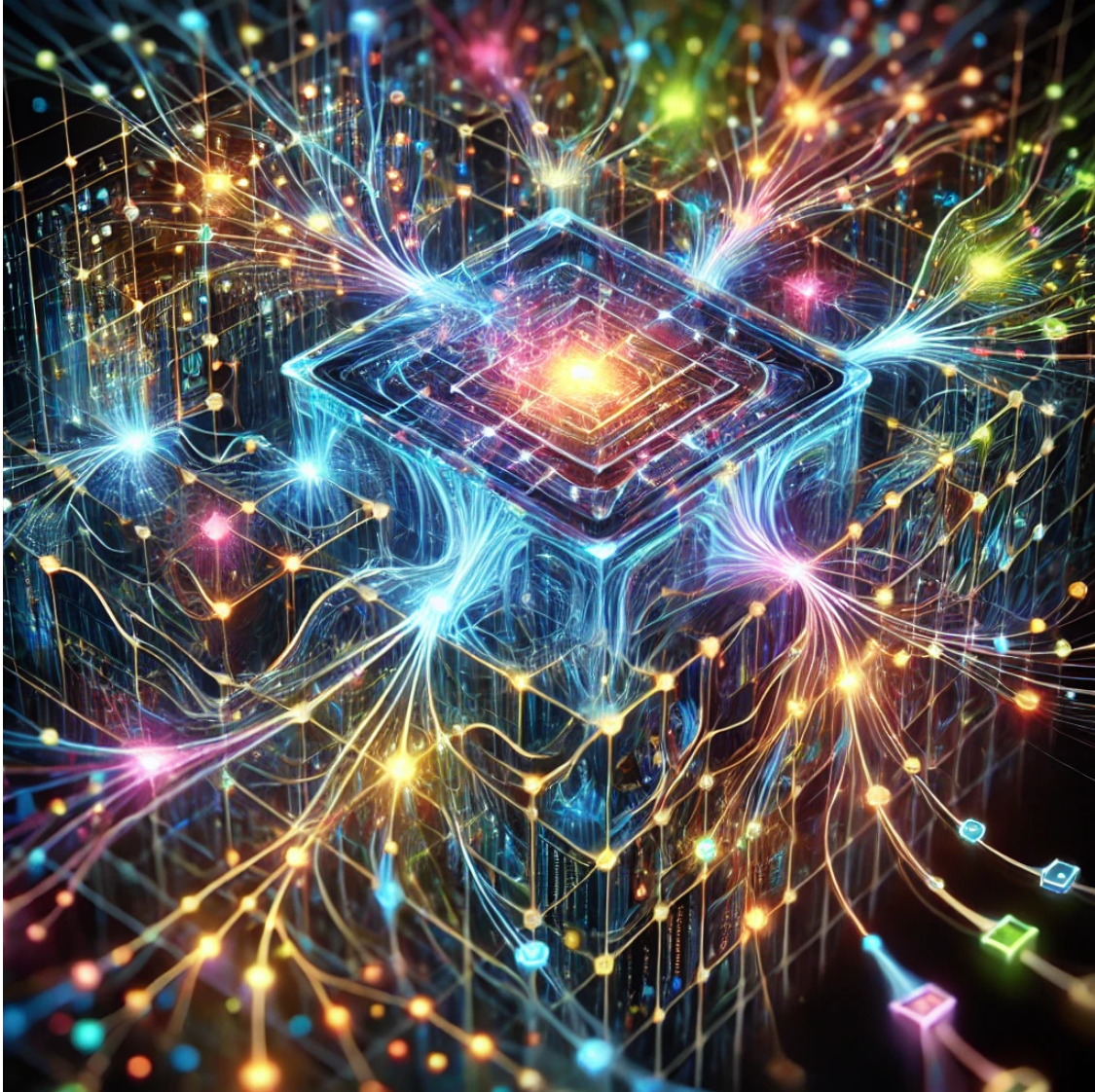


Neural Networks

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1 Lecture 1

Why Study Neural Networks?

- The human brain is very successful at certain tasks that prove to be difficult to accomplish on a classical computer, such as learning and pattern recognition. For example, consider the generic classification task: when you see a picture of an apple, you immediately recognize it as an apple. Similarly, you understand what someone says when they speak. Computers, however, often struggle with these tasks. For instance, even with modern technology, most commercial speech recognition systems cannot perfectly understand speech, especially with accents. This motivates us to study how the brain, or the human neural network, functions.
- Why is the human brain better at certain tasks compared to computers? Could it be about speed? No, because while a transistor can switch in nanoseconds, it takes a few milliseconds for your brain to process a stimulus, such as pricking your leg with a needle. Hence, the difference is not necessarily due to speed. What the brain lacks in speed, it compensates for in the number of processing elements and the massively interconnected and parallel nature of its operations, unlike the serial operations of classical computers. Additionally, the human brain is adaptive; its functionality evolves through learning, whereas the physical structure of a CPU in a computer is fixed and does not change over time.

Summary:

- **Classical computer:** Almost operates in serial (e.g., running a program while ignoring pipelining), but each operation can be performed extremely fast (less than nanoseconds per operation).
- **Neural network:** Operates massively in parallel, and may be more powerful than a classical computer, despite the fact that a given neuron is relatively “slow” (a neuron can fire at most hundreds of times per second, which translates to milliseconds per operation).
- **Motivation:** By imitating the brain’s operation (through artificial neural networks), we may be able to solve complex tasks that classical computers struggle to perform. This is arguably the most important motivation for advancements in this field.
- **Other motivation:** To understand how the brain and nervous system function.

Neural Networks Summary:

1. Neural networks are massively parallel distributed processors composed of simple processing units (mathematical neurons) and the connections between them.
2. Neural networks can store and utilize experiential knowledge through a learning process.

An artificial neural network resembles the brain in two key respects:

- Knowledge is acquired by the network from its environment through a learning process.
- Inter-neuron connection strengths, known as synaptic weights, are used to store the acquired knowledge.

2 Lecture 2

Neural Networks are characterized by **non linearity** and by a **learning component**.

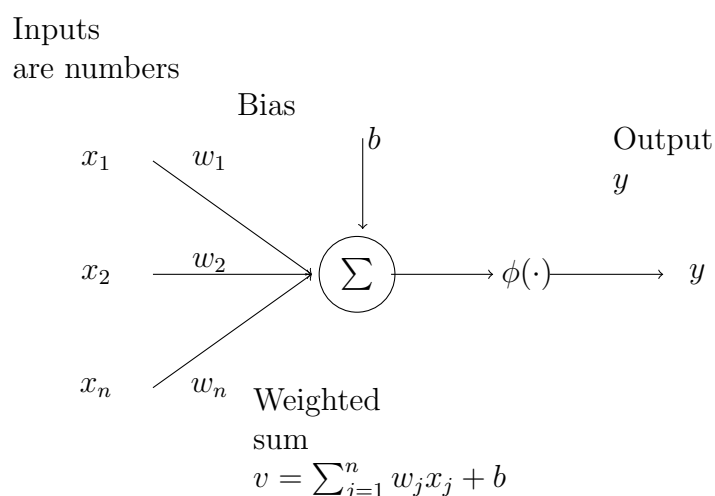
It is useful to understand what a **biological neuron** is, since Neural Networks get inspiration from them. To make it simple: some excitation comes in and the neuron may be activated, if there is enough excitation. The more it is used, the stronger it becomes (**Hebbian Algorithm**).

- n neurons in the brain: 10^{11}
- n neurons in the whole nervous system: 10^{12}
- n synapses: 10^{14}

In our body there are different types of neurons: sensory, internal, motor... and there are special architectures that adapt to different tasks.

Ex. when we born our brain is not fine tuned yet, we must give input data to it → **language learning**

2.1 Mathematical Model of the Neuron



The inputs are pulses, chemical perturbations (in biology). In our context, they are basically numbers.

The inputs are going to be weighted. They don't have all the same importance. **Weights** represent the strength of the connection (synaptic weight).

The **activation function** controls whether the input is strong enough.

The **Bias** is an internal excitation.

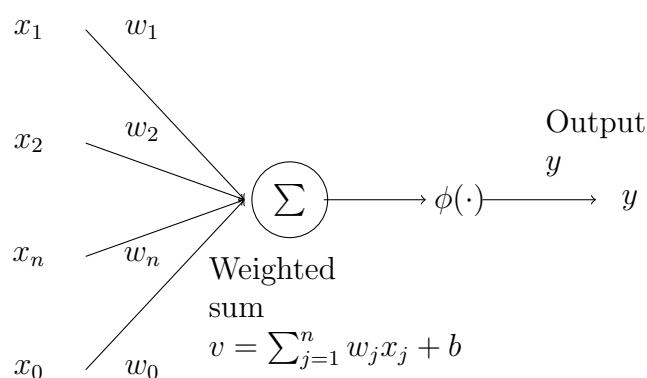
$$v = \sum_{j=1}^n w_j x_j + b$$

This is an affine equation (linear + constant).

A lot of nature is linear. If the input is scaled, the output is also scaled. If I add inputs, the output gets added up. This is the essence of linearity.

Inputs

are numbers



If we add an input x_0 we can remove the bias and have a linear equation, but usually, we keep the bias separated.

The output y is the result of a final **activation function** ($y = \phi(v)$).

$$y = \phi(v) = \phi\left(\sum_{j=1}^n w_j x_j + b\right) \quad (1)$$

This is what is doing the whole neuron.

2.2 Types of activation functions

For the moment, we will focus on "squashing" type of function, i.e. It limits the amplitude range of the neuron output.

1. **Step function:** threshold function, Heaviside. As long as the input is negative, nothing happens, then something happens.

$$\phi(v) = \begin{cases} 1 & \text{if } v \geq 0, \\ 0 & \text{if } v < 0. \end{cases} \quad (2)$$

We can have flexibility and change where the jump is happening. We do that with the bias: the threshold at which the bias occurs.

How do we define the weights? We should design them to create a specific behavior. The problem with the step function is that it becomes a discrete problem and finding the proper weights becomes a long process of trial and error → **optimization algorithms**, to gradually change the weights and see what happens. This is the motivation behind the **sigmoid function**.

2. Sigmoid function: "smoothed" threshold function

$$\phi(v) = \frac{1}{1 + e^{-aV}}, a > 0 \quad (3)$$

When a becomes larger, the function becomes sharper, while if a becomes closer to 0, we will have a smoother function.

Step and sigmoid function have outputs $\in [0, 1]$, but sometimes we may want to have outputs $\in [-1, 1]$ → we can change both the step and sigmoid function.

1. Sign:

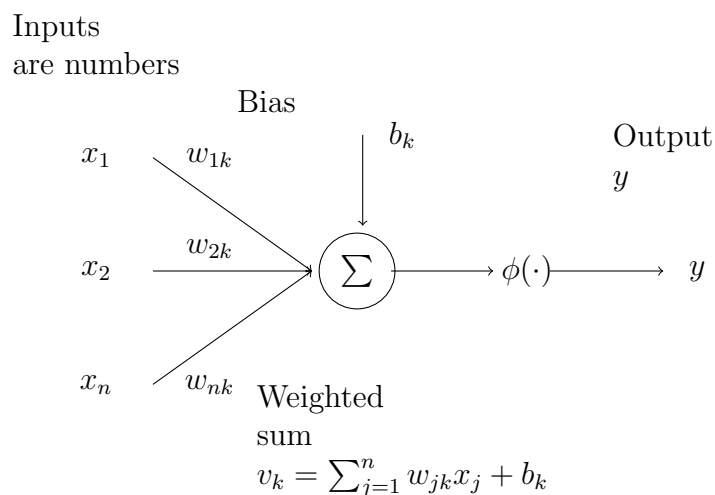
$$\phi(v) = \begin{cases} -1 & \text{if } v < 0, \\ 0 & \text{if } v = 0, \\ 1 & \text{if } v > 0 \end{cases} \quad (4)$$

2. Hyperbolic Tangent:

$$\phi(v) = \frac{e^{aV} - e^{-aV}}{e^{aV} + e^{-aV}} \quad (5)$$

2.3 Neural Network architectures

Up to now, we have talked about just one neuron, but we need to connect them. Let's name the first neuron, neuron K . We need to label all the neuron to book keep them



All the k 's tell you that these units belong to that neuron.

3 Lecture 3

In the neural network, the **activation function** is what is creating the **non-linearity**

$$\phi(v) = \begin{cases} 1 & \text{if } v \geq 0, \\ 0 & \text{if } v < 0. \end{cases} \quad (6)$$

This is an idealization of what is happening inside the neuron, but it is quite rough, sometimes it is better to use a smoother function, like the sigmoid:

$$\phi(v) = \sigma(v) = \frac{1}{1 + e^{-aV}} \quad (7)$$

a is a parameter that defines the sharpness of the transition.

There are also variants that bring the step and the sigmoid functions in the range $[-1, 1]$, and they are the **sign** and the **hyperbolic tangent**.

Also, there are **non squashing** types of functions:

Rectified Linear Unit (ReLu):

$$\phi(v) = \begin{cases} v & \text{if } v \geq 0, \\ 0 & \text{if } v < 0. \end{cases} \quad (8)$$

3.1 Feed Forward Neural Network

K different neurons, each one with its weights and biases. We index the neurons and the inputs are the same for each neuron. Each neuron produces its output y_k .

w_{ki} is the weight that connects neuron k to input i . All this bunch of neurons form a single layer. So this is a **single layer fully connected feed-forward neural network**, but we can have multiple layers.

The information flow is only over one direction, i.e. the input layer of source nodes projects directly onto the output layer of neurons. There is no feedback of the network's output to the network's input. We thus say that the network in Fig. 1 is of feed forward type.

If, for example, we have two layers, we no longer see the output of the first layer, so this becomes an **hidden layer**, while the second layer (in this case) is the output layer.

$$x = \begin{pmatrix} x1 \\ x2 \\ \dots \\ x_n \end{pmatrix}$$

it is a vector

$$W = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ w_{31} & \dots & & \\ \dots & & & \\ w_{k1} & \dots & w_{kn} & \end{pmatrix}$$

All the weights are inside a matrix, every row is associated to one neuron, while every column is associated with an input.

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_k \end{pmatrix}$$

One bias for each neuron

$$U = \begin{pmatrix} u_{11} & \dots & u_{1k} \\ u_{21} & \dots & u_{2k} \end{pmatrix}$$

In this case, 2 neurons, k inputs.

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

All these vectors and matrices are parameters, i.e. everything needed to define the neural network.

3.2 Linear Algebraic Representation

Any linear operator can be written as **inner products**.

$$\sum_{i=1}^n w_{1i}x_i + b_1 = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + b_1$$

Repeat per neuron 2 (second row of the matrix)

$$\sum_{i=1}^n w_{2i}x_i + b_2 = \begin{pmatrix} w_{21} & w_{22} & \dots & w_{2n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + b_2$$

...

Repeat per neuron k

$$\sum_{i=1}^n w_{ki}x_i + b_k = \begin{pmatrix} w_{k1} & w_{k2} & \dots & w_{kn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + b_k$$

Single Layer

$$z = \phi(Wx + b)$$

Two Layers

The second layer does the same thing, not to x , but to z

$$y = \phi(Uz + c) = \phi(U(\phi(Wx + b) + c))$$

It is extremely important to bookkeep dimensions:

$$x \in R^n \tag{9}$$

$$W \in R^{k \times n} \tag{10}$$

$$b \in R^k \tag{11}$$

$$z \in R^k \tag{12}$$

$$U \in R^{2 \times k} \tag{13}$$

$$c \in R^2 \tag{14}$$

In this case the **naming** is: n-k-2 fully connected feed forward neural network (each dash is a layer).

N.b A neural network is called fully-connected when we allow all weights; sometimes we can decide to force some of them to zero: in this case connections are removed and the network is not fully-connected anymore.

Example: Convolutional Neural Networks. If some outputs feed back (after some delay) as inputs, the network is not feed forward.

Example: LSTM (Long Short Term Memory)

3.3 Examples:

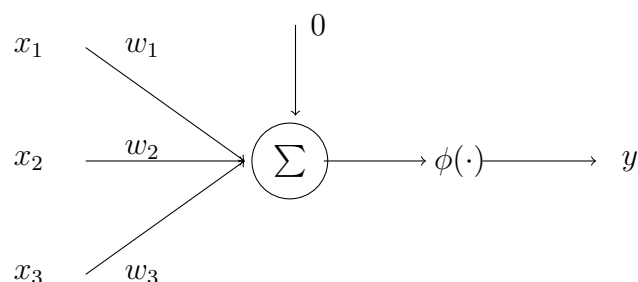
a) Majority Applications

- $n = 3$ inputs $x \in \{-1, +1\}^3$

$$\bullet \text{ Activation Function: sign: } \phi(v) = \begin{cases} +1 & \text{if } v > 0 \\ 0 & \text{if } v = 0 \\ -1 & \text{if } v < 0 \end{cases}$$

- Goal: design a simple neuron that outputs the value that most appears in the input.

E.g. $x = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \rightarrow y = -1$



$$y = \text{sign}(x_1 + x_2 + x_3) = \text{majority}(\{x_1, x_2, x_3\})$$

This neural network has 100% accuracy. If there are more +1 the result is going to be positive, otherwise it is going to be negative.

b) Logical Statements

- c) **Classification:** e.g. Image representing a 3, the neural network needs to provide 3 as output.

We need to break the image into sth that the NN can take as input \rightarrow we need to **vectorize** it.

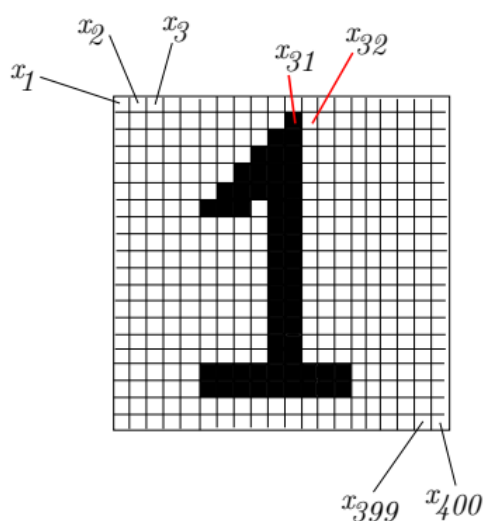


Figure 1: An image of a digit. Each pixel corresponds to one input node

One may design a feedforward neural network for digit classification. We are given 20 pixel by 20 pixel images of handwritten digits. Our job is to find, for each given

image, the actual digit corresponding to the image. Suppose that each pixel assumes binary values, i.e. the images are black and white only with no grayscale. We can then consider a feedforward neural network with 400 nodes in the input layer (labeled as x_1, x_2, \dots, x_{400}), some large number of nodes (say 1000 nodes) in a hidden layer, and 10 nodes in the output layer giving outputs y_0, \dots, y_9 . Ignoring the biases, this fully-connected network has:

$$400 \times 1000 + 1000 \times 10 = 410000$$

parameters (synaptic weights) that we can optimize.

Neurons in a neural network can act as shape-detecting units, extracting **salient features** from the input. To represent this layer, we might need 1000 weights. For example, we could have one neuron specialized in detecting the digit '1', similar to how specialized areas in the brain work. While multiple neurons might be activated, the network can only predict one number at a time. To resolve this, at the end of the neural network, we can apply an operation called ARGMAX. This produces a one-hot representation, where the '1' indicates which neuron 'won' the prediction.

In other scenarios, using a SOFTMAX layer might be beneficial, which outputs a probability distribution. This helps determine the network's confidence in its prediction, offering insight into how wrong or uncertain the network might have been.

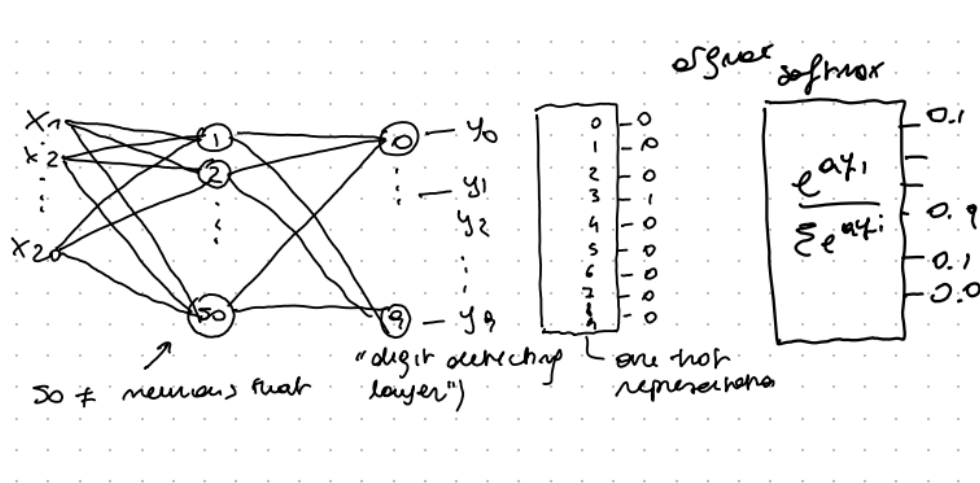


Figure 2: Graphical Representation of the neural network to perform digit classification

- Each node in the input layer will correspond to one pixel in the image, as illustrated in Fig. 2. For example, if the image in Fig. 2 is to be fed to the network as input, the inputs would be $x_0 = 0, x_1 = 0, \dots, x_{30} = 0, x_{31} = 1, x_{32} = 0, \dots, x_{399} = 0, x_{400} = 0$, i.e., we use 0 to represent a white pixel, and we use 1 to represent a black square.
- It is possible to design the network (choose the weights) such that when some image of digit 0 is fed to the network, the output will be $y_0 = 1, y_1 = \dots, y_9 = 0$. Likewise,

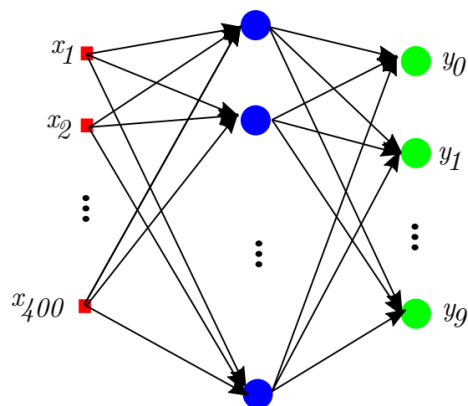


Figure 3: The 2-layer feedforward neural network for digit classification

for any $i \in \{0, \dots, 9\}$, when some image of digit i is fed to the network, the output will be $y_i = 1$ and $y_j = 0, \forall j \neq i$.

- This is usually done via supervised learning. We prepare (or usually acquire!) a large number of 20×20 training images, each of which looks like Fig. 2 but of course with different digits, and differently written versions of the same digit. See Fig. 4 for some sample portion of such a training set. Each training image has a label that specifies the actual digit corresponding to the image. For example, in Fig. 4, the images in the first row will be labeled as 0, the second row as 1, and so on.

4 Lecture 4

4.1 Approximation theorems and learning algorithms

- Can we always find weights and biases (parameters) to do what we want to do?
- Given enough neurons and layers, almost every function can be approximated (as accurately as we want)
- with logic (binary inputs/outputs) the **DNC/CNF** theorem shows that 2 layers are enough.

The DNF/CNF theorem states that every Boolean formula can be transformed into either a Disjunctive Normal Form (DNF) or a Conjunctive Normal Form (CNF). In DNF, the formula is expressed as a disjunction (OR) of conjunctions (AND) of literals, such as $(x_1 \wedge \neg x_2) \vee (x_3 \wedge x_4)$. Conversely, in CNF, the formula is expressed as a conjunction (AND) of disjunctions (OR) of literals, for example, $(x_1 \vee \neg x_2) \wedge (x_3 \vee x_4)$. This theorem is important because it guarantees that any Boolean function can be represented in one of these two canonical forms, which is useful in logic, computer science, and digital circuit design.

- But how do we set these parameters to achieve the desired behavior? Manual adjustment is impractical, except in simple cases. Instead, **learn using guidance from data**.

$$y_{\text{params}}(x) = \phi(U\phi(Wx + b) + c)$$

This output ideally is equal to our target y , but this is not always the case. **Goal:** Minimize the number of mistakes. Mathematically we have a mistake when $y_{\text{params}} \neq y$.

What we can do is to look at the whole dataset and count how many times mistakes are made.

$$\sum_{(x,y) \in \text{data}} \{y_{\text{params}} \neq y\}$$

Typical algorithm: Local search + gradual data.

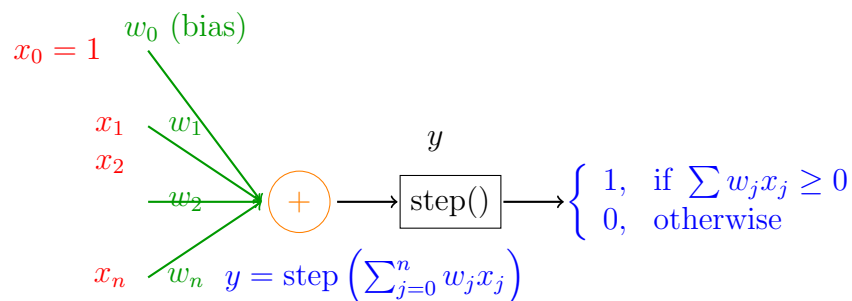
1. Initialize parameters **randomly**
2. for **each** (x,y) in the data
 - Feed x to NN and get $y_{\text{params}}(x)$
 - Look **in the neighborhood** of params to make $y_{\text{params}}(x)$ closer to y
3. Repeat 2. until parameters converge

Steps 2. and 3. represent an **epoch**, i.e. one full pass over the dataset.

4.2 Perceptrons

This is the simplest situation we are going to consider.

Consider the **McCullock-Pitt (1945)** neuron, with bias and weights.



What can it do?

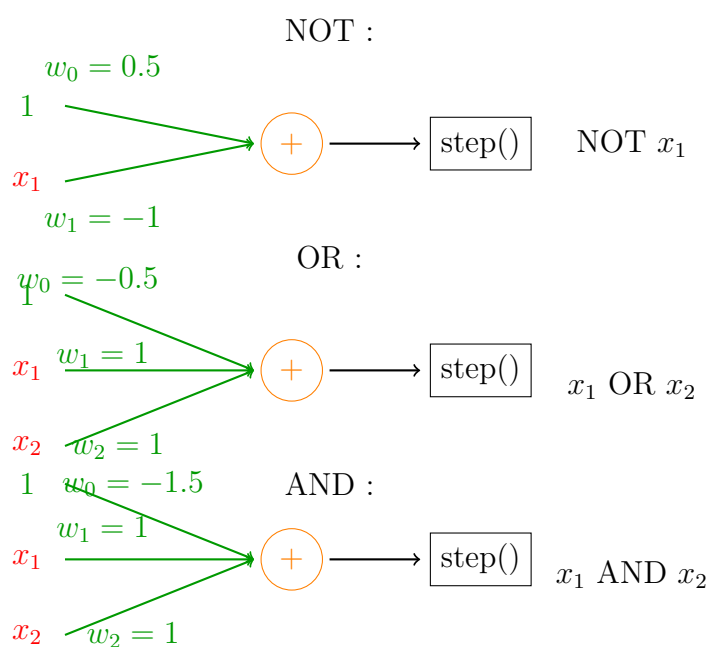
- **Recall:** $(w_0 \ w_1) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = 0$ This is a line that passes through the origin.

$$(w_0 \ w_1 \ w_2) \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0 \text{ This is a plane that passes through the origin}$$

- In higher dimensions, this is called **hyperplane**: $(\cdot \quad \cdot \quad w \quad \cdot \quad \cdot) \begin{pmatrix} \cdot \\ \cdot \\ x \\ \cdot \\ \cdot \end{pmatrix} = 0$ is an **hyperplane**.

$$(\cdot \quad \cdot \quad w \quad \cdot \quad \cdot) \begin{pmatrix} \cdot \\ \cdot \\ x \\ \cdot \\ \cdot \end{pmatrix} \geq 0 \text{ is a } \mathbf{halfspace}$$

- It can model **logic gate**



- It can classify using an hyperplane as **decision boundary**, which is the boundary with which we decide to switch from 0 to 1. This classifier is a linear separator

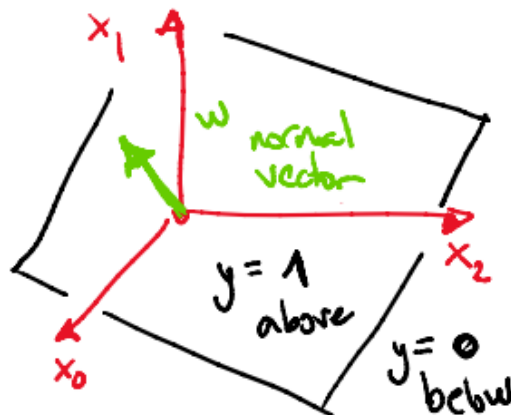


Figure 4: The image illustrates the decision boundary of a hyperplane in a 3D space. The axes x_0 , x_1 , and x_2 represent input features, and the green vector w is the normal vector to the plane, representing the weight vector of a binary classifier (e.g., a perceptron). Points above the plane are classified as $y = 1$, while points below it are classified as $y = 0$. The plane itself serves as the decision boundary between these two classes.

5 Lecture 5

Last time: Approximation theories and learning theories. Perceptrons → one of the simplest algorithms

5.1 Example

$$\neg x_1 \wedge \neg x_2 \dots \wedge \neg x_n \wedge x_{n+1} \wedge x_{n+2} \dots \wedge x_{n+m}$$

This is a logic description, but we can move towards an arithmetic description:

$$(1 - x_1) + (1 - x_2) + \dots + (1 - x_n) + (x_{n+1}) + \dots + (x_{n+m})$$

If I add these guys, depending on their values, I can obtain a different range of results: the smallest one is zero, while the biggest is $n+m$.

Now we want to **threshold** it (i.e. **adjust the bias**), to build an AND operator.

$$(1 - x_1) + (1 - x_2) + \dots + (1 - x_n) + (x_{n+1}) + \dots + (x_{n+m}) \geq n + m - \frac{1}{2}$$

If it is an AND the output should be 1 only when they are all 1, so when the sum is equal to $n+m$:

$$\underbrace{-(m + \frac{1}{2}) \cdot x_0}_{w_0} + \underbrace{(-x_1)}_{w_1} + \underbrace{(-x_2)}_{w_2} + \dots + \underbrace{(-x_n)}_{w_n} + \underbrace{x_{n+1}}_{w_{n+1}} + \dots + \underbrace{x_{n+m}}_{w_{n+m}}$$

Because NOT and either AND or OR together are **weighted gates**, **multilayer perceptron** can represent **any** logical operator. How many layers are sufficient to implement

any operator? **We need 2 layers**, thanks to the **DNF/CNF** formula. May it not be the most efficient way, to implement a table, that's why the concept of **depth** is then introduced.

If we make a **truth table**, we need many entries, while with depth, fewer neurons are needed.

Is 1 layer sufficient? Answer: NO.

5.2 Perceptron Learning Algorithm

Given a description of the desired behavior, $y = f(x)$, how do we find parameters such that $y_{params}(x) \cong f(x)$? (In this case, $params = w$). Manually find this pattern is out of question, we aim for an automatic way.

We assume that exists w such that the perception can do the job exactly:

$$y_w(x) = \text{step}(w^T x) = y$$

Jargon: we define **classes**, which are $y=0$ and $y=1$ regions; they represent higher-level concepts of what we should do.

In **supervised learning**, the description takes the form of **data**.

Algorithm: **Input:** Data = $\{(x, y), \dots\} \ x \in R^n, y \in \{0, 1\}$

Output: weight vector $w \in R^{n+1}$.

n.b. we have w_{n+1} weights because we also have w_0 .

1. Initialize w randomly. In general there are errors
2. While errors: $\exists(x, y) \in \text{data}$ such that $y_w(x) \neq y$
- 3.

For each $(x, y) \in \text{Data}$:

If $y_w(x) = y$: do nothing

Else if $y_w(x) = 0$ and $y = 1$: $\mathbf{w} \leftarrow \mathbf{w} + \eta \mathbf{x}$

Else if $y_w(x) = 1$ and $y = 0$: $\mathbf{w} \leftarrow \mathbf{w} - \eta \mathbf{x}$

$$\begin{cases} \mathbf{w} \leftarrow \mathbf{w} + \eta \mathbf{x} & \text{if } y_w(x) = 0 \text{ and } y = 1 \\ \mathbf{w} \leftarrow \mathbf{w} - \eta \mathbf{x} & \text{if } y_w(x) = 1 \text{ and } y = 0 \end{cases}$$

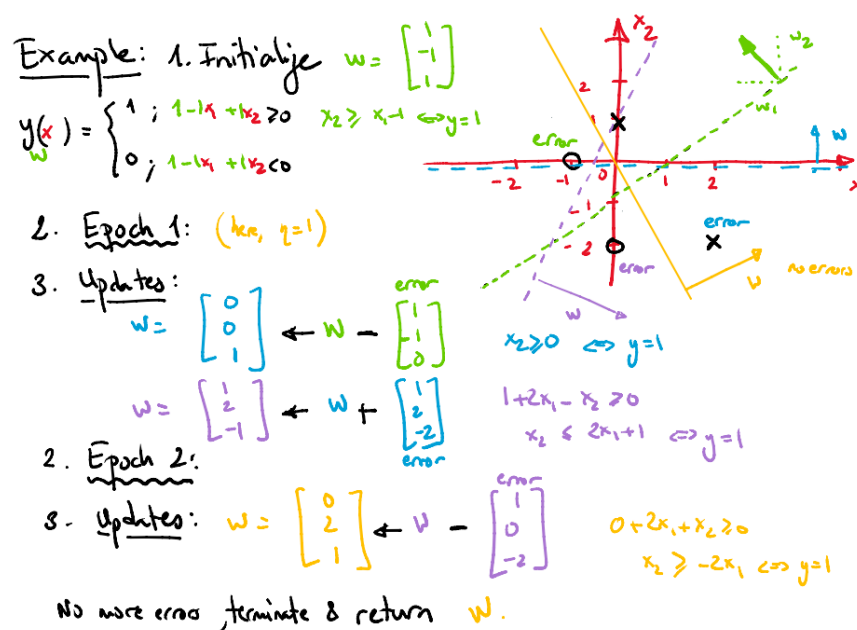
Simplifies to: $\mathbf{w} \leftarrow \mathbf{w} + \eta \mathbf{x}(y - y_w(x))$ We move the weights toward the example η represents the speed

Epoch: one full pass over data

Theorem: If classes are linearly separable, the algorithm will converge

6 Lecture 6

6.1 Example



In this example, weights are initialized to

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

In this case, the weight vector is pointing in the negative direction for x_1 and in the positive direction for x_2 . x_0 controls the shift from the origin.

This separator is making errors, so we have to change the weights every time we notice an error.

Epoch 1: weights are updated using the error point: the first error point considered is $(-1, 0)$, which has been classified as 1 but should have been classified as 0, so we subtract that point from the weight vector. We obtain

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

We obtain a vector that is still making errors. The point $(2, -2)$ is classified as 0 but should be classified as 1, so we add this point to the weight vector:

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

6.2 Supervised Learning

In **supervised learning** we are dealing with data (x, y) , where y is a sort of supervision. There is not a specific task, but it is defined according to how x behaves.

$x \in \chi$ (**feature space:** context where we perform the task)

$y \in Y$ (**label space:** target)

- y can be **discrete** and in this case we talk about **classification**, but it can also be **continuous** and in this case we talk about **regression**.
- **Task:** **predict/imitate** y using only x .
- **How?** Use a neural network: $y_w(x) = f(x, w)$ (function of x , with parameter w). This gives us a predictor, which is a function of x parametrized by w .
- **Loss:** measures how far y and $f(x, w)$ are:
 - **0-1 loss:** $l(y, f) = 1\{y \neq f\}$
 - **squared loss:** $l(f, y) = \|y - f\|^2 \rightarrow$ euclidean distance between them
- **Risk:** Average loss over a dataset

$$\text{Data} = \{(x, y), \dots\}$$

$$\text{Risk: } \frac{1}{\text{Data}} \sum_{(x, y) \in \text{Data}} l(y, f(x, w))$$

The risk is a property of the data and of the weights. The risk for the perception is the rate of mistakes and it represents the probability of making a mistake.

- **Goal:** Minimize (empirical) risk. Make as few mistakes as possible, remaining as close to y as possible \rightarrow search the space for w , to find the one that gives the smallest $R(w)$.
- **Types of algorithms:**
 - **Online:** We use these algorithms when data streams in. Weights are updated **along the way**. (usually) don't revisit data points. If we forgo epoch and get new data, the perceptron will be online.
 - **Batch:** data is available in whole, so we don't have to look at each point singularly, but we can have a global perspective. (Usually) pass over data multiple times (epochs).
 - **Mini batch:** Process data gradually in small batches.

7 Lecture 7

7.1 Linear Regression with Squared Loss

- **Data:** $\{(x, y), \dots\}$, $x \in R^n, y \in R^m$ (x is an n -dimensional vector to predict an m -dimensional vector).
- **Predictor:** $f(x, W) = Wx$, $W \in R^{n \times m}$ (this is a single neuron with $\phi(v) = v$, identity activation).
- **Goal:** Minimize $R(w) = \sum_{(x,y) \in \text{Data}} \|y - Wx\|^2$
How do we solve a problem like this? Analytic solution: Optimality condition, because we are unconstrained, we want:

$$\nabla_w R = 0$$

This is not an equation, but $n \cdot m$ equations.

Let's consider a single data point to start:

$$l = \left(\sum_{k=1}^m y_{k'} - \sum_{j'=1}^n w_{kj'} x_{j'} \right)^2$$

This is the loss of a single point, on m neurons

$$\frac{\partial l}{\partial w_{kj}} = -2y_k x_j + 2 \sum_{j=1}^n w_{kj} x_j x_j$$

We are making the derivative with respect to a particular j

in terms of vectorial operation is an **outer product**, so it takes two vectors as input and gives a matrix as output.

This matrix is multiplied by matrix W . For every w we find the gradient and put it in a matrix.

$$\nabla_w l = -2 \cdot y \cdot x^T$$

$$\begin{bmatrix} \dots \\ y \\ \dots \end{bmatrix} \begin{bmatrix} \dots & x^T & \dots \end{bmatrix}$$

The rank of this matrix is 1

$$\nabla_w l = -2 \cdot y \cdot x^T + 2 \cdot W \cdot x \cdot x^T$$

This is the gradient of the loss for a single data point.

The risk is the sum of the losses of the data points. We want to find the derivative of that. But since the derivative is a linear operator, the derivative of the sum is the sum of the derivatives \rightarrow the gradient of the sum is the sum of the gradients.

$$\begin{aligned} R(w) &= \sum_{(x,y) \in \text{Data}} l(y, Wx) \rightarrow \nabla_w R = \sum_{(x,y)} \nabla_w l \\ \nabla_w R &= \sum_{(x,y)} -2 \cdot y \cdot x^T + 2 \cdot W \cdot x \cdot x^T \\ -2 \cdot \begin{bmatrix} \dots & \dots & \dots \\ y & y & y \\ \dots & \dots & \dots \end{bmatrix} \cdot \begin{bmatrix} \dots & x^T & \dots \\ \dots & x^T & \dots \\ \dots & x^T & \dots \end{bmatrix} &+ 2 [W] \cdot \begin{bmatrix} \dots & \dots & \dots \\ x & x & x \\ \dots & \dots & \dots \end{bmatrix} \cdot \begin{bmatrix} \dots & x^T & \dots \\ \dots & x^T & \dots \\ \dots & x^T & \dots \end{bmatrix} \end{aligned}$$

In terms of data matrices:

$$\nabla_w R = 0 \iff -2 \cdot y \cdot x^T + 2 \cdot w \cdot x \cdot x^T = 0$$

$$-2 \cdot y \cdot x^T + 2 \cdot w \cdot x \cdot x^T = 0 \Rightarrow W = y \cdot x^T \cdot (x \cdot x^T)^{-1}$$

We are trying to solve $W \cdot x \sim y$, but the problem is that we can't always invert X , because it isn't always a square matrix. We can almost invert it with $X^T \cdot (X \cdot X^T)^{-1}$, which is the **Moon-Penrose Pseude-Inverse**.

8 Lecture 8

9 Gradient Descend