Simple Special and General Relativity

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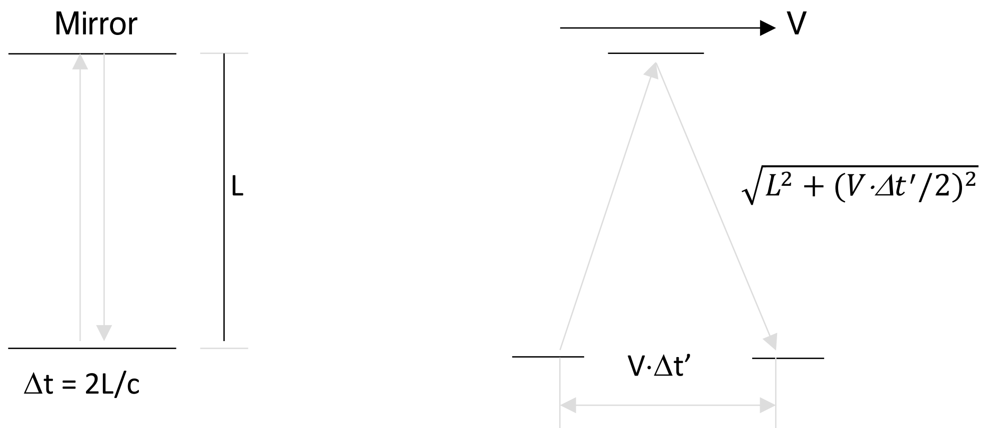
My introduction to special relativity was as an undergraduate engineering student. We were told that the speed of light was constant for all observers traveling at different velocities with respect to one another and then spent the rest of our time manipulating Lorentz transformations. Due to a combination of bad teaching and bad learning, I thought that moving clocks ran slow because it took time for the information to get back to the observer. Of course, this idea was wrong, but it was what I was left with. I will try to stay away from manipulating formulas but try to derive things as we go along (learning by thinking, I call it). I will try to do almost everything in special and general relativity using time dilation and length contraction. The aim is to get a good feel for relativity, without the complexity of tensors and all that fancy stuff the experts use. The serious reader needs to get out a pen and pencil and fill in some of the math I have left out.

Einstein's assumptions for special relativity were that in systems traveling with constant velocity with respect to each other (no acceleration), they are equal and the physics inside each of the systems is the same. He assumed that the speed of light in free space, with respect all observers, is a constant. This assumption was not well received because it really screws up space and time. This assumption came from Maxwell's equations in which one solution showed a traveling electromagnetic wave traveling at a velocity which seemed to be close to that of the speed of light, but it didn't say in what system this speed was measured. There were also experiments which showed that the velocity of light from a star was the same whether you were traveling, in the earth orbit, toward the star or away from the star.
General relativity is complicated. We will see how far we can get using what we learn in special relativity.
Other than some classical mechanics, everything will be done using only these assumptions. Relativity is not my field and so I'm afraid there's a lot of approaches that I haven't looked at and don't know about.

Special relativity

Let's start with the standard optical clock. This clock consists of an emitter/detector which emits pulses which go up to a mirror, spaced a distance L from the emitter, and then back down to the emitter/ detector where a new pulse is sent out, making one tick of the clock. There is an identical clock which is moving off to the right at a velocity V in another reference frame, which we will call the primed frame. So why the mirror? It turns out in relativity if you're going to measure a time interval, it is best to measure the start and end of the interval at the same place in whatever reference frame you are in. Time and space get mixed up so it is better if you can keep space out of it. We will see an example of this shortly.



What we want to calculate is the ratio of the time interval in the primed frame, as seen by an observer in the stationary frame, to the time interval in the stationary frame for one “tick” of the clocks. Now the velocity of light, c, is the same, $3\text{x}10^8$ m/sec, in both systems whether the system is moving with a velocity to the right or not. In our stationary system, one tick is just 2L divided by c. In the primed frame one tick is clearly going to be longer because the path is now longer than 2L. During the time interval Δ t’ (one tick of the moving clock) the detector has moved a distance V·Δt′. The distance the pulse must travel is, therefore, $2\sqrt{L^2 + (V\cdot\Delta t'/2)^2}$.

So, $\Delta t' = 2\sqrt{L^2 + (V\cdot\Delta t'/2)^2}$/c. With Δt = 2L/c, we get:

$$\Delta t' = \gamma\cdot\Delta t$$

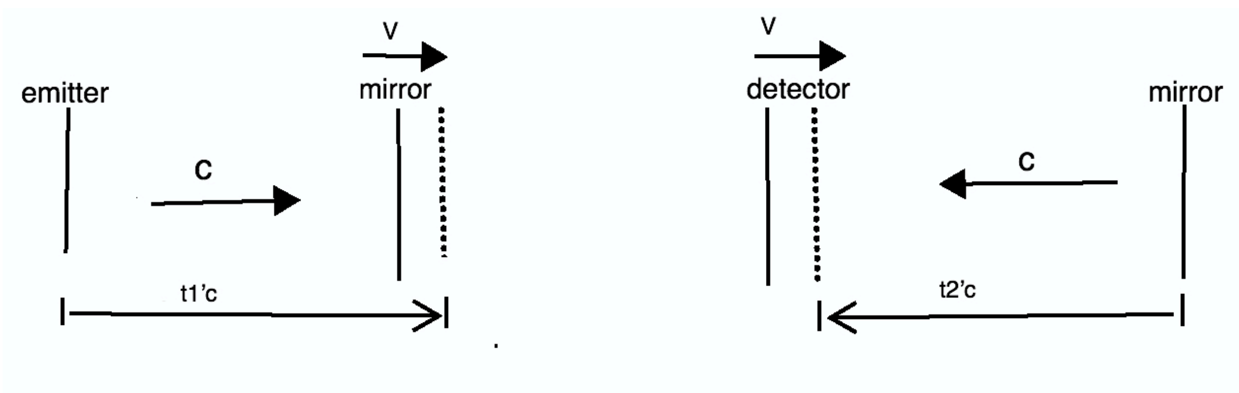
where $\gamma = 1/\sqrt{1 - V^2/c^2}$ a parameter greater than or equal to 1. In our everyday world where velocities are not much greater than a few hundred meters per second, this parameter is essentially 1 and so we rarely see any of this time dilation (GPS is the glaring exception-more on this later). If V$> c$, then the clock (and time) does not work anymore so the restriction is that V $\leq c$.

The takeaway from this is that the clock in the primed (moving) system is ticking slower than the stationary clock; Moving clocks run slow! time is slower in the moving system. By the way, if we were in the primed frame looking at the clock in the unprimed system, we would see it running slow by the same factor. Clocks moving by us run slower than our clock. Now Einstein said that without acceleration or gravity, these reference frames are equal. So, who is moving? The other guy, of course.

Back in my deep dark past I was part of an experiment at the Berkeley Bevatron to measure the lifetimes of the pi plus and pi minus mesons. If we could find a difference in the lifetimes, it would mean that a fundamental theory, CPT, was violated and therefore it would be a big deal. Alas (for us, but not for physics), we measured the lifetimes to be equal. The point of all this is that the pi mesons, which have an average lifetime at rest of 2.6×10^{-8}sec, travelling at the speed of light, without relativity, would only travel about 8 m on the average. But our spectrometer was about 30 m in length over which we measured the decay of the mesons. They were living about twice their “at rest” lifetime as they moved through the lab. Why? Moving clocks run slow and in our case the clock was the meson’s lifetime and γ was about 2.

Now let’s say that a particular pi meson decays halfway down the spectrometer. People in the lab frame and the meson in the meson’s frame both agree where this happens, halfway down the spectrometer. But in the meson’s frame it’s clock is not running slow and therefore the lifetime is the at-rest lifetime. In this case the meson, at best, can only see 8 m of spectrometer go by before it decays. How did it get to the 16 meter mark? The 16 meter mark in the lab frame is only 8 meters in the meson’s at rest frame. From the meson’s point of view, the spectrometer is contracted by about a factor of 2 so that in the meson’s normal at-rest lifetime, half of the spectrometer passes by. Objects in a reference frame moving by a stationary observer are length contracted in the direction of motion by γ i.e. $L' = L/\gamma$. This length contraction is only along the direction of motion. There is no transverse length contraction, as opposed to time dilation which occurs everywhere equally in the moving frame. So, the meson sees a distorted moving space where dimensions along the direction of the moving lab are contracted but the transverse dimensions are not. Time dilation and length contraction go hand-in-hand and the magnitude of both relativistic effects is the same.

With this length contraction idea, let’s go back and revisit our optical clock, but this time we will lay it along the X axis, the direction of motion, so that the pulses travels along the X axis. What we want to show is that length contraction is required in order to make the clock work properly. One might think offhand that, due to the length contraction of the clock, it will actually take less time for the pulses to go to the mirror and back than it does in the stationary frame and time would run faster in the moving frame. If that is true, we are in deep trouble. The rate time passes in the primed frame should not depend on the orientation of our clock.



Let’s break the round-trip of the path, as seen by the stationary observer, into two parts. The first part being from the emitter/detector to the mirror (time t1’) and the second part from the mirror back to the emitter/detector (t2’). In the first part the mirror is moving <u>away from</u> the pulse, which is travelling at velocity c, so the distance traveled is $(L/\gamma + V \cdot t1') = t1' \cdot c$, where the length of the clock has been length contracted to L/γ. We solve for t1' and get $t1' = L/(\gamma\cdot(c - V))$. In the second part the emitter/detector is <u>moving toward</u> the pulse and so the distance traveled is $(L/\gamma - V\cdot t2') = t2'\cdot c$ and so we get $t2' = L/(\gamma\cdot(c + V))$. The total time for one tick of the moving clock is t1' + t2' which is $2L/(\gamma c\cdot(1 - V^2/c^2)) = 2L\gamma/c$. The time for one tick of the stationary clock is 2L/c and so the clock in the moving frame, as seen from the stationary frame, is still running slow by the factor γ. Things are OK, but we get the correct answer <u>only</u> because the moving clock was length contracted in the direction of motion. In the stationary frame the times t1 and t2 are identical, so the time between two events, spaced apart along the x-axis, are different as seen by the two frames. Even events that are simultaneous in one frame will not be simultaneous as seen in the other frame. Here is a simple example. Let's say we synchronize two clocks in the primed (moving) frame, spaced a distance L apart, by placing a pulse emitter halfway between the clocks and sending out pulses of light which start the clocks running when the pulses arrive. In the primed frame the clocks are now synchronized. As seen from the stationary frame, the left clock is running into the pulse and therefore gets set before the right clock, which is moving away from the pulse, and so the left clock measures time which is ahead of the right clock as seen from the stationary frame. If we calculate how much it is ahead, it involves the same calculation we did for our clock which was laying down.

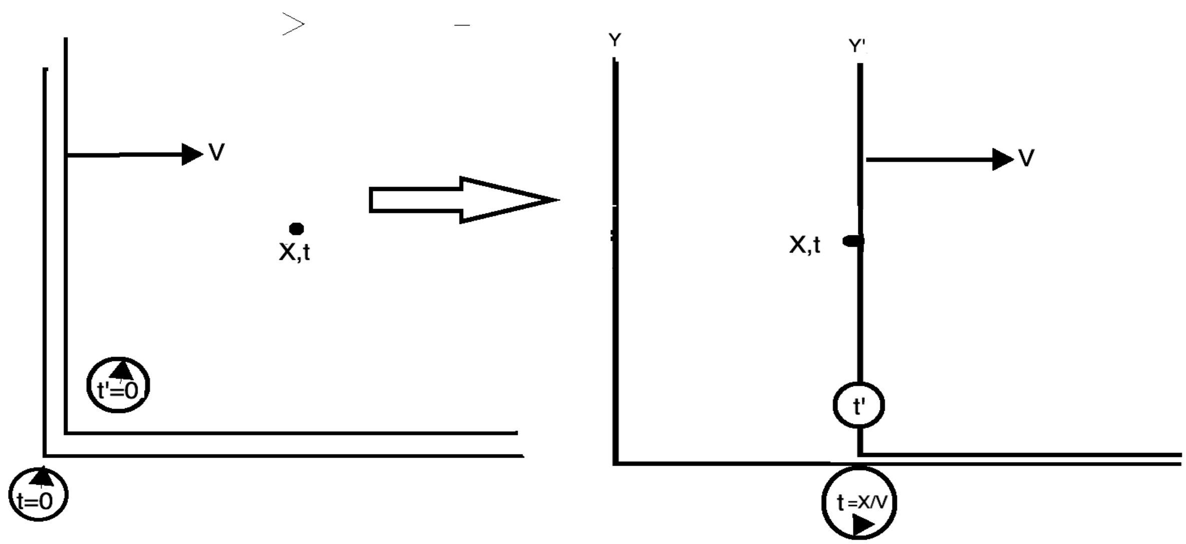
Lorentz Transformations

(I said I was going to stay away from this but some think it is needed)

Again, we consider two reference frames, one the primed frame moving with a velocity V to the right of the stationary frame. We have a point in the stationary frame with coordinates X and t and we now want to calculate the coordinates X' and t' in the primed frame. At time zero, all clocks are synchronized in each of the respective frames and set to zero and the origins of both coordinate systems are superimposed.

Length Contraction First we will derive the Lorentz transformations, for transforming the space and time coordinates from a stationary frame to a moving frame, by simply using only length contraction. Classically we have, as seen from the stationary frame, $X' = X - Vt$. With relativity, the moving frame is length contracted and, therefore, we see, from the stationary frame, that X' is now X'/γ . So, $X' = \gamma(X - Vt)$, and looking from the primed frame we see an equal relation, $X = \gamma(X' + Vt')$, except that V is now in the other direction. Eliminating X' between these relations we get; $t' = \gamma(t - X\cdot V/c^2)$ and, similarly, $t = \gamma(t' + X'\cdot V/c^2)$. The term $X\cdot V/c^2$means that the time on the X' axis, as seen from the stationary frame, depends on the position in the stationary frame. Therefore, the Lorenz transformation for transferring the coordinates in one frame to a moving frame are the following.

$$X' = \gamma(X - Vt) \quad \text{and } t' = \gamma(t - X\cdot V/c^2)$$
$$X = \gamma(X' + Vt) \quad \text{and } \quad t = \gamma(t' + X'\cdot V/c^2)$$
$$Y' = Y$$



Time Dilation

We will now derive the Lorentz transformations using only time dilation. We already found that it's dangerous to compare the time measured on two clocks in the moving frame that are separated in space, so we will use only one clock in the moving frame. The clocks in the stationary frame all read the same time.

Consider a clock at the origin of the moving frame and count the time on this clock, as viewed from the stationary frame, from time zero until the Y' axis of the moving frame is coincident with the point of interest, X,t. We have, $t' = t/\gamma$ (the moving clock is running slow).

Multiplying the top and bottom by γ, we get:

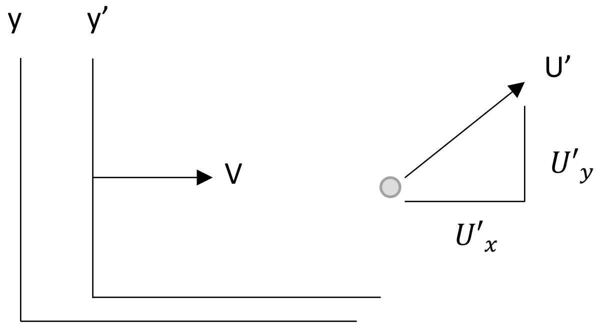
$$t' = \gamma/\gamma^2 \cdot t = \gamma t \cdot (1 - V^2/c^2) \text{ but, } t = X/V \; (V = X/t) \text{ so: } t' = \gamma(t - X \cdot V/c^2)$$

As before, the rest of the transformations follow from this. The Lorentz transformations can be derived from either length contraction or time dilation. A big difference is that length contraction is directional and time dilation is not. In general relativity, we will see that we have to use both to get the right answers, and the directional aspect of length contraction will be a big deal.

Velocity Addition

We now will calculate how to add velocities in relativity. We will do this two ways, one by thinking, and one using the Lorenz transformations. Let's do thinking first.

Consider a particle moving in the primed frame with a velocity U' and with the frame moving with a velocity V in the X direction. What we want to calculate is the velocity U of the particle in the stationary frame ie. we want to add U' and V to get U. This velocity U must satisfy several boundary conditions. Two are that when U' or V increases to the velocity of light(c), U also goes to c and cannot go above it. Also, if U' and V are very small compared to the velocity of light, we should get the classical result, which is just U' + V. Whatever relation we derive, it must be symmetrical in U' and V.



Consider first only the x component of U', the component in the direction of V. In the low velocity case we have:

U_x = U'_x+V, but as V ⇒ c $U_x = \alpha \cdot (U'_x + c)$ ⇒ c where α is some factor.

α therefore must have a form something like c/$(U'_x + c)$ but similarly must have the form c/(V+ c) when U'_x⇒c. To include all these conditions and the symmetry between U and V, we wind up with α= c/(c + V·$U'_x/c) = 1/(1 + V \cdot U'_x/c^2)$.

So: U_x =(U'_x+V) /(1+V·U'_x/c^2) ⇒c when V, U'_x ⇒c and ⇒U'_x+V when V/c ⇒0

Another way to find U_x is to calculate dX/dt from the two Lorentz transformations; X = γ(X' + Vt') and t = γ(t' + X'·V/c^2) dX/dt = γ(dX'+Vdt')/γ(dt'+dX'V/c^2)

Dividing by dt' top and bottom we get, as before

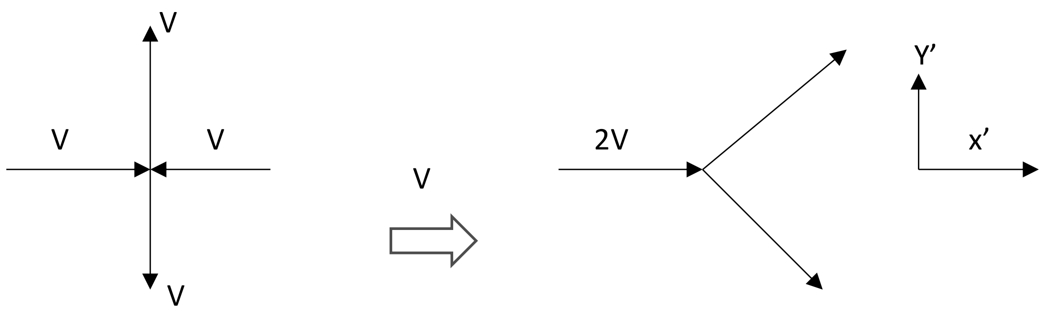
dX/dt = U_x = (U'_x+ V) / (1 + V·U'_x/c^2).

Now consider the Y component of U. You might assume that the velocity in the Y direction should not be affected by a transformation between two coordinate systems which have a relative velocity in the X direction. Let's see how that works out. Consider a particle travelling only along the Y' axis. When transformed to the stationary system, we will assume the Y velocity to be unchanged and the X velocity will just be V (U'_x=0). The resultant velocity will be $\sqrt{U'_y{}^2 + V^2}$. Well, you can see the problem. As V ⇒ c, this resultant velocity is clearly going to be larger than the velocity of light, and that is a not allowed. U_y must go to 0 as V⇒ c. The Y velocity measured in the stationary frame, U_y, must be smaller than the Y' velocity measured in the primed frame, U'_y, to avoid this problem. Let's take U_y to be κ·U'_y and see if we can figure out κ. We have U=$\sqrt{(\kappa \cdot U'_y)^2 + V^2}$. If this is to be c as V goes to c, we have U^2 = $\kappa^2 \cdot U'_y{}^2$ + c^2 = c^2. So κ must go to zero as V goes to c. Let U'_y go to c and the result must be the same. $\kappa^2 c^2$ + V^2 = c^2. Or: κ^2= 1 - V^2/c^2 = 1/γ^2. So, κ = 1/ γ. Now if time were running slower in the primed frame, by the factor γ ,that is, t' = t/γ, then dY/dt would be slower than dY'/dt' (velocity is inverse time). But, we already knew that time runs slower in the moving frame. So, you can derive time dilation, and therefore all the Lorentz transformations, by only insisting that things not go faster than the speed of light.

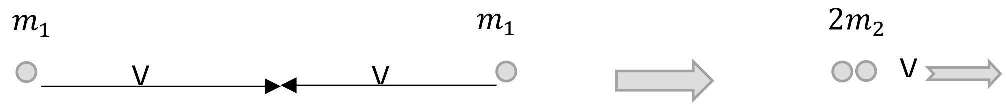
Conservation of Momentum

So far, special relativity has made a mess out of space and time. Let's see if we can save Newton's most famous law of motion, action equals reaction, also known as conservation of momentum. Some people feel that F = ma is his most famous law of motion but, to me, it is just the definition of force, and besides, the force is measured in Newtons! We will find that unless we do something with mass, momentum is not conserved because of our new velocity addition rule.

Consider the collision of two particles of mass m and velocity V travelling in opposite directions, colliding, and then going off at 90° with velocity V. In this frame, momentum is conserved because both the X and Y components of momentum are zero both before and after the collision. Call this the primed frame. We now let this frame move to the right with velocity V and observe the collision in a stationary frame. Is momentum still conserved?



Let's consider the classical case first. In this case we have a particle moving with the velocity 2V striking a stationary particle and then the particles go off at a 45 ° angle with each particle having an X component of velocity V. We have an incoming X momentum of 2V·m and outgoing momentum of mV·2. The total Y momentum is still zero, and so classically momentum is conserved in the collision. Let's now use our velocity addition formula to see what relativity thinks of this. The incoming velocity is <u>not</u> 2V but rather 2V/(1+V^2/c^2) which is less than 2. The X component of momentum after the collision is still 2·mV, and so momentum is ***<u>NOT</u>*** conserved. To make this work we need to have the mass of the incoming particle be larger than either of the masses of the 2 outgoing particles. The likely suspect is, of course, to multiply the rest mass of the particles by γ so that particles with a higher velocity would have a larger mass. This would also be convenient because it would mean that the transverse, (Y), momentum would stay constant when we transform between the moving frame and the stationary frame, the larger mass making up for the smaller transverse velocity. If you do this fudge to the mass <u>of all the mass's</u> by multiplying the rest masses by γ, then you'll find out, if you do the algebra, that in this example the momentum is conserved. Try not to make any mistakes because the algebra is sort of a mess. And be sure that the mass fudge factor involves the TOTAL velocity of the particle. We now want to show, with our correction to the mass, that we can create mass from energy.



Let's take a simpler example, an inelastic collision where we have two particles of rest mass $m_1$travelling with equal and opposite velocities V, colliding with each other and sticking together, making a rest mass 2·m_2. We call the masses different, because they may be different. Momentum is conserved in this frame because both the X and Y total momentum are zero before and after the collision. Let's again move this frame off to the right with a velocity V so that in a stationary frame we now have only one particle moving with a velocity given by relativity of 2V/(1+V^2/c^2) which we call V_{in}. After the collision, we have a particle of mass 2·m_2 moving to the right with a velocity V. Conserving momentum, we have.

$$2\cdot m_1 \cdot V_{in}/\sqrt{1 - V_{in}{}^2}/c^{\,2} = 2m_2 \cdot V/\sqrt{1 - V^2}/c^{\,2}$$

You can do the algebra (there's a lot of cancellation) and what you end up with is that the <u>rest mass</u> m_2 is NOT equal to the rest mass m_1 but rather:

$$m_2 = m_1/\sqrt{1 - V^2}/c^{\,2}$$ this is the REST MASS of m_2, not the moving mass!

We now have a problem. We either give up our idea of conservation of momentum or we need to do something about this thing we call mass(again!). We will stick with conservation of momentum.

Classically the loss in kinetic energy goes into heat of the final particles that are stuck together. Let's see what this has to do with the increase in rest mass. In the stationary system the kinetic energy before the collision is 1/2 $m \cdot (2\,V)^2$ and after the collision is $2\cdot 1/2mV^2$ so the loss in kinetic energy in the collision is, <u>classically</u>, 1/2 $m \cdot (2\,V)^2 - 2\cdot 1/2mV^2$ = mV^2 since the input and output masses are the same.

To first order in V^2/c^2 we have, from above, $2m_2 = 2m_1(1 + 1/2(V^2/c^2)) = 2m_1 + m_1V^2/c^2$, or in English; the outgoing particles (which are stuck together) have a total rest mass equal to the two incoming particle's rest mass + the loss in kinetic energy in the collision divided by c^2. This collision has turned kinetic energy into mass! Let's see what it looks like in the moving frame where it is easier to analyze things. Here we have two particles headed toward each other with the kinetic Energy of 1/2· mV^2. The final kinetic energy is zero. If this is turned into mass, then we get an increase in the rest mass/c^2 of mV^2, the same as see from the stationary frame.

We need to be careful. We may be adding apples and oranges. We combined "mass is energy", a relativistic concept, with kinetic energy, a Newtonian concept. It looks like the fudge factor gamma which we multiplied the mass with to make conservation of momentum work, may be a way to include in the mass, it's energy of motion. Rest energy (mc^2)+ energy of motion = γmc^2 =

E. This certainly works where V/c is much less than one. We get E= $\gamma mc^2 \cong$ (1+ $1/2 \cdot V^2/c^2$) mc^2 = mc^2+$1/2 \cdot mV^2$. It also works at higher velocities.
With total energy E = γmc^2 and momentum, P = γmV, we have that E = Pc^2/V. For a particle travelling at the speed of light (a photon) this gives E = Pc. In the limit of V going to 0 (P=mV), we have E = mc^2, the rest energy. Everything is consistent.

Work, energy and conservation of energy

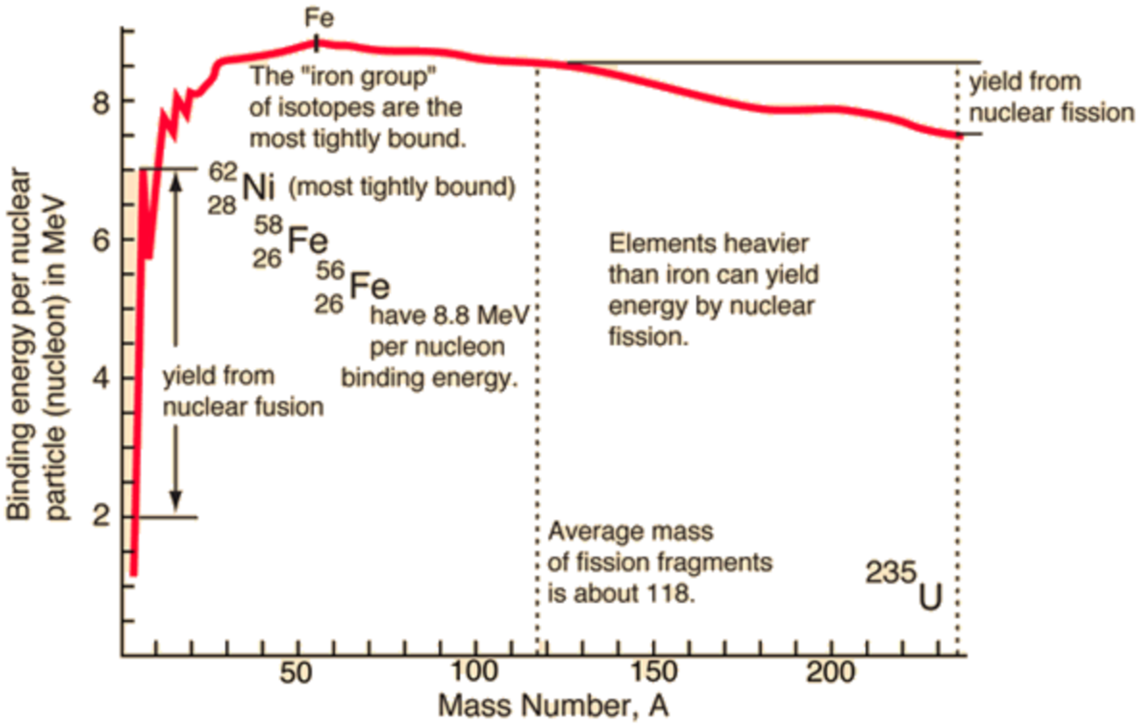
We can derive this mass-energy relationship by using the age-old concept of work (force times distance) equals the increase in energy. We have dE = Fdx = dP/dt· $dx =$ dP/dV· VdV. Integrating from 0 to V we have:

$$\text{Change in Energy} = \Delta\text{E} = \int_0^V dP/dV \cdot VdV.$$

The difference with relativity is that P involves γ and so the integral becomes more involved. You can convince yourself that dP/dV is just $\gamma^3\, m$ and that the integral, then, is $\gamma mc^2]_0^V$ = γmc^2 - mc^2. The increase in energy from rest to velocity V is the total energy (γmc^2) minus the rest energy (mc^2) .
Since energy is now mass, then our concept of conservation of energy is the conservation of total mass, γmc^2. Let's try this in our inelastic collision example. In the moving frame, the total mass before the collision is just 2γm and therefore that is the mass after the collision, which is just the rest mass of the final particles stuck together. This is the same answer we got when we used conservation of momentum.
Are there other examples of turning energy into mass? Of course, that's what they do at particle accelerators. I mentioned before the measurement of π^+ and π^- meson lifetimes. But if you want to measure those lifetimes, first you need to make the mesons. This is done with a proton-proton inelastic collision, p+p $\Longrightarrow$ p + p + $\pi^+ + \pi^-$, where the first proton comes out of a proton accelerator and the second proton is stationary in a liquid hydrogen target. In the center of mass system, it is clear that the threshhold kinetic energy of each proton must be such to create the rest energy of one pi meson, about 135 Mev (million electron volts). So, in the center of mass we have $\gamma m_p = m_p + m_\pi$ for each proton. Transforming to the laboratory frame where one of the protons is stationary, using our velocity transformation formula, we get that the total energy of the proton from the accelerator is E = $\gamma' m_p$ = m_p+4m_π+2m_π^2/m_p. The kinetic energy required, 4m_π+ 2m_π^2/m_p, is over twice that needed in the center of mass frame. Not very efficient. If we were making a proton- antiproton pair, each with a rest energy of 1000 Mev, instead of a pi meson pair in the collision, then the kinetic energy required in the lab frame would be three times that required in the center of mass. This ratio gets bigger the larger the masses that are created in the collision.
In order to create more massive exotic particles, the problem was solved by making the center of mass frame and the lab frame the same using colliding beam accelerators. The first proton colliding beam accelerator was the Intersecting Storage Rings (ISR) built in Geneva in 1971. This machine stored protons from a synchrotron into two counter-rotating beams and then steered them with magnets to collide with each other, making the lab frame the center of mass frame. The current (2018) most powerful colliding beam accelerator is the large hadron collider (LHC),

also at the CERN lab in Geneva. This machine has two counter rotating beams each with kinetic energy of 6.5 TeV, yes, that is with a T. This accelerator is huge and has a diameter of 27 km. Yes, that's kilometers. This accelerator was used to confirm the existence of the famous Higgs boson which was found to have a mass of 125 GEV, about 125 times the mass of each of the colliding protons.

We've shown how mass can be created with kinetic energy, so how about the reverse where we create kinetic energy from mass. Well, it happens every day in your (non-electric) automobile. In this case carbon and oxygen combine to make CO_2 plus 4.1 eV of energy. The reactants contain a total of 44 nuclei which have a total rest energy of about 44 billion electron volts. It would take a very good scale to notice that 4.1 eV of rest energy is missing in the final product. This reaction doesn't convert much of the rest energy into kinetic energy. These atomic reactions involve energies of the order of electron volts, the binding energy of electrons in atoms. Nuclear reactions, on the other hand, involve energies of millions of electron volts, the binding energy of nucleons in the nucleus. The method would be to take one or more nuclei with a small binding energy and somehow convert them into one or more nuclei that have a much higher binding energy. This binding energy might be thought of as a negative energy. If two things bind together then it takes energy to separate them so, therefore, when they bind together, they give off kinetic energy equal to the binding energy. From the diagram below we see that the most tightly bound nuclei are the iron group and so we could create kinetic energy by taking something very heavy, like uranium, and splitting it to make smaller nuclei or, we could take very light nuclei like hydrogen and deuterium and fuse them together to make heavier nuclei like helium. The first is easier to do but unfortunately some of the products in the fission reaction are very radioactive and last for tens of decades. The second fusion process is very hard to do because you need to get the light nuclei up to a high enough temperature so that their kinetic energy can overcome the repulsive Coulomb force between the positively charged nuclei, so they can react. This is usually done by containing the particles in a magnetic field. I worked one summer at the Oak Ridge National Laboratory and had a chance to tour the fusion facility. They told me that they were just turning the corner to success. It sounded impressive. The problem is, that was 1961. It has been a big corner but progress is still being made.



This brings special relativity to an end. Let's see what we have learned based on just two bold assumptions, one that the speed of light is the same for all observers and two, that reference frames travelling at a constant velocity with respect one another are equal.

1. Clocks moving with respect to us run slow
2. Objects moving with respect to us are length contracted but only in the direction of their motion.
3. Velocities don't add as easily as they used to.
4. Momentum can only be conserved if mass increases as a function of velocity.
5. The mass of a body contains both its rest energy and kinetic energy.
6. Conservation of energy is conservation of mass.
7. Energy can be turned into mass.
8. Mass can be turned into energy.

Quite a bit from simple assumptions. If this all seems strange, stranger things are yet to come.

General relativity

Under repair. It requires more thinking (and learning)