\theta \geq \theta_m) \times I(\theta \geq \max\{x_1, \dots, x_N\})\,d\theta$$

$$\hat{\theta}^{MAP} = \frac{1}{K} \times \frac{1}{\theta^{\alpha+N}} \times I(\theta \geq \theta_{max})$$

$$\theta_{max} = \max\left[\theta_m, \max\{x_1, \dots, x_N\}\right]$$

2.2 Sample size tends to infinity

$$\hat{\theta}^{ML} = \frac{1}{(\hat{\theta}^{MLE})^N} \; ; \; \hat{\theta}_{MLE} = \max\{x_1, \dots, x_N\}$$

$$\hat{\theta}^{MAP} = \frac{1}{K} \times \frac{1}{\theta^{\alpha+N}} \times I(\theta \geq \theta_{max}) \; ; \; \theta_{max} = \max[\theta_m, \max\{x_1, \dots, x_N\}]$$

When sample size N tends to infinity:

case-I: $\theta_m > \max\{x_1, \dots, x_N\}$

$$\hat{\theta}^{MAP} = \frac{1}{K} \times \frac{1}{\theta^{\alpha+N}} \times I(\theta \geq \theta_{max}) \; ; \; \theta_{max} = \theta_m$$

$$\hat{\theta}^{ML} = \frac{1}{(\hat{\theta}^{MLE})^N} \; ; \; \hat{\theta}^{MLE} = \max\{x_1, \dots, x_N\}$$

as θ_{max} and $\hat{\theta}^{MLE}$ are different $\hat{\theta}^{MAP}$ does not tend to $\hat{\theta}^{ML}$ when N tends to ∞

case-II: $\max\{x_1, \dots, x_N\} > \theta_m$

$$\hat{\theta}^{MAP} = \frac{1}{K} \times \frac{1}{\theta^{\alpha+N}} \times I(\theta \geq \theta_{max}) \; ; \; \theta_{max} = \max\{x_1, \dots x_N\} = \hat{\theta}^{MLE}$$

Here $\hat{\theta}^{MAP}$ tends to $\hat{\theta}^{ML}$ when N tends to ∞

Is this desirable?:

In case-I: Not desirable

In case-II: desirable

$\therefore$ overall it is not desirable.

3 Estimator of the mean of the posterior distribution

Estimator of mean of posterior distribution:

$$\hat{\theta}^{\text{Posterior mean}} = \int_0^\infty \theta \, P(\theta|D) \, d\theta$$

$$= \int_0^{\theta_{max}} \theta \times 0 \, d\theta + \int_{\theta_{max}}^\infty \theta \times \frac{1}{K} \times \frac{1}{\theta^{\alpha+N}} \, d\theta \quad ;$$

$\theta_{max} = \max[\theta_m, \max\{x_1, \ldots, x_N\}]$

$$= 0 + \frac{1}{K}\left[\frac{-1}{(\alpha+N-2)\,\theta^{\alpha+N-2}}\right]_{\theta_{max}}^{\infty}$$

$$\hat{\theta}^{\text{Posterior mean}} = \frac{1}{K(\alpha+N-2)(\theta_{max})^{\alpha+N-2}}$$

$K = \int_{\theta \in R} \frac{1}{\theta^{\alpha+N}} \times I(\theta \geq \theta_m) \times d\theta$
$I(\theta \geq \max(x_1, \ldots, x_N))$

$K = \frac{1}{\theta_{max}^{\alpha+N-1}(\alpha+N-1)}$

$\theta_{max} = \max\{\theta_m, \max(x_1, \ldots, x_N)\}$

$$\therefore \hat{\theta}^{\text{Posterior mean}} = \theta_{max} \times \frac{(\alpha+N-1)}{(\alpha+N-2)}$$

★ When sample size N tends to ∞:

case(i): $\theta_m > \max\{x_1, \ldots, x_N\}$

$$\lim_{N\to\infty} \hat{\theta}^{\text{Posteiror mean}} = \theta_m \times 1 = \theta_m$$

$$\hat{\theta}^{\text{Posterior mean}} \neq \hat{\theta}^{ML}$$

$\therefore$ It is not desirable

case(ii): $\max\{x_1, x_2, \ldots, x_N\} > \theta_m$

$$\lim_{N\to\infty} \hat{\theta}^{\text{Posterior mean}} = \hat{\theta}^{ML} \times 1 = \hat{\theta}^{ML}$$

$$\therefore \hat{\theta}^{\text{Posterior mean}} = \hat{\theta}^{ML}$$

In this case it is desirable.