Operations on Polynomials

I. Addition – combine like terms (same variables and same exponents for these variables) using the distributive property: $ba + ca = (b+c)a$

Examples: (1) $2x^2 + 3x^2 = (2+3)x^2 = 5x^2$

(2) $4xy + xy = 4xy + 1xy = (4+1)xy = 5xy$

(3) $(3x^2 + 7x + 8) + (5x^2 - 8x + 2) = 3x^2 + 7x + 8 + 5x^2 - 8x + 2 =$
$(3x^2 + 5x^2) + (7x - 8x) + (8+2) = 8x^2 + (-1x) + 10 = 8x^2 - x + 10$

(4)
$$\begin{array}{rr} & 3x^2 + 7x + 8 \\ (+) & 5x^2 - 8x + 2 \\ \hline & 8x^2 - 1x + 10 \end{array}$$
(column form)

II. Subtraction – add the opposite of the second polynomial

Examples: (1) $7x - 8x = 7x + (-8x) = [7 + (-8)]x = (-1)x = -x$

(2) $(3x^2 + 7x + 8) - (5x^2 - 8x + 2) = (3x^2 + 7x + 8) + (-5x^2 + 8x - 2) =$
$3x^2 + 7x + 8 - 5x^2 + 8x - 2 = (3x^2 - 5x^2) + (7x + 8x) + (8 - 2) =$
$-2x^2 + 15x + 6$

(3)
$$\begin{array}{rr} & 3x^2 + 7x + 8 \\ (-) & 5x^2 - 8x + 2 \\ \hline \end{array} \Rightarrow \begin{array}{rr} & 3x^2 + 7x + 8 \\ (+) & -5x^2 + 8x - 2 \\ \hline & -2x^2 + 15x + 6 \end{array}$$
(column form)

III. Multiplication – use the distributive property $a(b+c) = ab + ac$, laws of exponents, and addition and subtraction of polynomials

Examples: (1) $(-2x)(3x^2) = -6x^3$

(2) $3x(2x - 5) = (3x)(2x) - (3x)(5) = 6x^2 - 15x$

(3) $(2x+1)(3x-4) = (2x)(3x) - (2x)(4) + (1)(3x) - (1)(4) =$
$6x^2 - 8x + 3x - 4 = 6x^2 - 5x - 4$

(4)

$$\begin{array}{rr} & 2x+1 \\ (\times) & 3x-4 \\ \hline & 6x^2+3x \\ & -8x-4 \\ \hline & 6x^2-5x-4 \end{array}$$

(column form)

IV. Powers of polynomials and binomial expansion

Note the following patterns:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}$$

Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Rows start and end with 1, and each of the other numbers is equal to the sum of the two numbers above it.

Examples: (1) $(2x-3)^3 = [2x+(-3)]^3 = (2x)^3 + 3(2x)^2(-3) + 3(2x)(-3)^2 + (-3)^3 =$
$8x^3 - 36x^2 + 54x - 27$

(2) $(x^2+2x-3)^2 = (x^2+2x-3)(x^2+2x-3) = (x^2)(x^2) + (x^2)(2x) +$
$(x^2)(-3) + (2x)(x^2) + (2x)(2x) + (2x)(-3) + (-3)(x^2) + (-3)(2x) +$
$(-3)(-3) = x^4 + 2x^3 - 3x^2 + 2x^3 + 4x^2 - 6x - 3x^2 - 6x + 9 =$
$x^4 + 4x^3 - 2x^2 - 12x + 9$

V. Division

(1) Monomial $\div$ monomial – use laws of exponents

Example: $\dfrac{24x^4y^2}{18x^2y^3} = \dfrac{4x^2}{3y}$

(2) Polynomial $\div$ monomial – separate and divide individually

Example: $\dfrac{4x^5 - 8x^3 + 12x^2}{6x^2} = \dfrac{4x^5}{6x^2} - \dfrac{8x^3}{6x^2} + \dfrac{12x^2}{6x^2} = \dfrac{2x^3}{3} - \dfrac{4x}{3} + 2$

(3) Polynomial $\div$ polynomial – use the following long division algorithm

(i) Write dividend and divisor polynomials in standard polynomial form. Use zero coefficients for powers of the variable which are missing in the dividend and divisor.

(ii) Divide first term of the divisor into the first term of the dividend. Put this quotient above term in the dividend.

(iii) Multiply quotient by all terms of the divisor and put products under the appropriate terms of the dividend.

(iv) Subtract (change signs on bottom polynomial and add) and bring down remaining terms.

(v) Continue to divide first term by first term until the power of the divisor is larger than the power in the dividend.

Example: $(8 + 3x - x^3) \div (x - 2)$

$$
\begin{array}{rl}
 & -x^2 - 2x - 1 \\
x-2 & \overline{\big) -x^3 + 0x^2 + 3x + 8} \\
(-) & -x^3 + 2x^2 \\
\hline
 & -2x^2 + 3x + 8 \\
(-) & -2x^2 + 4x \\
\hline
 & -x + 8 \\
(-) & -x + 2 \\
\hline
 & 6 \text{ (remainder)}
\end{array}
$$

$\dfrac{-x^3}{x} = -x^2$

$\dfrac{-2x^2}{x} = -2x$

$\dfrac{-x}{x} = -1$

Thus, $(8+3x-x^3)\div(x-2)=-x^2-2x-1+\dfrac{6}{x-2}$.

Check:

$$
\begin{array}{lr}
 & -x^2-2x-1 \\
(\times) & x-2 \\
\hline
 & -x^3-2x^2-x \\
 & 2x^2+4x+2 \\
\hline
 & -x^3+0x^2+3x+2 \\
(+) & 6 \ \text{(remainder)} \\
\hline
 & -x^3+0x^2+3x+8
\end{array}
$$

Practice Sheet – Operations on Polynomials

Perform the indicated operations and simplify:

(1) $(x^5-x^3+x)+(3x^5-4x^4)+(x^3+2x+5)=$

(2) $(8x^2-5x)-(3x^2-3)+(3x-5)=$

(3) $(5x^2-4x+3)-[(2x^2+x)-(3x+4)]=$

(4) $(4x-1)(7x+2)=$

(5) $(6x-5)(6x+5)=$

(6) $(3x+4)^2=$

(7) $(2x-1)^3=$

(8) $(x-2)^4=$

(9) $(2x+3)(-x^2+5x-4)=$

(10) $(2x+1)(4x^2-2x+1)=$

(11) $(x^2+x-2)(x^2-3x-4)=$

(12) $(x^2+3x-1)^2=$

(13) $\dfrac{15x^4+30x^3+12x^2-9x}{3x}=$

(14) $\dfrac{25x^2y^4-15x^3y^3+40x^2y^2}{5x^2y}=$

(15) $5(3x+4)-x(2x-1)=$

(16) $(3x-1)(2x+3)-(6x+5)(x-2)=$

(17) $(3x+2)^2-(2x-5)(x+1)=$

(18) $(x^3+2x^2-17x-10)\div(x+5)=$

(19) $(3x^3+2x^2-2)\div(x-1)=$

(20) $(4x^3-8x^2-11x+18)\div(2x-3)=$

(21) $(8x^3+27)\div(2x+3)=$

(22) $(x^3+3x^2-11x-8)\div(x^2-2x-3)=$

(23) $(x^4+2x^3+x^2-1)\div(x^2+x-1)=$

(24) $(x^4-3x+4)\div(x^2+3)=$

(25) $(x^5-1)\div(x^2-1)=$

Solution Key for Operations on Polynomials

(1) $4x^5-4x^4+3x+5$

(2) $5x^2-2x-2$

(3) $3x^2-2x+7$

(4) $28x^2+x-2$

(5) $36x^2-25$

(6) $9x^2+24x+16$

(7) $8x^3-12x^2+6x-1$

(8) $x^4-8x^3+24x^2-32x+16$

(9) $-2x^3+7x^2+7x-12$

(10) $8x^3+1$

(11) $x^4-2x^3-9x^2+2x+8$

(12) $x^4+6x^3+7x^2-6x+1$

(13) $5x^3+10x^2+4x-3$

(14) $5y^3-3xy^2+8y$

(16) $14x+7$

(17) $7x^2+15x+9$

(18) x^2-3x-2

(19) $3x^2+5x+5+\dfrac{3}{x-1}$

(20) $2x^2-x-7+\dfrac{-3}{2x-3}$

(21) $4x^2-6x+9$

(22) $x+5+\dfrac{2x+7}{x^2-2x-3}$

(23) x^2+x+1

(24) $x^2-3+\dfrac{-3+13}{x^2+3}$

(25) $x^3+x+\dfrac{x-1}{x^2-1}=x^3+x+\dfrac{1}{x-1}$

(15) $-2x^2+16x+20$ 5