Exercises Guide to Economics 11
Summer Session A

Victoria Barone *
UCLA

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*Department of Economics, University of California, Los Angeles. Contact: mvbarone@ucla.edu. If you find any typo or error please shoot me an email.

Contents

I Optimization - basics

I.i Exercises

In this chapter I will focus on finding the optimal value of some function. This will be helpful because we will frequently adopt the assumption that an economic actor seeks to maximize or minimize some function. We will deal with the univariate case and with the multivariate case as well. You will also learn how to optimize both unconstrained and constrained functions. An unconstrained function allows any value for its variables, while a constrained function imposes some sort of restriction. For example, a constrain will be the amount of money a firm has to hire workers.

I.i.1 Single Variable Optimization

Consider the problem of finding the optimal value of the following function:

$$f(x) = \ln(15x) - 2x^2; x \geq 0$$

To find a maximum we will follow these steps:
Step 1: compute First Order Conditions (FOC) and get critical values

$$\begin{aligned}
f'(x) &= 0 \\
f'(x) &= \frac{1}{15x}15 - 4x = \frac{1}{x} - 4x \\
\frac{1}{x} - 4x &= 0 \\
\frac{1}{x} &= 4x \\
x^2 &= \frac{1}{4}
\end{aligned}$$

The critical values are $x^* = \frac{1}{2}$ and $x^* = -\frac{1}{2}$. But remember that we are interested in $x \geq 0$ so we need to consider only $x^* = \frac{1}{2}$. Is this a local maximum or a minimum?
Step 2: compute Second Order Condition, remember that:

$$f''(x^*) < 0 \Rightarrow \text{local maximum}$$
$$f''(x^*) > 0 \Rightarrow \text{local minimum}$$

In the case of a maximum, we required the function $f'(x)$ to be a decreasing function at x^*. That is, the derivative of $f'(x)$ (i.e. $f''(x)$), must be negative. The same kind of argument is true for the local minimum, we require the function $f'(x)$ to be an increasing function.

In our case $f''(x) = -\frac{1}{x^2} - 4$; but remember that SOC are about $f''(x^*)$, then:

$$\begin{aligned}
f''(x^*) &= -\frac{1}{\left(\frac{1}{2}\right)^2} - 4 \\
&= -\frac{1}{\frac{1}{4}} - 4 \\
&= -4 - 4 \\
f''(x^*) &= -8 \\
f''(x^*) &= -8 < 0
\end{aligned}$$

Then, $x^* = \frac{1}{2}$ is a local maximum. Is it a global maximum? To answer this question we need to compare the values of $f(x)$ at $x = 0$ and $x = \frac{1}{2}$. Why do we include $x = 0$ in this comparison?

$$\begin{aligned}
f(x) &= \ln(15x) - 2x^2 \\
f\left(\frac{1}{2}\right) &= \ln\left(15 \cdot \frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^2 \approx 1.5 \\
f(0) &= \ln(15 \cdot 0) = -\infty
\end{aligned}$$

That is: $f\left(\frac{1}{2}\right) > f(0)$ and $x^* = 1/2$ is a global maximum.

I.i.2 Multivariate Optimization and the Lagrangian Method

Find the critical value of $f(x_1,x_2)$ subject to $x_1 + 2x_2 = 10$. We will attempt this problem in two different ways.

$$\begin{aligned}
f(x_1,x_2) &= x_1^3 + 8x_2 \\
&\text{s.t. } x_1 + 2x_2 = 10
\end{aligned}$$

Method 1: simplifying the setting

Step 1: Solve for x_2 as a function of x_1

As the constraint must be satisfy we solve for one of the variables as a function of the other (*exercise:* you should get the same result if you choose to solve for x_1 as a function of x_2):

$$\begin{aligned} x_1 + 2x_2 &= 10 \\ 2x_2 &= 10 - x_1 \\ x_2 &= 5 - \frac{x_1}{2} \end{aligned}$$

Step 2: Transform the original multivariate optimization problem in a single optimization one

That is, we know that:

$$f(x_1,x_2) = x_1^3 + 8x_2$$
$$x_2 = 5 - \frac{x_1}{2}$$

We can plug in the second equation in the first one and follow the procedure of the previous exercise:

$$f(x_1,x_2) = x_1^3 + 8\left(5 - \frac{x_1}{2}\right)$$
$$f(x_1,x_2) = x_1^3 + 40 - 4x_1$$

And now we proceed as in the first exercise and compute $f'(x)$ and find the value of x that makes $f'(x) = 0$:

$$\begin{aligned} f'(x) &= 3x_1^2 - 4 \\ 3x_1^2 - 4 &= 0 \\ 3x_1^2 &= 4 \\ x_1^2 &= \frac{4}{3} \end{aligned}$$

Then: $x_1^* = \frac{2}{\sqrt{3}}$ and remember that the original problem has two variables **(don't forget to find x_2^*):**

$$x_2 = 5 - \frac{x_1}{2}$$
$$x_2 = 5 - \frac{\left(\frac{2}{\sqrt{3}}\right)}{2}$$

The critical values are $x_1^* = \frac{2}{\sqrt{3}}$ and $x_2^* = 5 - \frac{1}{\sqrt{3}}$ and both values are strictly greater than 0.

Method 2: Lagrangian Method

$$f(x_1,x_2) = x_1^3 + 8x_2$$
$$\text{s.t. } x_1 + 2x_2 = 10$$

Step 1: Rewrite the constraint as: $g(x_1,x_2) = 0$

$$0 = 10 - x_1 - 2x_2$$

Step 2: Form the Lagrangian

$$\mathcal{L} = f(x_1,x_2) + \lambda \cdot g(x_1,x_2)$$
$$\mathcal{L} = x_1^3 + 8x_2 + \lambda \cdot [10 - x_1 - 2x_2]$$

Step 3: To find critical values of $f(x_1,x_2)$ subject to the constraint $g(x_1,x_2)$ we need to solve a system of equations:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 0$$
$$g(x_1,x_2) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 3x_1^2 - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 8 - 2\lambda = 0$$
$$0 = 10 - x_1 - 2x_2$$

Let's do some algebra with the previous expression and get:

$$3x_1^2 = \lambda$$
$$\lambda = 4$$
$$2x_2 = 10 - x_1$$

$$x_1^2 = \frac{4}{3}$$
$$\lambda = 4$$
$$x_2 = 10 - \frac{x_1}{2}$$

Which are the same expressions we found using method 1. Note that there is one additional variable in the problem λ but that the critical values for x_1 and x_2 are the same.

II Optimization - Rules

Here you will find a short summary of optimization rules, please refer to the Math Review file to have examples and more detailed explanations.

II.i Single Variable Optimization

Given the objective function $y = f(x)$:
Step 1: compute First Order Conditions (FOC) $f'(x) = 0$ and get critical values.
Step 2: compute Second Order Condition, remember that:

$$f''(x^*) < 0 \Rightarrow \text{local maximum}$$
$$f''(x^*) > 0 \Rightarrow \text{local minimum}$$

II.ii Multivariate Optimization - two variables case

Given the objective function $y = f(x_1, x_2)$:
Step 1: compute First Order Conditions (FOC) and get critical values.

$$\frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} = \cdots = \frac{\partial y}{\partial x_k} = 0$$

Step 2: compute Second Order Condition, remember that:

$$f_{xx}, f_{yy} < 0 \text{ and } f_{xx} f_{yy} > f_{xy}^2 \Rightarrow \text{local maximum}$$
$$f_{xx}, f_{yy} > 0 \text{ and } f_{xx} f_{yy} > f_{xy}^2 \Rightarrow \text{local minimum}$$

III Optimization - canonical examples

III.i Firms Optimization Problem

The goal of the firm's optimization problem is to choose the amount of goods to produce that will maximize profits $\Pi(x_1, x_2)$. Where profits are defined as the difference of revenue and cost:

$$\Pi(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2)$$

That is, given

$$R(x_1, x_2) = 6x_1 + \left(\frac{99}{4}\right) x_2$$
$$C(x_1, x_2) = \frac{x_1 x_2}{4} + \ln\left(\frac{1}{x^{(1/8)}}\right) + \frac{(x_2)^2}{2}$$

the profit function of the firm is:

$$\Pi(x_1, x_2) = 6x_1 + \left(\frac{99}{4}\right) x_2 - \frac{x_1 x_2}{4} - \ln\left(\frac{1}{x_1^{(1/8)}}\right) - \frac{(x_2)^2}{2}$$

And the maximization problem of the firm is:

$$\max_{x_1, x_2} \Pi(x_1, x_2) = 6x_1 + \left(\frac{99}{4}\right) x_2 - \frac{x_1 x_2}{4} - \ln\left(\frac{1}{x_1^{(1/8)}}\right) - \frac{(x_2)^2}{2}$$

We will use the methods learned last week to find the solution. Note that we are interested on finding (x_1^*, x_2^*) that maximizes $\Pi(x_1, x_2)$. This is a multivariate optimization problem without constraints (just as the example we solved last week!). Then, we can use the following approach:

Step 1: compute First Order Conditions (FOC) and get critical values.

$$\frac{\partial\Pi(x_1,x_2)}{\partial x_1} = 0$$
$$\frac{\partial\Pi(x_1,x_2)}{\partial x_2} = 0$$

$$\frac{\partial\Pi(x_1,x_2)}{\partial x_1} = 6 - \frac{x_2}{4} - \frac{1}{\left(\frac{1}{x_1^{(1/8)}}\right)} \cdot \left(-\frac{1}{8}\right) x_1^{-\frac{1}{8}-1}$$
$$\frac{\partial\Pi(x_1,x_2)}{\partial x_2} = \frac{99}{4} - \frac{x_1}{4} - \frac{1}{2} \cdot 2x_2$$

With some manipulation the FOC conditions look like:

$$\frac{\partial\Pi(x_1,x_2)}{\partial x_1} = 6 - \frac{x_2}{4} + \frac{1}{8}\frac{1}{x_1}$$
$$\frac{\partial\Pi(x_1,x_2)}{\partial x_2} = \frac{99}{4} - \frac{x_1}{4} - x_2$$

Now we have a system of two equations and two unknowns, there are many ways to solve this system, here I present one of them:

$$0 = 6 - \frac{x_2}{4} + \frac{1}{8}\frac{1}{x_1}$$
$$0 = \frac{99}{4} - \frac{x_1}{4} - x_2$$

$$0 = 6 - \frac{x_2}{4} + \frac{1}{8}\frac{1}{x_1}$$
$$x_2 = \frac{99}{4} - \frac{x_1}{4}$$

$$
\begin{aligned}
0 &= 48 - 2 \cdot x_2 + \frac{1}{x_1} \\
x_2 &= \frac{99 - x_1}{4} \\
0 &= 96 - 4 \cdot x_2 + \frac{2}{x_1} \\
x_2 &= \frac{99 - x_1}{4}
\end{aligned}
$$

$$
\begin{aligned}
&0 = 96 - 4\left(\frac{99 - x_1}{4}\right) + \frac{2}{x_1} x_2) \\
&0 = 96 - 99 + x_1 + \frac{2}{x_1} \\
&0 = -3 + x_1 + \frac{2}{x_1} \\
&0 = x_1^2 - 3x_1 + 2
\end{aligned}
$$

Using the quadratic equation (let me state the general form first):

$$
\begin{aligned}
0 =& az^2 + bz + c \\
z =& \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

In our problem: $a = 1$, $b = -3$,$c = 2$ and we get that $x_1 = 1$ and $x_1 = 2$ are solutions of the quadratic equation.

Then, using $x_2 = \frac{99 - x_1}{4}$ we get that the pairs:

$$
\begin{aligned}
(x_1^*, x_2^*) = (1, \frac{98}{4}) \\
(x_1^*, x_2^*) = (2, \frac{97}{4})
\end{aligned}
$$

Let me also present another way to solve this system, begging with the FOC system and solving x_2 as a function of x_1:

$$
\begin{aligned}
0 &= 6 - \frac{x_2}{4} + \frac{1}{8}\frac{1}{x_1} \\
x_2 &= \frac{99}{4} - \frac{x_1}{4}
\end{aligned}
$$

$$\begin{aligned}\frac{1}{8x_1} &= -6 + \frac{x_2}{4} \\ \frac{1}{x_1} &= -48 + x_2 2 \\ \frac{2}{x_1} &= -96 + x_2 4\end{aligned}$$

And now I will plug in x_2:

$$\begin{aligned}\frac{2}{x_1} &= -96 + \left[\frac{99}{4} - \frac{x_1}{4}\right] 4 \\ 2 &= -96x_1 + 99x_1 - x_1^2 \\ x_1^2 - 3x_1 + 2 = 0\end{aligned}$$

And we can use the same formulta than before!

$$\begin{aligned}0 =& az^2 + bz + c \\ z =& \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\end{aligned}$$

In our problem: $a = 1$, $b = -3$,$c = 2$ and we get that $x_1 = 1$ and $x_1 = 2$ are solutions of the quadratic equation.

Then, using $x_2 = \frac{99-x_1}{4}$ we get that the pairs:

$$\begin{aligned}(x_1^*,x_2^*) &= (1, \frac{98}{4}) \\ (x_1^*,x_2^*) &= (2, \frac{97}{4})\end{aligned}$$

Which of the two pairs is the one that maximizes the profit function?

Step 2: compute Second Order Condition, remember that:

$$\Pi_{x_1x_1},\Pi_{x_2x_2} < 0 \text{ and } \Pi_{x_1x_1}\Pi_{x_2x_2} > \Pi^2_{x_1x_2} \Rightarrow \text{local maximum}$$
$$\Pi_{x_1x_1},\Pi_{x_2x_2} > 0 \text{ and } \Pi_{x_1x_1}\Pi_{x_2x_2} > \Pi^2_{x_1x_2} \Rightarrow \text{local minimum}$$

These derivatives are:

$$\begin{aligned}
\Pi_{x_1x_1} &= -\frac{1}{8x_1^2} && < 0 \text{ for every point} \\
\Pi_{x_2x_2} &= -1 && < 0 \text{ for every point} \\
\Pi_{x_1x_2} &= -\frac{1}{4} && < 0
\end{aligned}$$

Then, for $(x_1^*, x_2^*) = (1, \frac{98}{4})$

$$\Pi_{x_1x_1}(x_1^*, x_2^*) = -\frac{1}{8} < 0 \text{ ; } \Pi_{x_2x_2} = -1 < 0 \text{ and } \frac{1}{8} > \frac{1}{16}$$

Thus, $(x_1^*, x_2^*) = (1, \frac{98}{4})$ is a maximum.

Then, for $(x_1^*, x_2^*) = (2, \frac{97}{4})$

$$\Pi_{x_1x_1}(x_1^*, x_2^*) = -\frac{1}{32} < 0 \text{ ; } \Pi_{x_2x_2} = -1 < 0 \text{ and } \frac{1}{32} < \frac{1}{16}$$

Thus, $(x_1^*, x_2^*) = (2, \frac{97}{4})$ is a neither minimum or a maximum, it is called a saddle point.[1]

III.ii Consumer's Optimization Problem

Suppose we have a consumer whose preferences can be represented by the utility function:

$$u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$$

and the consumer face the budget constraint: $p_1x_1 + p_2x_2 = I$

III.ii.1 Consumer's utility maximization

The maximization problem that represents consumer's utility maximization is:

$$\max_{x_1, x_2} u(x_1, x_2) = x_1^{1/2} x_2^{1/2} \text{ subjetc to } p_1x_1 + p_2x_2 = I$$

This is a maximization problem with constraint and the solution method is the one we used in exercise 3 in the previous section: Lagrangian Method.

Step 1: Rewrite the constraint as: $g(x_1, x_2) = 0$

[1]Take this as an example of why checking the FOC is important, don't expend much time thinking about saddle points!

$$I - p_1 x_1 - p_2 x_2 = 0$$

Step 2: Form the Lagrangian

$$\mathcal{L} = u(x_1,x_2) + \lambda \cdot g(x_1,x_2)$$
$$\mathcal{L} = x_1^{1/2} x_2^{1/2} + \lambda \cdot [I - p_1 x_1 - p_2 x_2]$$

Step 3: FOC:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 0$$
$$g(x_1,x_2) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{2}\left(\frac{x_2}{x_1}\right)^{1/2} - \lambda p_1$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{2}\left(\frac{x_1}{x_2}\right)^{1/2} - \lambda p_2$$
$$I - p_1 x_1 - p_2 x_2 = 0$$

$$\frac{1}{2}\left(\frac{x_2}{x_1}\right)^{1/2} - \lambda p_1 = 0 \tag{1}$$
$$\frac{1}{2}\left(\frac{x_1}{x_2}\right)^{1/2} - \lambda p_2 = 0 \tag{2}$$
$$I - p_1 x_1 - p_2 x_2 = 0 \tag{3}$$

To get the critical values we need to solve the above system of equations: 3 equations and 3 unknowns. I will show one way to deal with this system.
Working with equations (1) and (2) we can see that:

$$\frac{1}{2}\left(\frac{x_2}{x_1}\right)^{1/2} = \lambda p_1$$
$$\frac{1}{2}\left(\frac{x_1}{x_2}\right)^{1/2} = \lambda p_2$$
$$I - p_1 x_1 - p_2 x_2 = 0$$

Then:

$$\lambda = \frac{1}{p_1}\frac{1}{2}\left(\frac{x_2}{x_1}\right)^{1/2} = \frac{1}{p_2}\frac{1}{2}\left(\frac{x_1}{x_2}\right)^{1/2}$$

That is, we have expressed λ as a function of x_1 and x_2. Now we need to find x_1 and x_2 as function of the p_1, p_2 and I. Taking the ratio of equations (1) and (2):

$$\begin{aligned}
\frac{\left(\frac{x_2}{x_1}\right)^{1/2}}{\left(\frac{x_1}{x_2}\right)^{1/2}} &= \frac{\lambda p_1}{\lambda p_2} \\
x_2^{1/2}x_2^{1/2}x_1^{-1/2}x_1^{-1/2} &= \frac{p_1}{p_2} \\
\frac{x_2}{x_1} &= \frac{p_1}{p_2} \\
\frac{x_2}{x_1} &= \frac{p_1}{p_2} \\
x_2 &= \frac{p_1}{p_2}x_1
\end{aligned}$$

Plug in this equation back in the budget constraint:

$$\begin{aligned}
I &= p_1x_1 + p_2\left[\frac{p_1}{p_2}x_1\right] \\
I &= p_1x_1 + p_1x_1 \\
I &= 2p_1x_1 \\
x_1^* &= \frac{1}{2}\frac{I}{p_1}
\end{aligned}$$

Note:

- $x_1 = \frac{1}{2}\frac{I}{p_1}$ is a solution of the problem because it expresses x_1 as a function of income and prices.
- $x_2 = \frac{p_1}{p_2}x_1$ is not a solution of the problem because it expresses x_2 as a function of income, prices, and x_1 and the consumer also picks x_1. But we can use this relation to solve for x_2 :

$$\begin{aligned}
x_2 &= \frac{p_1}{p_2}\frac{1}{2}\frac{I}{p_1} \\
x_2^* &= \frac{1}{2}\frac{I}{p_2}
\end{aligned}$$

And I will left you as an exercise to get the solution for λ. Check that you get the same result using both expressions!

The solution of this utility maximization problem is: $x_1^*(p_1,p_2,I) = \frac{1}{2}\frac{I}{p_1}$ and $x_2^*(p_1,p_2,I) = \frac{1}{2}\frac{I}{p_2}$
Note: these are functions that we can use to compute optimal quantities for different prices and income levels.

III.ii.2 Comparative Statics

What happens to the optimal quantity of x_1 chosen as p_2 increases?
Note that in this particular case, the optimal quantity of x_1 does not depends on p_2. Bear in mind that this is not the general case, in general x_1^* is a function of prices (p_1 and p_2) and income (I).

Then as p_2 changes there is no change on x_1^*. How we can show this formally?

$$\frac{\partial x_1^*}{\partial p_2} = 0$$

What happens to the optimal quantity of x_1 chosen as p_1 increases?

$$\frac{\partial x_1^*}{\partial p_1} = -\frac{1}{2} I \frac{1}{p_1^2} < 0$$

What happens to the optimal quantity of x_1 chosen as I increases?

$$\frac{\partial x_1^*}{\partial I} = \frac{1}{2p_1} > 0$$

III.ii.3 Welfare analysis

Is the consumer better off as prices increase *and income doesn't change*? To answer this question we will plug in the optimal quantities in the utility function:

$$\begin{aligned}
u^* &= u(x_1^*, x_2^*) \\
u^* &= \left(\frac{1}{2}\frac{I}{p_1}\right)\left(\frac{1}{2}\frac{I}{p_2}\right) \\
u^* &= \frac{I}{2p_1^{1/2}p_2^{1/2}}
\end{aligned}$$

If prices increase the consumer will be worse off, note that:

$$\frac{\partial u^*}{\partial p_1} = \frac{I}{2p_2^{1/2}} \left(-\frac{1}{2}\right) \frac{1}{p_1^{1/2}} < 0$$

What if income increases *and prices don't change*?

$$\frac{\partial u^*}{\partial I} = \frac{1}{2p_1^{1/2} p_2^{1/2}} > 0$$

The consumer is better off.

IV Extra practice I - Optimization

1. (Unconstrained Optimization-Single Variable) Consider the following function:

$$f(x) = 3x^3 - 5x^2 + x$$

where

$$x \in [0,2]$$

   (a) Find the x's that are critical value points of $f(x)$ (values of x that are a potential minimum or maximum). Are those critical values maxima or minima? (Hint: Check the second order derivative. Check also the value of the function at the endpoints 0 and 2 to make sure you have the correct answer.)

2. (Unconstrained Optimization-Two Variables) Consider the function:

$$f(x_1,x_2) = x_1^3 + 3x_2^3 - 9x_1x_2$$

   (a) Find a minimum given that $x_1,x_2 \geq 1$.

3. (Constrained Optimization) Julia maximizes the following utility function:

$$u(x_1,x_2) = x_1^{2/7}x_2^{5/7}$$

subject to the budget constraint $p_1x_1 + p_2x_2 = I$ where $p_1, p_2, x_1, x_2, I > 0$

   (a) Find the (x_1,x_2) that maximizes $u(x_1,x_2)$.

   (b) Show that the maximizer $x_2^*(p_1,p_2,I)$ is decreasing in p_2 and increasing in I (Hint: Use the partial derivatives).

   (c) Does $x_2^*(p_1,p_2,I)$ change if p_1 changes? (Hint: partial derivatives).

V Indifference Curves

An indifference curve is a representation of all the combinations of x and y which the consumer values equally. If the consumer has complete, transitive, and continuous preferences we can then represent her preferences with a utility function. To plot an indifference curve, set the utility function equal to a fixed value, $\bar{U}$, and chart all the values that satisfy this function.

$$U(x,y) = \bar{U}$$

a) How could we find the slope of the indifference curve? Implicit Function Theorem!

$$U(x,y) = \bar{U}$$

$$\Rightarrow g(x,y) = U(x,y) - \bar{U} = 0$$

$$\Rightarrow g_x dx + g_y dy = 0$$

$$\Rightarrow U_x dx + U_y dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{U_x}{U_y}$$

$-\frac{dy}{dx}$ or $\frac{U_x}{U_y}$ is know as the *marginal rate of substitution* (MRS). The MRS measures the individual's willingness to trade y for x, that is, *give up* y to *gain additional units* of x

Example: Let $U(x,y) = x^{2/3}\, y^{1/3}$. What is $\frac{dy}{dx}$?

$$\begin{aligned}\frac{dy}{dx} = -\frac{U_x}{U_y} &= -\frac{\frac{2}{3}x^{-1/3}\, y^{1/3}}{\frac{1}{3}x^{2/3}\, y^{-2/3}} \\ &= 2x^{(-1/3-2/3)}y^{1/3+2/3} \\ &= -2\frac{y}{x}\end{aligned}$$

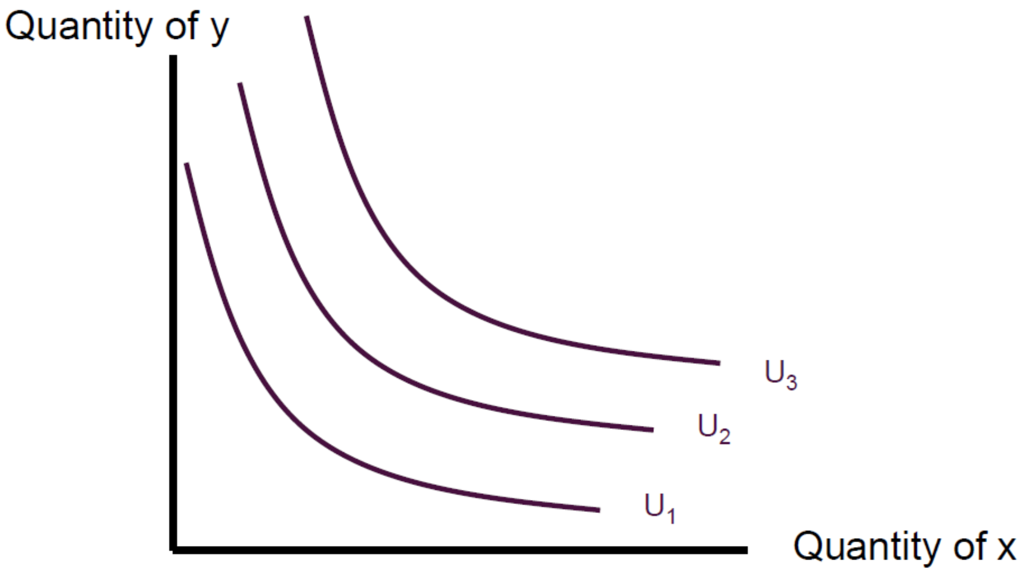
b) Draw the indifference curves and compute the MRS

i. Cobb-Douglas Preferences: $U(x,y) = x^{\alpha}y^{(1-\alpha)}$

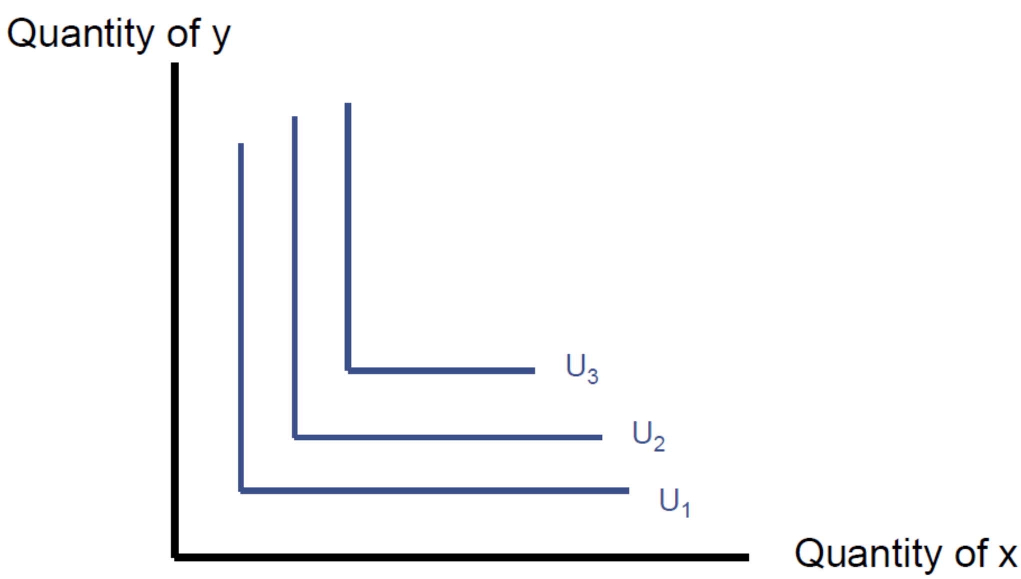
This is the general case of what we did in our previous example. The Cobb-Douglas preferences are extensively used in economics!

The MRS is the slope $MRS = \frac{U_x}{U_y} = \dfrac{\alpha x^{\alpha-1}y^{1-\alpha}}{(1-\alpha)\, x^{\alpha}y^{-\alpha}} = \frac{\alpha}{1-\alpha}x^{\alpha-1-\alpha}y^{1-\alpha+\alpha} = \frac{\alpha}{1-\alpha}\frac{y}{x}$

Graph:

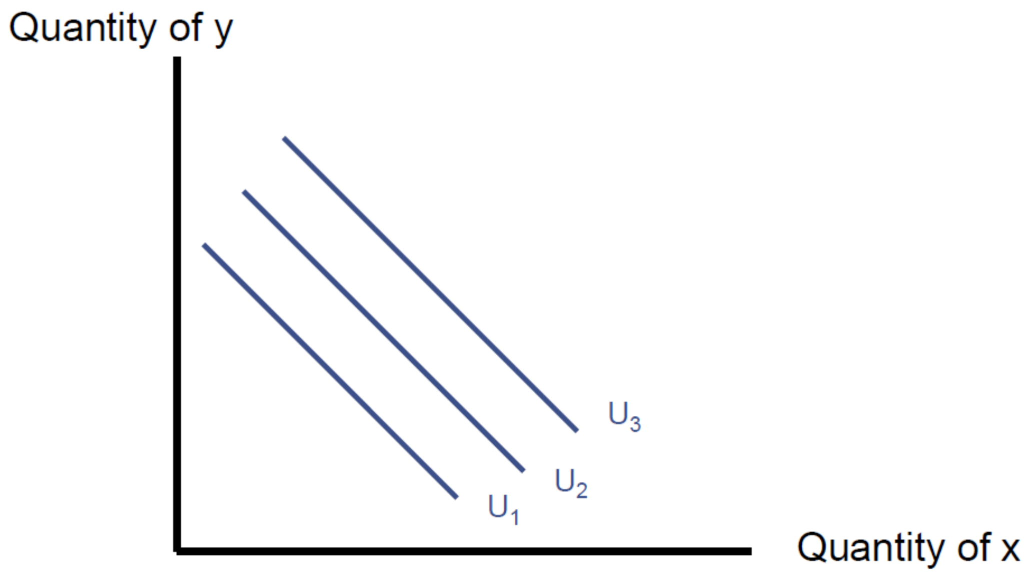


ii. Perfect complements: $U(x,y) = \min\{Ax, By\}$
This utility function is not differentiable, so we won't compute the MRS.
Graph:



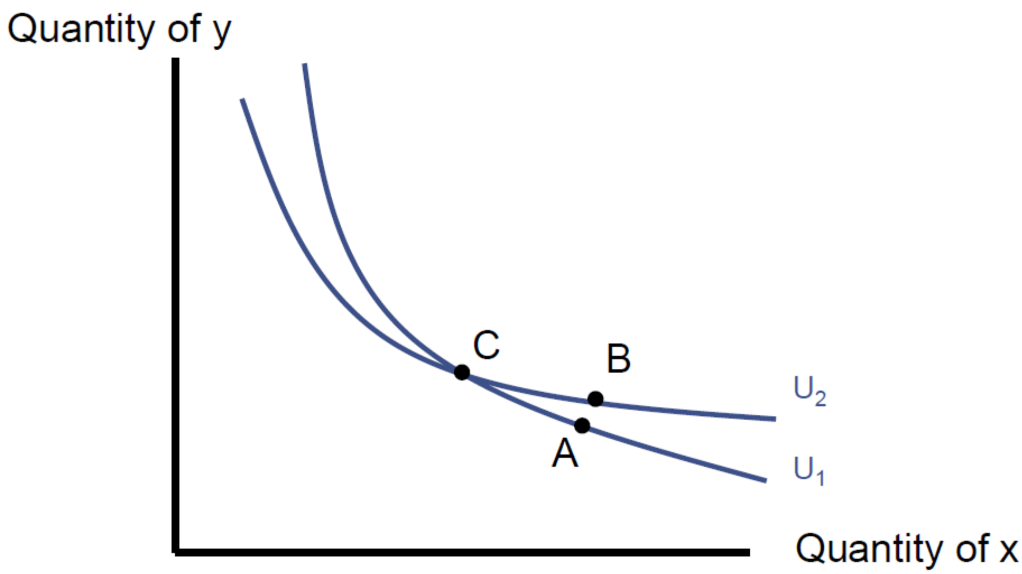
iii. Perfect substitutes: $U(x,y) = Ax + By$

$MRS = \frac{U_x}{U_y} = \frac{A}{B}$
Graph:



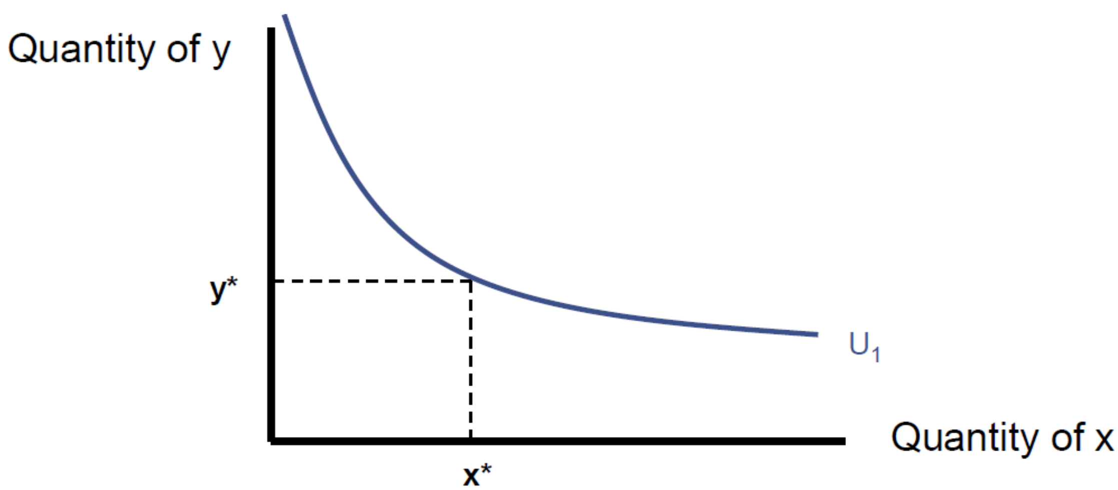
c.Transitivity If preferences are transitive, the indifference curves can not cross. How do we see this? When preferences are transitive, we know that if $A \succ B$ and $B \succ C$ then: $A \succ C$. For example, if you like chocolate ice-cream more than vanilla ice-cream and vanilla ice-cream more than strawberry ice-cream then you like chocolate ice-cream more than strawberry ice-cream!

How does this relates with not-crossing indifference curve:



The individual is indifferent between A and C. The individual is indifferent between C and B. Transitivity suggests that the individual should be indifferent between A and B. But B is preferred to A because B contains more x and y than A.

d. Convexity this property tell us that all lines segments are contained in the upper-level set (the area of the graph that is above a given indifference curve). This assumption is required for the uniqueness of the solution to the utility maximization problem. How do a indifference curve look like under this assumption? What if preferences are not convex?



VI Consumer's Problem and MRS

Sarah has a monthly income of 500 dollars that she spends buying food that costs 5 dollars, clothes that cost 20 dollars, and books that cost 50 dollars. Sarah's utility function is given by:

$$U(F,C,B) = 6\ln(F) + 2\ln(C) + \ln(B)$$

1. Write down Sarah's budget constraint.

   Remember from past-section notes where the budget constrain was: $p_1x_1 + p_2x_2 = I$. But now we have three goods and we also know their prices! Then, the budget constrain is: $5\ F + 20\ C + 50\ B = 500$

2. Suppose that Sarah wants to consume $F = 60$, $C = 5$, and $B = 1$. Is this a feasible allocation?

   Plug in the proposed quantities and see if Sarah can afford this allocation.
   $5\ 60 + 20\ 5 + 50\ 1 = 450 \leq 500$. Thus, the answer is Yes!

3. Compute the marginal rate of substitution of books for clothing and the MRS of clothing for books.

   $MRS_{C,B} = \frac{U_B}{U_C} = \frac{1/B}{2/C} = \frac{C}{2\ B}$
   $MRS_{B,C} = \frac{U_C}{U_B} = \frac{2/C}{1/B} = \frac{2\ B}{C}$

4. Is $F = 60$, $C = 5$, and $B = 1$ an optimal allocation given her income? Is so explain why, if not explain how her income could be changed to make this allocation optimal.

   Let's use the Lagrangian method to characterize the optimal allocations:

$$\mathcal{L} = u(F,C,B) + \lambda \cdot g(F,C,B)$$
$$\mathcal{L} = 6\ln(F) + 2\ln(C) + \ln(B) + \lambda \cdot [I - p_F\ F - p_C\ C - p_B\ B]$$

   The FOC are:

$$\frac{\partial \mathcal{L}}{\partial F} = \frac{6}{F} - \lambda p_F = 0$$
$$\frac{\partial \mathcal{L}}{\partial C} = \frac{2}{C} - \lambda p_C = 0$$
$$\frac{\partial \mathcal{L}}{\partial C} = \frac{1}{B} - \lambda p_B = 0$$
$$I = p_F\ F + p_C\ C + p_B\ B$$

$$\frac{6}{F} = \lambda p_F \tag{4}$$
$$\frac{2}{C} = \lambda p_C \tag{5}$$
$$\frac{1}{B} = \lambda p_B \tag{6}$$
$$I = p_F\ F + p_C\ C + p_B\ B \tag{7}$$

$$
\begin{aligned}
(1) : (3) \Rightarrow \frac{U_F}{U_B} = MRS_{B,F} = \frac{p_F}{p_B} \Rightarrow \frac{\frac{6}{F}}{\frac{1}{B}} &= 5/50 \\
6B &= F/10 \\
F &= 60B
\end{aligned}
$$

Then, we see that if $B = 1$ and $F = 60$ then $MRS_{B,F} = \frac{p_F}{p_B}$

$$
\begin{aligned}
(1) : (2) \Rightarrow \frac{U_C}{U_B} = MRS_{B,C} = \frac{p_C}{p_B} \Rightarrow \frac{\frac{2}{C}}{\frac{1}{B}} &= 20/50 \\
2B &= C\frac{2}{5} \\
C &= 5B
\end{aligned}
$$

Then, we see that if $B = 1$ and $C = 5$ then $MRS_{B,C} = \frac{p_C}{p_B}$

Now, use the budget constrain again:

$$
\begin{aligned}
I &= p_F\ F + p_C\ C + p_B\ B \\
I &= 5\,(60\ B) + 20\,(5\ B) + 50\ B \\
I &= 450\ B
\end{aligned}
$$

As $B = 1$ **the income that makes this allocation optimal is** $I = 450$

Remark: MRS measures the individual's willingness to trade y for x. That is, how many units the person would be willing to give up of y to gain additional units of x.

VII Cobb-Douglas Utility

- Utility Function: $U(x,y) = x^a y^b$
- Utility Maximization problem:

$$\max_{x,y} x^a y^b$$
$$\text{s.t. } p_x x + p_y y = I$$

- Lagrangian: $\mathcal{L}(x,y,\lambda) = x^a y^b - \lambda [I - p_x x - p_y y]$
- $MRS = \frac{U_x}{U_y} = \dfrac{ax^{a-1}y^b}{bx^a y^{b-1}} = \frac{a}{b}\frac{y}{x}$
- Tangency condition: $MRS = \frac{U_x}{U_y} = \frac{p_x}{p_y} \Rightarrow \frac{a}{b}\frac{y}{x} = \frac{p_x}{p_y}$
- Marshallian demand: I will derive the marshallian demand from the tangency condition, you can also use the lagrangian method, the FOC will boil down to the tangency condition. The two equations we want to combine are:

$$\frac{a}{b}\frac{y}{x} = \frac{p_x}{p_y} \tag{1}$$
$$I = p_x x + p_y y \tag{2}$$

Using equation (1):

$$\frac{a}{b}\frac{y}{x} = \frac{p_x}{p_y}$$
$$y = \frac{p_x}{p_y}\frac{b}{a}x$$

Plug-in this expression on equation (2):

$$
\begin{aligned}
I &= p_x x + p_y \frac{p_x}{p_y}\frac{b}{a}x \\
I &= p_x x + p_x x\frac{b}{a} \\
I &= \frac{a}{a}p_x x + \frac{b}{a}p_x x \\
I &= \frac{a+b}{a}p_x x \\
\frac{a}{a+b}I &= p_x x \\
x^*(p_x,p_y,I) &= \frac{a}{a+b}\frac{I}{p_x}
\end{aligned}
$$

Using equation (1) re-written as: $y = \frac{p_x}{p_y}\frac{b}{a}x$

$$
\begin{aligned}
y &= \frac{p_x}{p_y}\frac{b}{a}\frac{a}{a+b}\frac{I}{p_x} \\
y^*(p_x,p_y,I) &= \frac{b}{a+b}\frac{I}{p_y}
\end{aligned}
$$

- Indirect Utility Function:

$$
\begin{aligned}
V(p_x, p_y, I) &= x^{*a}y^{*b} \\
V(p_x, p_y, I) &= \left[\frac{a}{a+b}\frac{I}{p_x}\right]^a \left[\frac{b}{a+b}\frac{I}{p_y}\right]^b \\
&= \left[\frac{a}{a+b}\right]^a \left[\frac{b}{a+b}\right]^b I^{a+b} \left[\frac{1}{p_x}\right]^a \left[\frac{1}{p_y}\right]^b \\
&= a^a b^b \left[\frac{I}{a+b}\right]^{a+b} \left[\frac{1}{p_x}\right]^a \left[\frac{1}{p_y}\right]^b \\
&= \left[\frac{I}{a+b}\right]^{a+b} \left[\frac{a}{p_x}\right]^a \left[\frac{b}{p_y}\right]^b
\end{aligned}
$$

- Expenditure Minimization Problem:

$$
\begin{aligned}
&\min_{x,y} p_x x + p_y y \\
&\text{s.t. } x^a y^b = \bar{u}
\end{aligned}
$$

- Hicksian demand:

a) I will first show that the FOC from the expenditure minimization problem are "the same" than the utility maximization problem. Let's start with the utility maximization problem:

$$\mathcal{L}(x,y,\lambda) = x^a y^b - \lambda \left[I - p_x x - p_y y \right]$$

F.O.C.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= a x^{a-1} y^b - \lambda p_x \\
\frac{\partial \mathcal{L}}{\partial y} &= b x^a y^{b-1} - \lambda p_y \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_x x - p_y y
\end{aligned}$$

Using the first two equations:

$$\begin{aligned}
\frac{a x^{a-1} y^b}{b x^a y^{b-1}} &= \frac{p_x}{p_y} \\
\frac{a}{b} \frac{y}{x} &= \frac{p_x}{p_y}
\end{aligned}$$

Which is the tangency condition.

Now, lets write up the lagrangian for the expenditure minimization problem:

$$\mathcal{L}(x,y,\lambda) = p_x x + p_y y - \lambda \left[\bar{u} - x^a y^b \right]$$

F.O.C.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= p_x - \lambda a x^{a-1} y^b \\
\frac{\partial \mathcal{L}}{\partial y} &= p_y - \lambda b x^a y^{b-1} \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= \bar{u} - x^a y^b
\end{aligned}$$

Using the first two equations:

$$\begin{aligned}
\frac{p_x}{p_y} &= \frac{a x^{a-1} y^b}{b x^a y^{b-1}} \\
\frac{p_x}{p_y} &= \frac{a}{b} \frac{y}{x}
\end{aligned}$$

This last expression is again, the tangency condition.

b) The hicksian demand: now that we are convinced that the tangency condition summarizes the F.O.C. of the expenditure minimization problem, we can use it to find the hicksian demand. The two equations we want to combine are:

$$\frac{p_x}{p_y} = \frac{a}{b}\frac{y}{x}$$
$$x^a y^b = \bar{u}$$

Using equation (1)

$$\frac{a}{b}\frac{y}{x} = \frac{p_x}{p_y}$$
$$y = \frac{p_x}{p_y}\frac{b}{a}x$$

Plug-in this expression on equation (2):

$$x^a \left[\frac{p_x}{p_y}\frac{b}{a}x\right]^b = \bar{u}$$
$$x^{a+b} \left[\frac{p_x}{p_y}\frac{b}{a}\right]^b = \bar{u}$$
$$x^{a+b} = \bar{u}\left[\frac{p_y}{p_x}\frac{a}{b}\right]^b$$
$$h_x = \bar{u}^{\frac{1}{a+b}}\left[\frac{p_y}{p_x}\frac{a}{b}\right]^{\frac{b}{a+b}}$$

Using equation (1) re-written as: $y = \frac{p_x}{p_y}\frac{b}{a}x$

$$h_y = \frac{p_x}{p_y}\frac{b}{a}\bar{u}^{\frac{1}{a+b}}\left[\frac{p_y}{p_x}\frac{a}{b}\right]^{\frac{b}{a+b}}$$
$$h_y = \bar{u}^{\frac{1}{a+b}}\left[\frac{p_x}{p_y}\right]^{1-\frac{b}{a+b}}\left[\frac{b}{a}\right]^{1-\frac{b}{a+b}}$$
$$h_y = \bar{u}^{\frac{1}{a+b}}\left[\frac{p_x}{p_y}\frac{b}{a}\right]^{\frac{a}{a+b}}$$

- Expenditure function:

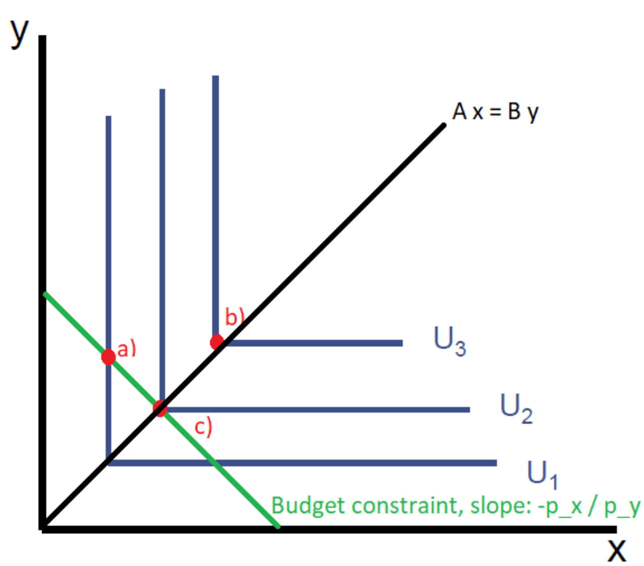
$$
\begin{aligned}
E(p_x, p_y, \bar{u}) &= p_x h_x + p_y h_y \\
&= p_x \bar{u}^{\frac{1}{a+b}} \left[\frac{p_y}{p_x}\frac{a}{b}\right]^{\frac{b}{a+b}} + p_y \bar{u}^{\frac{1}{a+b}} \left[\frac{p_x}{p_y}\frac{b}{a}\right]^{\frac{a}{a+b}} \\
&= \bar{u}^{\frac{1}{a+b}} p_x \left[\frac{p_y}{p_x}\right]^{\frac{b}{a+b}} \left[\frac{a}{b}\right]^{\frac{b}{a+b}} + \bar{u}^{\frac{1}{a+b}} p_y \left[\frac{p_x}{p_y}\right]^{\frac{a}{a+b}} \left[\frac{b}{a}\right]^{\frac{a}{a+b}} \\
&= \bar{u}^{\frac{1}{a+b}} p_x^{\frac{a}{a+b}} p_y^{\frac{b}{a+b}} \left[\frac{a}{b}\right]^{\frac{b}{a+b}} + \bar{u}^{\frac{1}{a+b}} p_y^{\frac{b}{a+b}} p_x^{\frac{a}{a+b}} \left[\frac{b}{a}\right]^{\frac{a}{a+b}} \\
E(p_x, p_y, \bar{u}) &= \bar{u}^{\frac{1}{a+b}} p_x^{\frac{a}{a+b}} p_y^{\frac{b}{a+b}} \left[\frac{a}{b}^{\frac{b}{a+b}} \frac{b}{a}^{\frac{a}{a+b}}\right]
\end{aligned}
$$

VIII Perfect Complements

- Utility Function: $U(x,y) = \min\{Ax, By\}$
- Utility Maximization problem:

$$\max_{x,y} \min\{Ax, By\}$$
$$\text{s.t. } p_x x + p_y y = I$$

- MRS: Remember that MRS is not defined in this case.
- Tangency condition: Since MRS is not defined, there is no tangency condition.
- Marshallian demand: I will rely on the following graph to derive the marshallian demand.



How to find x^* and y^*? The Utility Maximization problem has as a constraint that the budget constraint is satisfied with equality, that is: $p_x x^* + p_y y^* = I$. Thus, we know that the quantity of x and y that maximize the utility will also be on the budget line. Let's analyze three options:

a) This combination of x and y satisfies the budget constraint but it is not utility maximizing. Note that by buying more of x and less of y, the individual will spend all her money and get more utility.

b) This combination of x and y gives more utility to the consumer, but it is not feasible, that is, it does not satisfy the budget constraint.

c) This combination of x and y satisfies the budget constraint and it is utility maximizing. Note that this combination is on the edge of the indifference curve. This means that the combination is on the line that starts at the origin and joins all the edges of the IC.

So, we know that $A\,x^* = B\,y^*$ and $p_x x^* + p_y y^* = I$. Using this two equations:

$$A\,x = B\,y \Rightarrow y = \frac{A}{B}x$$

Using the budget constraint:

$$\begin{aligned}
p_x x + p_y \frac{A}{B}x &= I \\
x\left[p_x + p_y \frac{A}{B}\right] &= I \\
x\left[p_x \frac{B}{B} + p_y \frac{A}{B}\right] &= I \\
x\frac{1}{B}\left[Bp_x + Ap_y\right] &= I \\
x^* &= \frac{BI}{Bp_x + Ap_y}
\end{aligned}$$

Using the following equation: $y = \frac{A}{B}x$ we can get the marshallian demand for y:

$$\begin{aligned}
y^* &= \frac{A}{B}x^* \\
y^* &= \frac{A}{B}\frac{BI}{Bp_x + Ap_y} \\
y^* &= \frac{AI}{Bp_x + Ap_y}
\end{aligned}$$

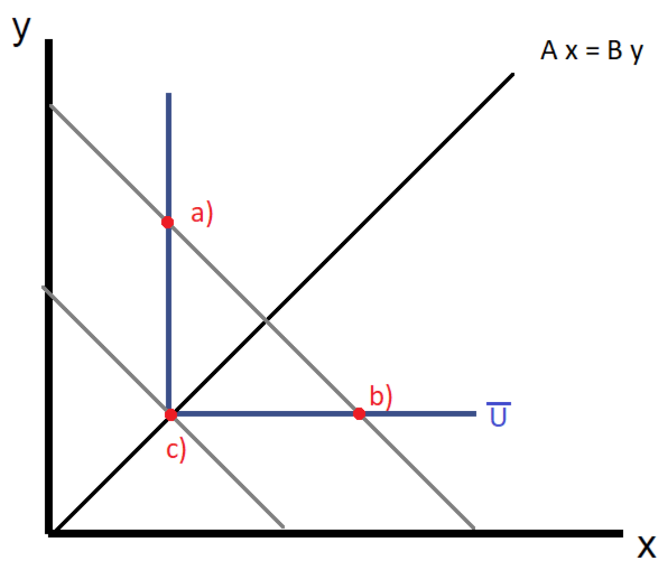
- Indirect Utility Function:

$$\begin{aligned}
V(p_X, p_y, I) &= \min\{Ax^*, By^*\} \\
V(p_X, p_y, I) &= \min\{A\frac{BI}{Bp_x + Ap_y}, B\frac{AI}{Bp_x + Ap_y}\} \\
V(p_X, p_y, I) &= \min\{\frac{ABI}{Bp_x + Ap_y}, \frac{ABI}{Bp_x + Ap_y}\} \\
V(p_X, p_y, I) &= \frac{ABI}{Bp_x + Ap_y}
\end{aligned}$$

- Expenditure Minimization Problem:

$$\min_{xy} p_x x + p_y y$$
$$\text{s.t. } \min\{Ax, By\} = \bar{u}$$

- Hicksian demand: I will rely on the following graph to derive the hicksian demand.



How to find x^* and y^*? The expenditure minimization problem has as a constraint that the individual gets an utility level equal to $\bar{u}$. Thus, we know that the quantity of x and y that minimize expenditure belong to this indiference curve. Let's analyze three options:

a) This combination of x and y gives the individual an utility level of $\bar{u}$, but it is not expenditure minimizing. In fact, less of y will give the consumer the same level of utility and reduce expenditure.

b) This combination of x and y gives the individual an utility level of $\bar{u}$, but it is not expenditure minimizing. In fact, less of x will give the consumer the same level of utility and reduce expenditure.

c) This combination of x and y gives the individual an utility level of $\bar{u}$, and it is expenditure minimizing. Note that this combination is on the edge of the indifference curve. This means that the combination is on the line that starts at the origin and joins all the edges of the IC.

So, we know that $A\,x^* = B\,y^*$ and $\min(Ax^*, By^*) = \bar{u}$. Then: $A\,x^* = \bar{u}$ and $B\,y^* = \bar{u}$ Now we solve for x and y:

$$h_x(p_x,p_y,\bar{u}) = \frac{\bar{u}}{A}$$
$$h_y(p_x,p_y,\bar{u}) = \frac{\bar{u}}{B}$$

- Expenditure function is:

$$E(p_x,p_y,\bar{u}) = p_x h_x + p_y h_y = p_x \frac{\bar{u}}{A} + p_y \frac{\bar{u}}{B}$$
$$E(p_x,p_y,\bar{u}) = \left[\frac{p_x}{A} + \frac{p_y}{B}\right]\bar{u}$$

IX Perfect Substitutes

- Utility Function: $U(x,y) = A\,x + B\,y$
- Utility Maximization problem:

$$\max_{x,y} A\,x + B\,y$$
$$\text{s.t. } p_x x + p_y y = I$$

- Lagrangian: $\mathcal{L}(x,y,\lambda) = A\,x + B\,y - \lambda\,[I - p_x x - p_y y]$
- $MRS = \frac{U_x}{U_y} = \frac{A}{B}$
- Tangency condition: $MRS = \frac{U_x}{U_y} = \frac{p_x}{p_y} \Rightarrow \frac{A}{B} = \frac{p_x}{p_y}$

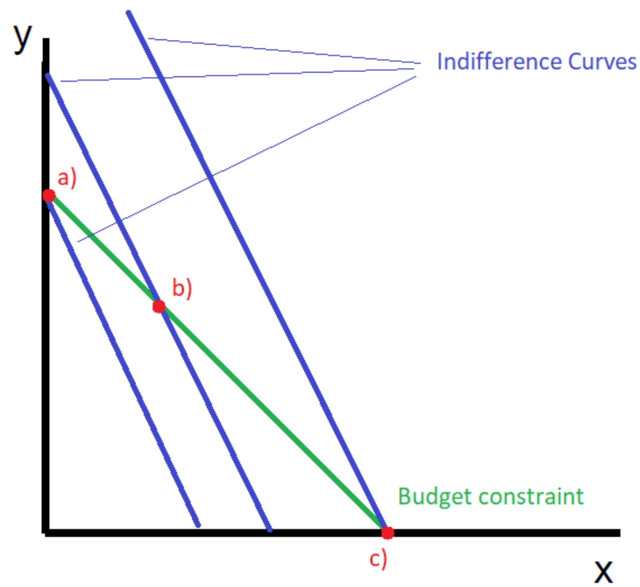
  Remark: The MRS is constant, so it is not diminishing with x. Thus we can easily run into problems if we try to use Lagrangian method or the tangency condition to find the optimal allocation. This is the reason why in this case we will use the graphical method.

- Marshallian demand: to derive the marshallian demand we consider two key facts:
  - the indifference curves are just straight lines and the slope of the indifference curve is: $-\frac{A}{B}$
  - the slope of the budget constraint is : $-\frac{p_x}{p_y}$

  The key thing is that the result will depend on what slope is steeper. We have 3 cases:

$$\text{CASE 1: } \frac{p_x}{p_y} < \frac{A}{B}$$
$$\text{CASE 2: } \frac{p_x}{p_y} > \frac{A}{B}$$
$$\text{CASE 3: } \frac{p_x}{p_y} = \frac{A}{B}$$

CASE 1: $\frac{p_x}{p_y} < \frac{A}{B}$



The green line is budget constraint. Blue lines are indifference curves. How to find x^* and y^*? The Utility Maximization problem has as a constraint that the budget constraint is satisfied with equality, that is: $p_x x^* + p_y y^* = I$. Thus, we know that the quantity of x and y that maximize the utility will also be on the budget line. Let's analyze three options:

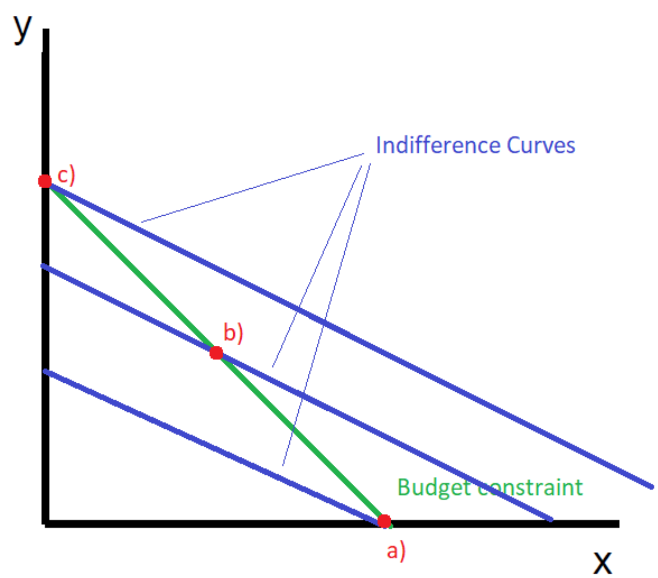
a) This combination of x and y satisfies the budget constraint but it is not utility maximizing. Note that by buying more of x and less of y, the individual will spend all her money and get more utility (will reach an indifference curve further to the origin).

b) This combination of x and y satisfies the budget constraint but it is not utility maximizing. Note that by buying more of x and less of y, the individual will spend all her money and get more utility (will reach an indifference curve further to the origin).

c) This combination of x and y satisfies the budget constraint and it is utility maximizing. Note that this combination is on the horizontal axis. This means that all the income is spent on good x.

   So, it must be that: $x^* = \frac{I}{p_x}$ and $y^* = 0$

CASE 2: $\frac{p_x}{p_y} > \frac{A}{B}$



The green line is budget constraint. Blue lines are indifference curves. How to find x^* and y^*? The Utility Maximization problem has as a constraint that the budget constraint is satisfied with equality, that is: $p_x x^* + p_y y^* = I$. Thus, we know that the quantity of x and y that maximize the utility will also be on the budget line. Let's analyze three options:

a) This combination of x and y satisfies the budget constraint but it is not utility maximizing. Note that by buying more of y and less of x, the individual will spend all her money and get more utility (will reach an indifference curve further to the origin).

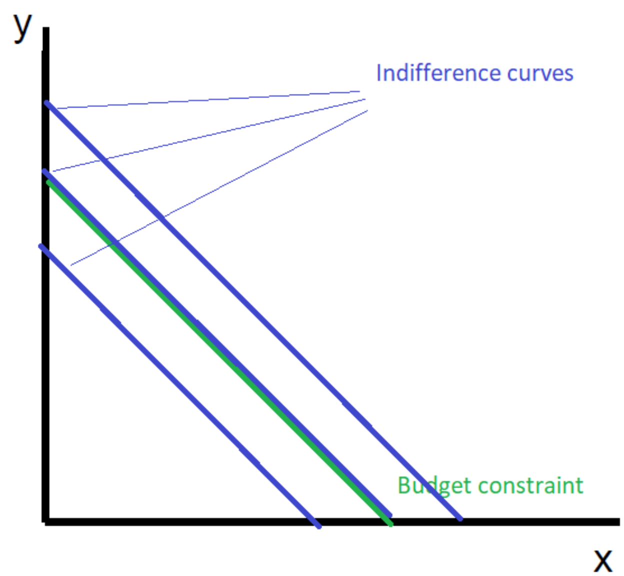
b) This combination of x and y satisfies the budget constraint but it is not utility maximizing. Note that by buying more of y and less of x, the individual will spend all her money and get more utility (will reach an indifference curve further to the origin).

c) This combination of x and y satisfies the budget constraint and it is utility maximizing. Note that this combination is on the vertical axis. This means that all the income is spent on good y.

   So, it must be that: $x^* = 0$ and $y^* = \frac{I}{p_y}$

CASE 3: $\frac{p_x}{p_y} = \frac{A}{B}$

In this case, the indifference curves and the budget constraint have the same slope. Thus, the indifference curve further from the origin the consumer can afford coincides in every point with the budget constraint, as shown in the following graph:



Any point on the budget line is optimal: $x^*(p_x,p_y,I) \geq 0$ and $y^*(p_x,p_y,I) \geq 0$ such that $p_x x + p_y y = I$

To summarize, the marshallian demand is defined as:

$$\begin{aligned}
&\text{CASE 1: } \frac{p_x}{p_y} < \frac{A}{B} &&\Rightarrow \quad x^*(p_x,p_y,I) = \frac{I}{p_x} \quad y^*(p_x,p_y,I) = 0\\
&\text{CASE 2: } \frac{p_x}{p_y} > \frac{A}{B} &&\Rightarrow \quad x^*(p_x,p_y,I) = 0 \quad y^*(p_x,p_y,I) = \frac{I}{p_y}\\
&\text{CASE 3: } \frac{p_x}{p_y} = \frac{A}{B} &&\Rightarrow \quad x^*(p_x,p_y,I) \geq 0, y^*(p_x,p_y,I) \geq 0 \text{ s.t. } p_x x + p_y y = I
\end{aligned}$$

- Indirect Utility Function:

$$
\begin{aligned}
&V(p_x, p_y, I) = Ax^* + By^* \\
\text{CASE 1: } \frac{p_x}{p_y} < \frac{A}{B} \quad \Rightarrow \quad & V(p_x, p_y, I) = A\frac{I}{p_x} \\
\text{CASE 2: } \frac{p_x}{p_y} > \frac{A}{B} \quad \Rightarrow \quad & V(p_x, p_y, I) = B\frac{I}{p_y} \\
\text{CASE 3: } \frac{p_x}{p_y} = \frac{A}{B} \quad \Rightarrow \quad & V(p_x, p_y, I) = Ax^* + By^* \text{ s.t.} \\
& x^*(p_x,p_y,I) \geq 0, y^*(p_x,p_y,I) \geq 0 \text{ and } p_x x + p_y y = I
\end{aligned}
$$

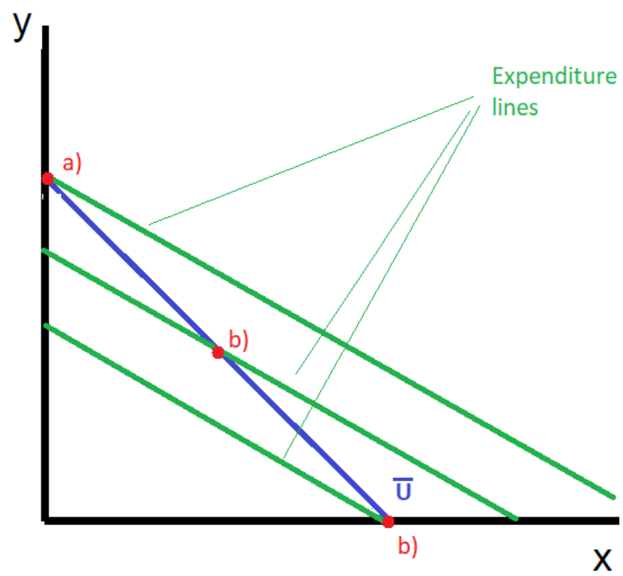
- Expenditure Minimization Problem:

$$
\min_{xy} p_x x + p_y y
$$
$$
\text{s.t. } Ax + By = \bar{u}
$$

- Hicksian demand: Now, utility level is fixed and what varies is expenditure. The slope of the budget constraint is $\frac{p_x}{p_y}$ and the MRS is $\frac{A}{B}$, then we should analyze the following cases:

$$
\begin{aligned}
\text{CASE 1: } & \frac{p_x}{p_y} < \frac{A}{B} \\
\text{CASE 2: } & \frac{p_x}{p_y} > \frac{A}{B} \\
\text{CASE 3: } & \frac{p_x}{p_y} = \frac{A}{B}
\end{aligned}
$$

CASE 1: $\frac{p_x}{p_y} < \frac{A}{B}$



The blue line is the indifference curve that provides a level of utility equal to $\bar{U}$. The green lines represent "expenditure lines", that is, lines with the slope $\frac{p_x}{p_y}$. Expenditure is minimize in a point that belongs to the indifference curve $\bar{U}$ and belongs to the expenditure line closest to the origin. Let's analyze three points:

a) This combination of x and y gives the individual an utility level of $\bar{u}$, but it is not expenditure minimizing. In fact, less of y and more x will give the consumer the same level of utility and reduce expenditure.

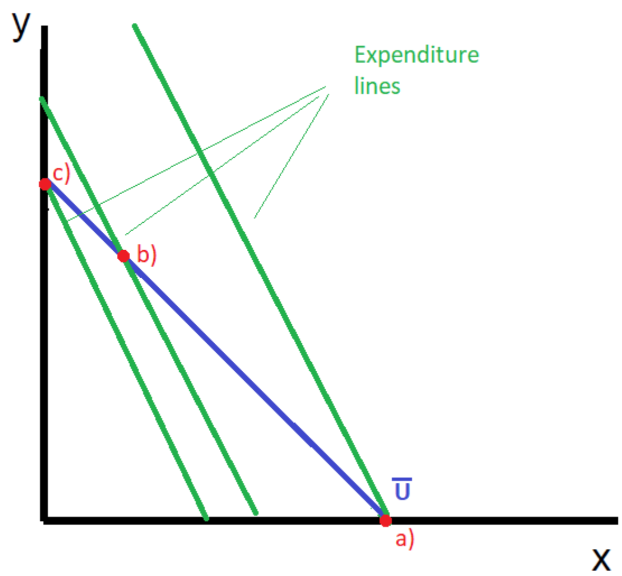
b) This combination of x and y gives the individual an utility level of $\bar{u}$, but it is not expenditure minimizing. In fact, less of y and more of x will give the consumer the same level of utility and reduce expenditure.

c) This combination of x and y gives the individual an utility level of $\bar{u}$, and it is expenditure minimizing. Note that this combination is no the horizontal axis. This implies: $h_y = 0$.

To derive the hicksian demand remember that the constraint of the expenditure minimization problem is: $\bar{U} = Ax + By$. Plug in $h_y = 0$ and get:

$$\bar{U} = Ax + B0$$
$$\bar{U} = Ax$$
$$h_x = \frac{\bar{U}}{A}$$

CASE 2: $\frac{p_x}{p_y} > \frac{A}{B}$

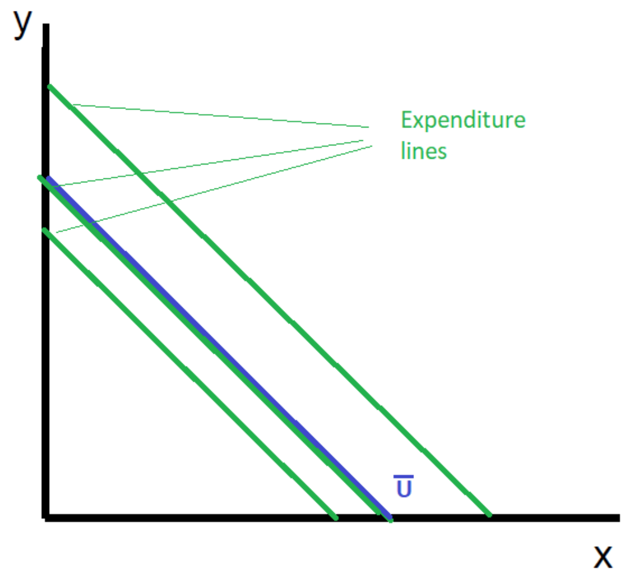


The blue line is the indifference curve that provides a level of utility equal to $\bar{U}$. The green lines represent "expenditure lines", that is, lines with the slope $\frac{p_x}{p_y}$. Expenditure is minimize in a point that belongs to the indifference curve $\bar{U}$ and belongs to the expenditure line closest to the origin. By similar arguments than the previous case, we can infer that expenditure is minimize at $c)$ and $h_x = 0$. So:

$$\bar{U} = A0 + By$$
$$\bar{U} = By$$
$$h_y = \frac{\bar{U}}{B}$$

CASE 3: $\frac{p_x}{p_y} = \frac{A}{B}$

In this case, the indifference curves and the expenditure lines have the same slope. Thus, the expenditure line closest to the the origin that gives the consumer an utility level $\bar{U}$ coincides in every point, as shown in the following graph. Any point on the indifference curve $\bar{U}$ is optimal : $h_x(p_x,p_y,\bar{U}) \geq 0$ and $h_y(p_x,p_y,\bar{U}) \geq 0$ such that $Ax + By = \bar{U}$



To summarize, the hicksian demand is defined as:

$$\text{CASE 1: } \frac{p_x}{p_y} < \frac{A}{B} \quad \Rightarrow \quad h_x(p_x,p_y,\bar{U}) = \frac{\bar{U}}{A} \quad y^*(h_x(p_x,p_y,\bar{U})) = 0$$

$$\text{CASE 2: } \frac{p_x}{p_y} > \frac{A}{B} \quad \Rightarrow \quad h_x(h_x(p_x,p_y,\bar{U})) = 0 \quad h_y(p_x,p_y,\bar{U}) = \frac{\bar{U}}{B}$$

$$\text{CASE 3: } \frac{p_x}{p_y} = \frac{A}{B} \quad \Rightarrow \quad h_x(p_x,p_y,\bar{U}) \geq 0 \text{ , } h_y(p_x,p_y,\bar{U}) \geq 0 \text{ s.t. } Ax + By = \bar{U}$$

- Expenditure Function:

$$E(p_x, p_y, \bar{U}) = p_x h_x + p_y h_y$$

$$\text{CASE 1: } \frac{p_x}{p_y} < \frac{A}{B} \quad \Rightarrow \quad E(p_x, p_y, \bar{U}) = p_x \frac{\bar{U}}{A}$$

$$\text{CASE 2: } \frac{p_x}{p_y} > \frac{A}{B} \quad \Rightarrow \quad E(p_x, p_y, \bar{U}) = p_y \frac{\bar{U}}{B}$$

$$\text{CASE 3: } \frac{p_x}{p_y} = \frac{A}{B} \quad \Rightarrow \quad E(p_x, p_y, \bar{U}) = p_x h_x + p_y h_y$$

$$\text{s.t. } h_x(p_x,p_y,\bar{U}) \geq 0 \text{ , } h_y(p_x,p_y,\bar{U}) \geq 0 \text{ and } Ax + By = \bar{U}$$

X Constant Elasticity of Substitution (CES) Utility

- Utility Function: $U(x,y) = \left(x^\delta + y^\delta\right)^{1/\delta}$
- Utility Maximization problem:

$$\max_{x,y} \left(x^\delta + y^\delta\right)^{1/\delta}$$
$$\text{s.t. } I = p_x x + p_y y$$

- MRS:

$$\begin{aligned} MRS = \frac{U_x}{U_y} &= \frac{\frac{1}{\delta}\left(x^\delta + y^\delta\right)^{1/\delta - 1} \delta x^{\delta-1}}{\frac{1}{\delta}\left(x^\delta + y^\delta\right)^{1/\delta - 1} \delta y^{\delta-1}} \\ &= \frac{x^{\delta-1}}{y^{\delta-1}} \end{aligned}$$

- Tangency condition: $\frac{x^{\delta-1}}{y^{\delta-1}} = \frac{p_x}{p_y}$
- Marshallian demand: Using the tangency condition

$$\begin{aligned} \frac{x^{\delta-1}}{y^{\delta-1}} &= \frac{p_x}{p_y} \\ \left(\frac{x}{y}\right)^{\delta-1} &= \frac{p_x}{p_y} \\ \left(\frac{y}{x}\right)^{1-\delta} &= \frac{p_x}{p_y} \\ y &= \left(\frac{p_x}{p_y}\right)^{\frac{1}{1-\delta}} x \end{aligned}$$

Now we plug in this expression for y as a function of x in the budget constraint:

$$\begin{aligned} I &= p_x x + p_y \left(\frac{p_x}{p_y}\right)^{\frac{1}{1-\delta}} x \\ I &= \left[p_x + p_y^{-\frac{\delta}{1-\delta}} p_x^{-\frac{1}{1-\delta}}\right] x \\ x^* &= \frac{I}{\left[p_x + p_y^{-\frac{\delta}{1-\delta}} p_x^{\frac{1}{1-\delta}}\right]} \end{aligned}$$

To make your life easier: multiply numerator and denominator by $p_x^{-\frac{1}{1-\delta}}$. Then, we can rewrite x^*:

$$x^* = \frac{p_x^{-\frac{1}{1-\delta}} I}{\left[p_x^{-\frac{1}{1-\delta}} p_x + p_y^{-\frac{\delta}{1-\delta}} p_x^{-\frac{1}{1-\delta}} p_x^{\frac{1}{1-\delta}}\right]}$$

Note that: $p_x^{-\frac{1}{1-\delta}} p_x = p_x^{-\frac{1}{1-\delta}+1} = p_x^{-\frac{1}{1-\delta}+\frac{1-\delta}{1-\delta}} = p_x^{-\frac{\delta}{1-\delta}}$. Then:

$$x^* = \frac{p_x^{-\frac{1}{1-\delta}} I}{\left[p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}\right]}$$

Now we can plug in this equation in $y = \left(\frac{p_x}{p_y}\right)^{-\frac{1}{1-\delta}} x$ and we get:

$$y^* = \left(\frac{p_x}{p_y}\right)^{\frac{1}{1-\delta}} \frac{p_x^{-\frac{1}{1-\delta}} I}{\left[p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}\right]}$$

$$y^* = \frac{p_y^{-\frac{1}{1-\delta}} I}{\left[p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}\right]}$$

- Indirect Utility Function:

$$V(p_x, p_y, I) = U(x^*, y^*) = \left(x^{*\delta} + y^{*\delta}\right)^{1/\delta}$$

$$V(p_x, p_y, I) = \left[\left(\frac{p_x^{-\frac{1}{1-\delta}} I}{p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}}\right)^{\delta} + \left(\frac{p_y^{-\frac{1}{1-\delta}} I}{p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}}\right)^{\delta}\right]^{1/\delta}$$

$$V(p_x, p_y, I) = \left[I^{\delta}\frac{p_x^{-\frac{\delta}{1-\delta}}}{\left(p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}\right)^{\delta}} + I^{\delta}\frac{p_y^{-\frac{\delta}{1-\delta}}}{\left(p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}\right)^{\delta}}\right]^{1/\delta}$$

$$V(p_x, p_y, I) = I\left[\left(p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}\right)\frac{1}{\left(p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}\right)^{\delta}}\right]^{1/\delta}$$

$$V(p_x, p_y, I) = I\left[\left(p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}\right)^{1-\delta}\right]^{1/\delta}$$

$$V(p_x, p_y, I) = I\left(p_x^{-\frac{\delta}{1-\delta}} + p_y^{-\frac{\delta}{1-\delta}}\right)^{\frac{1-\delta}{\delta}}$$

Note that: $\frac{\delta}{1-\delta} = -\frac{\delta}{\delta-1} = -r$ and $\frac{1-\delta}{\delta} = \frac{1}{r}$. Then:

$$V\left(p_x, p_y, I\right) = I\left(p_x^r + p_y^r\right)^{-1/r}$$

- Hicksian demand:
- Expenditure Function:

XI Extra practice II - Consumer's theory

1. Bill has a weekly endowment of 200 dollars that he spends on buying movies (M) and records (R). The price of each movie is 15 dollars and the price of each record is 10 dollars. Bill's utility is given by:
$$U\left(M,R\right) = 3M^{2/3} + 2R^{1/3}$$
   (a) Find Bill's Budget constraint.
   (b) Set up the Lagrangian and find his optimal consumption of movies and records (assume consumption levels don't need to be whole numbers).
   (c) What is his new consumption if the price of movies becomes 30 dollars?

2. Mary is the only person in an island, and she has an endowment of 8 units of X and 3 units of Y. Her utility function is:
$$U\left(X,Y\right) = 2\ln\left(X\right) + 7\ln\left(Y\right)$$
   (a) Find the marginal utility of X and of Y.
   (b) Find the marginal rate of substitution of good X for good Y when Mary consumes all her endowment. Interpret your findings.

3. The following utility function is known as CES (constant elasticity of substitution) function:
$$U(x,y) = (\alpha x^r + \beta y^r)^{1/r}, \quad where\ \alpha > 0\ \beta > 0$$
   (a) Define what it means for a function to be a *homogeneous of degree 1*. Is this function *homogeneous of degree 1*? Show your work.
   (b) How does the $MRS_{x,y}$ depend on the ratio x/y? Specifically, show that the $MRS_{x,y}$ is strictly decreasing in the ratio x/y for all values $r < 1$, increasing in the ratio x/y for all values $r > 1$ and constant for $r = 1$.
   (c) Show that if $x = y$, the MRS of this function depends only on the relatively sizes of α and β.

4. Suppose that an individual with income I cares about two goods, X and Y. The price of the two goods is p_X and p_Y. The individual has the following utility function:
$$U(X,Y) = X(4+Y) \tag{1}$$
   a) Find the Marshallian (uncompensated) demand for X and Y.

b) Find the indirect utility function.

5. Joseph likes roses (R) and tulips (T) equally, and views them as perfect substitutes in proportion 1 to 1. The price of a rose is \$4, the price of a tulip is \$2, and Joseph has \$20 to spend on flowers.

   a) How much of each flower will Joseph buy? (Hint: the first order conditions will not help; think about what you would do in this situation.)

   b) Now, suppose that the price of a tulip rises to \$10. How does the consumption of Joseph change?

   c) What are the Joseph's demands for roses and tulips as a function of prices and income $\{p_R,p_T,I\}$? You will have three cases depending on the relationship between p_R and p_T.

   d) How much should Joseph's income increase to compensate for the rise in the price of roses? (Hint: use the indirect utility function before and after the change)

6. Britney is very fashionable. When she buys a new dress (D), she also needs to buy a hat (H) to match that dress and vice-versa. So, she views the two goods as perfect complements. The price of a dress is \$15, the price of a hat is \$10, and she has \$85 to spend.

   a) Write down a utility function that represents Britney's preferences over dresses and hats.

   b) How many dresses and hats is she going to consume? (Hint: the first order conditions will not help; draw the budget constraint and the indifference curves and look at the highest one that intersects the budget constraint)

   c) What is the exact value of Britney's indirect utility?

7. The Army pays soldiers a salary of \$2000 per month. Suppose that all soldiers have the same utility function $U = \sqrt{HY}$, where H is the quantity of housing services demanded per month (i.e., the number of square feet of space) and Y is a composite of all other goods. The price of housing is p_H and the price of Y is p_Y. The Marshallian demand functions are $H^* = \frac{I}{2p_H}$ and $Y^* = \frac{I}{2p_Y}$. Soldiers have no income other than their Army salary.

   (a) Find the indirect utility function and the expenditure function for the typical soldier.

   (b) Suppose $p_H = 1$, $p_Y = 4$ and $I = 20$. What is the soldier's utility? How much housing does he purchase?

   (c) In addition to pay, the government provides a housing allowance that reimburses each soldier for 50 percent of their housing expenditures. How does the housing allowance affect the soldier's budget constraint? What is the utility level with the allowance? How much housing does the soldier purchase? How much does a soldier receive in housing subsidy?

(d) Suppose that the Army abolishes the housing allowance in part c. What salary would be required for the soldiers to be as well off as with the allowance?

8. Suppose United Airlines is deciding on purchasing engines from General Electric (GE) and Rolls-Royce (RR) for its new Boeing 787 fleet with budget I=\$3 billion. The airline company has utility function $U(g,r) = g^2 r$, where g are the engines produced by GE and r are the engines from RR. p_g and p_r are prices for each GE and RR engine respectively. The Marshallian demand functions are $g^* = \frac{2I}{3p_g}$ and $r^* = \frac{I}{3p_r}$.

   (a) Find the indirect utility function for United Airlines.

   (b) Suppose $p_g = \$25$ million and $p_r = \$20$ million. What is United's maximized utility? How many GE engines do they purchase?

   (c) Now suppose the U.S. Department of Commerce is trying to promote American engine manufacturers. Thus United Airlines will be given a 20% subsidy on price when purchasing from GE, and levied a 25% tax on price when purchasing from RR. How does this policy affect United's budget constraint? What is the utility level with this policy? How many GE engines are purchased? What is the total tax that United has to pay for purchasing the RR engines?

9. Fred won a lottery that pays him \$500 a month. While that's not a lot of cash, that's fine–Fred only consumes garlic and anchovies. While his breath may be rough, these two goods give him utility according to the function, $U(g,a) = g^{.5} a^{.5}$, where g represents the amount of garlic he consumes and a represents the amount of anchovies he consumes. The price of garlic is \$1 and the price of anchovies is \$2. Fred doesn't make any money other than his lottery check of \$500/month.

   (a) Find Fred's Marshallian (uncompensated) demands for anchovies and garlic.

   (b) Calculate the value of utility at the optimum for Fred.

   (c) In addition to pay, Fred's generous grandmother decides to give him an allowance. She likes fish too and is willing to reimburse 50% of Fred's anchovy expenditures. How does this allowance affect Fred's budget constraint? How many anchovies does Fred consume? What is the utility level with the allowance? How much of a subsidy is Fred's grandmother paying him?

XII Duality

XII.i Hicksian Demand Directly from Marshallian Demand

Suppose we have $U(x,y) = x^{\frac{1}{4}}y^{\frac{3}{4}}$. Thus, the utility maximization problem is:

$$\max_{x,y} x^{\frac{1}{4}}y^{\frac{3}{4}}$$
$$\text{s.t. } p_x x + p_y y = I$$

and the Marshallian demands are:

$$x^*(p_x,p_y,I) = \frac{1}{4}\frac{I}{p_x}$$
$$y^*(p_x,p_y,I) = \frac{3}{4}\frac{I}{p_y}$$

How can we get the hicksian demand without solving the expenditure minimization problem?

STEP 1 Get the indirect utility function

$$\begin{aligned}
V(p_x,p_y,I) &= x^{*\,1/4}y^{*\,3/4} \\
V(p_x,p_y,I) &= \left[\frac{1}{4}\frac{I}{p_x}\right]^{\frac{1}{4}}\left[\frac{3}{4}\frac{I}{p_y}\right]^{\frac{3}{4}} \\
&= I\left[\frac{1}{4}\right]^{\frac{1}{4}}\left[\frac{1}{p_x}\right]^{\frac{1}{4}}\left[\frac{3}{4}\right]^{\frac{3}{4}}\left[\frac{1}{p_y}\right]^{\frac{3}{4}} \\
&= I\frac{1}{4}3^{\frac{3}{4}}\left[\frac{1}{p_x}\right]^{\frac{1}{4}}\left[\frac{1}{p_y}\right]^{\frac{3}{4}}
\end{aligned}$$

STEP 2 Get the expenditure function from the indirect utility function by replacing: $\bar{U} = V(p_x,p_y,I)$ and $E(p_x,p_y,\bar{U}) = I$

$$\begin{aligned}
V(p_x,p_y,I) &= I\frac{1}{4}3^{\frac{3}{4}}\left[\frac{1}{p_x}\right]^{\frac{1}{4}}\left[\frac{1}{p_y}\right]^{\frac{3}{4}} \\
\bar{U} &= E(p_x,p_y,\bar{U})\frac{1}{4}3^{\frac{3}{4}}\left[\frac{1}{p_x}\right]^{\frac{1}{4}}\left[\frac{1}{p_y}\right]^{\frac{3}{4}} \\
E(p_x,p_y,\bar{U}) &= \bar{U}4\left(\frac{1}{3}\right)^{\frac{3}{4}}p_x^{\frac{1}{4}}p_y^{\frac{3}{4}}
\end{aligned}$$

STEP 3 Use the duality result: $g_x(p_x, p_y, E(p_x,p_y,\bar{U}) = h_x(p_x,p_y,\bar{U})$:

$$h_x(p_x,p_y,\bar{U}) = g_x(p_x,p_y,E(p_x,p_y,\bar{U})) = \frac{1}{4}\frac{E(p_x,p_y,\bar{U})}{p_x}$$
$$h_x(p_x,p_y,\bar{U}) = \frac{1}{4}\bar{U}4\ \left(\frac{1}{3}\right)^{\frac{3}{4}} p_x^{\frac{1}{4}} p_y^{\frac{3}{4}} \frac{1}{p_x}$$
$$= \bar{U}\left(\frac{1}{3}\right)^{\frac{3}{4}} \left(\frac{p_y}{p_x}\right)^{\frac{3}{4}}$$

Note: you can verify this result replacing $a = 1/4$, $b = 3/4$ and $a + b = 1$ in the section "Cobb-Douglas Utility".
Similarly for good y: $g_y(p_x, p_y, E(p_x,p_y,\bar{U}) = h_y(p_x,p_y,\bar{U})$:

$$h_y(p_x,p_y,\bar{U}) = g_y(p_x,p_y,E(p_x,p_y,\bar{U}))$$
$$= \frac{3}{4}\bar{U}4\ \left(\frac{1}{3}\right)^{\frac{3}{4}} p_x^{\frac{1}{4}} p_y^{\frac{3}{4}} \frac{1}{p_y}$$
$$= \bar{U}\left(\frac{1}{3}\right)^{\frac{1}{4}} \left(\frac{p_x}{p_y}\right)^{\frac{1}{4}}$$

Thus, the results that we have used in this section are:

$$\bar{U} = V(p_x, p_y, I) \text{ and } E(p_x, p_y, \bar{U}) = I$$

Marshallian to Hicksian $g_x(p_x, p_y, E(p_x,p_y,\bar{U}) = h_x(p_x,p_y,\bar{U})$ and $g_y(p_x, p_y, E(p_x,p_y,\bar{U}) = h_y(p_x,p_y,\bar{U})$
Hicksian to Marshallian $h_x(p_x,p_y,V(p_x, p_y, I)) = g_x(p_x, p_y,I)$ and $h_y(p_x,p_y,V(p_x, p_y, I)) = g_y(p_x, p_y,I)$

XIII Income and Substitution Effect

Suppose we have $U(x,y) = x^{\frac{1}{4}}y^{\frac{3}{4}}$. Thus, the utility maximization problem is:

$$\max_{x,y} x^{\frac{1}{4}}y^{\frac{3}{4}}$$
$$\text{s.t. } p_x x + p_y y = I$$

the Marshallian demands are:

$$g_x(p_x,p_y,I) = \frac{1}{4}\frac{I}{p_x}$$
$$g_y(p_x,p_y,I) = \frac{3}{4}\frac{I}{p_y}$$

and the Hicksian demands are:

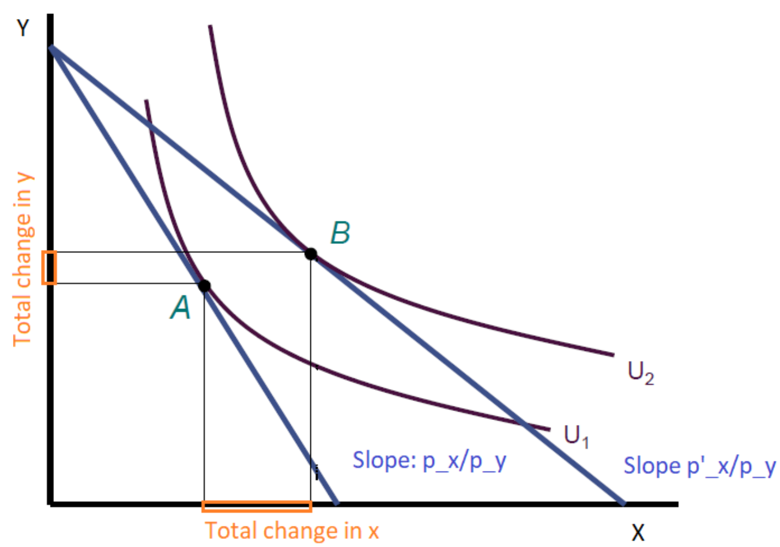
$$h_x(p_x,p_y,\bar{U}) = \bar{U}\left(\frac{1}{3}\right)^{\frac{3}{4}}\left(\frac{p_y}{p_x}\right)^{\frac{3}{4}}$$
$$h_y(p_x,p_y,\bar{U}) = \bar{U}\left(\frac{1}{3}\right)^{\frac{1}{4}}\left(\frac{p_x}{p_y}\right)^{\frac{1}{4}}$$

XIII.i "Big changes"

Now let's consider a change in prices. Suppose the initial price of x is p_x but x now become cheaper so that the new price of x is $p_x^{'} < p_x$. We are going to draw a series of graphs to try to understand the effects of this price change on the consumption of both goods, x and y. This is the procedure we will follow when considering big changes in prices. If the changes are small (e.g. 1%) we will use the Slutsky Equation, see next section for this case.

Total effect

The difference between the new consumption bundle and the old consumption bundle is called the total effect. The next graph shows that the budget constraint rotates to the right. The new consumption bundle is marked by point B.



$$\text{Total effect}_x = g_x(p_x^{'},p_y,I) - g_x(p_x,p_y,I)$$
$$\text{Total effect}_y = g_y(p_x^{'},p_y,I) - g_y(p_x,p_y,I)$$

In our example:

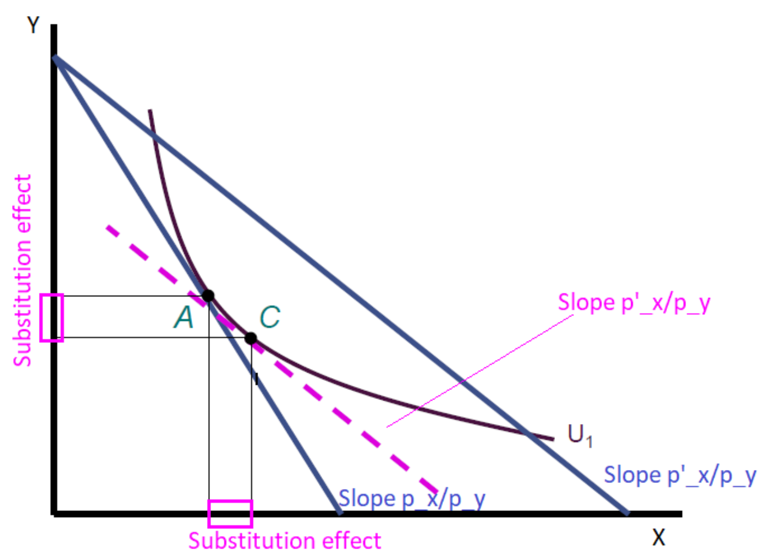
$$\begin{aligned}\text{Total effect}_x &= \frac{1}{4}\frac{I}{p_x^{'}} - \frac{1}{4}\frac{I}{p_x} \\ &= \frac{I}{4}\frac{p_x - p_x^{'}}{p_x^{'} p_x} \\ \text{Total effect}_y &= \frac{3}{4}\frac{I}{p_y} - \frac{3}{4}\frac{I}{p_y} = 0\end{aligned}$$

Remember that $p_x^{'} < p_x$, thus: $p_x - p_x^{'} > 0$ and we can see that the total effect is positive. In this particular effect, the total change in y equals 0. Note that the graph pictures the general case were both x and y depend directly on p_x and change as a result of the price change.

Remark: *A change in price causes two effects known as the substitution and income effect. As the price of x decreases the consumer becomes richer, this is known as the income effect. Also as the price of x decreases x becomes relatively cheaper than y, this is known as the substitution effect.*

Substitution effect

The substitution effect measures how much the consumer would substitute x for y given the price change holding utility constant. How do we compute demand while holding utility constant? The substitution effect is the movement from point A to point C.



Note that point A is the initial optimal consumption bundle. To get point C, we shift the new budget constraint (which slope is the new price ratio) towards the indifference curve with the initial level of utility. That is, towards the indifference curve where point A is located. Note that A and C belong to the same indifference curve, thus we are holding constant the utility level. The Hicksian demand functions do precisely this:

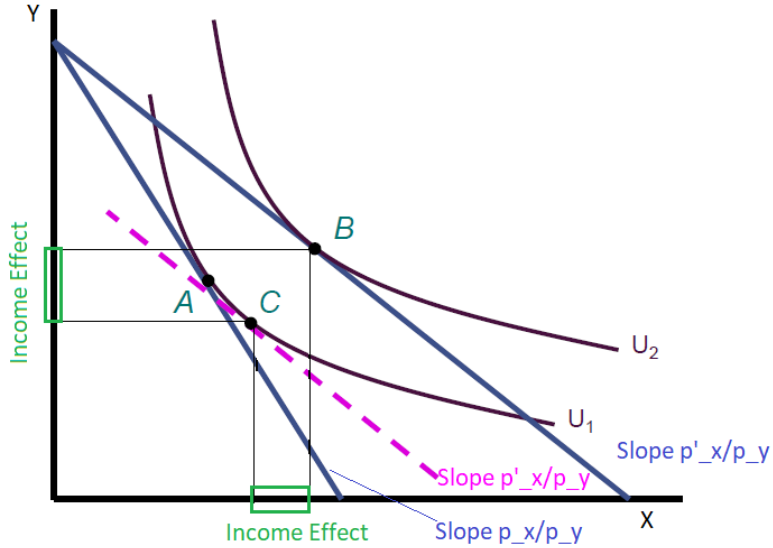
$$\text{Substitution effect}_x = h_x(p_x^{'}, p_y, \bar{U}) - h_x(p_x, p_y, \bar{U})$$
$$\text{Substitution effect}_y = h_y(p_x^{'}, p_y, \bar{U}) - h_y(p_x, p_y, \bar{U})$$

In our example:

$$\begin{aligned}
\text{Substitution effect}_x &= h_x(p_x^{'}, p_y, \bar{U}) - h_x(p_x, p_y, \bar{U}) \\
&= \bar{U}\left(\frac{1}{3}\right)^{\frac{3}{4}}\left(\frac{p_y}{p_x^{'}}\right)^{\frac{3}{4}} - \bar{U}\left(\frac{1}{3}\right)^{\frac{3}{4}}\left(\frac{p_y}{p_x}\right)^{\frac{3}{4}} \\
&= \bar{U}\left(\frac{p_y}{3}\right)^{\frac{3}{4}}\left[\frac{1}{p_x^{'}} - \frac{1}{p_x}\right]^{\frac{3}{4}} \\
&= \bar{U}\left(\frac{p_y}{3}\right)^{\frac{3}{4}}\left[\frac{p_x - p_x^{'}}{p_x^{'} p_x}\right]^{\frac{3}{4}} \\
&= \bar{U}\left(\frac{p_y}{3}\right)^{\frac{3}{4}}\left[\frac{p_x - p_x^{'}}{p_x^{'} p_x}\right]^{\frac{3}{4}} > 0 \text{ since } p_x^{'} < p_x \\
\text{Substitution effect}_y &= h_y(p_x^{'}, p_y, \bar{U}) - h_y(p_x, p_y, \bar{U}) \\
&= \bar{U}\left(\frac{1}{3}\right)^{\frac{1}{4}}\left(\frac{p_x^{'}}{p_y}\right)^{\frac{1}{4}} - \bar{U}\left(\frac{1}{3}\right)^{\frac{1}{4}}\left(\frac{p_x}{p_y}\right)^{\frac{1}{4}} \\
&= \bar{U}\left(\frac{1}{3\,p_y}\right)^{\frac{1}{4}}\left[p_x^{'} - p_x\right] \\
&= \bar{U}\left(\frac{1}{3\,p_y}\right)^{\frac{1}{4}}\left[p_x^{'} - p_x\right] < 0 \text{ since } p_x^{'} < p_x
\end{aligned}$$

Income effect

The income effect measures the change in demand given an increase in income holding the price ratio constant. How do we compute demand holding the price ratio constant? The income effect is the movement from point C to point B. Note that point B is determined by the tangency of the new budget constraint (slope $\frac{p_x^{'}}{p_y}$) to the indifference curve U_2. The point C is determined by the tangency of the auxiliary budget constraint that has slope $\frac{p_x^{'}}{p_y}$ and is tangent to the indifference curve U_1. That is, we are holding constant the price ratio, by comparing two points were the slope is equal to $\frac{p_x^{'}}{p_y}$.



$$\text{Income effect}_x = \text{Total effect}_x - \text{Substitution effect}_x$$

$$\text{Income effect}_x = g_x(p_x^{'},p_y,I) - g_x(p_x,p_y,I) - \left[h_x(p_x^{'},p_y,\bar{U}) - h_x(p_x,p_y,\bar{U})\right]$$

$$\text{Income effect}_y = \text{Total effect}_y - \text{Substitution effect}_y$$

$$\text{Income effect}_y = g_y(p_x^{'},p_y,I) - g_y(p_x,p_y,I) - \left[h_y(p_x^{'},p_y,\bar{U}) - h_y(p_x,p_y,\bar{U})\right]$$

In our example:

$$\begin{aligned}
\text{Income effect}_x &= \text{Total effect}_x - \text{Substitution effect}_x \\
&= g_x(p_x^{'},p_y,I) - g_x(p_x,p_y,I) - \left[h_x(p_x^{'},p_y,\bar{U}) - h_x(p_x,p_y,\bar{U})\right] \\
&= \frac{I}{4}\frac{p_x - p_x^{'}}{p_x^{'}p_x} - \bar{U}\left(\frac{p_y}{3}\right)^{\frac{3}{4}}\left[\frac{p_x - p_x^{'}}{p_x^{'}p_x}\right]^{\frac{3}{4}} \\
\text{Income effect}_y &= \text{Total effect}_y - \text{Substitution effect}_y \\
&= g_y(p_x^{'},p_y,I) - g_y(p_x,p_y,I) - \left[h_y(p_x^{'},p_y,\bar{U}) - h_y(p_x,p_y,\bar{U})\right] \\
&= 0 - \bar{U}\left(\frac{1}{3\,p_y}\right)^{\frac{1}{4}}\left[p_x^{'} - p_x\right] \\
&= -\bar{U}\left(\frac{1}{3\,p_y}\right)^{\frac{1}{4}}\left[p_x^{'} - p_x\right]
\end{aligned}$$

XIII.ii “Small changes” - Slutsky Equation

First, I will show the derivation of the Slutsky Equation. Next I will present a step-by-step example on how to use the derived results.

Derivation: own-price Slutsky Equation

Start from the duality result:

$$g_x(p_x, p_y, E(p_x,p_y,\bar{U})) = h_x(p_x,p_y,\bar{U})$$

Let's take the partial derivative with respect to p_x and use the fact : $I = E(p_x,p_y,\bar{U})$

$$\begin{aligned}
\frac{\partial g_x(p_x,p_y,I)}{\partial p_x} + \frac{\partial g_x(p_x,p_y,I)}{\partial I}\frac{\partial E(p_x,p_y,\bar{U})}{\partial p_x} &= \frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_x} \\
\text{By Shephard's Lemma:} \quad \frac{\partial g_x(p_x,p_y,I)}{\partial p_x} + \frac{\partial g_x(p_x,p_y,I)}{\partial I} h_x(p_x,p_y,\bar{U}) &= \frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_x} \\
\text{By duality:} \quad \frac{\partial g_x(p_x,p_y,I)}{\partial p_x} + \frac{\partial g_x(p_x,p_y,I)}{\partial I} g_x(p_x,p_y,I) &= \frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_x}
\end{aligned}$$

Re-arrange terms:

$$\underbrace{\frac{\partial g_x(p_x,p_y,I)}{\partial p_x}}_{\text{Total Effect}} = \underbrace{\frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_x}\,|_{\bar{U}=V}}_{\text{Substitution Effect}} \underbrace{- \frac{\partial g_x(p_x,p_y,I)}{\partial I} g_x(p_x,p_y,I)}_{\text{Income Effect}}$$

Note: *We evaluate the derivative of the Hicksian demand at the indirect utility function to make the equation consistent and to express everything in terms of prices and income.*

Derivation: cross-price Slutsky Equation

Start from the duality result:

$$g_x(p_x, p_y, E(p_x,p_y,\bar{U})) = h_x(p_x,p_y,\bar{U})$$

Let's take the partial derivative with respect to p_y and use the fact : $I = E(p_x,p_y,\bar{U})$

$$\begin{aligned}
\frac{\partial g_x(p_x,p_y,I)}{\partial p_y} + \frac{\partial g_x(p_x,p_y,I)}{\partial I}\frac{\partial E(p_x,p_y,\bar{U})}{\partial p_y} &= \frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_y} \\
\text{By Shephard's Lemma:} \quad \frac{\partial g_x(p_x,p_y,I)}{\partial p_y} + \frac{\partial g_x(p_x,p_y,I)}{\partial I} h_y(p_x,p_y,\bar{U}) &= \frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_y} \\
\text{By duality:} \quad \frac{\partial g_x(p_x,p_y,I)}{\partial p_y} + \frac{\partial g_x(p_x,p_y,I)}{\partial I} g_y(p_x,p_y,I) &= \frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_x y}
\end{aligned}$$

Re-arrange terms:

$$\underbrace{\frac{\partial g_x(p_x,p_y,I)}{\partial p_y}}_{\text{Total Effect}} = \underbrace{\frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_y}\,|_{\bar{U}=V}}_{\text{Substitution Effect}} \underbrace{-\frac{\partial g_x(p_x,p_y,I)}{\partial I}g_y(p_x,p_y,I)}_{\text{Income Effect}}$$

Application:

Suppose a consumer has the following utility function: $u(x,y) = x^2 + xy$. Therefore[2]

$$\text{Marshallian demands: } g_x(p_x,p_y,I) = \frac{I}{2}\frac{1}{(p_x-p_y)} \qquad g_y(p_x,p_y,I) = \frac{I}{2}\frac{(p_x-2p_y)}{(p_x-p_y)p_y}$$

$$\text{Indirect Utility Function: } V(p_x,p_y,I) = \frac{I^2}{4}\frac{1}{(p_x-p_y)p_y}$$

$$\text{Hicksian demands: } h_x(p_x,p_y,\bar{U}) = \bar{U}^{1/2}\frac{p_y^{1/2}}{(p_x-p_y)^{1/2}} \qquad h_y(p_x,p_y,\bar{U}) = \bar{U}^{1/2}\frac{p_x-2p_y}{[(p_x-p_y)p_y]^{1/2}}$$

$$\text{Expenditure function: } E(p_x,p_y,\bar{U}) = 2\bar{U}^{1/2}[(p_x-p_y)p_y]^{1/2}$$

Note that for $g_x > 0$ we need to impose $p_x > p_y$.

1. How does a small change in p_x affect the demand for x?

   i) What is the total effect? (easiest to compute it directly)

   $$\text{Total effect}_x = \frac{\partial g_x(p_x,p_y,I)}{\partial p_x} = -\frac{I}{2}\frac{1}{(p_x-p_y)^2}$$

   ii) What is the income effect? (easiest to compute it directly)

   $$\begin{aligned}\text{Income effect}_x &= -\frac{\partial g_x(p_x,p_y,I)}{\partial I}g_x(p_x,p_y,I)\\ &= -\frac{1}{2}\frac{1}{(p_x-p_y)}\frac{I}{2}\frac{1}{(p_x-p_y)}\\ &= -\frac{I}{4}\frac{1}{(p_x-p_y)^2}\end{aligned}$$

   iii) What is the substitution effect? (two ways to compute this: directly or use the Slutsky equation)
   Direct way:

[2] The detailed derivation is left as an exercise

$$
\begin{aligned}
\text{Substitution effect}_x &= \frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_x} \Big|_{\bar{U}=V} \\
&= -\frac{1}{2}\frac{p_y}{(p_x-p_y)^2}\sqrt{\bar{U}\frac{(p_x-p_y)}{p_y}} \\
&= -\frac{1}{2}\sqrt{\frac{\bar{U}p_y}{(p_x-p_y)^3}} \\
&= -\frac{1}{2}\sqrt{\frac{\frac{I^2}{4}\frac{1}{(p_x-p_y)p_y}p_y}{(p_x-p_y)^3}} \\
&= -\frac{1}{2}\sqrt{\frac{I^2}{4}\frac{1}{(p_x-p_y)^4}} \\
&= -\frac{I}{4}\frac{1}{(p_x-p_y)^2}
\end{aligned}
$$

Using Slutsky equation:

$$
\begin{aligned}
\text{Total effect}_x &= \text{Substitution effect}_x + \text{Income effect}_x \\
\text{Substitution effect}_x &= \text{Total effect}_x - \text{Income effect}_x \\
&= \frac{\partial g_x(p_x,p_y,I)}{\partial p_x} - \left(-\frac{\partial g_x(p_x,p_y,I)}{\partial I}g_x(p_x,p_y,I)\right) \\
&= -\frac{I}{2}\frac{1}{(p_x-p_y)^2} - \left(-\frac{I}{4}\frac{1}{(p_x-p_y)^2}\right) \\
&= -\frac{I}{2}\frac{1}{(p_x-p_y)^2} + \frac{I}{4}\frac{1}{(p_x-p_y)^2} \\
&= -\frac{I}{4}\frac{1}{(p_x-p_y)^2}
\end{aligned}
$$

2. How does a small change in p_y affect the demand for x?

   i) What is the total effect? (easiest to compute it directly)

$$
\text{Total effect}_x = \frac{\partial g_x(p_x,p_y,I)}{\partial p_y} = \frac{I}{2}\frac{1}{(p_x-p_y)^2}
$$

ii) What is the income effect? (easiest to compute it directly)

$$\begin{aligned}
\text{Income effect}_x &= -\frac{\partial g_x(p_x, p_y, I)}{\partial I} g_y(p_x,p_y,I) \\
&= -\frac{1}{2}\frac{1}{(p_x - p_y)}\frac{I}{2}\frac{(p_x - 2p_y)}{(p_x - p_y)p_y} \\
&= -\frac{I}{4}\frac{(p_x - 2p_y)}{(p_x - p_y)^2 p_y}
\end{aligned}$$

iii) What is the substitution effect? (two ways to compute this: directly or use the Slutsky equation)

Direct way:

$$\begin{aligned}
\text{Substitution effect}_x &= \frac{\partial h_x(p_x,p_y,\bar{U})}{\partial p_y} \,|_{\bar{U}=V} \\
&= \bar{U}^{1/2}\left[\frac{1}{2}\sqrt{\frac{1}{p_y(p_x - p_y)}} + \frac{1}{2}\sqrt{\frac{p_y}{(p_x - p_y)^3}}\right] \\
&= \bar{U}^{1/2}\frac{1}{2}\left[\sqrt{\frac{(p_x - p_y)^2}{p_y(p_x - p_y)^3}} + \sqrt{\frac{p_y^2}{p_y(p_x - p_y)^3}}\right] \\
&= \bar{U}^{1/2}\frac{1}{2}\sqrt{\frac{1}{p_y(p_x - p_y)^3}}(p_x - p_y + p_y) \\
&= \frac{p_x}{2}\sqrt{\frac{\bar{U}}{p_y(p_x - p_y)^3}} \\
&= \frac{p_x}{2}\sqrt{\frac{\frac{I^2}{4}\frac{1}{(p_x - p_y)p_y}}{p_y(p_x - p_y)^3}} \\
&= \frac{p_x}{2}\sqrt{\frac{I^2}{4}\frac{1}{(p_x - p_y)^4 p_y^2}} \\
&= \frac{p_x}{p_y}\frac{I}{4}\frac{1}{(p_x - p_y)^2}
\end{aligned}$$

Using Slutsky Equation:

$$\begin{aligned}
\text{Total effect}_x &= \text{Substitution effect}_x + \text{Income effect}_x \\
\text{Substitution effect}_x &= \text{Total effect}_x - \text{Income effect}_x \\
&= \frac{\partial g_x(p_x,p_y,I)}{\partial p_y} - \left(-\frac{\partial g_x(p_x,p_y,I)}{\partial I} g_y(p_x,p_y,I)\right) \\
&= \frac{I}{2}\frac{1}{(p_x-p_y)^2} - \left(-\frac{I}{4}\frac{(p_x-2p_y)}{(p_x-p_y)^2 p_y}\right) \\
&= \frac{I}{2}\frac{1}{(p_x-p_y)^2} + \frac{I}{4}\frac{(p_x-2p_y)}{(p_x-p_y)^2 p_y} \\
&= \frac{I}{4}\frac{2p_y}{(p_x-p_y)^2 p_y} + \frac{I}{4}\frac{(p_x-2p_y)}{(p_x-p_y)^2 p_y} \\
&= \frac{I}{4}\frac{2p_y+p_x-2p_y}{(p_x-p_y)^2 p_y} \\
&= \frac{I}{4}\frac{p_x}{p_y}\frac{1}{(p_x-p_y)^2}
\end{aligned}$$

iv) Is y a gross substitute or complement of x?

Definition:

- *two goods x and y are gross substitutes if:* $\frac{\partial g_x}{\partial p_y} > 0$
- *two goods x and y are gross complements if:* $\frac{\partial g_x}{\partial p_y} < 0$
- *It is possible for x to be a substitute for y and at the same time for y to be a complement of x. This occurs when the income effect dominates the substitution effect for one good but not for the other.*

Solution:

$$\frac{\partial g_x}{\partial p_y} = \frac{I}{2}\frac{1}{(p_x-p_y)^2} > 0 \Rightarrow \text{ gross substitutes}$$

3. How does a small change in I affect the demand for x?

$$\frac{\partial g_x(p_x,p_y,I)}{\partial I} = \frac{1}{2}\frac{1}{(p_x-p_y)} > 0 \text{ since we have assumed that: } p_x > p_y$$

i) Is x a normal or inferior good?

Definitions:

- *A good x for which $\frac{\partial g_x(p_x,p_y,I)}{\partial I} \geq 0$ over some range of income is a **normal** good in that range.*
- *A good x for which $\frac{\partial g_x(p_x,p_y,I)}{\partial I} < 0$ over some range of income is an **inferior** good in that range.*

Solution:

$$\frac{\partial g_x(p_x, p_y, I)}{\partial I} = \frac{1}{2}\frac{1}{(p_x - p_y)} > 0 \Rightarrow \text{ Normal good}$$

ii) What is the substitution effect of a change in I?

There is no substitution effect from a change in income. Remember that the substitution effect measures how much the consumer would substitute x for y **given the price change** holding utility constant. Since changes in income do not imply a price change, there is no substitution effect.

XIV Elasticities

XV Extra practice III - Duality and Demand Relationships among Goods

1. Consider the utility function

$$U(x,y) = \min\{x,3y\}$$

Let p_x, p_y and I denote the price of x, the price of y and the income level, respectively.

   (a) Find the Hicksian demand functions for x and y.

   (b) Find the expenditure function.

   (c) Without solving the utility maximization problem, recover the indirect utility function and the Marshallian demand functions.

   (d) Now suppose that $p_x = 2$, $p_y = 4$ and $I = 40$. Compute the value of the Marshallian demands for x and y and the corresponding optimal utility level, u^*.

   (e) Use the utility level computed in part (d) to verify that the Hicksian demands are equal to the Marshallian demands for x and y.

2. Jane's utility function has the following form:

$$U(x,y) = x^2 + 2xy$$

The prices of x and y are p_x and p_y respectively. Jane's income is I.

   (a) Find the Marshallian demands for x and y and the indirect utility function.

   (b) Without solving the cost minimization problem, recover the Hicksian demands for x and y and the expenditure function from the Marshallian demands and the indirect utility function.

   (c) Write down the Slutsky equation determining the effect of a change in p_x on the demand of x. Indicate which component represents the total price effect, which component represents the substitution effect, and which component represents the income effect.

   (d) Now assume that $p_x = 64$, $p_y = 2$, and $I = 100$. Using the Slutsky equation from part (c) compute the total, substitution, and income effect of a change in p_x on the demand of x.

3. Judy's Marshallian demand for oranges is $\dfrac{I^{0.5}(p_a+3)^{0.6}}{p_o}$, where p_a is the price of apples, p_o is the price of oranges, and I is Judy's income. Suppose $I = 100$, $p_a = 2$, and $p_o = 3$.

(a) Find and interpret the income elasticity for the demand for oranges. Are oranges an inferior or normal good?

(b) Find the own price elasticity of demand for oranges. Discuss how the price elasticity varies with p_o.

(c) Find the cross price elasticity for oranges. Are oranges and apples gross substitutes or gross complements?

4. Suppose the preferences of an individual are represented by a quasilinear utility function:

$$U(x,y) = 3\ln(x) + 6y$$

(a) Initially, p_x=1, p_y=2 and I=101. Then, the price of x increases to 2 (p_x=2). Calculate the changes in the demand for x. What can you say about the substitution and income effects of the change in p_x on x? (Hint: since the change in price is not small, you cannot use the Slutsky equation)

(b) What can you say about the substitution and income effects of the change in p_x on y?

(c) Instead of doubling to 2, suppose p_x is only increased by a small amount. Use the Slutsky equation to find the substitution and income effects of the change in the price of x on x. Compare your result to (a). Explain why there's no income effect of the change in p_x on x. Show your result on an indifference curve.

(d) Use the Slutsky equation to find the substitution and income effect of the change in p_x on y. Compare your result to (b).