Making Votes Count

A Mathematical Framework for Social Ranking and Electoral Systems

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Declaration

I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment.

Word Count: 12,513

Dedication

There have been countless individuals who have helped me over the course of this project. I have unending gratitude to Tom for pushing me to make this project as interesting as possible; to Maia and Emily, for giving me the motivation I needed to complete this; and to Cara, for her harsh but fair proof-reading skills. I must also thank and apologise to the many unwitting individuals who, over the last few months, were subject to my explanations of lesser-known paradoxes in voting systems. A special thanks must also be given to Dr. Friedrich Pukelsheim, for his gracious assistance with and interest in this project.

Contents

Disclaimer

The town of St Chad and all characters, locations, and events associated are, unless otherwise specified, purely fictional. No identification with actual persons (living or deceased) or places is intended or should be inferred.

1 Abstract

For as long as human society has existed, it has been necessary to make decisions that impact the whole of society. The premise of voting is to take the opinions of a group, and transform them into a single opinion. In this report, we explore the background of voting theory throughout history, the compatibility of different conditions upon voting systems, paradoxes that can occur, and the principle of voting tactically. We then investigate specific voting systems and their features, from the election of the Pope and the Holy Roman Emperor, to how we decide the composition of our legislatures and who wins Eurovision. We will also discover several concerning facts about the nature of voting.

2 History of Voting Theory

2.1 Classical Democracies

The first society to which the label 'democratic' can be reasonably applied is likely that of ancient Athens, although this would not be democracy as we recognise it today. Conversely, the ancient Athenians would view our democracy today as primitive. The concept of *election* was viewed as specifically oligarchic by the Athenians. Their thinking was that, in any system where candidates had to rely on others voting for them to be elected to office, this would quickly accrue power in the hands of a select few elites, who, by accumulating influence and awareness, would be able to remain in office for years [1, p.27]. In other words, the Athenians were opposed to politics as a career (save for a select few offices); instead, their officers of state were elected by lot, whereby any eligible citizen could enter his name into a lottery to serve on the institutions of Athens [1, p.20]

The Roman Republic also incorporated democratic elements, but was far from an equal system. The franchise was also limited to free male citizens, and votes were weighed based on property class, with an electoral college for each of these classes. There existed several different assemblies that served different electoral functions [2, p.268]. The *comitia centuriata* elected the highest offices in the Republic; the upper classes voted first, and if their vote was unanimous, then the assembly would disband as their block vote outweighed all other classes [2, p.284].

Much of Roman democracy is recognisably modern. Rome was one of the first instances of quasi-formal political parties being formed; many statesmen were connected to either the *optimates* or the *populares*, although factional labels tended to denote the constituency of support for each (the former largely being supported by noblemen, the latter by peasants) rather than any ideological leanings [3]. As political rallies were banned, candidates would often resort to other means to entice voters, such as giving out free tickets to games and hosting dinners, resulting in the only viable candidates being themselves wealthy or having a wealthy patron. Political campaigning tended to consist of *quid-pro-quo* deals, bribery, blackmail, and outright lying in order to secure election [4, p.15] [1].

2.2 Medieval Elections

The election of the Pope is one of the longest-running democratic procedures still in use. The electoral system used today was formalised in the 17th century. Prior to 1059, there was no formal election system, which contributed to the schism between the Eastern and Western churches, as multiple popes were able to claim legitimacy depending on their view on how the pope should be chosen [2] [5] . Several *ad hoc* methods were used instead. The first recorded method of election of a Pope was in 236 C.E., when Fabian ascended to the Chair of St Peter after a dove landed on him during the election proceedings, taken to be the divine intervention of the Holy Ghost [6].

The papal electorate consists of a number of high-ranking cardinals, all of whom meet in the one place (initially Rome, and today the Vatican City) and are made to select a candidate among themselves [7] [5]. Although the exact method has changed over time, they typically have an indefinite number of

[1] No parallels should be drawn with the present day.

[2] If nothing else, this is superb motivation for why voting theory is important.

rounds of balloting until one candidate gets a significant proportion of the votes (today set at two-thirds of electors) [5]. Given there is no guarantee of this threshold ever being met [3], various incentives have been introduced to force an eventual resolution. Cardinals are secluded from the outside world until the conclusion of election, are banned from receiving payment during the conclave, and receive diminishing amounts of food the longer the conclave lasts [5].

The Holy Roman Empire had a similarly complex electoral system. Emperors were elected by a body of powerful local rulers (known as *prince-electors*); this came to be a fixed body of seven. Similarly to the papal election system, electors had to choose a winner within thirty days or they would be reduced to rations of bread and water [8].

Nicolaus Cusanus (1401 - 1461 C.E.), a cleric and philosopher, proposed a method to reform this system. His seminal political work, *De concordantia catholica*, proposes a ranked-choice voting system that is equivalent to the Borda count (which will be discussed later) [1, p.99-101]. Cusanus also discusses the importance of secret voting, not just in the context of pressure being placed on electors to vote a certain way, but in terms of *tactical* voting; if one voter knows the preferences of all others, it is possible for them to cast a ballot insincerely (i.e. not in line with their true preference for what they want the outcome to be) in order to create a *more preferable* outcome according to their true preferences. Cusanus' work is a fascinating example of designing voting systems to mitigate the potential of tactical voting [1, p.99].

Another important figure in the development of voting theory was the theologian Ramon Llull (ca. 1252 - 1315 C.E.). Llull is a fascinating character for many reasons - aside from his contributions to voting theory, he has also been viewed as an early pioneer of computer science [9]. Llull conceived of two slightly different voting methods. In Llull's novel *Blanquerna* (1283), Llull describes a voting procedure in which the winner is the candidate who beats the most other candidates in a one-against-one contest, with additional discussion on resolving ties [1, p.95]. In another paper, *Ars eleccionis*, he describes another tournament-like system, in which two candidates are matched

[3] The election held after the 1268 death of Clement IV took three years to complete due to major disagreements between cardinals.

up against each other, and the winner of each round competes against a new candidate, until each candidate has had a chance to compete and a final winner is found [1, p.96-98]. Both methods will be detailed later in this report. Of interest here is Llull's implicit recognition of the advantage of reducing elections to a series of pairwise ballots; that with only two alternatives, then one can only vote for *or* against a given candidate in each round, reducing the potential for tactical voting.

2.3 Modern Voting Theory

As popular democracy began to flourish worldwide during the Enlightenment, the necessity for voting systems that would allow for representation of the masses increased. The French Revolution saw great opportunity for voting theorists, as new systems were in high demand for elections to the Assemblé Nationale. One major figure was the Marquis de Condorcet. Condorcet was active in politics and took a deep interest in constitutional affairs [1, p.147-149]. One of his most celebrated results is *Condorcet's jury theorem.* Suppose that we have a jury who are asked to pass verdict by majority vote on some question, and that only one verdict is *correct.* If each juror is (independently) more likely than not to reach the correct conclusion, then it is more likely overall that the correct verdict will be reached if more jurors are added. Otherwise, the correct verdict is more likely to be obtained if there are *fewer* jurors, and hence a single juror is optimal. This result is relatively easy to verify, but it has given rise to the further study of jury theory [1, p.150-151].

Condorcet made two important contributions to voting theory: the description of the *Condorcet paradox*, and the creation of the *Condorcet method* for voting [1, p.158-162]. The Condorcet paradox is a cornerstone of the philosophical argument that no voting system can ever be truly 'fair'. As an electorate can have cyclic preferences of candidates (for instance, preferring A to B, B to C, and C to A), there is no one way of determining which of the three should win overall. No matter *who* wins, one of the non-winning candidates can change the result by exiting the contest, thus making it non-obvious whether the original winner or the new winner is the 'true' winner [4]. The Condorcet method is based on Condorcet's analysis of the problem -

[4] This is a somewhat philosophical discussion, but if we are to reduce it to a mathemat-

it reduces elections to pairwise contests, in an equivalent manner to Llull's method.

Another notion that came of age during the Enlightenment was the concept of *proportional representation* [10, p.98-100]. Although parliamentary assemblies had existed for centuries, historical elections were usually decided by and for an elite few, and as such, political affiliation was relatively fluid. As mass democracy became the norm, as did more formal party systems; given the vastly greater level of accountability, it grew difficult to campaign and govern as an individual. Hence many polities opted to adopt voting systems for their legislatures that explicitly integrated a party system. Seats in the legislature would be decided based on party rather than (exclusively) personal votes. A party that received 20% of the vote in an election could be expected to obtain roughly 20% of the seats in the legislature. Outside elections, proportional representation is intangibly linked with *apportionment systems*; that is to say, methods of determining the number of seats in the legislature each region of a larger polity receives. Both the United States [11, p.132] and the European Union [11, p.23] are useful case studies for apportionment.

3 Introducing Voting Theory

3.1 Vote Early, Vote Often

The town of St Chad [5] (pop. 1,729) is a picturesque farming town, known for exporting handmade woolen jumpers. The town is still not quite up to speed with the industrial revolution - until a few years ago, it operated on a purely feudal basis. However, the townsfolk have enthusiastically adopted democracy, and regularly boast 100% turnout rates.

Unfortunately, St Chad's transition to democracy has been marred by a series of unusual election results. Madeline, the Electoral Commission's fraud spe-

ical problem, the true winner is whoever wins *under the voting system in place*, which is dependent on the set of candidates. Hence, the answer is arguably *both*!

[5] After the 2000 United States Presidential Election, there was a dispute over the validity of the result in part related to the paper 'chads' being read the vote counting machines, leading to the hitherto little-known St Chad being unofficially proclaimed the "patron saint of disputed elections"

cialist, was sent to St Chad to investigate. Madeline reviewed the results of the last election. The current mayor, Angela Keeting, had beaten landowner Robert Lemming by the narrowest of margins. There was also a less popular candidate, freelance tea-cosy designer Claire Monroe, who had drawn the ire of much of her fellow townsfolk in her bid to diversify St Chad's economy.

When Madeline met with Mayor Keeting, she immediately brought up her concerns about the new voting system: what was the change intended to do?

"We were just trying to make the process as fair as possible", said Mrs. Keeting.

"How is this system supposed to work?", asked Madeline.

"Well, we had a meeting of the town council, and we all agreed on two things our voting system should have. First of all, if everyone prefers me to Robert - they do, by the way - then I should rank above Robert in the final results."

"That makes sense."

"Secondly, if Claire thinks she somehow has a chance of beating me and runs for mayor, then that should have no impact on whether or not I am preferred to Robert in the final results."

"That also seems like a fair rule", Madeline murmured. "But there's just one problem-"

"Which is?", Mrs. Keeting exclaimed.

"That it seems one voter decides every single election."

Mrs. Keeting was aghast by this news. "But I was only trying to make the system *more* democratic!

"I know, Mrs. Keeting. But sometimes, if you want more democracy, you're going to need to accept less democracy. Let me explain."

Madeline sighed, took out her notebook, and turning to a fresh page, started scribbling away.

* * *

Briefly leaving our town of St Chad, we will now start investigating the mathematical background of voting theory. As voting theory is a relatively niche area of mathematics, it is difficult to find consistent terminology. Hence, for the sake of clarity, we will begin by explaining the basic concepts of voting theory with a set of notation that will be used throughout this report.

3.2 Set Orderings

Firstly, we take a brief interlude into the set-theoretic notion of *ordering*.

Definition 3.1 (Partially Ordered Set)**.** A *partially ordered set* $(P, \leq)$ is a set P and a relation on the elements of that set $\leq$ that satisfies [12]:

1. Reflexivity: For all $x \in P$, $x \leq x$.
2. Transitivity: For all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
3. Antisymmetry: For all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.

We can write partially ordered sets as a list of elements such that $b \not< a$ if b is to the left of a, and all elements $a = c$ are contained within a subset.

Definition 3.2 (Totally Ordered Set)**.** A *totally ordered set* $(T, \leq)$ a partially ordered set that additionally satisfies *totality*:

$$\forall a, b \in T : a \leq b \vee b \leq a$$

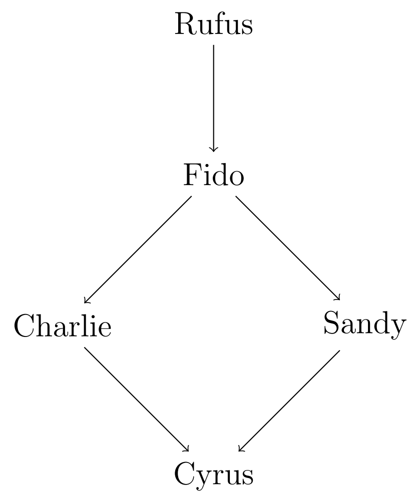
A *strictly totally ordered set*, $(S, <)$, satisfies transitivity, connectivity, and the negation of reflexivity (irreflexivity - for all $x \in S$, we have $x \not< x$).

Orders can be represented pictorially by a *Hasse diagram*. For instance, let us define a partially ordered set P. Hasse diagrams are not terribly useful for totally ordered sets, as they are simply a line of nodes with labels in the order of the set. As we will be using this in the context of *preferences* later, we can motivate this with a simple example.

Example 3.3. James is trying to name his dog. He comes up with a series of potential candidates: he prefers 'Rufus' overall, followed by 'Fido', and has equal preference for 'Charlie' and 'Sandy', both of which he prefers to 'Cyrus'. As a partially ordered set, this would be represented by:

$$(Rufus, Fido, \{Charlie, Sandy\}, Cyrus)$$

The equivalent Hasse diagram would be:



3.3 Elections and Social Rankings

Definition 3.4 (Voters)**.** The set of *voters*, or *players* (for game-theoretic contexts) N, is a set $[n] = \{1, 2, ..., n\}$.

The notion of voters is fairly intuitive. It assigns a label to each member of a given electorate, as in real-world contexts. For instance, each voter on a real-life voter roll could be given a single ID number from 1 to n to fit with this theory.

We will also restrict the set of voters to be of size at least 2, as having one voter is not particularly interesting, negates the need for a voting system in the first place, and is technically a dictatorship (as will be later demonstrated).

We now introduce the concept of *alternatives*. An alternative can be thought of as a candidate in an election, or an option in a referendum. An alternative is simply something that the voters are choosing between; the set of alternatives is the complete set of choices. Although this will not be analysed here, this could also include the 'empty alternative' or 'none of the above' option, where voters can choose explicitly to vote for *none* of the other alternatives; allowing voters to write in an alternative would essentially allow for infinite alternatives and so will not be considered here either for simplicity.

Definition 3.5 (Alternatives)**.** The set of *alternatives*, A, is a set $A = \{a_1, a_2, ..., a_m\}$. A piece of notation will become useful later is $A!$, which denotes the set of possible permutations (or total orderings) of A, and has the nice property that $(|A!| = |A|!$. For instance, for $A = \{a_1, a_2, a_3\}$:

$$A! = \{(a_1, a_2, a_3), (a_1, a_3, a_2), (a_2, a_1, a_3), (a_2, a_3, a_1), (a_3, a_1, a_2), (a_3, a_2, a_1)\}$$

Also useful here is the possible *weakly ordered sets* on A; this is similar to $A!$, but allows for ties between two or more alternatives. The number of weakly-ordered sets for a set of size m is the m^{th} *ordered Bell number* [13]. Unfortunately, there does not appear to be any conventional definition for the set of weak orderings of a set A, so we will use $A?$, with $|A?|$ equal to the m^{th} ordered Bell number for $|A| = m$.

We say that each voter has a *preference relation* ranking each alternative:

Definition 3.6 (Preference Relation)**.** A *preference relation* $(p_i, \succsim_{p_i})$ designates the order of preference for a voter $i \in N$. If $a \succsim_{p_i} b$, then i prefers a to b or ranks them equally. This relation is complete, reflexive, and transitive. A similar notion is the *total preference relation*, $(p_i, \succ_{p_i})$, which is as above but irreflexive; i.e. a and b cannot be ranked equally. Additionally, we write $a \approx_{p_i} b$ if a and b are ranked equally by p_i.

Within tuples where it is obvious which voter's preference we are talking about, we will forego the subscript on $\succsim$ and write $a/b/...$ for preferences ranked equally.

Each p_i can be represented as a tuple up to a isomorphism (where $p_i = (a \succ b \succ c/d \succ ...) \simeq (a, b, c/d, ...)$. We will use these notations interchangeably and, for all intents and purposes, assume that these are the same object.

We will use $\mathcal{P}^*(A)$ to describe the set of all preference relations over A, and $\mathcal{P}(A)$ to refer to the set of all *strict* preference relations over A. There will be no ambiguity as any given election will only be dealing with one case at a time.

To explain in real-world terms, a preference relation for a voter is the order of preference that a voter has for an alternative. For instance, in the game rock-paper-scissors, if a player has chosen to play 'rock', then their order of preference for the other player's choice is 'scissors', 'rock', then 'paper', resulting in a win, a draw, and a loss respectively.

Definition 3.7 (Preference Profiles)**.** The set of *preference profiles*, P, for voters $[n]$ and alternatives A, is a list $P = (p_i \in A! \text{ for } i \in [n])$. We will usually assert that each p_i is a *strict* preference order, but the definition is easily adapted as $P = (p_i \in A? \; \forall i \in [n])$ if we allow ties.

When writing preference profiles, we will later use the notion of a *preference class*; that is, a set of voters $N_j \subseteq N$ such that their preference relations are equal. This helps to simplify preference profiles, as we can re-write it as a *preference class profile*. This is a set $C = (a_1 p_1, a_2 p_2, ...)$, where each a_i is the total number of times $p_i \in \mathcal{P}^*(A)$ appears in P. Note that $\sum_i a_i = |N|$.

Preference profiles can be thought of as the preference relations of all voters.

3.4 Voting Functions

Bringing these definitions together, we can now define the primary topic of the paper: *voting functions*. There are two types of voting function: social welfare functions and social choice functions. We can think of social welfare functions as mapping the preference of society as a whole onto a single preference, and social choice functions as designating a subset of alternatives as 'winners' and all others as 'losers'. Although they are quite different (especially as the outputs of these functions are different objects), we can map from social welfare functions to social choice functions - for instance, the first n preferences in the output of a social welfare function could be declared the winners in a social choice function. However, this mapping is one-way.

Definition 3.8 (Social Welfare Function)**.** The *social welfare function* f_W maps from the set of preference profiles P to a single preference relation $W \subseteq \mathcal{P}^*(A)$.

Definition 3.9 (Social Choice Function)**.** The *social choice function* f_C maps from the set of preference profiles P to a subset $C \subseteq A$.

Definition 3.10 (Social Ranking)**.** A *social ranking*, R, is a tuple (n, A, P, f_R). R is a totally or partially ordered set defined by f_R.

Definition 3.11 (Election)**.** An *election*, E, is a tuple (n, A, P, f_E). The set of *winners* of the election will be denoted as E in shorthand.

Definition 3.12 (Voting Functions & Voting Systems)**.** Where appropriate, both social welfare functions and social choice functions shall be referred to as *voting functions*, f_V, and both social rankings and elections as *voting systems*, V.

Definition 3.13 (Societal Preference)**.** The *societal preference* of R is the preference relation generated by f_R, and we write $a \succ_R b$ if a is preferred to b in f_R.

Definition 3.14 (Pairwise Preference)**.** For alternatives $a, b \in A$ and a social welfare function f_W, we write that a is *pairwise preferred* to b if limiting the set of alternative to $A' = \{a, b\}$ implies $a \succ_W b$. (Later, in the language of tournaments, we will equivalently say that a *dominates* b.)

Note that that this is not necessarily equivalent to the preference for a over b in W. To avoid confusion, we will denote a pairwise preference for a over b by aWb.

If we have a pairwise contest, then each voter can only choose between two alternatives, and hence order of preference does not matter; it is a binary vote. Hence if we have more voters that rank a above b, then a will be the pairwise winner. It is theoretically possible to design a more exotic voting system where the alternative with *more* pairwise votes loses, but for the sake of simplicity and - frankly - sanity, we will exclude this case.

We can also extend this notion to an electoral function f_E if the set of winners is a singleton (or at least ensures that only one of a set of two alternatives is in the winning set). In this case, aEb if the set of alternatives is reduced to a and b, and $f_E(P) = \{a\}$. (We can, of course, modify f_E to produce a single winner to extend the idea of pairwise preferences to all voting functions.)

Definition 3.15 (Ranking x and y the same)**.** For alternatives a and b, the preference profiles $P = (p_1, ..., p_n)$ and $Q^N = (q_1, ..., q_n)$ *rank* a *and* b *the same* if for $i = 1, ..., n, x \succ_{P_i} y \Leftrightarrow x \succ_{Q_i} y$ (and, if the the preferences are weakly ordered, $x \approx_{P_i} y \Leftrightarrow x \approx_{Q_i} y$).

Definition 3.16 (Decisiveness)**.** A voter $i \in N$ with ballot p_i is *decisive for* x *over* y if, for alternatives x and y, $x \succ_{p_i} y$ implies $x \succ_{f_C(P)} y$). If this holds for *any* alternatives x and y (i.e. the preference of d determines the group preference), then d is a *dictator* [6].

Trivially, dictators are unique. If two voters are dictators and have different preference profiles, then at least one of them must *not* match the social preference, and so the voter with that profile cannot be a dictator.

4 Arrow's Impossibility Theorem

4.1 Unintended Consequences

We return once more to St Chad. Madeline had finished scribbling in her notebook and was now ready to explain the situation to Mayor Keeting.

"Let's say we have three candidates: Angela, Robert, and Claire. Imagine everyone in St Chad ranks Angela above Robert. Now, we get each voter to switch around their preference between you and Robert. At some stage, Robert is going to have to rank above Angela overall because of your first rule."

"Regrettably, yes"

"So now Robert is ahead because of one voter who's flipped their ballot. Let's call them Mr. X. If Claire decides to run too, then let's say that every voter who prefers Angela to Robert ranks Claire last, and every voter who prefers Robert to you ranks Claire in-between you."

"Why can we just determine what the voters think?", a confused Mayor Keeting queried.

[6] This does also mean that a voting system with a single voter is a dictatorship.

Madeline was somewhat pleased that she hadn't asked for more technical details (yet). "We're not saying what they think. We're saying what they *could* think. If your rules don't work for every possible way people could case their ballots, then they don't work." Madeline paused for a moment to allow Mayor Keeting to take this in, then continued.

"Because Mr. X determines whether or not the electorate prefers Robert or you, we can also say something about Claire. Let's suppose that Mr. X prefers you to either Robert or Claire. Then the group still prefers you overall-"

"Good"

"-and if the group prefers Robert to Claire, then it must also prefer you to Claire. Now let's try another scenario. We set up the voters again and get them to cast their ballots in the same order, so Mr. X will be in the same position we left them in last time. Now, suspend your disbelief just for a moment, and imagine that everyone prefers Robert to anyone else. Suppose that every voter that cast a vote before Mr. X still prefers you to Claire. So what do we know about the overall outcome?"

"Well, it's the same as the scenario where everyone prefers Robert over me, so Robert must be preferred over me overall."

"There's more. If you look at how Robert is ranked relative to Claire, it's exactly the same as the last scenario."

"So because the group prefers Robert to Claire, they also prefer me to Claire."

"Exactly. Well, *almost* exactly. That depends on one thing - who does Mr. X prefer? If they rank you above Claire, then the group must rank you above Claire." Madeline turned away for a minute and scribbled a few more scenarios down in her notebook. Turning her hastily-drawn diagrams around to Mayor Keeting, she continued. "In fact, you can use this same logic to show that how Mr. X ranks *any* two candidates relative to each other will always be the same as how the group overall votes. Which means one of two things."

"Go on"

"Either Mr. X is the world's greatest empath and knows exactly what the people want. Or - much more likely - they determine *exactly* what the outcome of the election will be. In other words, Mayor Keeting, I'm afraid that Mr. X has turned St Chad into a dictatorship."

* * *

Let us now formalise this as a mathematical proof.

4.2 Proving Impossibility

Theorem 4.1 (Arrow's Impossibility Theorem)**.** *An election $E = (n, A, P, f_C)$, with $n \geq 3$ voters, m alternatives and the ordering of winners $W = f_C(P)$, may satisfy at most two of the following three axioms (with all terms as defined above) [14] [15]:*

1. ***Unanimity:*** *For all $x, y \in A$ and for all $i \in N, x \succ_{p_i} y$ implies that $W = x \succ_{f_C(P)} y$.*

2. ***Independence of Irrelevant Alternatives:*** *For all $x, y \in A$ and for all preference profiles P, Q^n on A, such that $P = (p_1, ..., p_n)$ and $Q^n = (q_1, ..., q_n)$ rank x and y the same, $x \succ_{f_C(P)} y)$ if and only if $x \succ_{f_C(Q^N)} y)$.*

3. ***Non-dictatorship:*** *There is no $d \in [n]$ such that for all $x, y \in A$ with x and y not equal, $x \succ_{p_i} y$ implies that $f_C(P)$ has $x \succ y$.*

As the above is quite intense, we will also explain in natural language.

1. **Unanimity:** If *every* voter prefers x over y, then the group must prefer x over y. This feels inherently true of a good voting system - if a group of people rank episodes of a TV show, and everyone individually ranks, say, Episode 3 over Episode 5, then it would appear fair to assume that the group must rank Episode 3 over Episode 5.

2. **Independence of Irrelevant Alternatives:** Treating the two preference profiles as the same group of voters getting the chance to vote twice, if each voter keeps the same order of preference for x and y as before but arbitrarily changes any other preferences, then the group

ranking of x and y should be the same in both cases. Although this is wordy, it makes intuitive sense. For instance, if this group of people rank the episodes of a TV show, then a new episode coming out shouldn't make any difference either to each individual's ranking or the group's overall ranking, by whatever method the group use to decide their ranking.

3. **Non-dictatorship:** There is no single voter who is able to decide the group's overall preference; continuing our example, no single member of this group of TV watchers changing their mind should be able to force the group's order of TV preference match their own, rendering everyone else's opinion redundant. If we are to consider the real-world applications of voting systems, then in a democracy - in which *everyone* is supposed to get a say - this is an evidently necessary axiom for an electoral function.

In some sense, it seems obvious that all of the above axioms would hold in a democratic electoral system. However, Arrow's Impossibility Theorem demonstrates this is not the case by way of contradiction: enforcing the first two axioms will imply that the third axiom, that of non-dictatorship, cannot hold.

Proof. For shorthand, we will label our axioms as: Unanimity (1), Independence of Irrelevant Alternatives (2), and Non-Dictatorship (3).

By way of contradiction, we are going to prove Arrow's Impossibility Theorem. Assume we are living in a society with an electoral system that is characterised using the terms defined above, and that further we assume that axioms (1), (2), and (3) hold. We are going to create several voter profiles and compare them.

Part 1: There is a pivotal voter
Consider a social ranking $W = (n, A, P, f_R)$ with $|A| = m$ and f_R such that (1), (2), and (3) are satisfied. For simplicity, we further assume that ties are not allowed and that voters cannot have equal preference for two candidates. Let a, b, and c be alternatives in A. Set:

$$P_\alpha = (p_1, \dots, p_n : a \succ_{p_i} b)$$

$$P_\beta = (p_1, \ldots, p_n : b \succ_{p_i} a)$$

where other alternatives are arbitrary, but each p_j is equal except for its preference of a over b. By (1), R_α ranks $a \succ b$ and R_β ranks $b \succ a$.

We will then transform P_α into P_β by swapping the preference of a over b in each tuple in the preference profile. As R_α ranks $a \succ b$ and R_β ranks $b \succ a$, there must be a point in this process where, for this election, $a \succ b$ to $b \succ a$. Suppose we initiate this process:

$$P_\gamma = (p_1, \ldots, p_{d-1} : b \succ_{p_i} a;\ a \succ_{p_d} b;\ p_{i+1}, \ldots, p_n : a \succ_{p_i} b))$$
$$P_\delta = (p_1, \ldots, p_{d-1} : b \succ_{p_i} a;\ b \succ_{p_d} a;\ p_{d+1}, \ldots, p_n : a \succ_{p_i} b)$$

(I have used semicolons to separate sequences of elements of this list to avoid clutter.)

For some voter d, we will have that $a \succ_{p_d} b$ implies $a \succ_{R_\alpha} b$ and $b \succ_{p_d} a$ implies $b \succ_{R_\beta} a$, for some preference profile P as we swap between P_α and P_β. Hence d is a *pivotal voter*; i.e., the outcome of the election changes if they change their preference.

Part 2: The pivotal voter's preference of a, b, c determines the group preference of a, b, c

Now we will introduce c.

$$P_\epsilon = (p_1, \ldots, p_{d-1} : b \succ_{p_i} a \succ_{p_i} c;\ b \succ_{p_d} a/c;\ p_{d+1}, \ldots, p_n : a \succ_{p_i} b/c)$$

By the above, P_ϵ ranks a and b the same as P_β, so given that $b \succ_{R_\beta} a$, we have that $b \succ_{R_\epsilon} a$, as the only difference between this and $P_{b,a}$ is that we have specified c. By (1), $a \succ_{R_\epsilon} c$ implies that $b \succ_{R_\epsilon} c$.

Part 2.1: d is decisive for b over c

$$P_\zeta = (p_1, \ldots, p_{d-1} : a \succ_{p_i} b \succ_{p_i} c;\ a \succ_{p_d} b/c;\ p_{d+1}, \ldots, p_n : a \succ_{p_i} b/c)$$

As P_ζ ranks a and b the same as P_α and a and c the same as P_ϵ, by (2), we have that $b \succ_{R_\zeta} c$. Therefore for any profile, if d ranks b over c, then the group preference will be the same as R_ζ. Hence d is decisive for b over c.

Part 2.2: d is decisive for a over c

$$P_\eta = (p_1, \ldots, p_{d-1} : a/c \succ_{p_i} b; a \succ_{p_d} b \succ_{p_d} c;\ p_{i+1}, \ldots, p_n : a/c \succ_{p_i} b)$$

d is decisive for b over c, so we have $b \succ_{R_\eta} c$. By (1), as every voter prefers a to b, we have $a \succ_{R_\eta} b$. Then the social ranking is $a \succ_{R_\eta} b \succ_{R_\eta} c$, and in particular $a \succ_{R_\eta} c$, so d is decisive for a over c.

Part 2.3: d is decisive for c over a

$$P_\theta = (p_1, \ldots, p_{d-1} : b \succ_{p_i} c \succ_{p_i} a;\ c \succ_{p_d} a \succ_{p_d} b;\ p_{d+1}, \ldots, p_n : c \succ_{p_i} a \succ_{p_i} b)$$

P_θ ranks a and b the same as $P^n_{a,b}$, so by (II), we have that $a \succ_{R_\theta)} b$. By (1), we have that $c \succ_{R_\theta)} a$, and so $c \succ_{R_\theta)} b$ must hold.

$$P_\iota = (p_1, \ldots, p_{d-1} : b \succ_{p_i} a/c;\ c \succ_{p_d} b \succ_{p_d} a;\ p_{d+1}, \ldots, p_n : a/c \succ_{p_i} b)$$

By the above, $c \succ_{R_\iota} b$. However, as P_γ ranks a and b the same as P_β, we have that $c \succ_{R_\iota} a$, hence d is decisive for c over a.

Part 2.4: d is decisive for c over b

$$P_\kappa = (p_1, \ldots, p_{d-1} : a \succ_{p_i} b/c;\ c \succ_{p_d} a \succ_{p_d} b;\ p_{d+1}, \ldots, p_n : a \succ_{p_i} b/c)$$

P_κ ranks a and b the same as P_α, so $a \succ_{R_\kappa} b$. As d is decisive for c over a, we have that $c \succ_{R_\kappa} a$, and so $c \succ_{R_\kappa} b$.

$$P_\lambda = (p_1, \ldots, p_{d-1} : b \succ_{p_i} a \succ_{p_i} c;\ c \succ_{p_d} b \succ_{p_d} a;\ p_{d+1}, \ldots, p_n : a \succ_{p_i} b \succ_{p_i} c))$$

P_λ ranks a and c the same as P_κ, so $c \succ_{R_\lambda} a$. Hence, as d is decisive for c over a, $c \succ_{R_\lambda} b$.

Part 2.5: d is decisive for a over b and b over a

$$P_\mu = (p_1, \ldots, p_{d-1} : a/b \succ c;\ a \succ_{p_d} b \succ_{p_d} c;\ p_{d+1}, \ldots, p_n : a/b \succ c)$$
$$P_\nu = (p_1, \ldots, p_{d-1} : a/b \succ_{p_i} c;\ b \succ_{p_d} a \succ_{p_d} c;\ p_{d+1}, \ldots, p_n : a/b \succ_{p_i} c)$$

d is decisive for a over c, c over b, b over c, and c over a for all profiles, so it follows that $a \succ_{R_\mu} b$ and $b \succ_{R_\nu} a$. By (2), we conclude that d is decisive for a over b and for b over a.

Part 3: We are living in a dictatorship We now just need to show that i is decisive for an arbitrary number of alternatives. If we now consider alternatives x and y:

$$P_\xi = (p_1, \ldots, p_{d-1} : y \succ_{p_i} x;\ x \succ_{p_d} a \succ_{p_d} c;\ p_{d+1}, \ldots, p_n : y \succ_{p_i} a \succ_{p_i} x)$$

As we never specified which each alternative was initially, d's ranking is decisive for *any* alternative. Hence if x is ranked above y for p_d, $x \succ_{R_\xi} y$. Hence by our definition, i is a dictator. This contradicts (3) and completes the proof.

□

The first thing to note from this proof is how bizarre the conclusion seems. Only two very simple and seemingly reasonable rules were used to construct this voting system - (3) was not invoked until the very end - and yet we end up with a preference profile in which *every voter but one* could prefer alternative x to y, and y would still win. What's more, this does not depend on the specific x and y; only on voter d's personal whims.

If no voter is a dictator, we cannot have both (1) and (2) hold. If we agree that these are both desirable qualities for a voting system to have, it is therefore impossible for any voting system to be totally satisfactory.

* * *

The more Madeline explained, the more horrified Mayor Keeting looked. Had she accidentally created a dictatorship? What could she do now? She turned to Madeline for advice.

"You need to make a choice. You can either have elections obey unanimity, or the independence of irrelevant alternatives. If you have both, then Mr. X decides *every* result."

"But just who *is* this Mr. X?"

"Leave that with me. In the meantime, you should decide: what are your priorities for a good voting system?"

As Madeline left to continue her investigation, Mayor Keeting sat down and began to ponder this question. After a few hours, she had come up with a plethora of desirable features for voting systems, but one question remained: were they *compatible*?

In the next section, we will look at the voting criteria that St Chad could use in its future elections without accidentally creating a dictatorship.

5 Voting Criteria

5.1 Democracy and Utility

In this section, we will explore various criteria that can be applied to voting methods and their mutual compatibility. We will use the term *single-winner election* to refer to an electoral system in which only one alternative is chosen. This also encompasses a single alternative being chosen indirectly. For instance, parliamentary elections in which the alternatives are choices for which party leads the government also fall into this category.

As we demonstrated in Section 4, even relatively trivial axioms for voting can be incompatible. This motivates investigating a range of voting criteria and their mutual compatibility. As a consequence of Arrow's Impossibility Theorem, there is an inescapable philosophical question that arises in the study of voting systems - given that, in some sense, *no* voting system can ever be fair, how do we translate the opinions of individuals to the opinion of society as a whole?

The traditional approach to this in the design of voting systems has been to determine the *utility* of a given voting system. Roughly speaking, this means to create a measure of how satisfied society will be overall with the results produced by a system. As utility is a relatively elusive philosophical and economic concept, its precise definition outwith the scope of this report. It is nonetheless important to keep this notion of utility in mind. While we will not define what we mean by 'satisfied', producing the result that is the most acceptable to the most people will suffice.

Before going forwards, let us define some very basic principles of what we mean by a *democratic* voting system.

Definition 5.1 (Viability Criterion)**.** An alternative a is *viable* in a voting system V if there is some preference profile such that a is in the winning set of V. A social ranking or electoral system satisfies the *viability criterion* if every alternative is viable.

We can strengthen the viability criterion somewhat:

Definition 5.2 (Neutrality Criterion)**.** A voting system is *neutral* if the relabelling of preferences gives the same result up to relabelling.

Similarly, for voters, we have the *anonymity criterion*:

Definition 5.3 (Anonymity Criterion)**.** A social ranking or electoral system is *anonymous* if the result is the same for all scenarios where the preference profiles are the same up to relabelling of associated voter.

It is easy to see that anonymity is equivalent to the result not depending on the order in which votes are counted.

Definition 5.4 (Democratic Voting Systems)**.** We will define a *democratic voting system* as one in which:

1. No voter is a dictator

2. The voting system is impartial

3. Ballots are anonymous

Before moving forwards, we should verify that such systems exist. Take first-past-the-post, where the winner is the alternative that has the greatest number of first preferences compared with all other alternatives.

This satisfies non-dictatorship: if a would-be dictator d is in a position to make any alternative win, then all alternatives must have received the same number of votes, and therefore *any* other voter is also pivotal, and hence d is not unique and cannot be a dictator.

Impartiality is easy to verify. As the candidate with the highest vote total wins without any reference to the candidate themselves, the voting system must be impartial. Likewise, ballots are anonymous as switching labels of who cast which ballot will not affect the vote totals. We conclude that first-past-the-post is a democratic voting system.

5.2 A Selection of Voting Criteria

We have already met three possible criteria for a voting system: Unanimity, Independence of Irrelevant Alternatives, and Non-Dictatorship. From now on, we will take the latter to be a given; there is only so much analysis one can do on a decision-making system in which one person alone influences the outcome.

This already creates a trichotomy: for any voting system, it can *either* satisfy Unanimity, the Independence of Irrelevant Alternatives, or neither. This motivates research of other possible criteria and their mutual compatibility.

5.2.1 The Condorcet Paradox

Definition 5.5. We define alternative $x \in A$ as the *Condorcet winner* (W_{Cond}) of an election E if $\forall y \in A$, taking $A' = \{x, y\}$, we have $x \succ_E y$. In other words, an alternative is the Condorcet winner if and only if it beats every other alternative in a head-to-head matchup.

Likewise, we can define the Condorcet Loser (L_{Cond}) as an alternative that loses to every other alternative in a head-to-head matchup: $\forall y \in A$, taking $A' = \{x, y\}$, we have $y \succ_E x$.

Theorem 5.6. *An election can have at most one Condorcet winner and at most one Condorcet loser, if ties are disallowed.*

Proof. Suppose an election has two Condorcet winners and disallows ties, x and y. Then $x \succ_E y$ and $y \succ_E x$, and so $x \approx y$, a contradiction. The same reasoning shows there is at most one Condorcet loser. $\square$

Note that it is entirely possible that there is *no* Condorcet winner. A simple example would be Rock-Paper-Scissors. If we view these as alternatives, then Rock beats Scissors, Scissors beats Paper, and Paper beats Rock. Hence no alternative beats all other alternatives and there can be no Condorcet winner. If we introduced any number of alternatives that would lose to at least one of Rock, Paper, or Scissors, then we can quickly see that there is no guaranteed Condorcet winner for more than two alternatives. We will elaborate upon this possibility shortly.

Definition 5.7. A electoral system E satisfies the *Condorcet Winner criterion* if for all elections, if there is a Condorcet winner, then the Condorcet winner must win the election.

Definition 5.8. A *Condorcet cycle* in a social ranking R is a subset $C \subseteq A$ such that for each $c \in C$, there exists $d \in C$ such that $c \succ_R d$; every alternative in C must beat at least one other alternative.

If we have an example of a Condorcet cycle, then we observe the *Condorcet paradox*: we have a chain of alternatives, each of which beats the next, resulting in there being *no* Condorcet winner.

Although the Condorcet paradox can be seen in games such as rock-paper-scissors (as we will examine in the next section), it is difficult to find concrete examples of this in real-world votes. However, in the 2017 – 2019 U.K. Parliament, we find a very good example. When faced with the question of the United Kingdom's future relationship with the European Union, the House of Commons seemed to prefer:

1. Leaving without a deal to leaving with a deal (Vote on the Withdrawal Agreement Bill, 29$^{\text{th}}$ March 2019)
2. Remaining to leaving without a deal (Spelman Amendment, 13$^{\text{th}}$ March 2019)

3. Leaving with a deal to remaining (Vote on the Withdrawal Agreement Bill, 22^{nd} October 2019)

In the first case, the default alternative to voting down the Withdrawal Agreement Bill would have been leaving without a deal. In the second, the House explicitly ruled out leaving without a deal. In the third, the House endorsed the Withdrawal Agreement Bill (although later refused to allow the Government to proceed with its legislative program). Although somewhat convoluted, this does nonetheless show that the Condorcet paradox is entirely possible and not just a theoretical matter in the design of voting systems.

5.2.2 Winning Sets

Given that the Condorcet winner may not always exist, we want to investigate winning criteria that are guaranteed under any preference profile. We can generalise the existence of the Condorcet winner to a set of alternatives that will always win a head-to-head matchup; this is the*Smith set*.

Definition 5.9. The *Smith set*, $S \subseteq A$, is the minimal subset of A such that for all $s \in S$ and for all $a \in A/S$, $s \succ a$ pairwise.

Theorem 5.10. *For any social ranking E, the Smith set S exists and is unique.*

Proof. Let $S := A$. Then, for each element x of S, check which alternatives in S beat it. If x is beaten by every alternative in S, then remove x from S. Repeat until there are no elements that can be removed.
Suppose S' that is a Smith set for A. Now assume that S and S' intersect, that there is some $x \in S$ that is not in S'. Then each $s \in S'$, we must have that $s \succ_R x$.
Hence x only beats members of S that are not in S'. Then by our definition of S, all of these members are redundant as they will be beaten by every alternative S'/S and hence should be removed from the set. Furthermore, as x no longer beats any alternative in S, x cannot be in S. Hence S consists only of elements in S', and we conclude that $S \subseteq S'$.
Now assume S'/S is non-empty. Then by our construction of S, every $s \in S$ beats every element in S'/S. However, as S'/S consists only of alternatives in S', this would imply that these alternatives do not belong in S'. This is

a contradiction, and so the intersection must be empty. Hence $S = S'$, and we conclude that the Smith set exists and is unique, and therefore must be minimal. $\square$

Definition 5.11 (Dominating Set)**.** A *dominating set* is a subset $D \subseteq A$ where for each $d \in D$ and each $a \in A/D$, we have $d \succ_R a$.

Lemma 5.12. *For any two dominating sets D and E such that, without loss of generality, $|D| \geq |E|$, then we have $E \subseteq D$.*

Proof. Suppose there exists some alternative $e \in E/D$. Then for every $x \in D$, we have $x \succ_R e$. As D is at least as large as E, there must be at least one alternative d in D but not E. As $d \succ_R e$, we conclude that e does not beat every candidate outwith E and therefore E cannot be a dominating set, contradicting our original assumption. Hence the set E/D is empty and $E \subseteq D$. $\square$

Corollary 5.13. *The Smith set is the smallest non-empty dominating set.*

Definition 5.14 (Smith Criterion)**.** The *Smith Criterion* is satisfied by an electoral system if the set of winners is always a subset of the Smith set.

Definition 5.15 (Landau set)**.** For a social ranking R, the *Landau set* is a subset $L \subseteq A$ such that for all $p \in L$ and all $q \in A/L$, there exists alternative s such that we have $p \succ s$ pairwise and $s \succ q$ pairwise. It is no restriction on which alternative s can be; it can be either p or q, reducing it to $p \succ q$.

Lemma 5.16. *The Landau set always exists and is a subset of the Smith set.*

Proof. To see that the Landau set always exists, suppose that there are no alternatives such that the conditions for the Landau set are satisfied. Let a be the highest-ranked alternative in R. By our assumption that the Landau set does not exist, then for some other $c \in A$, there is no $x \in A$ such that $a \succ x$ pairwise and $x \succ c$ pairwise. This also implies that a does not beat c pairwise for any alternative c.

However, if we apply this logic to the second-highest-ranked alternative in R, say b, then we find that as b does not appear in the Landau set then it *also* cannot pairwise beat any candidate. As one of a or b must prevail in a pairwise matchup, this is a clear contradiction and we conclude that the

Landau set must exist.

For $x \in L$ and $y \in S$, then either $x \succ y$ pairwise (forcing x to be in S), or there exists z such that $x \succ z$ and $z \succ y$ pairwise (which forces z to be in S, and consequently x to be in S. Hence $L \subseteq S$. $\square$

St Chad has recently had a re-vote for its mayoral election. The results are curious:

St Chad			
Number	First Pref.	Second Pref.	Third Pref.
316	Robert	Claire	Angela
341	Robert	Angela	Claire
245	Angela	Robert	Claire
171	Claire	Robert	Angela
436	Claire	Angela	Robert
220	Angela	Claire	Robert

This is a perfect example of the Condorcet paradox in an election. If we remove *any* alternative, and assume that voters will move to their next-available preference, we obtain a different result for each candidate removed, as we can trivially resolve pairwise contests [7]. In this case, removing Robert means that Angela wins, removing Angela means that Claire wins, and removing Claire means that Robert wins.

However, the residents of St Chad neglected one very important factor: they still haven't settled on a voting system. Let us look at the considerations they will need to make for any given system.

5.2.3 Restrictions on Voting Behaviour

Definition 5.17. Suppose R is a social ranking with preferences P, and that $f_W(P)$ is such that $a \succ_R b$ for alternatives a and b. Let v be some additional voter that we append to N. R satisfies the *Participation Criterion* if for all

[7] More accurately, we can resolve them up to a definition of 'democratic'. If all ballots and alternatives are treated the same, then it comes down to whichever alternative has the highest number of votes.

choices of a and b, with p_v such that $a \succ_{p_v} b$, for $R' = f_W(P^{n \cup \{v\}})$, we have $a \succ_R b$.

The participation criterion means that if a voter ranks a and b the same as R before adding their vote, then R' should rank a and b the same as R. In other words, enforcing the Participation Criterion aims to remove a voter creating a tactical advantage by *not* voting.

For example, any voting system with a turnout threshold automatically fails the participation criterion. Referendums occasionally have these thresholds. The Italian electoral reform referendum in 1999 provides one good example of problems with the participation criterion not being enforced. The alternative 'Yes' beat 'No' by a wide margin, but because the turnout was just below 50% (the threshold for the referendum to be valid, and therefore the *de facto* threshold for the alternative 'Yes' to win), 'No' ended up winning. Voters who preferred 'No' were better off *not* voting, as this would mean 'Yes' would have to have won 50% of the total electorate [8] [16].

Often, the existence of two alternatives that are viewed as similar by voters can have an impact on the outcome of a voting function. To motivate us here, we can take a real-world example. In a by-election in the Australian electoral division of Swan in 1918, the Labor Party won with 34.4% of the vote, with the Country Party on 31.4% of the vote, and the Nationalist Party with 29.6% [17]. As the election was conducted under First-Past-The-Post (FPTP), which designates the unique winner as the candidate with the largest number of votes, the Labor Party candidate won.
f The Country Party and the Nationalist Party were ideologically similar, and hence it is likely that had *either* of them stood, then they would have won the seat. For virtually all Country Party voters, their second preference would have been the Nationalist Party, and *vice versa*. This allows us to roughly define what we mean by *clones*: alternatives that always appear together in preference relations.

This specific by-election had further implications. As a direct result, Australia introduced Instant Runoff Voting (IRV), a modified version of Single

[8] This is a counterexample to pairwise elections always being decided by the side with more votes, and is an example of asymmetric alternatives, meaning - by our previous definition - this system is *not* democratic, failing impartiality.

Transferable Vote, which is much less likely to be affected by clones than the previous FPTP system [17].

Definition 5.18 (Clones)**.** We say that alternatives $C = \{c_1, c_2, ...\}$ are *clones* if there is no $x \in N$ such that for some $a \in A$, $c_i \succ_{p_x} a \succ_{p_x} c_j$ for all pairs in C. In other words, alternatives are clones if each preference relation has a sub-tuple with exactly the elements of C.

Definition 5.19. Suppose we have a social ranking R for alternatives $A = \{a_1, a_2, ..., a_m\}$, and label one of these alternatives c. Let c' be such that if added to A and preferences were updated accordingly, c' would be a clone of A. R satisfies the *independence of clones criterion* if adding c' to A does not change the output of $f_W(P)$.

Definition 5.20 (Consistency Criterion)**.** Suppose we take arbitrary partitions of the set of voters $[n]_1, [n]_2, \ldots [n]_m$ such that $\bigcup_i [n]_i = [n]$. An social ranking R is *consistent* if $K = f_R(P^{[n]_i}) = f_R(P^{[n]_j})$ for all i and $j \neq i$ implies that $f_R(P) = K$. In other words, whenever we are able to partition the set of voters such that their societal sub-rankings are the same, then the voting system is consistent if this sub-ranking is also the overall ranking.

Definition 5.21 (Homogenity Criterion)**.** A social ranking or electoral system S is said to be *homogeneous* if replicating each ballot the same number of times produces the same result. In other words, if we let xP denote x duplicates of P (amending the membership of $[n]$ accordingly), then $f_S(xP) = f_S(P)$.

Definition 5.22 (Monotony Criterion)**.** A social ranking or electoral system S is *monotonous* if for all voters v and for all pairs of alternatives a_i, a_j where $a_i \succ_{p_v} a_j$ and $a_j \succ_S a_i$, switching the preference between a_i and a_j in p_v does not change the preference of S for a_i and a_j.

Definition 5.23 (Down-Monotony)**.** An electoral system E with a single winner, i.e. $|f_E(P)| = 1$ is *down-monotonous for singleton winners* if for a voter v, switching the preference relation for any two alternatives $a_i, a_j \notin f_E(P)$ in a revised preference profile P'^n, we have that $f_E(P) = f_E(P'^n)$ [18].

5.3 Tournaments

We will now investigate a graph-theoretic way of viewing voting systems.

Definition 5.24 (Tournament)**.** A *tournament* is a complete oriented graph. In the context of voting systems, we can define the nodes by the alternatives A and the direction of each edge by saying that if a dominates b, the arrow is directed from a to b, denoted $a \to b$.

Definition 5.25 (Transitive Tournaments)**.** A tournament with at least 3 nodes is *transitive* if for all $a, b, c \in A$, then $a \to b, b \to c$ implies $a \to c$.

We can rewrite a few results in the context of tournaments. For example, the Condorcet paradox occurring is equivalent to the tournament being non-transitive. Likewise, a Condorcet winner a exists if and only if for all other alternatives b, $a \to b$.

Definition 5.26 (Indegrees and Outdegrees)**.** In a directed graph, the *indegree* of a node is the number of edges that are directed into the node; likewise, the *outdegree* of a node is the number of edges that originate from the node.

Theorem 5.27. *In a transitive tournament with n alternatives, the set of outdegrees is* $\{0, 1, \ldots, n-1\}$.

Proof. Let D be a directed complete graph such that D is a transitive tournament. Suppose that there is no node with an outdegree of $n-1$. Hence there must be two nodes with the same outdegree; assume, initially, that two nodes a and b have an outdegree of $n-2$, and assume without loss of generality that $a \to b$. As there are only $n-2$ nodes other than a and b, b must dominate all of them, and a must dominate all but one, say c. However, this means that b dominates c which dominates a, contradicting transitivity.

The above establishes that we must have some alternative which dominates $n-1$ alternatives and therefore has an outdegree of $n-1$. If we remove this alternative, then we are left with a subgraph which must also be transitive. Following from the above argument, assuming that there are two nodes with an outdegree of $n-3$ results in a contradiction. If we proceed in this way, we conclude that no two nodes can have the same outdegree and the largest outdegree if $n-1$, resulting in $\{0, 1, ..., n-1\}$ being the only possible outdegree sequence. □

Corollary 5.28. *In a transitive tournament, there is exactly one Hamiltonian path (a walk that passes through each node exactly once).*

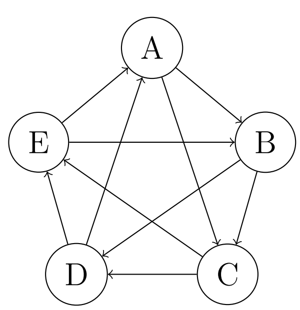
Proof. As there is exactly one node, a, with an outdegree of $n-1$, and therefore an indegree of 0, the Hamiltonian path must begin there as it cannot be reached from any other node. There is exactly one node, b, with an outdegree of $n-2$, which can only be reached from a. Examining the subgraph induced by removing a, we reach the same argument; any Hamiltonian path must begin at b as there it can no longer be reached from any other node, and as there is a unique node with an outdegree of $n-2$, which equates to being dominated by only one alternative in this subgraph, there is only one route this path can take. Hence, by induction, the Hamiltonian path is unique, and is defined by the path along the nodes corresponding to the outdegrees in the ordered list $(n-1, n-2, ..., 1, 0)$. □

Corollary 5.29. *Transitive tournaments are acyclic.*

Proof. Suppose we have a sequence of nodes in a transitive tournament that forms a cycle. As each node has a unique outdegree, let a be the node in this cycle with the largest outdegree. As a is in the cycle, it must be dominated by at least one alternative, as otherwise a would not be reachable. Now take the subgraph induced by removing all nodes with a larger outdegree than a, and let m be the number of nodes remaining. This subgraph must also be transitive, and therefore a has an outdegree of $m-1$, and therefore is not dominated by any alternative, a contradiction. □

Definition 5.30 (Balanced tournament)**.** A tournament is *balanced* if every alternative pairwise wins against and is pairwise defeated by the same number of alternatives.

Below we have an example of a balanced tournament, in which each of the five alternatives beats - and is beaten by - two other alternatives.



Theorem 5.31. *A tournament of size n can only be balanced if n is odd; if so, then there is always a way to construct a balanced tournament.*

Proof. Suppose a tournament with n alternatives is balanced. If n is even, then each vertex v neighbours an odd number of vertices. Hence the indegree and outdegree of v must be different. Suppose that the outdegree is greater than the indegree; then each of the n alternatives v must beat k alternatives, where $k > \frac{n-1}{2}$. Then the total number of contests in which we need to proclaim a winner is $nk > \frac{n(n-1)}{2}$. However, the total number of edges in a complete graph is $\frac{n(n-1)}{2}$, a contradiction. If we make the opposite assumption that each indegree is greater than each outdegree and then count the total number of losses we observe, then we can proceed similarly and obtain a contradiction. Thus n cannot be even.

To create a balanced tournament on n vertices for odd n, number the vertices $1, \dots, n$. Then we want to achieve an outdegree of $m = \frac{n-1}{2}$ for each vertex. For each v_i, let v_i beat each v_j where j is in the set $\{i+1, i+2, \dots, i+m\}(mod(n))$.

Clearly, this makes the outdegree of each vertex m as required, but does it result in contradicting winners? If we arrange the vertices $1, \dots, n$ clockwise, each vertex v beats the first m vertices travelling clockwise, and loses to the first m vertices travelling anticlockwise. As $2m = n-1$, we have accounted for all edges of v, and there is no vertex that v both beats and loses to. Hence the proof is complete. $\square$

6 Tactical Voting

In this section we shall examine the notion of *tactical voting*; loosely speaking, how an individual voter can change their vote to be different to their preference profile in order to make the overall outcome more similar to their preference profile.

6.1 Preference Classes and Metrics

We can return to the earlier notion of *preference classes*, and formally define them. In particular, preference classes are an equivalence relation.

Definition 6.1 (Preference Class)**.** Suppose we have a preference profile P, a set of alternatives A, and a set of voters $[n]$. For each $i \in [n]$ and for some $A' \in A$?, we say that i is in *preference class* $P(A')$ if and only if $p_i = A'$. This is an equivalence relation under equality. Only transitivity is non-trivial to prove; if $[p_i] = [p_j]$ and $[p_j] = [p_k]$, then p_i has the same preference ordering A' as p_j which has in turn the same ordering as p_k, and hence $[p_i] = [p_k]$.

We can also easily extend this to look only at the order between *subsets* of alternatives. This is equivalent to removing alternatives from the set of alternatives and from all profiles until we have the set of alternatives of interest, and hence do not need a separate proof that this is also an equivalence class.

We'll develop some further notation here. As we are primarily concerned with the size of each class, we'll write $[a, b, c, \dots |m]$ to indicate that we have a preference class of size m with order of preference $a, b, c, \dots$. If we're looking at the relation between specific preferences, then we will write $[a \succ b \succ c|m]$ for the class of relations $[p_i]$ that have $a \succ_{[p_i]} b \succ_{[p_i]} c$, without regard for other preferences.

Definition 6.2 (Preference Metric)**.** The metric between two preference profiles x and y, $d(x, y)$, is defined by the total difference between the indices of each metric. Using $I(x_i)$ to denote the index value of the i^{th} preference of x:

$$d(x, y) = \Sigma_{i \in A} \frac{|I_(x_i) - I_(y_i)|}{2}$$

The sum of all index differences will always be even; if we take identical preference relations x, y and swap two alternatives in y, if we then denote the displacement between these alternatives in x by s, then $d(x, y) = 2s$. Resetting $x := y$ and then swapping two more pairs of alternatives, we can create any preference profile. As this cumulative distance will always be even, we reduce the sum by a factor of two to allow $d(x, y)$ to take all positive integer values.

We can use this metric to quantify tactical voting in social rankings; there are different ways of quantifying tactical voting for elections, which we will meet later.

We can now move on to studying tactical voting.

6.2 Tactical Voting in Social Rankings

We will make a slight clarification to preference relations here. When we talk about the *preference relation* for voter i, we have so far been specifically referring to the order of preference that each voter has for each alternative *in isolation* of the specific voting system. However, as we will see, there are situations where - paradoxically - a voter can obtain a result closer to this 'true' preference relation by submitting a different preference relation.

We will denote a voter i's *true preference relation* by the usual notation, p_i. Their *tactical preference relation*, $\hat{p}_i$, is a preference relation that, when substituted in place of p_i in P, makes f_V 'more similar' to p_i.

Definition 6.3 (Tactical preference relation)**.** Let p_i be the (true) preference relation for voter i and R the social ranking under welfare function W. A *tactical preference relation* for voter i, is any preference relation $\hat{p}_i$ with corresponding with new social ranking $\hat{R}$ where $d(p_i, \hat{R}) \leq d(p_i, R)$.

It is, of course, unlikely that a *single* tactical vote will change the social ranking. Hence we can use the notion of preference classes once more. As all preference relations in a preference class are the same, we can replace $\hat{p}_i$ in the above definition with $[\hat{p}_i]$, and likewise $\hat{R}$ is the social ranking corresponding to changing the preference relations $[p_i]$ to $[\hat{p}_i]$.

An intuitive way to think about this is that the order of preferences becomes closer to $[p_i]$ after voters in class $[i]$ change their ballots.

For elections, the notion of tactical voting is much simpler. It asks only whether a voter can manipulate their ballot to change the set of preferences *either* to make the set of winners contain someone that the voter prefers to at least one of the members of the previous set of winners, *or* removes an alternative from the set of winners that the voter prefers less than any other alternative in the set of winners. In the former case, we name the tactical voter an *optimistic voter*, and in the latter a *pessimistic voter*.

In its simplest (and most useful) form, this can be reduced to assuming the set of winners is singleton; i.e. there is a unique winner. A tactical vote in this instance must be one where the voter is both optimistic and pessimistic, which we will designate as a *realist voter*.

Definition 6.4 (Imposed Systems)**.** An electoral system E is *imposed* if there exists $a \in A$ such that $a \notin f_E(P)$ for any choice of P. This is to say an imposed system is one in which some candidate or candidates will never win.

We define two types of tactical voter.

Definition 6.5 (Optimistic Voters)**.** For a voter i_t and an election E, i_t votes *optimistically* if for the tactical preference relation p'_{i_t}, for all $w \in f_E(P')$ and for all $v \in f_E(P)$, we have $w \succ_b v$.

Definition 6.6 (Pessimistic Voters)**.** For a voter i_t and an election E, i_t votes *pessimistically* if for the tactical preference relation p'_{i_t}, there exists at least one $w \in f_E(P')$ and at least one $v \in f_E(P)$ such that $w \succ_b v$.

We will also define the concept of *manipulability*: this refers to the possibility of a voter to vote tactically. We will say that a voting system is *manipulable* if there is a preference profile such that for at least one voting class, there exists a tactical voting strategy.

Theorem 6.7. *In any election either without a Condorcet winner, or with a Condorcet winner that does not win the truthful election, at least one voting class has an optimal tactical voting strategy.*

Proof. We will prove this by contrapositive: if there is no optimal tactical voting strategy for any voting class, then the election must have a Condorcet winner that wins the truthful election.

Suppose there is no optimal tactical voting strategy for any voting class $B_i \in P$. This implies that for all B_i, the overall winner, W, will not be changed by B_i changing votes to another class. In this way, we can eliminate alternatives until we arrive at a choice between only two alternatives. If we consider only choices between W and one other alternative, then by the previous reasoning, W must win each of these elections. Hence W is the Condorcet winner. □

As Condorcet winners that win truthful elections are rare in the real world, there is an important implication here: in practice, there is usually an optimal tactical voting strategy for at least one voting bloc.

6.3 Theorems of Tactical Voting

We are now going to demonstrate something profound about voting: democracy is incompatible with non-manipulability.

The Duggan-Schwartz Theorem gives us another restriction on democratic voting systems. This time, we will that tactical voting cannot be fully eliminated.

Theorem 6.8 (Duggan-Schwartz Theorem)**.** *Suppose V is a non-imposed voting system that cannot be manipulated by an optimistic or a pessimistic voter. Then there exists at least one $i \in [n]$ such that p_i's top preference is always in $f_E(P)$, and is therefore a dictator.*

We will give an outline of the proof, as shown in [18].

Proof. First of all, we can show that a voting system is down-monotonic if it cannot be manipulated by optimists or pessimists.

Next, we can show that the set of all voters is a dictating set under V, and furthermore, that for any two disjoint subsets of $[n]$, exactly one is a dictating set.

As a result of these two statements, we can continuously partition $[n]$ into a chain of subsets, each of which must be a dictating set for V, until we reach a subset consisting of one voter; this voter must therefore be a dictator. □

The last part of this proof is encapsulated in the Gibbard-Satterthwaith Theorem [19]:

Theorem 6.9 (Gibbard-Satterthwaith Theorem)**.** *Let V be a non-dictatorial voting system with a singleton winner with at least three alternatives, all of which are viable. Then the voting system is manipulable.*

7 Voting Methods

With all this considered, St Chad finally needs to settle on a voting system. With a reformed town charter, they can no longer simply elect a mayor, but must also choose a town council to properly scrutinise the mayor's activities.

This means that they need to consider not just voting systems for electing a single individual, but also ones that will work for electing several individuals, or potentially political parties.

7.1 Independent Alternatives

7.1.1 Llull's Method

The Llull method of election, f_{Llull}, is outlined in a short paper titled *De Arte Eleccionis* [20]. Initially proposed for the election of members of the clergy, the Llull method is a pairwise electoral system that produces a single winner. Llull's description only explicitly outlines a system with nine candidates, but we can generalise it to more candidates with relative ease. Here, we will briefly explain Llull's original system.

Llull begins by taking a set of alternatives, $\{b, c, d, e, f, g, h, i, k\}$ [9]. He then creates an upper-triangular array which details every possible pairing of these alternatives:

$$\begin{array}{cccccccc}
bc & cd & de & ef & fg & gh & hi & ih \\
bd & ce & df & eg & fh & gi & hk & \\
be & cf & dg & eh & fi & gh & & \\
bf & cg & dh & ei & fk & & & \\
bg & ch & di & ek & & & & \\
bh & ci & dh & & & & & \\
bi & ck & & & & & & \\
bk & & & & & & &
\end{array}$$

We begin in the top-left corner, and hold a vote among the electorate as to who is preferred between the two candidates listed in the pairing: in this case, b and c. If $b \succ c$, then we evaluate the relation between b and d; likewise, the relation between c and d if $c \succ b$. We continue in this way until we reach a comparison with the last alternative, k. Either k wins, or the other alternative that has reached the last round wins.

This method of pairwise comparisons is non-exhaustive. Not all of the $\frac{n(n-1)}{2}$ possible comparisons between n alternatives are evaluated; in fact, we only

[9]These were the alternatives Llull used in his original paper.

need $n-1$ comparisons, allowing this method to run in linear time for n alternatives.

However, there is an obvious paradox that we encounter: what if we switch the orders of the candidates around? For instance, we could create a Condorcet cycle between b, c, and d by letting the pairwise comparisons bc, cd, and bd be such that $b \succ c$, $c \succ d$, and $d \succ b$. If we evaluate bc first, then next we evaluate bd, resulting in d winning. However, were we to switch the order of c and d, so we would evaluate bd initially, and next evaluate bc, then b would win [21].

This method satisfies the Condorcet criterion, in that if a Condorcet winner exists, then they must win. This is trivial, as a Condorcet winner wins every pairwise matchup, which is a sufficient condition for victory under the Llull method. It does not satisfy the Smith criterion:

Proof. Suppose that the last alternative, x, wins in a pairwise contest with exactly one other alternative, y.

x is in the Smith set if y is in the Smith set; if y is in the Smith set, then x must be in the Smith set as otherwise y would not be preferred over every alternative outside the Smith set.

If y is not in the Smith set, then x is in the Smith set if and only if y is the only alternative *not* in the Smith set. If there were some other alternative z such that $y \succ z$, then x would not be in the Smith set but would still win. Hence by this construction, the Llull method is shown not to satisfy the Smith criterion. □

Definition 7.1 (Llull's Method)**.** We define f_{Llull} by means of an algorithm. Say we have $E = \{N, A, P, f_C\}$, and $|N| = n$. Then we define $f_C(P)$ by:

1. Define $x = (a_i, a_j)$ for $a_i, a_j \in A$.

2. Set $i = 1, j = 2$.

3. Evaluate $f_C(P'^N)$ for $E' = (N, a_i, a_j, P, f_C)$.

4. For $j < n$: If $a_i \succ a_j$ then set $j := j + 1$. Otherwise [10] set $i := j$ and $j := j + 1$. Repeat step (4). For $j = n$: if $a_i \succ a_j$, then $f_C(P) = a_i$. Otherwise, $f_C(P) = a_j$

or in pseudocode:

```
N = [1,2,...,n]
A = [a_1,a_2,...,a_m]
P = [p_1,p_2,...,p_n] #i.e. p_1 = [3,1,2,...], etc

pairwise_comp(P,a,b) {
  a_counter = 0; b_counter = 0
  for i in length(P){
    for j in length(P[i]){
      if j = a {
        a_counter += 1
        break
      }
      if j = b {
        b_counter += 1
        break
      }
    }
  }
}

f_C(N,A,P) {
  i = 1; j = 2
  for i in n-1 do:
  x = (A[i],A[j])
    if pairwise_comp(P,x) = x[1] {
      j += 1
    } else {
      i = j
      j += 1
    }
  return A[i]
```

[10] We assume, as Llull did in his original paper, that there are no ties.

Let us now investigate the possible winners under the Llull method if we switch the order in which we evaluate the alternatives. To do so, it will be easier to model the election as a tournament.

Although we will nominally model this as a complete directed graph, we have an additional restriction on which paths we can take. We can only move from a_i to a_j if $i < j$, where each alternative is in some predetermined order.

Let us denote the winner of a Llull tournament by a_w for $1 \leq w \leq m$ for m alternatives. Then it is easy to see that a_w satisfies:

1. $a_w \rightarrow a_{w+i}$ where $w + i \leq m$;
2. There exists $j < w$ such that $a_w \rightarrow a_j$.

These conditions are necessary, but not sufficient. It must also be the case that such a value for a_j is paired with a_w. If there is some a_k before a_w such that $a_k \rightarrow a_w$, then a_w is only the winner of the tournament if we do not have the pairing $a_w a_k$. However, it is trivial to see that the order of pairings after a_w is irrelevant; if a_w dominates whichever alternative wins the subtournament T' for $(a_1, a_2, ..., a_{w-1})$, then a_w wins overall.

Let us denote two classes of alternatives in T': those that dominate a_w, D, and those that do not, S. Further assume that D is nonempty, as otherwise a_w dominates every alternative and trivially wins regardless of the order of other alternatives. It is clear that some alternative in S must win this subtournament if a_w wins the tournament; however, so long as the alternative is in S, we do not have any further restrictions.

Lemma 7.2. *If a_w wins the tournament, then either a_w is preceded by (s_1, s_2) for $s_q, s_2 \in S$, or there must be at least one $s \in S$ and at least one $d \in D$ such that $s \rightarrow d$ where the pairwise contest (s, d) is evaluated directly before any pairwise contest with a_w.*

Proof. a_w dominates both s_1 and s_2 so the winner of this pairwise contest is irrelevant. As $d \rightarrow a_w$, we cannot have the pairing (d, a_w). Hence for some s, we must have the pairing (s, a_w). Thus for some values of d and s, we have $s \rightarrow d$ to avoid the pairing (d, a_w). □

Let us strengthen the statement: what are the conditions for a_w to win *regardless* of the order of alternatives in T'?

Let B be the set of alternatives such that $a_b \rightarrow a_w$ for $a_b \in B$. B cannot be equal to the Smith set, as this would imply that alternatives in B beat every other alternative in T', meaning a_w does not win overall. So there must be at least one alternative a_s that is in the Smith set but not in B (call this subset S/B) where a_s, and hence a_s beats at least one alternative a'_b in B.

Necessarily, all alternatives in B are in T'. If we evaluate the tournament of B, we only need to ensure that the winners a_b of each pairwise contest have an alternative a_s in S/B such that $a_s \rightarrow a_b$ that succeeds it in T. Hence we reach the following condition:

Theorem 7.3 (Theorem of Llull Tournaments)**.** *An alternative a_w wins a Llull tournament T if and only if the following is true:*

1. *a_w dominates every successive alternative;*

2. *each alternative that dominates a_w precedes it;*

3. *and each alternative a_b that dominates a_w is succeeded by an alternative in the Smith set S of T that does not dominate a_w, a_s, that dominates a_b.*

7.1.2 First-Past-The-Post

We will now consider First-Past-The-Post (FPTP), which has traditionally been used in the United Kingdom and the United States for their elections [22, p.35-38].

Definition 7.4 (FPTP)**.** The social choice function *FPTP*, f_{FPTP}, for $f_{FPTP} : P \rightarrow A$, is defined by:

$$f_{FPTP}(P) = a_i \in A : i = max\{x_1, ..., x_m : x_j = \sum_{r=1}^{n} x_j \succ_{p_r} x_k : j \neq k\}$$

Note that this can easily be transformed into a social welfare function if we rank each x_i from largest to smallest and apply that same ordering to A:

$$f_{FPTP}(P) = (a_i)^N \in A : a_1 = max(\mathbf{x}), x_i > x_{i+1} \forall i \in A :$$

$$\{\mathbf{x} = x_1, ..., x_m : x_j = \sum_{r=1}^{n} x_j \succ_{p_r} x_k : j \neq k\}$$

This can be easily written in pseudocode:

```
N = [1,2,...,n]
A = [a_1,a_2,...,a_m]
P = [p_1,p_2,...,p_n] #i.e. p_1 = [3,1,2,...], etc

f_FPTP(N,A,P) {
  counter = [x for x in length(A)]
  for i in length(P){
    j = P[i,1]
    counter[j] += 1
  }
  return A[max(counter)]
}
```

7.1.3 Single Transferable Vote

Single Transferable Vote (STV) is unique, in the sense that it can be naturally be extended to multi-member systems. We will define the algorithm as follows [23, p.12-16]:

1. Calculate the quota, q: $q = \frac{n}{s+1} + 1$ [11]

2. The *first preference* on each ballot is tallied.

3. If any alternative a has q or more votes, then a is elected and can be removed from each preference relation. Let x be the number of votes more than q that a received. (Note that multiple alternatives may reach the quota on one round, in which case we repeat for each such alternative.)

4. Examine all profiles where a had previously been the first preference. If y is the total number of profiles that now alternative b as the first preference, multiply these preferences by a weight of $\frac{xy}{x+q}$ and remove all preferences before b.

[11] Alternative quotas may be used. Specifically, this is the *Droop quota*.

5. If all seats have been filled, terminate the algorithm. Otherwise, tally all (new) first preferences and eliminate the alternative with the fewest. Remove this alternative from each preference relation.

6. Repeat steps 2-6 until all seats have been filled.

Note that we assume that all voters fill out all preferences, so the algorithm will eventually terminate; if voters do not have to submit a preference for all candidates, then auxiliary methods must be used to decide who wins the final seat.

In pseudocode:

```
def stv(P,A,s){
    #P: preference profile
    #A: set of alternatives
    #s: number of seats

    #Make a copy of P so we're not editing P itself
    P_copy := P

    #Create weights for P to use later
    P_weights = [1 for p in P]

    #Track seats distributed
    seat_track = 0

    #Check if we've managed to calculate a winner
    #Basically a winner needs to get quota + 1 votes
    filled = False
    quota = sum(P[1])/(s+1)

    #Begin a loop - this keeps going until we have a winner
    while not filled{

        #votes keeps track of how many profiles currently
        #rank each candidate as their first pref
        votes = [0 for i in A]
```

```
W = [] #set of winners

#Populate votes
for i in range(len(P_copy)){
    #if i has first pref ’a’, add 1 to the
    #total number of votes for ’a’
    votes(A.index(P_copy[i][0])) += P_weights[i]
    }

#Check if we have winner
for a in A {
    if votes[A.index(a)] >= quota:
        W.append(a)
        #Now re-adjust weights
        x = votes[A.index(a)] - quota
        y = sum(p in P_copy where p[0] = a
        for p in P_copy if p[0] == a{
            P_weights.index(p) *= (x*y)/(x + quota)
        }
        # Now remove ’a’ as an alternative
        for p in P_copy{
            P_copy.remove(a)
        }
        seat_track += 1
    }


#Check seats
if seat_track = s {
    filled = True
}

#Eliminate lowest-polling candidate
if not filled{
    #This function will always choose someone
    #so we don’t need to be that careful
    elim = A[tabs.index(min(tabs))]
    #Now remove this alternative from every profile in P
```

```
            for i in P_copy{
                P_copy.remove(elim)
                }
            }

        return W
    }
}
```

7.1.4 Borda Count

Definition 7.5 (Borda Count)**.** The social welfare function *the Borda count*, f_{BC}, for $f_{BC} : P \to A$, is defined by the algorithm [1, p.132-137]:

1. Establish a tally of points for each alternative, starting with zero each.
2. For each tuple in P, go from left to right, and add $n-1$ points first alternative's total number of points, $n-2$ to the second alternative, and so on, down to adding 1 point to the second-to-last alternative.
3. Order the alternatives from largest to smallest number of points. This determines the social ranking.

In pseudocode:

```
N = [1,2,...,n]
A = [a_1,a_2,...,a_m]
P = [p_1,p_2,...,p_n] #i.e. p_1 = [3,1,2,...], etc

f_BC(N,A,P) {
  c = length(A)
  counter = [0 for x in length(A)]
  for i in (1, length(P)){
    for j in (1, length(P[i])){
      counter[P[i,j]] += c-j
    }
  }
  ranking = ["" for x in length(A)]
  for i in (1, length(A)){
```

```
      index = arg_large(counter,i) #arg_large(A,i) returns i-th
          largest element of array A
      ranking[i] = A[index]
    }
  }
```

Initially devised by Nicolas de Cusa for the election of Holy Roman Emperors, this method was later reinvented by Jean-Charles de Borda for elections of members of the French Academy of Sciences [1, p.111, p.132-133]. What makes this method different from those we have seen before is that it utilises *pegged voting*; each ballot is converted into 'points' for each alternative, based on the relative positions in each preference relation. This allows for the creation of an overall social ranking from the largest to the smallest number of points. We can now consider what is likely the largest use case of a pegged voting system: the Eurovision Song Contest.

7.1.5 Eurovision

The Eurovision Song Contest is notable not only for its vast range of performances - from heart-wrenching ballads to unashamedly camp pop with questionable outfit choices - but also for its spectacular number of different voting systems over time [24].

Although there have been several Eurovision voting systems (specifically, social welfare functions) used over the years, they all have a few commonalities. Firstly, they employ a system that uses several tiers - the public [12] and a panel of jurors in each country participating in the contest all cast votes, which are weighed in some manner, and then collated in some way across all countries to form a final result. Secondly, the public and jurors cannot vote for their own country; as this means ballots are not anonymous (it *does* matter who casts which ballot), the voting system is not democratic.

The votes from the public and jurors are transformed into a social ranking via some social welfare function. Each alternative then receives 'points' based on its position in the ranking; this allows for the creation of a supranational social ranking, with countries ranked in descending order by number of points

[12] Public 'televoting' was introduced for all countries in 1998 [24].

[13]. The top ten alternatives in each ranking are given, in ascending order, $1, 2, 3, \ldots, 8, 10, 12$ points respectively [25]. The sum of the points given to each alternative gives a total which can then be used to create this new social ranking. The winner of the contest is the alternative with the greatest number of points.

7.2 Weighed Alternatives

Given St Chad's wish to elect a town council, we now introduce a class of voting methods that is different to the voting systems we have described thus far. In many electoral systems - especially ones used to elect most modern legislatures - the idea of adding *weight* to each alternative is inbuilt. This is to say that each alternative is not just a single candidate to be elected, but in fact a *number* of candidates who will receive greater representation among members of the winning set dependent on the number of votes cast for this alternative.

This is more commonly known as *proportional representation*. In a proportional election, we change our definition of the social choice function f_E; the output of this function is now a multiset, meaning that it can include repeats of alternatives. For instance, for a set of alternatives (a, b, c), the output of $f_E(P)$ could be $\{a, b, a, c, b, a\}$. For our purposes, we will simplify this by quantifying each alternative; for instance, the previous set would be $\{3a, 2b, c\}$.

7.2.1 The Alabama Paradox

The *Alabama Paradox* illustrates why it is necessary to have rigorously-defined algorithms of proportional representation rather than relying on crude formulaic allocation methods.

During the late 19$^{\text{th}}$ century, the United States allocated congressional seats to states by the *largest remainder method*. To implement this method, we follow a simple procedure. Firstly, we multiply the proportion of each state's population by the number of seats in House of Representatives. We take

[13] Although we won't consider ties, the mechanism for resolving a tie is that the alternatives that performed earlier in the contest are ranked higher than those that performed later.

the whole number part of this result, and give each state at minimum this number of seats. For n remaining seats, we order the states from largest to smallest remainder, and give the first n states an additional seat.

Although this method seemed watertight, there is one significant issue: the decimal expansions are as good as random for working out where the additional seats should go. In 1880, for instance, calculations show that increasing the total number of seats in the House of Representatives would lead to Alabama's number of seats *decreasing* [11, p.179-181].

Modern systems are susceptible to it as well. In 2009, the Federal Constitutional Court of Germany ruled that the German electoral system that the country had been using violated their constitution [26]. Article 38 of the Basic Law reads [27]:

> Members of the German Bundestag shall be elected in general, direct, free, equal and secret elections.

The German electoral system was guilty of violating *directness*; it was possible in certain conditions that voting for a party would result in that party *losing* seats [14] (and hence it fails to satisfy the monotonicity criterion).

7.2.2 D'Hondt/Jefferson Method

Although now commonly known as the d'Hondt method, after the Belgian mathematician who formally defined the algorithm [11, p.306], this method is equivalent to one devised by Thomas Jefferson in the 1790s for the allocation of congressional seats [11, p.92] [11, p.297-298]. Crucially, it avoids the Alabama Paradox, and also satisfied one of George Washington's other concerns about the Hamilton method: that the divisors used for each state remained the same.

Definition 7.6 (D'Hondt Method)**.** The D'Hondt method, for a set of voters N, a set of alternatives A, a singleton preference P, and s seats is defined by [28, p.24-36]:

[14] Note that, in practice, this is not a major issue; the German electoral system is designed to guarantee overall proportionality of seats for each party, and modifies the total number of seats in the Bundestag to ensure this.

1. Total the number of votes for each alternative and label this $V = \{v_1, v_2, ..., v_m\}$.
2. Let $W = \{w_1, w_2, ..., w_m\}$ be the number of seats allocated to each alternative.
3. Find the largest value $\frac{v_i}{w_i+1} : v_i \in V$.
4. Set $w_i := w_i + 1$.
5. Repeat steps 3-4 s times.
6. Each alternative a_i receives w_i seats.

In pseudocode, we define this algorithm as:

```
N = [1,2,...,n]
A = [a_1,a_2,...,a_m]
P = [p_1,p_2,...,p_n] #i.e. p_1 = [6], p_2 = [3], etc


#For s seats being allocated
f_dHondt(N,A,P,s){
  seats = [0 for a in A]

  votes = [0 for a in A]
  for i in (1,length(P)){
    votes[P[i]] += 1
  }

  quota := votes
  div = [1 for a in A]

  for i in (1,s){
    index = arg_large(votes,i)
    seats[index] += 1 ; div[index] += 1
    quota[index] = votes[index]/div[index]
  }
  return seats
}
```

8 Conclusion and Final Thoughts

St Chad remains unsatisfied by its discoveries in the world of voting theory. Mr. X has thus far remained elusive, likely as a figment of Madeline's mathematical imagination. Mrs. Keeting, Mr. Lemming, and Ms. Monroe have begrudgingly agreed to rotate the mayorship between them.

The townsfolk of St Chad, unable to create a voting system that will satisfy them all, return to the simplest possible system: whenever they need to reach consensus on an issue, they simply bicker among themselves until some opinion prevails. Is it democratic, efficient, or fair? No, but it *does* mean that no longer will St Chad be burdened with the inevitable consequences of voting mechanics.

This does, however, raise a rather serious point. We've shown throughout this report that, in several ways, voting systems are inherently unfair under some metric. There is no way to summarise the opinions of a diverse society into a single opinion without upsetting somebody.

Arrow's Impossibility Theorem places a hard restriction on the criteria voting systems can satisfy. Duggan-Schwartz and Gibbard-Satterthwaith both mean that we cannot eliminate tactical voting. Perhaps most profound is the simplest result of them all - Condorcet Paradox, which demonstrates that even *without* a voting system in place, we will occasionally derive contradictory and confusing results. All this shows that voting is *necessarily* imperfect. Even with all the technology in the world, we would not be able to create a totally satisfactory system.

Although we have learned many problems with voting systems, we have gained an insight into how these systems function, and as such, we *know* the problems we will encounter with them. As participants in democratic society, if we cannot mitigate the inherent issues with voting systems, we should at least be aware of their limitations.

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