EE101: RLC Circuits (with DC sources)

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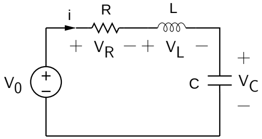
Series RLC circuit



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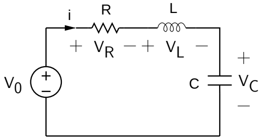

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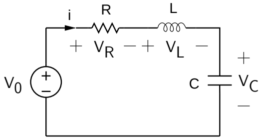

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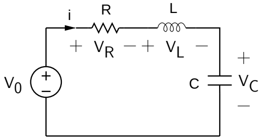

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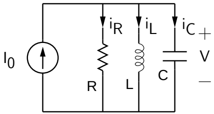
i.e., $\dfrac{d^2 i}{dt^2} + \dfrac{R}{L}\,\dfrac{di}{dt} + \dfrac{1}{LC}\,i = 0\,,$

a second-order ODE with constant coefficients.

Parallel RLC circuit



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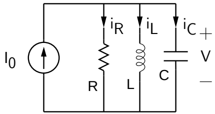

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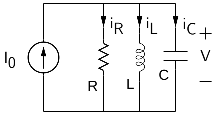

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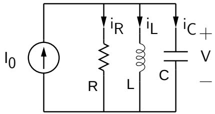

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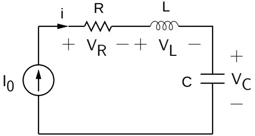
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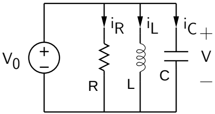
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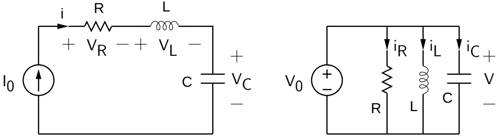

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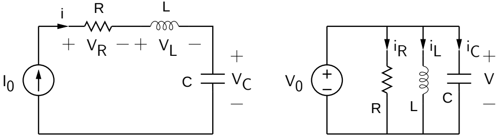

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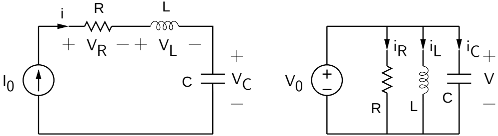

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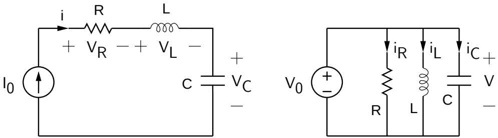

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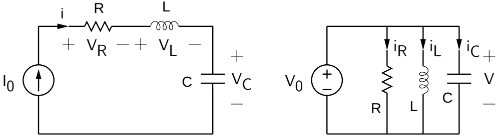

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* The above equations hold even if the applied voltage or current is not constant, and the variables of interest can still be easily obtained without solving a differential equation.

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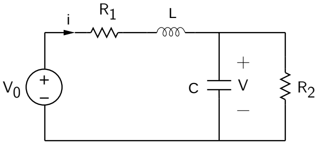

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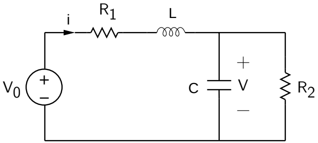

$$V_0 = R_1\, i + L\, \frac{di}{dt} + V \tag{1}$$

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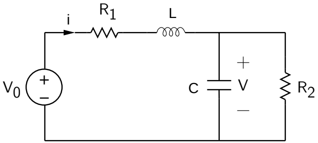

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Substituting (2) in (1), we get

$$V_0 = R_1\left[CV' + V/R_2\right] + L\left[CV'' + V'/R_2\right] + V\,, \tag{3}$$

$$V''\,[LC] + V'\,[R_1 C + L/R_2] + V\,[1 + R_1/R_2] = V_0\,. \tag{4}$$

General solution

Consider the second-order ODE with constant coefficients,

$$\frac{d^2y}{dt^2} + a\,\frac{dy}{dt} + b\,y = K \text{ (constant)}\,.$$

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$$y(t) = y^{(h)}(t) + y^{(p)}(t)\,,$$

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In the context of RLC circuits, $y^{(p)}(t)$ is the steady-state value of the variable of interest, i.e.,

$$y^{(p)} = \lim_{t\to\infty} y(t),$$

which can be often found by inspection.

For the homogeneous equation,

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we first find the roots of the associated *characteristic equation*,

$$r^2 + a\,r + b = 0\,.$$

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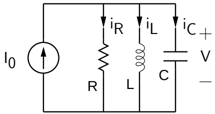

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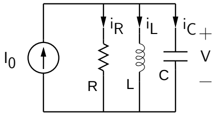
C=1 μF

L=0.44 mH

I_0 = 100 mA

Parallel RLC circuit



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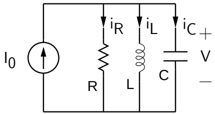

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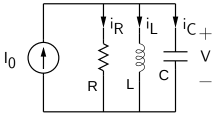

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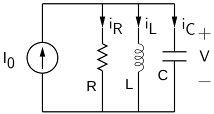

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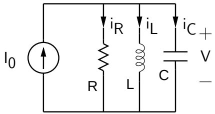

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where $\tau_1 = -1/r_1 = 15.4\,\mu s$, $\tau_2 = -1/r_1 = 28.6\,\mu s$.

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$$A + B = 0\,. \tag{1}$$

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From (1) and (2), we get the values of A and B, and

$$V(t) = -3.3\,[\exp(-t/\tau_1) - \exp(-t/\tau_2)]\ \ V\,. \tag{3}$$

(SEQUEL file: `ee101_rlc_1.sqproj`)

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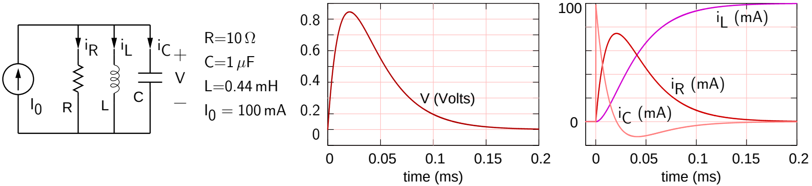
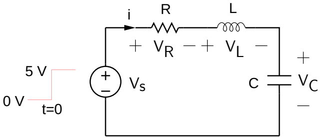
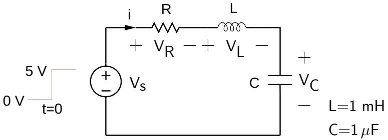


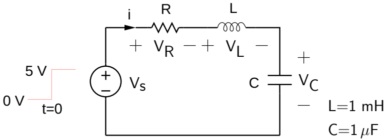


L=1 mH

C=1 μF



(a) Show that the condition for critically damped response is $R = 63.2\,\Omega$.



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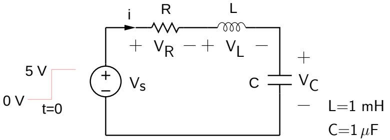
(b) For $R = 20\,\Omega$, derive expressions for $i(t)$ and $V_L(t)$ for $t > 0$ (Assume that $V_C(0^-) = 0\ V$ and $i_L(0^-) = 0\ A$). Plot them versus time.



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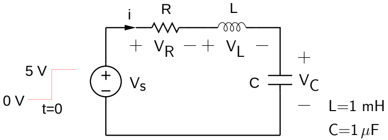

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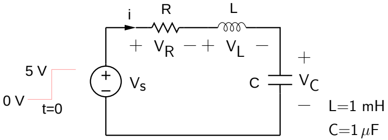
(d) Compare your results with the following plots.
(SEQUEL file: `ee101_rlc_2.sqproj`)



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