SUMS OF POWERS OF THREE

JUHO HÄPPÖLÄ

Abstract. A formula for summing positive powers of three is derived.

Define the difference equation

$$x_{n+1} = 3x_n + 1$$

then

$$\begin{aligned} x_{n+2} =& 3x_{n+1} + 1, \\ x_{n+1} =& 3x_n + 1, \\ x_{n+1} - x_{n+2} =& 3x_n - 3x_{n+1}, \\ x_{n+2} - 4x_{n+1} + 3x_n =& 0 \end{aligned}$$

take ansatz $x_n = r^n$:

$$r^2 - 4r + 3 = 0$$
$$(r-3)(r-1) = 0$$

which gives that the difference equation has a solution

$$x_n = 3^n C + D$$

for some constants C and D. (cf. `https://en.wikipedia.org/w/index.php?title=Recurrence_relation&oldid=602802902`) These constants are defined as

$$C + D = x_0$$
$$3C + D = 3x_0 + 1,$$

giving us $C = x_0 + \frac{1}{2}$, $D = -\frac{1}{2}$ On the other hand, we have

$$\begin{aligned} x_1 =& 3x_0 + 1 \\ x_2 =& 9x_0 + 3 + 1 \\ x_3 =& 27x_0 + 9 + 3 + 1 \\ x_n =& 3^n x_0 + \sum_{j=0}^{n-1} 3^j \end{aligned}$$

giving us that

$$3^n x_0 + \sum_{j=0}^{n-1} 3^j = 3^n x_0 + \frac{3^n - 1}{2}$$
$$\sum_{j=0}^{n-1} 3^j = \frac{3^n - 1}{2}.$$

This derivation perhaps enables one to compute sums of powers of other numbers too, but I need only powers of three right now.