Chebyshev Filter Design

Chebyshev filters exhibit equiripple behavior over a band of frequencies and monotonic characteristic in other band. Depending upon the band of frequencies over which the behavior is equiripple the filter designs are called Type I Chebyshev or Type II Chebyshev (Chebyshev or Inverse Chebyshev).

Type I Chebyshev Filters

These filters are all-pole designs that exhibit equiripple passband behavior and monotonic stopband response. The squared magnitude response of an N^{th} order Type I filter can be expressed as

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega)} \tag{1}$$

where $T_N(\Omega)$ is the N^{th} order Chebyshev polynomial defined as

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega), & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}\Omega), & |\Omega| > 1 \end{cases} \tag{2}$$

and ε is a parameter that is related to the passband ripple.

Chebyshev polynomials can be generated by the recursive equation

$$T_{N+1}(\Omega) = 2\Omega T_N(\Omega) - T_{N-1}(\Omega),\ N = 1, 2, \ldots.. \tag{3}$$

Let the passband edge of the design is at $\Omega = 1$ (where $|H(1)|^2 = \frac{1}{1+\varepsilon^2}$) and the stopband edge is at $\Omega = \Omega_r$ (where $|H(\Omega)|^2 = \frac{1}{A^2}$).

The poles ($s_k = \sigma_k + j\Omega_k$, k = 1, 2, 3, …., N) of a Type I filter are simple and lie on an ellipse in the s plane given by

$$\frac{\sigma_k^2}{\sinh^2\varphi} + \frac{\Omega_k^2}{\cosh^2\varphi} = 1 \tag{4}$$

where $$\sigma_k = -\sinh\varphi \sin\left[\frac{(2k-1)\pi}{2N}\right],\ \ \Omega_k = \cosh\varphi\cos\left[\frac{(2k-1)\pi}{2N}\right]$$

$$\sinh\varphi = \frac{\gamma-\gamma^{-1}}{2},\ \ \cosh\varphi = \frac{\gamma+\gamma^{-1}}{2},\ \gamma = \left(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)^{1/N}$$

Type II Chebyshev Filters

These filters have both poles and zeros and exhibit a monotonic behavior in the passband and equiripple behavior in stopband. Zeros of this class of filters lie on the imaginary axis in the *s*-plane.
The squared magnitude response of an N^{th} order Type I filter can be expressed as

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[T_N(\Omega_r)/T_N(\Omega_r/\Omega)\right]^2} \tag{5}$$

The poles ($s_k = \sigma_k + j\Omega_k,\ k = 1, 2, 3, \ldots, N$) can be shown to be located at,

$$\sigma_k = \frac{\Omega_r \alpha_k}{\alpha_k^2+\beta_k^2},\ \beta_k = \frac{-\Omega_r \beta_k}{\alpha_k^2+\beta_k^2} \tag{6}$$

where

$$\alpha_k = -\sinh\varphi \sin\left[\frac{(2k-1)\pi}{2N}\right],\ \beta_k = \cosh\varphi \cos\left[\frac{(2k-1)\pi}{2N}\right]$$

$$\sinh\varphi = \frac{\gamma-\gamma^{-1}}{2},\ \cosh\varphi = \frac{\gamma+\gamma^{-1}}{2},\ \gamma = \left(A+\sqrt{A^2-1}\right)^{1/N}$$

The zeros are located at

$$s_k = j\frac{\Omega_r}{\cos\{[(2k-1)/2N]\pi\}}\ \text{k=1, 2,…,}\ N \tag{7}$$

Chebyshev filters (both Type I and II) are completely specified by selecting values for any three of the following four parameters.

1. N, the filter order
2. ε, the parameter related to passband ripple.
3. Ω_r, the lowest frequency at which the stopband loss attains the prescribed attenuation.
4. A, the parameter related to stopband loss.

The Chebyshev filter degree N required to achieve given values of ε, A, and Ω_r is given by

$$N = \frac{\log_{10}\left(g+\sqrt{g^2-1}\right)}{\log_{10}\left(\Omega_r+\sqrt{\Omega_r^{\,2}-1}\right)} \tag{8}$$

where $$g = \sqrt{\frac{A^2-1}{\varepsilon^2}}$$

If the desired filter is High-pass or Band-pass or Band-stop then the following analog transformations can be applied on low-pass analog filter transfer function.

$$s \to \frac{\Omega_p \Omega_p^{'}}{s} \quad \text{(Low-pass to High-pass)}$$

$$s \to \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \quad \text{(Low-pass to Band-pass)}$$

$$s \to \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u} \quad \text{(Low-pass to Band-stop)}$$

Ω_p is passband edge cutoff frequency of low pass filter.

$\Omega_p^{'}$ is new pass band edge frequency and Ω_l, Ω_u are lower and upper cutoff frequencies.

For analog to digital domain transformation $s \to (z-1)/(z+1)$

Example 1

Problem

Find the minimum-order Chebyshev filter required to meet the following specifications
Passband ripple = 2 dB, Transition ratio = 0.781, Stopband loss = 40 dB.

Solution

$$20\log_{10}\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right) = \text{-}2 \;\Rightarrow\; \varepsilon = 0.764$$

$$20\log_{10}\left(\frac{1}{A}\right) = \text{-}30 \;\Rightarrow\; A = 31.62$$

Ω_r=1/0.781=1.28

Substituting these values in (8) we will get $N = 6.03$.

Characteristics

Some important characteristics of Chebyshev filters are described below.

1. Chebyshev Type I filters are all pole filters that exhibit equiripple behavior in the passband and a monotonic characteristic in the stopband.
2. Chebyshev Type II filters contain both poles and zeros and exhibit a monotonic behaviour in the passband and an equiripple behaviour in the stopband. Zeros of this class of filters lie on imaginary axis in the s-plane.
3. A Chebyshev Type II filter, of the same order, has a more constant magnitude response in the passband, a more nearly linear phase response, a more nearly constant phase delay and group delay, and less ringing in the impulse and step responses, than does a Chebyshev Type I filter.

References

1. Digital signal processing Principles, Algorithms, and Applications' fourth edition, John G. Proakis and Dimitris G. Manolakis.
2. 'Theory and Applications of Digital Signal Processing', L. R. Rabiner and B. Gold.
3. Www.dspguide.com