Q4 3.)

$$\frac{d^2u}{dx^2} + u = x^2 \quad \text{for } 0 < x < 1,$$

$$u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} = 1$$

3a) (1) $u(x) \approx a_0 + a_1 x$. – two term polynomial approx.

(2) $u(x) = a_0 + a_1 x + a_2 x^2$
↳ three term polynomial approximation

3b) $u(x) = a_0 + a_1 x$

Two term

$$\frac{d^2u}{dx^2} + u = x^2$$

$$\frac{d^2u}{dx^2} = 0, \qquad a_0 + a_1 x = x^2$$

At $x = 1/3 \rightarrow a_0 + a_1/3 = (1/3)^2$

$x = 2/3 \rightarrow a_0 + \frac{2a_1}{3} = (2/3)^2$

Solve to get a_0, a_1

Q3. Three-term Collocation

say, $u(x) = a_0 + a_1 x + a_2 x^2$

$$\frac{d^2u}{dx^2} = 2a_2$$

$$2a_2 + a_0 + a_1 \tfrac{1}{4} + \tfrac{a_2}{16} = \left(\tfrac{1}{4}\right)^2$$

$$\left\{\begin{array}{l} 2a_2 + a_0 + a_1 \tfrac{1}{2} + \tfrac{a_2}{4} = \left(\tfrac{1}{2}\right)^2 \\ 2a_2 + a_0 + \tfrac{3a_1}{4} + \tfrac{9a_2}{16} = \left(\tfrac{3}{4}\right)^2 \end{array}\right.$$

$\rightarrow$ 3 eq. 3 unknowns

C. Least squares Method.

$$R(a_0, a_1, a_2) = \int_0^1 \left(\frac{d^2u}{dx^2} + u - x^2 \right)^2 dx$$

$$u(x) = a_0 + a_1 x$$

$$R(a_0, a_1, a_2) = \int_0^1 (2a_2 + a_0 + a_1 x + a_2 x^2 - x^2)^2$$

[differentiate R w.r.t. a_0, a_1, a_2 to get 3 equations, equate all to zero as to solve, minimize residue.

3. d.) Galerkin.

$R = \frac{d^2u}{dx^2} + u - x^2$

$\int_0^1 R \cdot w_i \, dx = 0$

Choose $w_1 = 1$, $w_2 = x$.

$u(x) = a_0 + a_1 x + a_2 x^2$

$\int_0^1 R \cdot w_i \, dx = 0$

for 3 term choose $w_1 = 1$, $w_2 = x$, $w_3 = x^2$

1) $\frac{d^2u}{dx^2} = -\cos(\pi x)$ for $0 < x < 1$,

$u(0) = 0$, $u(1) = 0$

4a.) 3 term polynomial solution,

$u(x) = a_1 x + a_2 x^2 + a_3 x^3$

$u(1) = a_1 \cdot 1 + a_2 \cdot 1^2 + a_3 \cdot 1^3 = a_1 + a_2 + a_3 = 0$

given: $u(1) = 0$.

$u(0) = 0$

3 term Trigonometric:

$u(x) = b_1 \sin(\pi x) + b_2 \sin(2\pi x) + b_3 \sin(3\pi x)$

∴ This satisfies $u(0) = 0$ & $u(1) = 0$

Q(4.) b.)

$u(x) \approx a_1 x + a_2 x^2 + a_3 x^3$

$$\frac{d^2u}{dx^2} = 2a_2 + 6a_3 x.$$

$$2a_2 + 6a_3 x = -\cos \pi x.$$

↳ compute at $x = 1/4, 1/2, 3/4$

↳ 3 equations, 3 unknowns solve.

For trigonometric solution

$$u(x) = b_1 \sin(\pi x) + b_2 \sin(2\pi x) + b_3 \sin(3\pi x)$$

$$\frac{d^2u}{dx^2} = -\pi^2 b_1 \sin(\pi x) - 4\pi^2 b_2 \sin(2\pi x) - 9\pi^2 b_3 \sin(3\pi x)$$

↳ $x = 1/4, 1/2, 3/4$ put & solve 3 eq^n 3 unknowns

Q(4.) c) Galerkin method.

Polynomial.

$$\int_0^1 \left(\frac{d^2u}{dx^2} + \cos(\pi x)\right) \phi_i(x)\, dx = 0$$

$\phi_1(x) = x$, $\phi_2(x) = x^2$ & $\phi_3(x) = x^3$,

$\phi_1(x) = x$, $\phi_2(x) = x^2$, $\phi_3(x) = x^3$

$$\int_0^1 (2a_2 + 6a_3 x + \cos \pi x)\, x\, dx = 0$$

$$\int_0^1 (2a_2 + 6a_3 x + \cos(\pi x))\, x^2\, dx = 0$$

$$\int_0^1 (2a_2 + 6a_3 x + \cos(\pi x))\, x^3\, dx = 0$$

Trignometric :

weights, $\varphi_1(x) = \sin(\pi x)$

$\varphi_2(x) = \sin(2\pi x)$

$\varphi_3(x) = \sin(3\pi x)$

solve 3 equation & 3 unknowns

Q 4.7 d Applying Galerkin method to weak form would lead to similar set of equations but boundary conditions that are weakly enforced.

Q 5.7
$$\frac{d}{dx}\left(u\frac{du}{dx}\right) - f(x) = 0 \quad \text{for } 0 < x < L$$

$$u\frac{du}{dx}\Big|_{x=0}, \quad u(L) = u_0$$

1.) multiply by test function / weight $= v(x)$

$$\int_0^L v(x)\left[\frac{d}{dx}\left(u\frac{du}{dx}\right) - f(x)\right]dx = 0$$

Using integration by parts

$$\int_0^L v(x)\frac{d}{dx}\left(u\frac{du}{dx}\right)dx = \left[v(x)\cdot u\cdot\frac{du}{dx}\right]_0^L - \int_0^L \frac{dv}{dx}\left(u\frac{du}{dx}\right)dx$$

$$\left[v(x)\, u\frac{du}{dx}\right]\Big|_{x=0} = 0 \quad \text{as } \cancel{\ldots}\ \text{at } u\frac{du}{dx}\Big|_{x=0} = 0$$

$u(L) = u_0$ so other boundary remains

Q(i)

$$2u\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 + 4 = 0$$

$u(0) = 1, \quad u(1) = 0$

Q.1(a) Weak form of weighted residual equation

$$\int_0^1 v(x)\left[2u\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 + 4\right]dx = 0$$

$$\int_0^1 v(x)\cdot\frac{d^2u}{dx^2}\,dx = \left[v(x)\frac{du}{dx}\right]_0^1 - \int_0^1 \frac{dv}{dx}\cdot\frac{du}{dx}\cdot dx$$

$$\left[v(x)\frac{du}{dx}\right]_0^1 = v(1)\frac{du}{dx}\Big|_{x=1} - v(0)\frac{du}{dx}\Big|_{x=0}$$

$x=1,\ u(1)=0$ so $v(1)\frac{du}{dx}\Big|_{x=1} = 0$

$x=0,\ v(0)\frac{du}{dx}\Big|_{x=0} = 0$, b'coz $v(0)=0$

test function boundary condition

Combining terms & simplifying,

$$\int_0^1 2u\frac{du}{dx}\cdot\frac{dv}{dx}\cdot dx + \int_0^1 v(x)\left(\frac{du}{dx}\right)^2 dx = \int_0^1 4v(x)\,dx$$

Q6) b)

(1) polynomial Approximation

$u(x) \approx a_0 + a_1 x + a_2 x^2$

$u(0) = 1 \Rightarrow a_0 = 1$, $u(1) = 0 \Rightarrow a_0 + a_1 + a_2 = 0$

(2) $v(x)\ \frac{du}{da_i}$, weighting factor $i = 1, 2$.

$v_1(x) = x$, $v_2(x) = x^2$

↳ substitute in weak form to get 2 equations

$$\int_0^1 2(1 + a_1 x + a_2 x^2)(a_1 + 2a_2 x)\,dx + \int_0^1 x(a_1 + 2a_2 x)^2\,dx = \int_0^1 4x\,dx$$

$$\int_0^1 2(1 + a_1 x + a_2 x^2)(a_1 + 2a_2 x) \cdot x^2\,dx + \int_0^1 x^2(a_1 + 2a_2 x)^2\,dx = \int_0^1 4x^2\,dx$$

Q6) c) $u(x) = b_1 \sin(\pi x) + b_2 \sin(2\pi x)$

$u(0) = 1$, $u(1) = 0$.

Apply Galerkin: $v(x) = \sin(\pi x)$, $v(x) = \sin(2\pi x)$

Residuals calc:

$u(x) = b_1 \sin(\pi x) + b_2 \sin(2\pi x)$

$\frac{du}{dx} = \pi b_1 \cos(\pi x) + 2\pi b_2 \cos(2\pi x)$

Solve for b_1, b_2 by writing weak form