Rachunek macierzowy

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Jeśli używasz fragmentów tego wykładu, zacytuj źródło matrix cookbook

https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf

$$\boxed{(\mathbf{AB})^\mathbf{T} = \mathbf{B}^\mathbf{T}\mathbf{A}^\mathbf{T}} \qquad \left(\underbrace{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}\right)^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^T \tag{1}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5*1+6*3 & 5*2+6*4 \\ 7*1+8*3 & 7*2+8*4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$$L = \left(\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)^T = \left(\begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}\right)^T = \begin{bmatrix} 23 & 31 \\ 34 & 46 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} =$$

ok

$$= \begin{bmatrix} 1*5+3*6 & 1*7+3*8 \\ 2*5+4*6 & 2*7+4*8 \end{bmatrix} = \begin{bmatrix} 23 & 31 \\ 34 & 46 \end{bmatrix} = L$$

$$\boxed{(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}} \tag{2}$$

Dowód

Z własności A^T mamy

$$\boxed{(Ax, y) = (x, A^T y)} \tag{3}$$

Wówczas

$$(ABx, y) = (A(Bx), y) = (\text{traktujemy } Bx \text{ jak wektor}) = \tag{4}$$

$$= \cancel{(\text{z def } A^T)} = (Bx, A^T y) = \cancel{(\text{z def } B^T)} = (x, \boxed{B^T A^T} y) \tag{5}$$

stąd $(AB)^T = B^T A^T$

Stosuję teraz ten wzór dla AB

$(AB)^T = B^T A^T$

LICZBY ZESPOLONE

sprzężenie hermitowskie

$$(\mathbf{A}+\mathbf{B})^{\mathbf{H}} = \mathbf{A}^{\mathbf{H}} + \mathbf{B}^{\mathbf{H}} \tag{6}$$

A B

$$L = \left(\begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix} + \begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix}\right)^H =$$

Re Im

TRANSPOZYCJA HERMITOWSKA = T + ZMIANA ZNAKU CZĘŚCI Im

$$R = \begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix}^H + \begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix}^H$$

$$L = \left(\begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix} + \begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix}\right)^H =$$

$$\begin{bmatrix} (11+1)+(12+2)i & (13+3)+(14+4)i \\ (15+5)+(16+6)i & (17+7)i+(18+8)i \end{bmatrix}^H =$$

$$\begin{bmatrix} 12+14i & 16+18i \\ 20+22i & 24+26i \end{bmatrix}^H = \begin{bmatrix} 12-14i & 20-22i \\ 16-18i & 24-26i \end{bmatrix}$$

$$(\mathbf{A}+\mathbf{B})^{\mathbf{H}} = \mathbf{A}^{\mathbf{H}} + \mathbf{B}^{\mathbf{H}}$$

$$\left(\begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix} + \begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix}\right)^{H} =$$

$$\begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix}^{H} + \begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix}^{H}$$

$$R = \begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix}^{H} + \begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix}^{H} =$$

$$\begin{bmatrix} 1-2i & 5-6i \\ 3-4i & 7-8i \end{bmatrix} + \begin{bmatrix} 11-12i & 15-16i \\ 13-14i & 17-18i \end{bmatrix} = \begin{bmatrix} 12-14i & 20-22i \\ 16-18i & 24-26i \end{bmatrix} = L$$

$L = R$

$$(\mathbf{A}+\mathbf{B})^{\mathbf{H}} = \mathbf{A}^{\mathbf{H}} + \mathbf{B}^{\mathbf{H}} \tag{7}$$

$(x,y) = x \cdot \bar{y}$

Dowód

Z własności A^H mamy

$$\boxed{(Ax, y) = (x, A^H y)} \tag{8}$$

Wówczas

$$((A+B)x, y) = (x, (A+B)^H y) = \text{(z własności iloczyny skalarnego)} =$$

$$(Ax, y) + (Bx, y) = (x, A^H y) + (x, B^H y) = \tag{9}$$

$$(x, \underbrace{(A^H + B^H)}\, y)$$

stąd $(A+B)^H = A^H + B^H$

$$(\mathbf{AB})^{\mathbf{H}} = \mathbf{B}^{\mathbf{H}}\mathbf{A}^{\mathbf{H}} \tag{10}$$

$$\left(\begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix} \begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix}\right)^H =$$

$$\begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix}^H \begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix}^H$$

$$L = \left(\begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix} \begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix}\right)^H =$$

$$\begin{bmatrix} (11+12i)(1+2i)+(13+14i)(5+6i) & (11+12i)(3+4i)+(13+14i)(7+8i) \\ (15+16i)(1+2i)+(17+18i)(5+6i) & (15+16i)(3+4i)+(17+18i)(7+8i) \end{bmatrix}^H =$$

$$= \begin{bmatrix} (11-12i)(1-2i)+(13-14i)(5-6i) & (15-16i)(1-2i)+(17-18i)(5-6i) \\ (11-12i)(3-4i)+(13-14i)(7-8i) & (15-16i)(3-4i)+(17-18i)(7-8i) \end{bmatrix}$$

$$(\mathbf{AB})^{\mathbf{H}} = \mathbf{B}^{\mathbf{H}}\mathbf{A}^{\mathbf{H}}$$

A $$\begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix}^H = \begin{bmatrix} 1-2i & 5-6i \\ 3-4i & 7-8i \end{bmatrix}$$

B $$\begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix}^H = \begin{bmatrix} 11-12i & 15-16i \\ 13-14i & 17-18i \end{bmatrix}$$

$$R = \begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix}^H \begin{bmatrix} 11+12i & 13+14i \\ 15+16i & 17+18i \end{bmatrix}^H =$$

$$\begin{bmatrix} 1-2i & 5-6i \\ 3-4i & 7-8i \end{bmatrix} \begin{bmatrix} 11-12i & 15-16i \\ 13-14i & 17-18i \end{bmatrix} =$$

$$\begin{bmatrix} (1-2i)(11-12i)+(5-6i)(13-14i) & (1-2i)(15-16i)+(5-6i)(17-18i) \\ (3-4i)(11-12i)+(7-8i)(13-14i) & (3-4i)(15-16i)+(7-8i)(17-18i) \end{bmatrix}$$

$$= L$$

Podstawy - teoretycznie

$$\mathbf{(AB)^H = B^H A^H} \tag{11}$$

Dowód

Z własności A^H mamy

$$Ax\bar{y} = \boxed{(Ax, y) = (x, A^H y)} \quad x\overline{A^H y} \tag{12}$$

Wówczas

$$(ABx, y) = (A(\underbrace{Bx}_{*}), y) = (\text{traktujemy } Bx \text{ jak wektor}) = \tag{13}$$

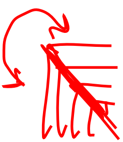
$$= (\cancel{\text{z własności } A^H}) = (Bx, A^H y) = (\cancel{\text{z własności } B^H}) = (x, B^H A^H y) \tag{14}$$

stąd $\boxed{(AB)^H = B^H A^H}$

Ślady

$$Tr(\mathbf{A}) = \sum_i A_i i \qquad Tr\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 + 4 = 5 \tag{15}$$

T.



$$\boxed{Tr(\mathbf{A}) = Tr\left(\mathbf{A}^{\mathbf{T}}\right)} \qquad Tr\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = Tr\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T \tag{16}$$

$$L = Tr\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 + 4 = 5$$

$$R = Tr\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = Tr\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 1 + 4 = 5 = L$$

Ślady - teoretycznie

$$Tr(\mathbf{A}) = Tr\left(\mathbf{A}^{\mathbf{T}}\right) \tag{17}$$

Dowód

$(A^T)_{ii} = A_{ii}$

$$diag(A) = diag(A^T) \tag{18}$$

$$Tr(\mathbf{A}) = L = \sum_i A_{ii} = R = Tr\left(\mathbf{A}^{\mathbf{T}}\right) \tag{19}$$

$$Tr(\mathbf{AB}) = Tr(\mathbf{BA}) \tag{20}$$

$$Tr\left(\overset{A}{\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}}\overset{B}{\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}}\right) = Tr\left(\overset{B}{\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}}\overset{A}{\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}}\right)$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = \begin{bmatrix}1*5+2*7 & 1*6+2*8\\3*5+4*7 & 3*6+4*8\end{bmatrix} = \begin{bmatrix}19 & 22\\43 & 50\end{bmatrix}$$

$$L = Tr\left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\right) = Tr\left(\begin{bmatrix}19 & 22\\43 & 50\end{bmatrix}\right) = 19 + 50 = 69$$

$$\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}5*1+6*3 & 5*2+6*4\\7*1+8*3 & 7*2+8*4\end{bmatrix} = \begin{bmatrix}23 & 34\\31 & 46\end{bmatrix}$$

$$R = Tr\left(\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right) = Tr\left(\begin{bmatrix}23 & 34\\31 & 46\end{bmatrix}\right) = 23 + 46 = 69 = L$$

$$Tr(\mathbf{AB}) = Tr(\mathbf{BA}) \tag{21}$$

Dowód

$$(AB)_{ij} = \sum_l a_{il} b_{lj}, \quad (AB)_{ii} = \sum_l a_{il} b_{li} \tag{22}$$

$$(BA)_{ij} = \sum_l b_{il} a_{lj}, \quad (BA)_{ii} = \sum_l b_{il} a_{li} \tag{23}$$

$$Tr(AB) = \sum_i (AB)_{ii} = \sum_i (\sum_l a_{il} b_{li}) = \sum_i \sum_l a_{il} b_{li} \tag{24}$$

$$Tr(BA) = \sum_i (BA)_{ii} = \sum_i (\sum_l b_{il} a_{li}) = \sum_i \sum_l b_{il} a_{li} = \sum_i \sum_l a_{li} b_{il} =$$

$$= \sum_l \sum_i a_{li} b_{il} (\text{zmiana literek i na l}) \tag{25}$$

$i - l$

Ślady

$$Tr(\mathbf{A} + \mathbf{B}) = Tr(\mathbf{A}) + Tr(\mathbf{B}) \tag{26}$$

$$Tr\left(\overbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}}\right) = Tr\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + Tr\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$L = Tr\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\right) = Tr\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} = 6 + 12 = 18$$

$$R = Tr\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + Tr\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = (1 + 4) + (5 + 8) = 18 = L$$

$$Tr(\mathbf{A}+\mathbf{B}) = Tr(\mathbf{A}) + Tr(\mathbf{B}) \tag{27}$$

Dowód

$$Tr(A) = \sum_i a_{ii}; \quad Tr(B) = \sum_i b_{ii} \tag{28}$$

$$Tr(A+B) = \sum_i (a_{ii} + b_{ii}) = \sum_i (a_{ii}) + \sum_i (b_{ii}) = Tr(A) + Tr(B) \tag{29}$$

Ślady

$$\mathbf{a}^{\mathbf{T}}\mathbf{a} = \mathbf{Tr}\left(\mathbf{a}\mathbf{a}^{\mathbf{T}}\right) \qquad \begin{bmatrix}1\\2\\3\end{bmatrix}^T \begin{bmatrix}1\\2\\3\end{bmatrix} = Tr\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}^T\right) \tag{30}$$

$$L = \begin{bmatrix}1\\2\\3\end{bmatrix}^T \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}123\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = 1*1+2*2+3*3 = 14$$

$$R = Tr\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\right) = Tr \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix}1*1 & 1*2 & 1*3\\ 2*1 & 2*2 & 2*3\\ 3*1 & 3*2 & 3*3\end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} =$$

$$Tr\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{bmatrix} = 1+4+9 = 14 = L$$

Ślady - teoretycznie

$$\mathbf{a}^\mathbf{T}\mathbf{a} = \mathbf{Tr}\left(\mathbf{a}\mathbf{a}^\mathbf{T}\right) \tag{31}$$

Dowód

$$L = \sum_i a_i^2 \tag{32}$$

$$\left(\mathbf{a}\mathbf{a}^\mathbf{T}\right)_{ij} = a_i a_j \tag{33}$$

$$R = Tr\left(\mathbf{a}\mathbf{a}^\mathbf{T}\right) = \sum_i \left(\mathbf{a}\mathbf{a}^\mathbf{T}\right)_{ii} = \sum_i a_i a_i = \sum a_i^2 = L \tag{34}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad L = \left(\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1} = R \tag{35}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \qquad \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 3 \\ 7/2 & -10/4 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 7/2 & -10/4 \end{bmatrix} =$$

$$= \begin{bmatrix} -2 * (-4) + 1 * (7/2) & -2 * 3 + 1 * (-10/4) \\ 3/2 * (-4) + (-1/2) * 7/2 & 3/2 * 3 + (-12/) * (-10/4) \end{bmatrix} =$$

$$= \begin{bmatrix} 23/2 & -17/2 \\ -31/4 & 23/4 \end{bmatrix}$$

Macierze odwrotne

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad L = \left(\underbrace{\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}}\right)^{-1} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}^{-1}\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}^{-1} = R$$

$$\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}5*1+6*3 & 5*2+6*4\\7*1+8*3 & 7*2+8*4\end{bmatrix} = \begin{bmatrix}23 & 34\\31 & 46\end{bmatrix}$$

$$L = \left(\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right)^{-1} = \left(\begin{bmatrix}23 & 34\\31 & 46\end{bmatrix}\right)^{-1}$$

$$L = \begin{bmatrix}23 & 34\\31 & 46\end{bmatrix}^{-1} = \begin{bmatrix}23/2 & -17/2\\-31/4 & 23/4\end{bmatrix} = R$$

$MM^{-1} = I$ definicja macierzy odwrotnej $M^{-1}M = I$

$M = AB$ $\quad M^{-1} = B^{-1}A^{-1}$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{36}$$

Dowód

Z definicji A^{-1} to taka macierz, że $\boxed{AA^{-1} = I}$ oraz $A^{-1}A = I$.
Sprawdzamy czy $(AB)(B^{-1}A^{-1}) = I$ oraz $(B^{-1}A^{-1})(AB) = I$

$$MM^{-1} = A\overbrace{BB^{-1}}A^{-1} = A(\overbrace{BB^{-1}}^{I})A^{-1} = A\overbrace{I}A^{-1} = AA^{-1} = I \tag{37}$$

$$M^{-1}M = B^{-1}A^{-1}AB = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}\underbrace{IB} = \underbrace{B^{-1}B}_{I} = I \tag{38}$$

$$\left(\mathbf{A}^{\mathbf{T}}\right)^{-\mathbf{1}} = \left(\mathbf{A}^{-\mathbf{1}}\right)^{\mathbf{T}} \quad L = \left(\overset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}^{T}\right)^{-1} = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}\right)^{T} = R \tag{39}$$

$AX = I$

$IX = A^{-1}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}_{A^T}^{-1} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$$

$$R = \left(\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A}^{-1}\right)^{T} = \left(\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\right)^{T} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$$

$$L = \left(\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A}^{T}\right)^{-1} = \left(\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}_{A^T}\right)^{-1} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix} = R$$

$M M^{-1} = I$ $\qquad$ $M^{-1} M = I$

$M = A^T$

$M^{-1} = (A^{-1})^T$

$$\underbrace{\left(\mathbf{A}^{\mathbf{T}}\right)}_{M}^{-1} = \overbrace{\left(\mathbf{A}^{-1}\right)^{\mathbf{T}}}^{M^{-1}} \tag{40}$$

Dowód

Z definicji A^{-1} to taka macierz, że $AA^{-1} = I$ oraz $A^{-1}A = I$.
Sprawdzamy czy $(A^T)(A^{-1})^T = I$ oraz $(A^{-1})^T(A^T) = I$
Z własności transpozycji $A^T B^T = (BA)^T$

$$M M^{-1} = \overbrace{(A^T)}^{M}\overbrace{(A^{-1})^T}^{M^{-1}} = (\overset{A^{-1}A}{\cancel{\quad}})^T = I^T = I \tag{41}$$

$$M^{-1} M = \underbrace{(A^{-1})^T}_{M^{-1}}\underbrace{(A^T)}_{M} = (\underset{AA^{-1}}{\cancel{\quad}})^T = I^T = I \tag{42}$$

Ok

$C^T D^T = (DC)^T$ $\qquad$ $C = A^T$

$D = A^{-1}$

Obliczenie macierzy odwrotnej $\mathcal{O}(N^3)$ $AX = I \to IX = A^{-1}$

Uruchamiamy eliminacje Gaussa z macierzą identycznościową z prawej strony. Odejmujemy każdy wiersz od wszystkich innych

$A^{-1}AX = A^{-1}I$

3×1st

$$AX = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$IX = A^{-1}$

Pierwszy wiersz = pierwszy / $A(1,1)$ = pierwszy / 1
Drugi wiersz = drugi - $A(2,1)$ * pierwszy = drugi - 3 * pierwszy

2×2nd

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Drugi wiersz = drugi / $A(2,2)$ = drugi / (-2)
Pierwszy wiersz = pierwszy - $A(1,2)$*drugi = pierwszy - 2* drugi

$$IX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = A^{-1}$$

Obliczenie macierzy odwrotnej (zapis skrócony)

$4 - 3 \cdot 2 = -2 \qquad 0 - 3 \cdot 1 = -3$

$$\cdot 3 - \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \Big/ -2 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \quad \cdot 2 -$$

$AX = I$

$A | I$

$IX = A^{-1}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad 2 - 2 \cdot 1 = 0$$

$0 - \cancel{2} \cdot \left(-\frac{1}{\cancel{2}}\right) = 1 \qquad 1 - \cancel{2} \cdot \frac{3}{\cancel{2}} = -2$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 6/5 \\ 0 & -4/10 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ -7/5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 7/2 & -10/4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 3 \\ 7/2 & -10/4 \end{bmatrix}$$

Wyznaczniki

$$\mathbf{det(cA) = c^n det(A)} \qquad det\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = 2^2 det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{43}$$

$$\underbrace{\begin{bmatrix} 2\cdot 1 & 2\cdot 2 \\ 2\cdot 3 & 2\cdot 4 \end{bmatrix}}_{2\times 2}$$

$$L = det\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = 2*8 - 4*6 = 16 - 24 = -8$$

$$(2\cdot 1\cdot 2\cdot 4 - 2\cdot 2\cdot 2\cdot 3)$$

$$R = 4det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4(\underbrace{1*4 - 3*2}_{-2}) = -8 = L$$

$$\mathbf{det(A^T) = det(A)} \qquad det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = det\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \tag{44}$$

A

$$L = det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1*4 - 2*3 = -2$$

$$R = det\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 1*4 - 3*2 = -2 = L$$

$$\mathbf{det(AB) = det(A)det(B)}$$

A B

$$det\left(\cancel{\begin{bmatrix}2 & 4\\ 6 & 8\end{bmatrix}}\cancel{\begin{bmatrix}2 & 4\\ 6 & 8\end{bmatrix}}\right) = det\begin{bmatrix}19 & 22\\ 43 & 50\end{bmatrix} \tag{45}$$

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix} = \begin{bmatrix}1*5+2*7 & 1*6+2*8\\ 3*5+4*7 & 3*6+4*8\end{bmatrix} = \begin{bmatrix}19 & 22\\ 43 & 50\end{bmatrix}$$

$$L = det\begin{bmatrix}19 & 22\\ 43 & 50\end{bmatrix} = 19*50 - 43*22 = 4$$

$$R = det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} det\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix} =$$

$$= (1*4-3*2)(5*8-6*7) = (-2)(-2) = 4 = L$$

$$\mathbf{det}(\mathbf{A}^{-1}) = \mathbf{1}/\mathbf{det}(\mathbf{A}) \qquad L = det\left(\underbrace{\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}^{-1}}_{A^{-1}}\right) = \frac{1}{det\underbrace{\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}}_{A}} = R \tag{46}$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}^{-1} = \begin{bmatrix}-2 & 1\\3/2 & -1/2\end{bmatrix}$$

$$L = det\left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}^{-1}\right) = det\begin{bmatrix}-2 & 1\\3/2 & -1/2\end{bmatrix} =$$

$$= -2 * (-1/2) - 1 * 3/2 = -1/2$$

$$R = \frac{1}{det\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}} = \frac{1}{1*4 - 3*2} = \frac{1}{-2} = -1/2 = L$$

Wyznaczniki

RANK 1 UPDATE

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix} = \begin{bmatrix}1\cdot 3 & 1\cdot 4\\2\cdot 3 & 2\cdot 4\end{bmatrix}$$

$$\det(\mathbf{I} + \underbrace{\mathbf{u}\mathbf{v}^\mathbf{T}}_{\text{MAGICZE}}) = \mathbf{1} + \mathbf{u}^\mathbf{T}\mathbf{v} \tag{47}$$

$$det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}^T\right) = 1 + \begin{bmatrix}1\\2\end{bmatrix}^T\begin{bmatrix}3\\4\end{bmatrix}$$

$$L = det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix}\right) = det\left(\begin{bmatrix}1 + 1*3 & 0 + 1*4\\0 + 2*3 & 1 + 2*4\end{bmatrix}\right) =$$

$$= det\left(\begin{bmatrix}4 & 4\\6 & 9\end{bmatrix}\right) = 12$$

$36 - 24$

$$1 + \begin{pmatrix}1 & 2\end{pmatrix}\begin{bmatrix}3\\4\end{bmatrix}$$

$$R = 1 + \begin{bmatrix}1\\2\end{bmatrix}^T\begin{bmatrix}3\\4\end{bmatrix} = 1 + (1 * 3 + 2 * 4) = 12 = L$$

TAŃSZA 2×2

$$\det(\underbrace{\mathbf{I}}_{} + \mathbf{A}) = \mathbf{1} + \det(\mathbf{A}) + \mathbf{Tr}(\mathbf{A}) \text{ (tylko dla n=2)} \qquad (48)$$

A

$$L = \quad det \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) = 1 + det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + Tr \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

4 flops 3 flops = 7 flops

$$L = det \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right) = det \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} = 2 * 5 - 2 * 3 = 4$$

10 - 6

$$R = 1 + det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + Tr \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 + (1 * 4 - 2 * 3) + (1 + 4) = 4 = L$$

2 flops 3 flops 1 flop = 6 flops

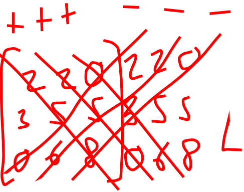
Wyznaczniki

TAŃSZA

$$\mathbf{det(I + A) = 1 + det(A) + Tr(A) + \frac{1}{2}Tr(A)^2 - \frac{1}{2}Tr(A^2)} \quad (49)$$

(tylko dla n=3)

3 × 3



SARRUS

17 flops

$$L = det\Bigg(\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}1&2&0\\3&4&5\\0&6&7\end{bmatrix}}_{9+17 \text{ flops}}\Bigg) =$$

$$1 + \underbrace{det\begin{bmatrix}1&2&0\\3&4&5\\0&6&7\end{bmatrix}}_{17} + \underbrace{Tr\begin{bmatrix}1&2&0\\3&4&5\\0&6&7\end{bmatrix}}_{2 \text{ flops}} +$$

$$\frac{1}{2}\Bigg(\underbrace{Tr\begin{bmatrix}1&2&0\\3&4&5\\0&6&7\end{bmatrix}}_{2 \text{ flops}}\Bigg)^2 - \frac{1}{2}Tr\Bigg(\begin{bmatrix}1&2&0\\3&4&5\\0&6&7\end{bmatrix}^2\Bigg) = R$$

Pochodne "macierzy" względem "skalarów"

$X \in \mathcal{R}^{n\times m}$ to "macierz", , $y \in \mathcal{R}$ to "skalar"
Wówczas

$$\frac{\partial X}{\partial y} = \begin{bmatrix} \frac{\partial X_{11}}{\partial y} & \frac{\partial X_{12}}{\partial y} & \dots & \frac{\partial X_{1m}}{\partial y} & \\ \frac{\partial X_{21}}{\partial y} & \frac{\partial X_{22}}{\partial y} & \dots & \frac{\partial X_{2m}}{\partial y} & \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial X_{n1}}{\partial y} & \frac{\partial X_{n2}}{\partial y} & \dots & \frac{\partial X_{nm}}{\partial y} & \end{bmatrix}$$

Przykład $X = \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}$, policzyć $\frac{\partial X}{\partial y}$

$$\frac{\partial X}{\partial y} = \begin{bmatrix} \frac{\partial(y)}{\partial y} & \frac{\partial(y^2)}{\partial y} \\ \frac{\partial(2y)}{\partial y} & \frac{(y^3)}{\partial y} \end{bmatrix} == \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}$$

Pochodna macierzy stałej (operatora liniowego)

$$\frac{\partial \mathbf{A}}{\partial \mathbf{y}} = \mathbf{0} \qquad \frac{\partial \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}}{\partial y} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{50}$$

$$L = \frac{\partial \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}}{\partial y} = \begin{bmatrix} \frac{\partial 1}{\partial y} & \frac{\partial 2}{\partial y} \\ \frac{\partial 4}{\partial y} & \frac{\partial 3}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = R$$

Pochodna przeskalowanej macierzy

$$\frac{\partial \alpha \mathbf{X}}{\partial \mathbf{y}} = \alpha \frac{\partial \mathbf{X}}{\partial \mathbf{y}} \qquad \frac{\partial \left(2 \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \right)}{\partial y} = 2 \frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y} \tag{51}$$

$$L = \frac{\partial \left(2 \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \right)}{\partial y} = \frac{\partial \begin{bmatrix} 2y & 2y^2 \\ 4y & 2y^3 \end{bmatrix}}{\partial y} = \begin{bmatrix} \frac{\partial(2y)}{\partial y} & \frac{\partial(2y^2)}{\partial y} \\ \frac{\partial(4y)}{\partial y} & \frac{(2y^3)}{\partial y} \end{bmatrix} = \begin{bmatrix} 2 & 4y \\ 4 & 6y^2 \end{bmatrix}$$

$$R = 2 \frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y} = 2 \begin{bmatrix} \frac{\partial(y)}{\partial y} & \frac{\partial(y^2)}{\partial y} \\ \frac{\partial(2y)}{\partial y} & \frac{(y^3)}{\partial y} \end{bmatrix} == 2 \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix} = \begin{bmatrix} 2 & 4y \\ 4 & 6y^2 \end{bmatrix} = L$$

Pochodna sumy macierzy

$$\frac{\partial \mathbf{X}+\mathbf{Y}}{\partial \mathbf{y}} = \frac{\partial \mathbf{X}}{\partial \mathbf{y}} + \frac{\partial \mathbf{Y}}{\partial \mathbf{y}} \tag{52}$$

$$\frac{\partial\left(\begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} + \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}\right)}{\partial y} = \frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y} + \frac{\partial \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}}{\partial y}$$

$$L = \frac{\partial\left(\begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} + \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}\right)}{\partial y} = \frac{\partial \begin{bmatrix} 1+y & y+y^2 \\ 2y+y^2 & 2y^3 \end{bmatrix}}{\partial y} =$$

$$= \begin{bmatrix} \frac{\partial 1+y}{\partial y} & \frac{\partial y+y^2}{\partial y} \\ \frac{\partial 2y+y^2}{\partial y} & \frac{\partial 2y^3}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1+2y \\ 2+2y & 6y^2 \end{bmatrix}$$

Pochodna sumy macierzy

$$\frac{\partial \mathbf{X} + \mathbf{Y}}{\partial \mathbf{y}} = \frac{\partial \mathbf{X}}{\partial \mathbf{y}} + \frac{\partial \mathbf{Y}}{\partial \mathbf{y}} \tag{53}$$

$$\frac{\partial \left(\begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} + \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} \right)}{\partial y} = \frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y} + \frac{\partial \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}}{\partial y}$$

$$R = \frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y} + \frac{\partial \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}}{\partial y} = \begin{bmatrix} \frac{\partial y}{\partial y} & \frac{\partial y^2}{\partial y} \\ \frac{\partial 2y}{\partial y} & \frac{\partial y^3}{\partial y} \end{bmatrix} + \begin{bmatrix} \frac{\partial 1}{\partial y} & \frac{\partial y}{\partial y} \\ \frac{\partial y^2}{\partial y} & \frac{\partial y^3}{\partial y} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2y & 3y^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 + 2y \\ 2 + 2y & 6y^2 \end{bmatrix} = L$$

Pochodna śladu macierzy

$$\frac{\partial \mathbf{tr}(\mathbf{X})}{\partial \mathbf{y}} = \mathbf{tr}\frac{\partial \mathbf{X}}{\partial \mathbf{y}} \qquad \frac{\partial \left(tr \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \right)}{\partial y} = tr\left(\frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y} \right) \tag{54}$$

$$L = \frac{\partial \left(tr \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \right)}{\partial y} = \left(\frac{\partial y + y^3}{\partial y} \right) = 1 + 3y^2$$

$$R = tr\left(\frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y} \right) = tr\begin{bmatrix} \frac{\partial(y)}{\partial y} & \frac{\partial(y^2)}{\partial y} \\ \frac{\partial(2y)}{\partial y} & \frac{\partial(y^3)}{\partial y} \end{bmatrix} = tr\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix} = 1+3y^2 = L \tag{55}$$

Pochodna iloczynu macierzy

$$\frac{\partial(\mathbf{XY})}{\partial \mathbf{y}} = \frac{\partial \mathbf{X}}{\partial \mathbf{y}}\mathbf{Y} + \mathbf{X}\frac{\partial \mathbf{Y}}{\partial \mathbf{y}} \tag{56}$$

$$\frac{\partial\left(\begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}\begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}\right)}{\partial y} = \frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y}\begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} + \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}\frac{\partial \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}}{\partial y}$$

$$\begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}\begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} = \begin{bmatrix} y*1+y^2*y^2 & y*y+y^2*y^3 \\ 2y*1+y^3*y^2 & 2y*y+y^3*y^3 \end{bmatrix} = \begin{bmatrix} y+y^4 & y^2+y^5 \\ 2y+y^5 & 2y^2+y^6 \end{bmatrix}$$

$$L = \frac{\partial\left(\begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}\begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}\right)}{\partial y} = \frac{\partial \begin{bmatrix} y+y^4 & y^2+y^5 \\ 2y+y^5 & 2y^2+y^6 \end{bmatrix}}{\partial y} =$$

$$\begin{bmatrix} \frac{\partial(y+y^4)}{\partial y} & \frac{\partial(y^2+y^5)}{\partial y} \\ \frac{\partial(2y+y^5)}{\partial y} & \frac{\partial(2y^2+y^6)}{\partial y} \end{bmatrix} = \begin{bmatrix} 1+4y^3 & 2y+5y^4 \\ 2+5y^4 & 4y+6y^5 \end{bmatrix}$$

Pochodna iloczynu macierzy

$$R = \frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y} \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} + \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \frac{\partial \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}}{\partial y} =$$

$$\begin{bmatrix} \frac{\partial(y)}{\partial y} & \frac{\partial(y^2)}{\partial y} \\ \frac{\partial(2y)}{\partial y} & \frac{\partial(y^3)}{\partial y} \end{bmatrix} \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} + \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \begin{bmatrix} \frac{\partial(1)}{\partial y} & \frac{\partial(y)}{\partial y} \\ \frac{\partial(y^2)}{\partial y} & \frac{\partial(y^3)}{\partial y} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix} \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} + \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2y & 3y^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 * 1 + 2y * y^2 & 1 * y + 2y * y^3 \\ 2 * 1 + 3y^2 * y^2 & 2 * y + 3y^2 * y^3 \end{bmatrix} +$$

$$\begin{bmatrix} y * 0 + y^2 * 2y & y * 1 + y^2 * 3y^2 \\ 2y * 0 + y^3 * 2y & 2y * 1 + y^3 * 3y^2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 + 2y^3 & y + 2y^4 \\ 2 + 3y^4 & 2y + 3y^5 \end{bmatrix} + \begin{bmatrix} 2y^3 & y + 3y^4 \\ 2y^4 & 2y + 3y^5 \end{bmatrix} = \begin{bmatrix} 1 + 4y^3 & 2y + 5y^4 \\ 2 + 5y^4 & 4y + 6y^5 \end{bmatrix} = L$$

Pochodna iloczynu skalarnego

$$\frac{\partial (\mathbf{X} \cdot \mathbf{Y})}{\partial \mathbf{y}} = \frac{\partial \mathbf{X}}{\partial \mathbf{y}} \cdot \mathbf{Y} + \mathbf{X} \cdot \frac{\partial \mathbf{Y}}{\partial \mathbf{y}} \tag{57}$$

$$\frac{\partial \left(\begin{bmatrix} y & y^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \end{bmatrix} \right)}{\partial y} = \frac{\partial \begin{bmatrix} y & y^2 \end{bmatrix}}{\partial y} \cdot \begin{bmatrix} 1 & y \end{bmatrix} + \begin{bmatrix} y & y^2 \end{bmatrix} \cdot \frac{\partial \begin{bmatrix} 1 & y \end{bmatrix}}{\partial y}$$

$$\begin{bmatrix} y & y^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \end{bmatrix} = y * 1 + y^2 * y$$

$$L = \frac{\partial \left(\begin{bmatrix} y & y^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \end{bmatrix} \right)}{\partial y} = \frac{\partial (y * 1 + y^2 * y)}{\partial y} = 1 + 3y^2$$

Pochodna iloczynu skalarnego

$$\frac{\partial\left(\begin{bmatrix} y & y^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \end{bmatrix}\right)}{\partial y} = \frac{\partial \begin{bmatrix} y & y^2 \end{bmatrix}}{\partial y} \cdot \begin{bmatrix} 1 & y \end{bmatrix} + \begin{bmatrix} y & y^2 \end{bmatrix} \cdot \frac{\partial \begin{bmatrix} 1 & y \end{bmatrix}}{\partial y}$$

$$\begin{bmatrix} y & y^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \end{bmatrix} = y * 1 + y^2 * y$$

$$\begin{aligned} R = \frac{\partial \begin{bmatrix} y & y^2 \end{bmatrix}}{\partial y} \cdot \begin{bmatrix} 1 & y \end{bmatrix} + \begin{bmatrix} y & y^2 \end{bmatrix} \cdot \frac{\partial \begin{bmatrix} 1 & y \end{bmatrix}}{\partial y} &= \\ \begin{bmatrix} \frac{\partial(y)}{\partial y} & \frac{\partial(y^2)}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} 1 & y \end{bmatrix} + \begin{bmatrix} y & y^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial(1)}{\partial y} & \frac{\partial(y)}{\partial y} \end{bmatrix} &= \\ \begin{bmatrix} 1 & 2y \end{bmatrix} \cdot \begin{bmatrix} 1 & y \end{bmatrix} + \begin{bmatrix} y & y^2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} &= \\ 1 * 1 + 2y * y + y * 0 + y^2 * 1 = 1 + 3y^2 &= L \end{aligned} \tag{58}$$

Pochodna produktu Kroneckera

$$\frac{\partial(\mathbf{X} \otimes \mathbf{Y})}{\partial \mathbf{y}} = \frac{\partial \mathbf{X}}{\partial \mathbf{y}} \otimes \mathbf{Y} + \mathbf{X} \otimes \frac{\partial \mathbf{Y}}{\partial \mathbf{y}} \tag{59}$$

$$\frac{\partial \left(\begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \otimes \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} \right)}{\partial y} =$$

$$\frac{\partial \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}}{\partial y} \otimes \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} + \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \otimes \frac{\partial \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}}{\partial y}$$

$$\underset{A}{\begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix}} \otimes \underset{B}{\begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}} = \begin{bmatrix} y\begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} & y^2\begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} \\ 2y\begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} & y^3\begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} y & y^2 & y^2 & y^3 \\ y^3 & y^4 & y^4 & y^5 \\ 2y & 2y^2 & y^3 & y^4 \\ 2y^3 & 2y^4 & y^5 & y^6 \end{bmatrix}$$

$$L = \frac{\partial\left(\begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \otimes \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix}\right)}{\partial y} \frac{\partial \begin{bmatrix} y & y^2 & y^2 & y^3 \\ y^3 & y^4 & y^4 & y^5 \\ 2y & 2y^2 & y^3 & y^4 \\ 2y^3 & 2y^4 & y^5 & y^6 \end{bmatrix}}{\partial y} =$$

$$\begin{bmatrix} \frac{\partial y}{\partial y} & \frac{\partial y^2}{\partial y} & \frac{\partial y^2}{\partial y} & \frac{\partial y^3}{\partial y} \\ \frac{\partial y^3}{\partial y} & \frac{\partial y^4}{\partial y} & \frac{\partial y^4}{\partial y} & \frac{\partial y^5}{\partial y} \\ \frac{\partial 2y}{\partial y} & \frac{\partial 2y^2}{\partial y} & \frac{\partial y^3}{\partial y} & \frac{\partial y^4}{\partial y} \\ \frac{\partial 2y^3}{\partial y} & \frac{\partial 2y^4}{\partial y} & \frac{\partial y^5}{\partial y} & \frac{\partial y^6}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 2y & 2y & 3y^2 \\ 3y^2 & 4y^3 & 4y^3 & 5y^4 \\ 2 & 4y & 3y^2 & 4y^3 \\ 6y^2 & 8y^3 & 5y^4 & 6y^5 \end{bmatrix}$$

Pochodna produktu Kroneckera

$$R = \begin{bmatrix} \frac{\partial y}{\partial y} & \frac{\partial y^2}{\partial y} \\ \frac{\partial 2y}{\partial y} & \frac{\partial y^3}{\partial y} \end{bmatrix} \otimes \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} + \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \otimes \begin{bmatrix} \frac{\partial 1}{\partial y} & \frac{\partial y}{\partial y} \\ \frac{\partial y^2}{\partial y} & \frac{\partial y^3}{\partial y} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix} \otimes \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} + \begin{bmatrix} y & y^2 \\ 2y & y^3 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 2y & 3y^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} & 2y \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} \\ 2 \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} & 3y^2 \begin{bmatrix} 1 & y \\ y^2 & y^3 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} y \begin{bmatrix} 0 & 1 \\ 2y & 3y^2 \end{bmatrix} & y^2 \begin{bmatrix} 0 & 1 \\ 2y & 3y^2 \end{bmatrix} \\ 2y \begin{bmatrix} 0 & 1 \\ 2y & 3y^2 \end{bmatrix} & y^3 \begin{bmatrix} 0 & 1 \\ 2y & 3y^2 \end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & y & 2y & 2y^2 \\ y^2 & y^3 & 2y^3 & 2y^4 \\ 2 & 2y & 3y^2 & 3y^3 \\ 2y^2 & 2y^3 & 3y^4 & 3y^5 \end{bmatrix} + \begin{bmatrix} 0 & y & 0 & y^2 \\ 2y^2 & 3y^3 & 2y^3 & 3y^4 \\ 0 & 2y & 0 & y^3 \\ 4y^2 & 6y^3 & 2y^4 & 3y^5 \end{bmatrix} = \begin{bmatrix} 1 & 2y & 2y & 3y^2 \\ 3y^2 & 4y^3 & 4y^3 & 5y^4 \\ 2 & 4y & 3y^2 & 4y^3 \\ 6y^2 & 8y^3 & 5y^4 & 6y^5 \end{bmatrix}$$

$$= L$$

Pochodna macierzy odwrotnej

$$\frac{\partial \mathbf{X}^{-1}}{\partial \mathbf{y}} = -\mathbf{X}^{-1}\frac{\partial \mathbf{X}}{\partial \mathbf{y}}\mathbf{X}^{-1} \tag{60}$$

$$\frac{\partial \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}^{-1}}{\partial y} = -\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}^{-1} \frac{\partial \left(\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix} \right)}{\partial y} \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2y \\ 0 & 3y^2-4y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2y \\ 0 & 3y^2-4y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1+\frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y} \\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1+\frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y} \\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y} \end{bmatrix}$$

Pochodna macierzy odwrotnej

$$L = \frac{\partial \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}^{-1}}{\partial y} = \frac{\partial \begin{bmatrix} 1 + \frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y} \\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y} \end{bmatrix}}{\partial y} = \begin{bmatrix} \frac{\partial(1+\frac{4y}{3y^2-4y})}{\partial y} & \frac{\partial(\frac{-2y}{3y^2-4y})}{\partial y} \\ \frac{\partial(\frac{-2}{3y^2-4y})}{\partial y} & \frac{\partial(\frac{1}{3y^2-4y})}{\partial y} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{(4y)'(3y^2-4y)-(4y)(3y^2-4y)'}{(3y^2-4y)^2} & \frac{(-2y)'(3y^2-4y)-(-2y)(3y^2-4y)'}{(3y^2-4y)^2} \\ \frac{(-2)'(3y^2-4y)-(-2)(3y^2-4y)'}{(3y^2-4y)^2} & \frac{1'(3y^2-4y)-1(3y^2-4y)'}{(3y^2-4y)^2} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{4(3y^2-4y)-(4y)(6y-4)}{(3y^2-4y)^2} & \frac{-2(3y^2-4y)+2y(6y-4)}{(3y^2-4y)^2} \\ \frac{2(6y-4)}{(3y^2-4y)^2} & \frac{-(6y-4)}{(3y^2-4y)^2} \end{bmatrix}$$

Pochodna macierzy odwrotnej

$$R = -\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}^{-1} \frac{\partial\left(\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}\right)}{\partial y} \begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}^{-1} =$$

$$-\begin{bmatrix}1+\frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y}\\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y}\end{bmatrix} \begin{bmatrix}\frac{\partial(1)}{\partial y} & \frac{\partial(2y)}{\partial y}\\ \frac{\partial(2)}{\partial y} & \frac{\partial(3y^2)}{\partial y}\end{bmatrix} \begin{bmatrix}1+\frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y}\\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y}\end{bmatrix} =$$

$$\left(-\begin{bmatrix}1+\frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y}\\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y}\end{bmatrix} \begin{bmatrix}0 & 2\\ 0 & 6y\end{bmatrix}\right) \begin{bmatrix}1+\frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y}\\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y}\end{bmatrix} =$$

$$\begin{bmatrix}0 & (-2-\frac{8y}{3y^2-4y})+(\frac{12y^2}{3y^2-4y})\\ 0 & (\frac{4}{3y^2-4y})2-(\frac{6y}{3y^2-4y})\end{bmatrix} \begin{bmatrix}1+\frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y}\\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y}\end{bmatrix} =$$

$$\begin{bmatrix}[(-2-\frac{8y}{3y^2-4y})+(\frac{12y^2}{3y^2-4y})](\frac{-2}{3y^2-4y}) & [(-2-\frac{8y}{3y^2-4y})+(\frac{12y^2}{3y^2-4y})](\frac{1}{3y^2-4y})\\ (\frac{4}{3y^2-4y})2-(\frac{6y}{3y^2-4y})(\frac{-2}{3y^2-4y}) & [(\frac{4}{3y^2-4y})2-(\frac{6y}{3y^2-4y})](\frac{1}{3y^2-4y})\end{bmatrix} =$$

$$(\ldots)\ \text{i o ile się nie pogubiłem}\ = L$$

Pochodna wyznacznika macierzy

$$\frac{\partial\left(\det\mathbf{X}\right)}{\partial\mathbf{y}} = \det\mathbf{X}\,\mathrm{tr}\left(\mathbf{X}^{-1}\frac{\partial\mathbf{X}}{\partial\mathbf{y}}\right) \tag{62}$$

$$\frac{\partial\left(det\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}\right)}{\partial y} = det\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix} tr\left(\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}^{-1}\frac{\partial\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}}{\partial y}\right)$$

$$L = \frac{\partial\left(det\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}\right)}{\partial y} = \frac{\partial\left(3y^2-4y\right)}{\partial y} = 6y-4$$

$$\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}^{-1} = \begin{bmatrix}1+\frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y}\\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y}\end{bmatrix}$$

Pochodna wyznacznika macierzy

$$\frac{\partial \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}}{\partial y} = \begin{bmatrix} \frac{\partial(1)}{\partial y} & \frac{\partial(2y)}{\partial y} \\ \frac{\partial(2)}{\partial y} & \frac{\partial(3y^2)}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 6y \end{bmatrix}$$

$$R = det \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix} tr \left(\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}^{-1} \frac{\partial \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}}{\partial y} \right) =$$

$$(3y^2 - 4y) tr \left(\begin{bmatrix} 1 + \frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y} \\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 6y \end{bmatrix} \right) =$$

$$(3y^2 - 4y) tr \begin{bmatrix} 0 & 2 + \frac{8y}{3y^2-4y} + \frac{-12y^2}{3y^2-4y} \\ 0 & \frac{-4}{3y^2-4y} + \frac{6y}{3y^2-4y} \end{bmatrix} = (3y^2 - 4y) \frac{6y-4}{3y^2-4y} =$$

$$= 6y - 4 = L$$

Pochodna logarytmu wyznacznika macierzy

$$\frac{\partial\left(\mathbf{ln}(\mathbf{det}\mathbf{X})\right)}{\partial\mathbf{y}} = \mathbf{tr}\left(\mathbf{X}^{-1}\frac{\partial\mathbf{X}}{\partial\mathbf{y}}\right) \tag{63}$$

$$\frac{\partial\left(ln(det\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix})\right)}{\partial y} = tr\left(\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}^{-1}\frac{\partial\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}}{\partial y}\right)$$

$$L = \frac{\partial\left(ln\left(det\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}\right)\right)}{\partial y} = \frac{1}{det\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}}\frac{\partial\left(det\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}\right)}{\partial y}$$

$$= \frac{1}{3y^2-4y}\frac{\partial\left(3y^2-4y\right)}{\partial y} = \frac{6y-4}{3y^2-4y}$$

Pochodna logarytmu wyznacznika macierzy

$$\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 + \frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y} \\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y} \end{bmatrix}$$

$$\frac{\partial \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}}{\partial y} = \begin{bmatrix} \frac{\partial(1)}{\partial y} & \frac{\partial(2y)}{\partial y} \\ \frac{\partial(2)}{\partial y} & \frac{\partial(3y^2)}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 6y \end{bmatrix}$$

$$R = tr\left(\begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}^{-1} \frac{\partial \begin{bmatrix} 1 & 2y \\ 2 & 3y^2 \end{bmatrix}}{\partial y} \right) = tr\left(\begin{bmatrix} 1 + \frac{4y}{3y^2-4y} & \frac{-2y}{3y^2-4y} \\ \frac{-2}{3y^2-4y} & \frac{1}{3y^2-4y} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 6y \end{bmatrix} \right)$$

$$= tr \begin{bmatrix} 0 & 2 + \frac{8y}{3y^2-4y} + \frac{-12y^2}{3y^2-4y} \\ 0 & \frac{-4}{3y^2-4y} + \frac{6y}{3y^2-4y} \end{bmatrix} = \frac{6y-4}{3y^2-4y} = L$$

Pochodna transpozycji

$$\frac{\partial(\mathbf{X}^\mathbf{T})}{\partial \mathbf{y}} = \left(\frac{\partial \mathbf{X}}{\partial \mathbf{y}}\right)^\mathbf{T} \tag{64}$$

$$\frac{\partial\left(\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}^T\right)}{\partial y} = \frac{\partial\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}^T}{\partial y}$$

$$L = \frac{\partial\left(\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}^T\right)}{\partial y} = \frac{\partial\begin{bmatrix}1 & 2\\ 2y & 3y^2\end{bmatrix}}{\partial y} = \begin{bmatrix}\frac{\partial(1)}{\partial y} & \frac{\partial(2)}{\partial y}\\ \frac{\partial(2y)}{\partial y} & \frac{\partial(3y^2)}{\partial y}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 2 & 6y\end{bmatrix}$$

$$R = \frac{\partial\begin{bmatrix}1 & 2y\\ 2 & 3y^2\end{bmatrix}^T}{\partial y} = \begin{bmatrix}\frac{\partial(1)}{\partial y} & \frac{\partial(2y)}{\partial y}\\ \frac{\partial(2)}{\partial y} & \frac{\partial(3y^2)}{\partial y}\end{bmatrix}^T = \begin{bmatrix}0 & 2\\ 0 & 6y\end{bmatrix}^T = \begin{bmatrix}0 & 0\\ 2 & 6y\end{bmatrix} = L$$

Pochodne "skalarów" względem "wektorów" i "wektorów" względem "skalarów"

$x \in \mathcal{R}^n$ "wektor", y "skalar", wówczas $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}$

POCHODNA "SKALARA" WZGLĘDEM "WEKTORA"

Przykład $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $y = x_1 x_2 x_3$, policzyć

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \frac{\partial y}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial(\cancel{x_1} x_2 x_3)}{\partial x_1} \\ \frac{\partial(x_1 x_2 x_3)}{\partial x_2} \\ \frac{\partial(x_1 x_2 x_3)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} x_2 x_3 \\ x_1 x_3 \\ x_1 x_2 \end{bmatrix}$$

POCHODNA "WEKTORA" WZGLĘDEM "SKALARA"

$y \in \mathcal{R}^n$ "wektor", x "skalar", wówczas $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} & \cdots & \frac{\partial y_n}{\partial x} \end{bmatrix}$

Przykład $y = \begin{bmatrix} x & x^2 & x^3 \end{bmatrix}$, policzyć

$y = [y_1 \; y_2 \; y_3]$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} & \frac{\partial y_3}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial(x)}{\partial x} & \frac{\partial(x^2)}{\partial x} & \frac{\partial(x^3)}{\partial x} \end{bmatrix} = \begin{bmatrix} 1 & 2x & 3x^2 \end{bmatrix}$$

$y_1 = x \quad y_2 = x^2 \quad y_3 = x^3$

$x \in \mathcal{R}^n$, $y \in \mathcal{R}^m$ to "wektory"

$\partial = \begin{Bmatrix} y_1 \\ \vdots \\ y_n \end{Bmatrix}$

$x = \begin{Bmatrix} x_1 \\ \vdots \\ x_m \end{Bmatrix}$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$$

Przykład $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2 \\ x_3^2 + 3x_2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, policzyć

$n = 2$ $\quad$ $m = 3$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_1}{\partial x_3} & \frac{\partial y_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial(x_1^2 - x_2)}{\partial x_1} & \frac{\partial(x_3^2 + 3x_2)}{\partial x_1} \\ \frac{\partial(x_1^2 - x_2)}{\partial x_2} & \frac{\partial(x_3^2 + 3x_2)}{\partial x_2} \\ \frac{\partial(x_1^2 - x_2)}{\partial x_3} & \frac{\partial(x_3^2 + 3x_2)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 & 0 \\ -1 & 3 \\ 0 & 2x_3 \end{bmatrix}$$

$y_1 = x_1^2 - x_2$

$y_2 = x_3^2 + 3x_2$

Pochodne "skalarów" względem "macierzy"

$X \in \mathcal{R}^{nxm}$ "macierz", y "skalar" (funkcja skalarna której argumentem jest macierz), wówczas

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \frac{\partial y}{\partial x_{11}} & \vdots & \frac{\partial y}{\partial x_{1m}} \\ \vdots & \vdots & \vdots \\ \frac{\partial y}{\partial x_{n1}} & \vdots & \frac{\partial y}{\partial x_{nm}} \end{bmatrix}$$

WYNIK MA TAKI SAM KSZTAŁT JAK MACIERZ ZMIENNYCH

Przykład $x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$, $y = x_{11}x_{12}x_{21}x_{22}$, policzyć

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \frac{\partial x_{11}x_{12}x_{21}x_{22}}{\partial x_{11}} & \frac{\partial x_{11}x_{12}x_{21}x_{22}}{\partial x_{12}} \\ \frac{\partial x_{11}x_{12}x_{21}x_{22}}{\partial x_{21}} & \frac{\partial x_{11}x_{12}x_{21}x_{22}}{\partial x_{22}} \end{bmatrix} = \begin{bmatrix} \boxed{x_{12}x_{21}x_{22}} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}$$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_{11}} & \frac{\partial y}{\partial x_{12}} \\ \partial y / \partial x_{21} & \partial y / \partial x_{21} \end{bmatrix}$$

$$\frac{\partial x_{11} x_{12} x_{21} x_{22}}{\partial x_{11}}$$

"STAŁA"

$$\frac{\partial (2 \sin x)}{\partial x}$$

Y to macierz której wyrazy zależą od zmiennych tworzących macierz **X**, np.

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \qquad \boxed{Y = \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}$$

WZNIAR

$$\frac{\partial \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}{\partial x_{11}} = \begin{bmatrix} \frac{\partial x_{12}x_{21}x_{22}}{\partial x_{11}} & \frac{\partial x_{11}x_{21}x_{22}}{\partial x_{11}} \\ \frac{\partial x_{11}x_{12}x_{22}}{\partial x_{11}} & \frac{\partial x_{11}x_{12}x_{21}}{\partial x_{11}} \end{bmatrix} = \boxed{\begin{bmatrix} 0 & x_{21}x_{22} \\ x_{12}x_{22} & x_{12}x_{21} \end{bmatrix}}$$

WYNIK

$$= \frac{\partial Y}{\partial X_{11}}$$

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \qquad \boxed{Y = \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}$$

$$\frac{\partial Y}{\partial x_{11}} = \frac{\partial \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}{\partial x_{11}} = \begin{bmatrix} \frac{\partial x_{12}x_{21}x_{22}}{\partial x_{11}} & \frac{\partial x_{11}x_{21}x_{22}}{\partial x_{11}} \\ \frac{\partial x_{11}x_{12}x_{22}}{\partial x_{11}} & \frac{\partial x_{11}x_{12}x_{21}}{\partial x_{11}} \end{bmatrix} = \boxed{\begin{bmatrix} 0 & x_{21}x_{22} \\ x_{12}x_{22} & x_{12}x_{21} \end{bmatrix}}$$

$$\frac{\partial Y}{\partial x_{12}} = \frac{\partial \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}{\partial x_{12}} = \begin{bmatrix} \frac{\partial x_{12}x_{21}x_{22}}{\partial x_{12}} & \frac{\partial x_{11}x_{21}x_{22}}{\partial x_{12}} \\ \frac{\partial x_{11}x_{12}x_{22}}{\partial x_{12}} & \frac{\partial x_{11}x_{12}x_{21}}{\partial x_{12}} \end{bmatrix} = \boxed{\begin{bmatrix} x_{21}x_{22} & 0 \\ x_{11}x_{22} & x_{11}x_{21} \end{bmatrix}}$$

$$\frac{\partial Y}{\partial x_{21}} = \frac{\partial \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}{\partial x_{21}} = \begin{bmatrix} \frac{\partial x_{12}x_{21}x_{22}}{\partial x_{21}} & \frac{\partial x_{11}x_{21}x_{22}}{\partial x_{21}} \\ \frac{\partial x_{11}x_{12}x_{22}}{\partial x_{21}} & \frac{\partial x_{11}x_{12}x_{21}}{\partial x_{21}} \end{bmatrix} = \boxed{\begin{bmatrix} x_{12}x_{22} & x_{11}x_{22} \\ 0 & x_{11}x_{12} \end{bmatrix}}$$

$$\frac{\partial Y}{\partial x_{22}} = \frac{\partial \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}{\partial x_{22}} = \begin{bmatrix} \frac{\partial x_{12}x_{21}x_{22}}{\partial x_{22}} & \frac{\partial x_{11}x_{21}x_{22}}{\partial x_{22}} \\ \frac{\partial x_{11}x_{12}x_{22}}{\partial x_{22}} & \frac{\partial x_{11}x_{12}x_{21}}{\partial x_{22}} \end{bmatrix} = \boxed{\begin{bmatrix} x_{12}x_{21} & x_{11}x_{21} \\ x_{11}x_{12} & 0 \end{bmatrix}}$$

Pochodne "macierzy" względem "macierzy"

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \qquad Y = \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}$$

$$\frac{\partial Y}{\partial X} = \begin{bmatrix} \frac{\partial \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}{\partial x_{11}} & \frac{\partial \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}{\partial x_{12}} \\ \frac{\partial \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}{\partial x_{21}} & \frac{\partial \begin{bmatrix} x_{12}x_{21}x_{22} & x_{11}x_{21}x_{22} \\ x_{11}x_{12}x_{22} & x_{11}x_{12}x_{21} \end{bmatrix}}{\partial x_{22}} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial Y}{\partial x_{11}} & \frac{\partial Y}{\partial x_{12}} \\ \frac{\partial Y}{\partial x_{21}} & \frac{\partial Y}{\partial x_{22}} \end{bmatrix}$$

$$\begin{bmatrix} 0 & x_{21}x_{22} & x_{21}x_{22} & 0 \\ x_{12}x_{22} & x_{12}x_{21} & x_{11}x_{22} & x_{11}x_{21} \\ x_{12}x_{22} & x_{11}x_{22} & x_{12}x_{21} & x_{11}x_{21} \\ 0 & x_{11}x_{12} & x_{11}x_{12} & 0 \end{bmatrix}$$

Pochodne macierzy względem macierzy

$$\boxed{\frac{\partial(\mathbf{det X})}{\partial \mathbf{X}} = -\mathbf{det}(\mathbf{X})\mathbf{X}^{-\mathbf{T}}} \quad X = \begin{bmatrix} x & y \\ r & z \end{bmatrix} \tag{65}$$

$$\boxed{\begin{array}{c|c} X & I \\ \hline I & X^{-1} \end{array}}$$

$$\frac{\partial\left(det\left(\begin{bmatrix} x & y \\ r & z \end{bmatrix}\right)\right)}{\partial X} = -det\left(\begin{bmatrix} x & y \\ r & z \end{bmatrix}\right)\begin{bmatrix} x & y \\ r & z \end{bmatrix}^{-T}$$

$$\begin{bmatrix} x & y \\ r & z \end{bmatrix}^{-1} = \frac{1}{ry - xz}\begin{bmatrix} -z & y \\ r & -x \end{bmatrix} \qquad det\begin{bmatrix} x & y \\ r & z \end{bmatrix} = xz - ry$$

$$L = \frac{\partial det\begin{bmatrix} x & y \\ r & z \end{bmatrix}}{\partial X} = \frac{\partial(xz - ry)}{\partial X} = \begin{bmatrix} \frac{\partial(xz-ry)}{\partial x} & \frac{\partial(xz-ry)}{\partial y} \\ \frac{\partial(xz-ry)}{\partial r} & \frac{\partial(xz-ry)}{\partial z} \end{bmatrix} = \begin{bmatrix} z & -r \\ -y & x \end{bmatrix}$$

$$R = -det\begin{bmatrix} x & y \\ r & z \end{bmatrix}\left(\begin{bmatrix} x & y \\ r & z \end{bmatrix}^{-T}\right) = \frac{-\cancel{(xz - ry)}}{\cancel{ry - xz}}\begin{bmatrix} -z & y \\ r & -x \end{bmatrix}^{T} = \begin{bmatrix} z & -r \\ -y & x \end{bmatrix} = L$$

$$\frac{\partial(\boxed{\mathbf{det}(\mathbf{X}^{-1})})}{\partial \mathbf{X}} = -\boxed{\mathbf{det}(\mathbf{X}^{-1})}\mathbf{X}^{-\mathbf{T}} \quad X = \begin{bmatrix} x & y \\ r & z \end{bmatrix} \tag{66}$$

$$\frac{\partial\left(det\left(\begin{bmatrix} x & y \\ r & z \end{bmatrix}^{-1}\right)\right)}{\partial X} = -det\left(\begin{bmatrix} x & y \\ r & z \end{bmatrix}^{-1}\right)\begin{bmatrix} x & y \\ r & z \end{bmatrix}^{-T}$$

$$\begin{bmatrix} x & y \\ r & z \end{bmatrix}^{-1} = \begin{bmatrix} \frac{-z}{ry-xz} & \frac{y}{ry-xz} \\ \frac{r}{ry-xz} & \frac{-x}{ry-xz} \end{bmatrix}$$

$$det\left(\begin{bmatrix} x & y \\ r & z \end{bmatrix}^{-1}\right) = det\begin{bmatrix} \frac{-z}{ry-xz} & \frac{y}{ry-xz} \\ \frac{r}{ry-xz} & \frac{-x}{ry-xz} \end{bmatrix} = \frac{xz-ry}{(ry-xz)^2} = \frac{1}{xz-ry}$$

Pochodne macierzy względem macierzy

$$L = \frac{\partial(det(X^{-1}))}{\partial X} = \frac{\partial \frac{1}{xz-ry}}{\partial X} = \begin{bmatrix} \frac{\partial \frac{1}{xz-ry}}{\partial x} & \frac{\partial \frac{1}{xz-ry}}{\partial y} \\ \frac{\partial \frac{1}{xz-ry}}{\partial r} & \frac{\partial \frac{1}{xz-ry}}{\partial z} \end{bmatrix} = \begin{bmatrix} -\frac{z}{(xz-ry)^2} & \frac{r}{(xz-ry)^2} \\ \frac{y}{(xz-ry)^2} & -\frac{x}{(xz-ry)^2} \end{bmatrix}$$

$$R = -det(X^{-1})X^{-T} = -\boxed{\frac{1}{xz-ry}} \begin{bmatrix} \frac{-z}{ry-xz} & \frac{y}{ry-xz} \\ \frac{r}{ry-xz} & \frac{-x}{ry-xz} \end{bmatrix} = \begin{bmatrix} \frac{-z}{(xz-ry)^2} & \frac{y}{(xz-ry)^2} \\ \frac{r}{(xz-ry)^2} & \frac{-x}{(xz-ry)^2} \end{bmatrix} = L$$

$det(X^{-1})$

$a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $a^T X^{-1} b$ = skalar

$b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\frac{\partial(\mathbf{a}^{\mathsf{T}}\mathbf{X}^{-1}\mathbf{b})}{\partial \mathbf{X}} = -\mathbf{X}^{-\mathsf{T}}\mathbf{a}\mathbf{b}^{\mathsf{T}}\mathbf{X}^{-\mathsf{T}} \quad X = \begin{bmatrix} x & y \\ r & z \end{bmatrix} \tag{67}$$

NALICZ

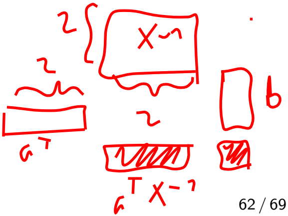
$$\frac{\partial\left(\begin{bmatrix}1\\2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}x & y\\ r & z\end{bmatrix}^{-1}\begin{bmatrix}3\\4\end{bmatrix}\right)}{\partial \mathbf{X}} = -\begin{bmatrix}x & y\\ r & z\end{bmatrix}^{-\mathsf{T}}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}x & y\\ r & z\end{bmatrix}^{-\mathsf{T}}$$

$X \mid I$

$I \mid X^{-1}$

$$\begin{bmatrix}x & y\\ r & z\end{bmatrix}^{-1} = \begin{bmatrix}\frac{-z}{ry-xz} & \frac{y}{ry-xz}\\ \frac{r}{ry-xz} & \frac{-x}{ry-xz}\end{bmatrix}$$

X^{-1}



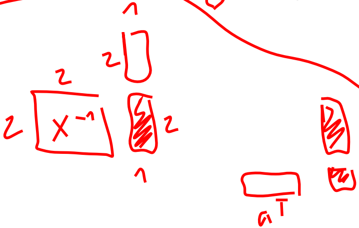
$$L = \frac{\partial \left(\overset{a}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}^T \overset{X^{-1}}{\begin{bmatrix} x & y \\ r & z \end{bmatrix}^{-1}} \overset{b}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \right)}{\partial X} = \frac{\partial \left(\begin{bmatrix} 1 & 2 \end{bmatrix} \overbrace{\begin{bmatrix} \frac{-z}{ry-xz} & \frac{y}{ry-xz} \\ \frac{r}{ry-xz} & \frac{-x}{ry-xz} \end{bmatrix}}^{X^{-1}} \overset{b}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \right)}{\partial X} =$$

$$\frac{\partial \left(\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \overbrace{\frac{-3z}{ry-xz} + \frac{4y}{ry-xz}}^{\text{WYNIK}} \\ \frac{3r}{ry-xz} - \frac{4x}{ry-xz} \end{bmatrix} \right)}{\partial X} = \frac{\partial \left(\frac{4y-3z}{ry-xz} + \frac{2(3r-4x)}{ry-xz} \right)}{\partial X} =$$

POCHODNA SKALARA WZGLĘDEM MACIERZY

$$\frac{\partial \left(\frac{4y-3z+6r-8x}{ry-xz} \right)}{\partial X} = \begin{bmatrix} \frac{\partial \left(\frac{4y-3z+6r-8x}{ry-xz} \right)}{\partial x} & \frac{\partial \left(\frac{4y-3z+6r-8x}{ry-xz} \right)}{\partial y} \\ \frac{\partial \left(\frac{4y-3z+6r-8x}{ry-xz} \right)}{\partial r} & \frac{\partial \left(\frac{4y-3z+6r-8x}{ry-xz} \right)}{\partial z} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{-3z^2+6rz+4yz-8ry}{(ry-xz)^2} & \frac{-6r^2+8rx+3rz-4xz}{(ry-xz)^2} \\ \frac{-4y^2+8xy+3yz-6xz}{(ry-xz)^2} & \frac{-8x^2+6rx+4xy-3ry}{(ry-xz)^2} \end{bmatrix}$$



WYNIK = MACIERZ

RANK 1 UPDATE

$$\begin{matrix} & \begin{bmatrix} 3 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \underbrace{\begin{bmatrix} 1\cdot 3 & 1\cdot 4 \\ 2\cdot 3 & 2\cdot 4 \end{bmatrix}} \end{matrix}$$

$$R = -X^{-T} a b^T X^{-T} = \overset{X^{-1}}{\begin{bmatrix} \frac{-z}{ry-xz} & \frac{y}{ry-xz} \\ \frac{r}{ry-xz} & \frac{-x}{ry-xz} \end{bmatrix}}^T \overset{a}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \overset{b^T}{\begin{bmatrix} 3 & 4 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} \frac{-z}{ry-xz} & \frac{y}{ry-xz} \\ \frac{r}{ry-xz} & \frac{-x}{ry-xz} \end{bmatrix}}^T =$$

$$\begin{bmatrix} \frac{-z}{ry-xz} & \frac{r}{ry-xz} \\ \frac{y}{ry-xz} & \frac{-x}{ry-xz} \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}} \begin{bmatrix} \frac{-z}{ry-xz} & \frac{r}{ry-xz} \\ \frac{y}{ry-xz} & \frac{-x}{ry-xz} \end{bmatrix} = \dots$$

Pochodne macierzy względem macierzy

$a = \begin{bmatrix}1\\2\end{bmatrix} \quad b = \begin{bmatrix}3\\4\end{bmatrix}$

$$\frac{\partial(\mathbf{a}^\mathbf{T}\mathbf{X}\mathbf{b})}{\partial\mathbf{X}} = \mathbf{a}\mathbf{b}^\mathbf{T} \quad X = \begin{bmatrix} x & y \\ r & z \end{bmatrix} \tag{68}$$

$\left\{\begin{bmatrix}x & y\\ r & z\end{bmatrix}\right.$

$[1 \;\; 2]$

$$L = \frac{\partial\left(\begin{bmatrix}1\\2\end{bmatrix}^T \begin{bmatrix} x & y \\ r & z \end{bmatrix} \begin{bmatrix}3\\4\end{bmatrix}\right)}{\partial X} = \frac{\partial\left(\begin{bmatrix}1 & 2\end{bmatrix} \begin{bmatrix} x & y \\ r & z \end{bmatrix} \begin{bmatrix}3\\4\end{bmatrix}\right)}{\partial X} =$$

„shalom"

$$\frac{\partial\left(\begin{bmatrix}1x+2r & 1y+2z\end{bmatrix} \begin{bmatrix}3\\4\end{bmatrix}\right)}{\partial X} = \frac{\partial\left(3x+6r+4y+8z\right)}{\partial X} =$$

$$\begin{bmatrix} \frac{\partial(3x+6r+4y+8z)}{\partial x} & \frac{\partial(3x+6r+4y+8z)}{\partial y} \\ \frac{\partial(3x+6r+4y+8z)}{\partial r} & \frac{\partial(3x+6r+4y+8z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

RANK 1 UPDATE

$\begin{bmatrix}3 & 4\end{bmatrix}$

$\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1\cdot 3 & 1\cdot 4\\ 2\cdot 3 & 2\cdot 4\end{bmatrix}$

$$R = ab^T = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = L$$

Pochodne macierzy względem macierzy

$a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\frac{\partial(\mathbf{a}^\mathsf{T}\mathbf{X}^\mathsf{T}\mathbf{b})}{\partial \mathbf{X}} = \boxed{\mathbf{b}\mathbf{a}^\mathsf{T}} \quad X = \begin{bmatrix} x & y \\ r & z \end{bmatrix} \tag{69}$$

$$L = \frac{\partial\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} x & y \\ r & z \end{bmatrix}^T \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)}{\partial X} = \frac{\partial\left(\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x & r \\ y & z \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)}{\partial X} =$$

WEKTOR

$$\frac{\partial\left(\begin{bmatrix} 1x + 2y & 1r + 2z \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)}{\partial X} = \frac{\partial(3x + 6y + 4r + 8z)}{\partial X} =$$

POCHODNA SKALARA WZGLĘDEM MACIERZY

$$\begin{bmatrix} \frac{\partial(3x+6y+4r+8z)}{\partial x} & \frac{\partial(3x+6y+4r+8z)}{\partial y} \\ \frac{\partial(3x+6y+4r+8z)}{\partial r} & \frac{\partial(3x+6y+4r+8z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$$

$$R = ba^T = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix} = L$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 4 \cdot 1 & 4 \cdot 2 \end{bmatrix}$$

RANK 1 UPDATE

Jakobian

TEN SŁAJD DO POPRAWY

$x, y \in \mathcal{R}^n$ to "wektory" tego samego rozmiaru, wówczas możemy policzyć Jakobian odwzorowania y

$$\frac{\partial 2}{\partial x} = 0 \qquad \frac{\partial (2x)}{\partial x} = 2$$

$$Jac(y) = det(\frac{\partial y}{\partial x}) = \left| \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \dots & \frac{\partial y_n}{\partial x_n} \end{bmatrix} \right|$$

Przykład $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2 \\ x_1^2 + 3x_2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, policzyć

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} \begin{bmatrix} y_1 & y_2 \end{bmatrix}$$

$Jac(y) = det(\frac{\partial y}{\partial x}) =$

$y_1 = x_1^2 - x_2, \quad y_2 = x_1^2 + 3x_2$

$$\left| \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} \right| = \left| \begin{bmatrix} \frac{\partial (x_1^2 - x_2)}{\partial x_1} & \frac{\partial (x_1^2 + 3x_2)}{\partial x_1} \\ \frac{\partial (x_1^2 - x_2)}{\partial x_2} & \frac{\partial (x_1^2 + 3x_2)}{\partial x_2} \end{bmatrix} \right| = \left| \begin{bmatrix} 2x_1 & 1 \\ -1 & 3 \end{bmatrix} \right| = 6x_1 + 1$$

Własności 1/2

1)$y = x^T A x$

wówczas

$\frac{\partial y}{\partial x} = Ax + A^T x$

Jeśli A jest symetryczna, czyli $A = A^T$, wówczas

$\frac{\partial y}{\partial x} = Ax + A^T x = 2Ax$

Ponadto

$\frac{\partial}{\partial x}\frac{\partial y}{\partial x} = 2A^T$

Jeśli A jest symetryczna, czyli $A = A^T$, wówczas

$\frac{\partial}{\partial x}\frac{\partial y}{\partial x} = 2A$

Własności 2/2

2) $\quad y = Ax$ wówczas $\frac{\partial y}{\partial x} = A^T$
Jeśli A jest symetryczna, czyli $A = A^T$, wówczas $\frac{\partial y}{\partial x} = A$

3) $\quad y = x^T A$ wówczas $\frac{\partial y}{\partial x} = A$

4) $\quad y = x^T x$ wówczas $\frac{\partial y}{\partial x} = 2x$

5)$x \in \mathcal{R}^n$ wektor, $y \in \mathcal{R}^r$ wektor, $z \in \mathcal{R}^m$ wektor, oraz $z = y(x)$
wówczas $\frac{\partial z}{\partial x} = \frac{\partial y}{\partial x} \frac{\partial x}{\partial y}$