Modeling and Simulation, MC312
Lab-6 (2025)

Random Walks and Monte Carlo Simulation

Due: October 28, 2025

Objective

In this lab we will numerically explore one of the simplest stochastic processes, the discrete one-dimensional random walk. The objective is to understand how random motion can lead to systematic statistical regularities such as the linear growth of the mean square displacement and the emergence of the normal distribution through repeated sampling. We will simulate ensembles of random walkers, examine the distribution of their positions, and verify how ensemble averages compare with theoretical expectations.

Model Setup

Consider a walker that starts at the origin $X_0 = 0$ and moves on an integer lattice. At every time step $k = 1, 2, \ldots, n$, the walker moves either one step to the right or one step to the left according to the following rule:

$$X_k = X_{k-1} + S_k, \quad S_k = \begin{cases} +1, & \text{with probability } p, \\ -1, & \text{with probability } q = 1 - p. \end{cases}$$

The walk is called *unbiased* when $p = q = 1/2$, and *biased* otherwise.

Tasks

1. Simulating a Single Trajectory

Write a short program to simulate a random walk of length $n = 100$ starting from $X_0 = 0$. At each time step, generate a random number and update the position according to the rule above. Plot X_k as a function of step number k. Repeat the experiment with different random seeds and overlay a few sample trajectories. Comment briefly on the variability among realizations and on what quantity you would expect to become stable when many realizations are averaged.

2. Ensemble Average and Mean Square Displacement

Repeat the simulation R times, where each realization represents an independent walker. Compute, at every step k, the ensemble-averaged position $\langle X_k \rangle$ and the mean square displacement $\langle X_k^2 \rangle$. Compare your numerical estimates with the theoretical predictions. Plot both the simulation results and the theoretical lines on the same figure for the unbiased and a chosen biased case (for example, $p = 0.7$). Discuss the extent to which the averages from the Monte Carlo simulation match the theoretical expressions and how the error decreases as R increases.

3. Distribution of Final Positions

Fix the number of steps n (for example, $n = 100$). From your ensemble of R walkers, record their final positions X_n and plot the probability mass function $P_n(m) = \Pr(X_n = m)$ as a histogram. Superimpose the theoretical distribution on it. For large n, overlay the normal approximation with the correct mean and variance. Comment on how the shape of the empirical distribution approaches a bell curve as n increases.

4. Biased Walks and Drift

Investigate how the bias parameter p affects the trajectory and the distribution. Perform the same analysis for $p = 0.55$ and $p = 0.7$ and find the dependence of the shit in emsemble mean on p and q. Compare the growth of the variance for different p values and verify that the variance remains proportional to n even when the mean is nonzero.

5. Monte Carlo Estimation and Error

Treat the ensemble simulation as a Monte Carlo experiment. Explain how repeated random sampling allows us to estimate expectations. Examine how the accuracy of the estimate for $\langle X_n^2 \rangle$ depends on the number of realizations R. Plot the estimated standard error of the mean as a function of R and verify that it scales approximately as $1/\sqrt{R}$. Briefly discuss how increasing R improves the confidence of the numerical estimate without changing the expected value itself.

6. Gambler's Ruin Problem

As an extension, consider a random walk constrained to the interval $[0, N]$ with absorbing boundaries at both ends. The walker starts at site i $(0 < i < N)$ and moves right with probability p and left with probability $q = 1 - p$. The game ends when the walker reaches either 0 (ruin) or N (success). Use Monte Carlo simulation to estimate the probability $P_{\text{ruin}}(i)$ that the walker hits 0 before N.

For the unbiased case $p = q = 1/2$, verify that the probability of ruin is

$$P_{\text{ruin}}(i) = 1 - \frac{i}{N}.$$

For biased walks $(p \neq q)$, derive or verify the general expression

$$P_{\text{ruin}}(i) = \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N}.$$

Plot the numerical estimates and theoretical curves together for representative values of p and N. Comment on how the bias changes the likelihood of reaching either boundary. Estimate the mean time to absorption and describe its dependence on the starting position. Outside the lab, you should obtain the two expressions analytically for the ruin problem.