Butterworth Filter Design

Lowpass Butterworth filters are characterized by the magnitude-squared frequency response

$$|H(\Omega)|^2 = 1/(1+ (\Omega/ \Omega_c)^{2N}) \qquad (1)$$

where,
Ω_c is the unnormalized -3dB frequency in rad/sec,
N is the order of the filter.

Since $H(s)\ H(-s)$ evaluated at $s = \mathrm{j}\ \Omega$ equal to $|H(\Omega)|^2$ it follows that

$$H(s)\ H(-s) = 1/(1+(-s^2/ \Omega_c^2)^N) \qquad (2)$$

The poles of $H(s)\ H(-s)$ occur on a circle of radius Ω_c at equally spaced points, which can be found by using (2) as

$$(-s^2/ \Omega_c^2) = (-1)^{1/N} = e^{j\,(2k+1)\,\pi\,/\,N}, \qquad k = 0,1, 2, 3, \ldots\ldots, N\text{-}1$$

$$s_k = \Omega_c\ e^{j\,\pi\,/\,2}\ e^{j\,(2k+1)\,\pi\,/\,N},\ k = 0,1, 2, 3, \ldots\ldots, N\text{-}1 \qquad (3)$$

from which we can write

$$H(s) = \frac{k_0}{\prod_{k=0}^{N-1} s - s_k} \qquad (4)$$

From (4), we can get the following information

N	Normalized Denominator Polynomials in Factored Form
1	(1+s)
2	$(1+1.414s+s^2)$
3	$(1+s)(1+s+s^2)$
4	$(1+0.765s+s^2)(1+1.848s^2)$
5	$(1+s)(1+0.618s+s^2)(1+1.618s^2)$
6	$(1+0.518s+s^2)(1+1.414s+s^2)(1+1.932s+s^2)$
7	$(1+s)(1+0.445s+s^2)(1+1.247s+s^2)(1+1.802s+s^2)$
8	$(1+0.390s+s^2)(1+1.111s+s^2)(1+1.663s+s^2)(1+1.962s+s^2)$
9	$(1+s)(1+0.347s+s^2)(1+s+s^2)(1+1.532s+s^2)(1+1.879s+s^2)$
10	$(1+0.313s+s^2)(1+0.908s+s^2)(1+1.414s+s^2)(1+1.782s+s^2)(1+1.975s+s^2)$

Example

Now let us find the coefficients of IIR Butterworth filter of 1st order (same procedure can be followed for higher orders) with cutoff frequency of 1 kHz. It is given that the signal being passed through the filter is sampled at 10 kHz.

If x(n) represents the input and y(n) represents the filtered output then they are related as

$$y(n) = b_0 x(n) + b_1 x(n-1) + a_1 y(n-1)$$

where b_0 b_1 a_1 are the filter coefficients to be found out.
As $N = 1$, from (4) we get

$H(s) = 1/(s+1)$ (normalized cutoff frequency is 1 rad/sec)

Normalized cutoff frequency in rad / sample (ω) = $(1000/10000)2\pi$
$= 0.2\pi$

Unnormalized cutoff frequency in rad / sec (Ω)= $\tan(\omega/2)$
$= \tan(0.1\pi)$
$= 0.3249$
For the required low pass filter, analog domain transformation is $s \rightarrow s/0.3249$.

Hence $H(s) = 1/(3.0779s+1)$ (unnormalized cutoff frequency is 0.3249 rad/sec)

For analog to digital domain transformation $s \rightarrow (z-1)/(z+1)$

Hence

$$H(z) = \frac{0.2495 + 0.2495z^{-1}}{1 - 0.51z^{-1}}$$

$$\frac{y(z)}{x(z)} = \frac{0.2495 + 0.2495z^{-1}}{1 - 0.51z^{-1}}$$

from which we can get

b_0=0.2495,
b_1=0.2495,
a_0=0.51.

In the same way we can get the coefficients for higher order filters.

If the design requires some other parameters like stop-band edge frequency (Ω_s), pass-band tolerance(δ_1), stop-band tolerance (δ_2) then the filter order can be found out using the relation

$$N \geq \frac{\log\left(\frac{D_2}{D_1}\right)^{0.5}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

where

$$D_1 = \frac{1}{(1-\delta_1)^2} - 1$$

$$D_2 = \frac{1}{\delta_2^{\ 2}} - 1$$

After finding the filter order the above procedure can be used to find the filter coefficients.

If the desired filter is High-pass or Band-pass or Band-stop then the following analog transformations can be applied on low-pass analog filter transfer function.

$$s \rightarrow \frac{\Omega_p \Omega_p^{'}}{s} \quad \text{(Low-pass to High-pass)}$$

$$s \rightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \quad \text{(Low-pass to Band-pass)}$$

$$s \rightarrow \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u} \quad \text{(Low-pass to Band-stop)}$$

Ω_p is passband edge cutoff frequency of low pass filter.

$\Omega_p^{'}$ is new pass band edge frequency and Ω_l, Ω_u are lower and upper cutoff frequencies.

Characteristics

Some important characteristics of Butterworth filters are described below.

1. Lowpass Butterworth filters are all-pole filters.
2. Maximally flat magnitude response and it is monotonic in passband as well as in stopband.
3. Roll-off is slower compared to chebyshev filters. So to get the given transition, order of Butterworth filter required is higher than the order of chebyshev filter.
4. The phase response of a Butterworth filter is more nearly linear than that of a Chebyshev Type II filter of same order.
5. No great differences in the unit impulse and step responses of Butteraworth and Cheyshev Type II filters.

References

1. 'Digital signal processing Principles, Algorithms, and Applications' fourth edition, John G. Proakis and Dimitris G. Manolakis.
2. 'Theory and Applications of Digital Signal Processing', L. R. Rabiner and B. Gold.
3. Www.dspguide.com